Spectral and scattering theory for the J-selfadjoint operators associated with the perturbed Klein-Gordon type equations

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§ 1. Introduction

In the present paper we shall investigate the spectral properties of the generators of the Klein-Gordon type equations and then study the asymptotic behavior of the generated semigroups, i.e., scattering problems.

The perturbed Klein-Gordon equation, to which we shall apply the abstract theory to be developed in the following, is

$$\left\{\left(\frac{\partial}{\partial t}-ib_0(x)\right)^2+\left(-\sum_{j=1}^{3}\left(\frac{\partial}{\partial x_j}-ib_j(x)\right)^2+m^2+q(x)\right)\right\}\phi(x,t)=0\,, \tag{1.1}$$

where $b_0(x)$, $b_j(x)$, $\frac{\partial}{\partial x_j}b_j(x)$ and $q(x)$ are real functions which behave like $0(|x|^{-2-\varepsilon})$ at infinity. The first order equation in t, which is associated with (1.1), is in an abstract form

$$\frac{d}{dt}\begin{pmatrix} f^1(t) \\ f^2(t) \end{pmatrix}=i\begin{pmatrix} 0 & -i \\ iH & K \end{pmatrix}\begin{pmatrix} f^1(t) \\ f^2(t) \end{pmatrix}. \tag{1.2}$$

In the case of the Klein-Gordon equation, the operators H and K in the above equation are specialized as

$$H=-\sum_{j=1}^{3}\left(\frac{\partial}{\partial x_j}-ib_j(x)\right)^2+m^2+q(x)-b_0(x)^2 \text{ and } K=2b_0(x)\,,$$

and $f^1(t)$ corresponds to $\phi(x,t)$ and $f^2(t)$ to $\frac{\partial}{\partial t}\phi(x,t)$. When H is positive or K is zero, we can construct a nice Hilbert space in which the generator of the equation (1.2) is selfadjoint (see [2], [3], [5], [9], [12], [17], [19], [23], [26], [27] and [28]). In general we can not find out such a space a priori and the generator may have the non-real spectrum (see [9], [17] and [24]). Therefore, we handle the equation (1.2) in the space for the unperturbed equation

$$\frac{d}{dt}\begin{pmatrix} f_0^1 \\ f_0^2 \end{pmatrix}=i\begin{pmatrix} 0 & -i \\ iH_0 & 0 \end{pmatrix}\begin{pmatrix} f_0^1 \\ f_0^2 \end{pmatrix},\ H_0\geqq c>0\,. \tag{1.3}$$

In the case of the Klein-Gordon equation, we take H_0 as $-\Delta+m^2$. Now, we must treat a non-selfadjoint problem, while we have a nice Hermitian symmetric form (not necessarily positive definite) for which the generator of (1.2) is symmetric, i.e., J-selfadjoint[1]. Then, using the limiting absorption method, we construct the perturbed spectral measure and the invariant subspaces which reduce the equation (1.2). The above mentioned form becomes positive definite in these spaces and we can develop the scattering theory with two Hilbert spaces.

§ 2. The unperturbed equation

Let X be a Hilbert space with an inner product (f, g) and the corresponding norm $\|f\|$, $f, g \in X$, and let H_0 be a positive definite selfadjoint operator in $X: H_0 \geqq c > 0$. We denote the square root of H_0 by h_0. The Hilbert space $\mathfrak{Y}_0$ is then defined as the direct sum of spaces $\mathfrak{D}(h_0)$ and X, where $\mathfrak{D}(h_0)$ is the domain of h_0[2] which is a Hilbert space with an inner product $(h_0 f, h_0 g)$, $f, g \in \mathfrak{D}(h_0)$. We consider the following equation

$$\frac{d}{dt}\begin{pmatrix} f_0^1(t) \\ f_0^2(t) \end{pmatrix} = i\begin{pmatrix} 0 & -i \\ iH_0 & 0 \end{pmatrix}\begin{pmatrix} f_0^1(t) \\ f_0^2(t) \end{pmatrix}, \quad f_0(t) = \begin{pmatrix} f_0^1(t) \\ f_0^2(t) \end{pmatrix} \in \mathfrak{Y}_0 = \begin{pmatrix} \mathfrak{D}(h_0) \\ X \end{pmatrix}. \tag{2.1}$$

PROPOSITION 2.1. *The operator B_0 which is defined as*

$$\mathfrak{D}(B_0) = \begin{pmatrix} \mathfrak{D}(H_0) \\ \mathfrak{D}(h_0) \end{pmatrix},$$

$$B_0 f_0 = \begin{pmatrix} 0 & -i \\ iH_0 & 0 \end{pmatrix} f_0, \quad f_0 = \begin{pmatrix} f_0^1 \\ f_0^2 \end{pmatrix} \in \mathfrak{D}(B_0),$$

is selfadjoint in $\mathfrak{Y}_0$.

Using this proposition, we can integrate the equation (2.1) and obtain a unitary group $V_0(t)$, $-\infty < t < \infty$, with the generator iB_0. A simple proof of this proposition is given as follows. We transform the equation (2.1) into a "diagonal" form. Let $\mathfrak{X}_0$ be another Hilbert space which is the direct sum of two copies of X, and let T be a unitary operator from $\mathfrak{Y}_0$ to $\mathfrak{X}_0$ given as

$$T f_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} h_0 & -i \\ h_0 & i \end{pmatrix} f_0 .$$

Then the equation (2.1) is transformed into

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[1] For the notion of J-selfadjointness, see the book of Bognár [4].

[2] In this paper we denote the domain and the range of an operator A by $\mathfrak{D}(A)$ and $\mathfrak{R}(A)$ respectively.

$$\frac{d}{dt}g_0(t)=i\begin{pmatrix} h_0 & 0 \\ 0 & -h_0 \end{pmatrix}g_0(t),\ g_0(t)=Tf_0(t)\ . \tag{2.2}$$

We denote the operator $\begin{pmatrix} h_0 & 0 \\ 0 & -h_0 \end{pmatrix}=TB_0T^{-1}$ by A_0. Then $\mathfrak{D}(A_0)=T\mathfrak{D}(B_0)=\begin{pmatrix}\mathfrak{D}(h_0)\\ \mathfrak{D}(h_0)\end{pmatrix}$, and A_0 is selfadjoint in $\mathfrak{X}_0$, so that B_0 is also selfadjoint in $\mathfrak{Y}_0$. We denote the unitary group $TV_0(t)T^{-1}$ by $U_0(t)$.

§ 3. The perturbed equation and the indefinite inner products

3.1. Generation of the perturbed semigroup

We shall now investigate the perturbed equation (1.2) under the following assumption.

ASSUMPTION 3.1. (1) H is a selfadjoint operator in X with domain $\mathfrak{D}(H)=\mathfrak{D}(H_0)$ and bounded from below; (2) K is a closed symmetric operator in X with domain $\mathfrak{D}(K)\supset\mathfrak{D}(h_0)$.

We then define an operator V as

$$\mathfrak{D}(V)=\mathfrak{D}(H_0) \text{ and } Vf=Hf-H_0f,\ f\in\mathfrak{D}(V)\ .$$

Under the above assumption we obtain the following theorem.

THEOREM 3.2. *The operator B, which is defined as*

$$\mathfrak{D}(B)=\mathfrak{D}(B_0) \text{ and } Bf=\begin{pmatrix} 0 & -i \\ iH & K \end{pmatrix}f,\ f=\begin{pmatrix} f^1 \\ f^2 \end{pmatrix}\in\mathfrak{D}(B)\ ,$$

is a closed operator in $\mathfrak{Y}_0$ and generates a C^0-group $V(t)$, $-\infty<t<\infty$.

In order to prove this theorem, we need the next lemma.

LEMMA 3.3. *If H is positive definite: $H\geqq c>0$, B is selfadjoint in $\mathfrak{Y}_0$ which is equipped with the inner product $(f,g)_{\mathfrak{Y}}=(\sqrt{H}f^1,\sqrt{H}g^1)+(f^2,g^2)$ and the corresponding norm.*

PROOF. First we remark that each of the inner products $(f,g)_{\mathfrak{Y}_0}$ and $(f,g)_{\mathfrak{Y}}$ defines the same topology in $\mathfrak{Y}_0$ by the closed graph theorem. Now let $\{f_n\}\subset\mathfrak{D}(B)$ and

$$f_n\to f \text{ and } Bf_n\to g,\ \text{as } n\to\infty\ .$$

Then, since $\{f_n^2\}\subset\mathfrak{D}(K)$ and

$$\|Kf_n^2\|\leqq c(\|h_0f_n^2\|+\|f_n^2\|)\leqq c'(\|Bf_n\|_{\mathfrak{Y}_0}+\|f_n\|_{\mathfrak{Y}_0})\ ,$$

we have that $f^2\in\mathfrak{D}(K)$ and $Kf_n^2\to Kf^2=iKg^1$. On the other hand, since H is closed and

$$iHf_n^1=(iHf_n^1+Kf_n^2)-Kf_n^2 \to g^2-iKg^1 ,$$

we have that $f^1 \in \mathfrak{D}(H)$ and $iHf^1=g^2-iKg^1$. These facts imply that $f \in \mathfrak{D}(B)$ and $Bf=g$. This proves the closedness of B. The symmetric property of B is the consequence of an easy calculation and we omit the proof. Then, to conclude the proof, we have only to show that the ranges of $B \pm i$ are dense in $\mathfrak{Y}_0$. If $\mathfrak{R}(B+i)$ is not dense, there is a non-zero element g such that, for all $f \in \mathfrak{D}(B)$,

$$0=((B+i)f, g)_{\mathfrak{Y}}=i(\sqrt{H}(f^1-f^2), \sqrt{H}g^1)+(iHf^1+(K+i)f^2, g^2) .$$

We first use this equation for f with $f^1 \in \mathfrak{D}(H)$ and $f^2=0$. Then

$$(Hf^1, g^1+g^2)=0 ,$$

which implies that $g^1=-g^2$. Next we put $f^1=0$ and $f^2=g^2 \in \mathfrak{D}(h_0)$. Then we have

$$i(\sqrt{H}g^2, \sqrt{H}g^2)+(Kg^2, g^2)+i(g^2, g^2)=0 .$$

Taking the imaginary part of this equation, we have that $g^2=0$ and consequently $g^1=0$. The case of $B-i$ is treated in the same way. q.e.d.

PROOF OF THEOREM 3.2. Let $e(\lambda)$ be the spectral family associated with $H: H=\int \lambda de(\lambda)$, and let $|H|=\int |\lambda| de(\lambda)$. Then the operator B^+ which is defined as

$$\mathfrak{D}(B^+)=\mathfrak{D}(B) \text{ and } B^+f=\begin{pmatrix} 0 & -i \\ i(|H|+1) & K \end{pmatrix} f,\ f \in \mathfrak{D}(B^+) ,$$

is closed and generates a C^0-group $V^+(t)$ in $\mathfrak{Y}_0$ (by Lemma 3.3). On the other hand, B is represented as $B=B^+ + C^-$ where

$$C^-=\begin{pmatrix} 0 & 0 \\ i\left(2\int_{-\infty}^{0} \lambda de(\lambda)-1\right) & 0 \end{pmatrix},$$

which is a bounded operator since H is bounded from below. This shows that B is closed and generates a C^0-group $V(t)$ in $\mathfrak{Y}_0$ (see Kato [11]; IX §2).

By the unitary operator T in §2, the operator B is transformed into the operator A which is given in $\mathfrak{X}_0$ as

$$\mathfrak{D}(A)=\mathfrak{D}(A_0) \text{ and } Af=TBT^{-1}f=A_0 f+Gf,\ f \in \mathfrak{D}(A)$$

where

$$G=\frac{1}{2}Vh_0^{-1}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}+\frac{1}{2}K\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Then we obtain the next theorem which corresponds to Theorem 3.2.

THEOREM 3.4. *The operator A is closed in $\mathfrak{X}_0$ and generates a C^0-group $U(t)$, $-\infty<t<\infty$.*

3.2. Indefinite inner products

We shall now introduce "inner products" for which the operator B or A is "symmetric". The Hermitian symmetric forms $(f, g)_{\mathfrak{Y}}$ in $\mathfrak{Y}_0$ and $(f, g)_{\mathfrak{X}}$ in $\mathfrak{X}_0$, which we call (indefinite) inner products, are defined as follows. Let $h[f, g]$ be a Hermitian symmetric form on $X\times X$ defined as

$$\mathfrak{D}[h]=\mathfrak{D}(h_0)\times\mathfrak{D}(h_0)^{3)},$$
$$h[f^1, g^1]=(\sqrt{H+\gamma}f^1, \sqrt{H+\gamma}g^1)-\gamma(f^1, g^1),\quad f^1\times g^1\in\mathfrak{D}[h],$$

where $-\gamma$ is a lower bound of H, $H\geqq-\gamma$. Then the desired form $(f, g)_{\mathfrak{Y}}$ is defined as

$$(f, g)_{\mathfrak{Y}}=h[f^1, g^1]+(f^2, g^2),\quad f, g\in\mathfrak{Y}_0.$$

We also define the form $(f, g)_{\mathfrak{X}}$ in $\mathfrak{X}_0$ as

$$(f, g)_{\mathfrak{X}}=(T^{-1}f, T^{-1}g)_{\mathfrak{Y}}=(f, g)_{\mathfrak{X}_0}+V[f, g],\quad f, g\in\mathfrak{X}_0,$$

where

$$V[f, g]=\frac{1}{2}\{h-h_0\}[h_0^{-1}(f^1+f^2), h_0^{-1}(g^1+g^2)] \tag{3.1}$$

with $h_0[f^1, g^1]=(h_0f^1, h_0g^1)$.

PROPOSITION 3.5. *The forms $(f, g)_{\mathfrak{Y}}$ and $(f, g)_{\mathfrak{X}}$ are bounded in $\mathfrak{Y}_0$ and $\mathfrak{X}_0$ respectively.*

PROOF. These facts are the direct consequences of the fact $\mathfrak{D}(h_0)=\mathfrak{D}(\sqrt{H+\gamma})$ and the closed graph theorem. q.e.d.

Furthermore, we have the following proposition, which shows that B is symmetric with respect to the form $(f, g)_{\mathfrak{Y}}$ and so is A with respect to the form $(f, g)_{\mathfrak{X}}$.

PROPOSITION 3.6. *For $f, g\in\mathfrak{D}(B)$, we have*

$$(Bf, g)_{\mathfrak{Y}}=(f, Bg)_{\mathfrak{Y}},$$

3) We denote the domain of a sesqui-linear form $h[f, g]$ by $\mathfrak{D}[h]$.

and for $f, g \in \mathfrak{D}(A)$

$$(Af, g)_{\mathfrak{X}} = (f, Ag)_{\mathfrak{X}} .$$

PROOF. The proof for B is an easy calculation:

$$\begin{aligned}(Bf, g)_{\mathfrak{Y}} &= h[-if^2, g^1] + (iHf^1 + Kf^2, g^2) \\ &= (f^2, iHg^1) + h[f^1, -ig^2] + (f^2, Kg^2) \\ &= h[f^1, -ig^2] + (f^2, iHg^1 + Kg^2) = (f, Bg)_{\mathfrak{Y}} .\end{aligned}$$

The proof for A is the direct consequence of this fact. q.e.d.

§ 4. Structure of the perturbed operator

4.1. Discrete spectrum

First we state a basic condition which we shall always assume from now on.

ASSUMPTION 4.1. The operators V and K are compact from $\mathfrak{D}(H_0)$ to X and from $\mathfrak{D}(h_0)$ to X respectively, where $\mathfrak{D}(H_0)$ and $\mathfrak{D}(h_0)$ are equipped with the corresponding graph norms.

From this assumption, using the interpolation theorem (see Hayakawa [7]), we obtain the following lemma.

LEMMA 4.2. *The operators* $H_0^{-s} V H_0^{-(1-s)}$, $0 \leqq s \leqq 1$, *have bounded extensions* $(H_0^{-s} V H_0^{-(1-s)})^a$, $0 \leqq s \leqq 1$, *which are compact.*

PROOF. The assumption implies that VH_0^{-1} is compact in X. Since V is symmetric, taking the adjoint of VH_0^{-1}, we have that $(H_0^{-1}V)^a = (VH_0^{-1})^*$ is also compact. Hence, V has a compact extension from X to $\mathfrak{D}(H_0)^*$, the dual space of $\mathfrak{D}(H_0)$ which is identified with the completion of X with respect to the norm $\|H_0^{-1}f\|$. Therefore, by the interpolation theorem, V has a compact extension from $\mathfrak{D}(H_0^{(1-s)})$ to $\mathfrak{D}(H_0^s)^*$ which is identified with the completion of X with respect to the norm $\|H_0^{-s}f\|$. This implies that $H_0^{-s} V H_0^{-(1-s)}$ has a compact extension in X. q.e.d.

REMARK. From this lemma, putting $s = 1/2$, we have that

$$V[f, g] = \frac{1}{2}((h_0^{-1} V h_0^{-1})^a (f^1 + f^2), g^1 + g^2) .$$

Now we shall proceed to the investigation of the discrete spectrum of B. If we denote by $R(z; B)$ the resolvent of B: $R(z; B) = (B - z)^{-1}$, we have for non-real z that

$$B-z=(1+C^{-}R(z\,;\,B^{+}))(B^{+}-z)\,.$$

This implies that $R(z\,;\,B)$ exists for z with a sufficiently large imaginary part. On the other hand, $R(z\,;\,B)$ is rewritten as

$$R(z\,;\,B)=R(z\,;\,B_0)(1+WR(z\,;\,B_0))^{-1}$$

with $W=B-B_0$, and $WR(z\,;\,B_0)$, $z\in \boldsymbol{C}\setminus R$, is the analytic family of compact operators. Then, if we denote by $\sigma(B)$ the spectrum of B and $\rho(B_0)$ the resolvent set of B_0, we have the next theorem.

THEOREM 4.3. *The intersection of* $\sigma(B)$ *and* $\rho(B_0)$ *is a discrete set and* $B-z$ *is a Fredholm operator with an index* 0 *at such a point. Furthermore, the non-real points of the spectrum of* B *are finite and appear symmetrically with respect to the real axis* R *and are included in the circle* $\{z\,;\,|z|\leqq\sqrt{\gamma}\}$.

PROOF. Since $WR(z\,;\,B_0)$ is compact and analytic, $1+WR(z\,;\,B_0)$ is either nowhere invertible or invertible except at a discrete set (see Steinberg [25]). Furthermore, since $R(z\,;\,B)$ exists for z with a sufficiently large imaginary part, the second alternative is realized. The properties of $B-z$ are derived from the theory of the relatively compact perturbation (see Kato [11]). The finiteness of the non-real eigenvalues of B is established as follows (see Bognár [4]; IX, Theorem 4.6). Since H_0 is positive definite and V is H_0-compact, the negative part of H, $e(0)X$, is finite dimensional. Let $\{\lambda_j\}$, $j=1,\cdots,r$, be the non-real eigenvalues of B with positive (or negative) imaginary parts, and $\{f_j\}$, $j=1,\cdots,r$, be the corresponding eigenvectors. We define M as the finite dimensional subspace which is spanned by $\{f_j\}$, $j=1,\cdots,r$, and let Π be the operator from M to $\mathfrak{Y}_0$ which is defined as

$$\Pi f=\begin{pmatrix} e(0)f^1 \\ 0 \end{pmatrix},\ f=\begin{pmatrix} f^1 \\ f^2 \end{pmatrix}\in M\,.$$

Since $(f,f)_{\mathfrak{Y}}=0$ for $f\in M$, $\Pi f=0$ implies $f=0$. Hence, we have that $\dim M\leqq \dim e(0)X<\infty$. The symmetric location of the non-real spectrum is proved as follows. Let $Bg-zg=0$ with some non-real z and non-zero g, then

$$((B-z)g,f)_{\mathfrak{Y}}=0,\ f\in\mathfrak{D}(B)\,.$$

Hence

$$(g,(B-\bar{z})f)_{\mathfrak{Y}}=0\,.$$

If $R(B-\bar{z})$ is dense in $\mathfrak{Y}_0$, we have

$$(g,f)_{\mathfrak{Y}}=0,\ f\in\mathfrak{Y}_0\,,$$

and g takes the form $g=\begin{pmatrix} g^1 \\ 0 \end{pmatrix}$, where $g^1 \in \mathfrak{N}(H)=$the null space of H. This implies that $Bg=0$, $g\neq 0$. Accordingly $\bar{z}$ belongs to the residual spectrum of B and also belongs to the point spectrum since the index of $B-\bar{z}$ is zero. The boundedness of the absolute values of the non-real spectrum of B is proved as follows. Let z belong to the non-real spectrum of B with an eigenvector $f\neq 0$. Then

$$(Bf, f)_{\mathfrak{H}}=z(f, f)_{\mathfrak{H}} .$$

Taking the imaginary part of this equation, we have

$$(f, f)_{\mathfrak{H}}=(Hf^1, f^1)+(f^2, f^2)=0 .$$

On the other hand, as $Bf=zf$, we have $f^2=izf^1$. Therefore

$$\gamma\|f^1\|^2 \geqq -(Hf^1, f^1)=|z|^2\|f^1\|^2 .$$

Since $f^1\neq 0$, this implies that $|z|\leqq\sqrt{\gamma}$. q.e.d.

In the case of A, we have the next theorem in $\mathfrak{X}_0$.

THEOREM 4.4. *If B_0 is replaced by A_0 and B by A in Theorem 4.2, all the statements in the theorem are still valid.*

4.2. Construction of the perturbed spectral measure

In the following part of this paper we use the $\mathfrak{X}_0$-representation almost exclusively and use the notations: $R_0(z)=R(z; A_0)$, $R(z)=R(z; A)$ and $r_0(z)=R(z; h_0)$. In order to investigate the continuous spectrum, we state here another basic assumption.

ASSUMPTION 4.5. There is a bounded selfadjoint operator d in X which has the following properties: (1) d is one to one and has the dense range; (2) $\mathfrak{R}(V)$, $\mathfrak{R}(K)\subset\mathfrak{R}(d)=\mathfrak{D}(d^{-1})$; (3) the operators $dr_0(\lambda\pm i\varepsilon)d$, $dr_0(0)r_0(\lambda\pm i\varepsilon)d$, $d^{-1}Kr_0(\lambda\pm i\varepsilon)d$ and $d^{-1}Vr_0(0)r_0(\lambda\pm i\varepsilon)d$ with real λ and ε are compact, and have boundary values in the operator norm topology as $\varepsilon\downarrow 0$, where the convergence is uniform for λ which belongs to any fixed compact interval of the real axis R; (4) the operators d, $d^{-1}K$ and $d^{-1}Vr_0(0)$ are h_0-smooth in the sense that

$$\int_R \|\tilde{d}r_0(\lambda+i\varepsilon)f\|^2 d\lambda < c\|f\|^2 ,$$

where $\tilde{d}$ stands for any one of d, $d^{-1}K$ and $d^{-1}Vr_0(0)$, and c does not depend on ε.

Then we define the operator D in $\mathfrak{X}_0$ as

$$Df=\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} f, \quad f \in \mathfrak{X}_0 .$$

The following lemma is a consequence of Assumption 4.5 and the definition of D.

LEMMA 4.6. *Under Assumption 4.5, we have*: (1) $\Re(GR_0(z)) \subset \Re(D)$; (2) *the operator* $Q(z)$, *which is defined as* $Q(z)=D^{-1}GR_0(z)D$, *is bounded for non-real* z *and has a boundary value*

$$Q(\lambda \pm i0)=\lim_{\varepsilon \downarrow 0} Q(\lambda \pm i\varepsilon)$$

in the operator norm topology and $Q(\lambda \pm i0)$ *is continuous in* λ; (3) *there exists a closed set* $\Gamma \subset R$ *with Lebesgue measure zero such that* $1+Q(\lambda \pm i\varepsilon)$ *is invertible and continuous on the closure of* $\Pi_{\pm}(\Delta\,;\delta)$, *here* $\Pi_{\pm}(\Delta\,;\delta)$ *is defined for a compact set* Δ *which does not intersect with* Γ *and for sufficiently small* $\delta>0$, *as*

$$\Pi_{\pm}(\Delta\,;\delta)=\{z=\lambda \pm i\varepsilon\,;\ \lambda \in \Delta,\ \varepsilon \in (0, \delta)\} .$$

PROOF. The statements (1) and (2) are the direct consequences of Assumption 4.5. The invertibility of $1+Q(\lambda \pm i\varepsilon)$ on $\Pi_{\pm}(\Delta\,;\delta)$ is the consequence of the finiteness of the non-real spectrum of B and the resolvent equation

$$R(z)D(1+Q(z))=R_0(z)D . \tag{4.1}$$

We obtain the existence of Γ from the results of Kato-Kuroda (see [14], Lemma 4.20). q.e.d.

REMARK. The concrete expression of Γ is

$$\Gamma=\{\lambda\,;\ \lambda \in R,\ 1+Q(\lambda+i0) \text{ or } 1+Q(\lambda-i0) \text{ is not invertible}\} .$$

After these preparatory works we can define the sesqui-linear form $\mathfrak{E}(\Delta\,;\varepsilon)[f, g]$ on $\mathfrak{X}_0 \times \mathfrak{X}_0$ for a compact Δ with $\Delta \cap \Gamma=\emptyset$ as

$$\begin{aligned}
\mathfrak{E}(\Delta\,;\varepsilon)[f, g] &= \frac{1}{2\pi i}\int_{\Delta} (\{R(\lambda+i\varepsilon)-R(\lambda-i\varepsilon)\}f, g)_{\mathfrak{X}_0} d\lambda \\
&= \frac{1}{2\pi i}\int_{\Delta} (\{R_0(\lambda+i\varepsilon)-R_0(\lambda-i\varepsilon)\}f, g)_{\mathfrak{X}_0} d\lambda \\
&\quad - \frac{1}{2\pi i}\int_{\Delta} (R(\lambda+i\varepsilon)GR_0(\lambda+i\varepsilon)f, g)_{\mathfrak{X}_0} d\lambda \\
&\quad + \frac{1}{2\pi i}\int_{\Delta} (R(\lambda-i\varepsilon)GR_0(\lambda-i\varepsilon)f, g)_{\mathfrak{X}_0} d\lambda .
\end{aligned}$$

The first term of these integrals is bounded and has a limit as $\varepsilon \downarrow 0$, which equals to $(E_0(\Delta)f, g)_{\mathfrak{X}_0}$ with $E_0(\Delta)$ a spectral measure of the selfadjoint operator A_0. We remark here that A_0 is absolutely continuous by Assumption 4.5, (3). The second and the third integrals are estimated in the following manner by the smoothness condition (see Kato [10] and Mochizuki [20], [21], [22]). From the resolvent equation (4.1), the integrand in the second integral is written as

$$\begin{aligned}&(R(\lambda+i\varepsilon)GR_0(\lambda+i\varepsilon)f, g)_{\mathfrak{X}_0}\\&\quad=((1+Q(\lambda+i\varepsilon))^{-1}D^{-1}GR_0(\lambda+i\varepsilon)f, DR_0(\lambda-i\varepsilon)g)_{\mathfrak{X}_0}\,.\end{aligned}$$

Hence, the integral is estimated as

$$\begin{aligned}(4.2)\qquad &\left|\frac{1}{2\pi i}\int_{\Delta}(R(\lambda+i\varepsilon)GR_0(\lambda+i\varepsilon)f, g)_{\mathfrak{X}_0}d\lambda\right|\\&=\frac{1}{2\pi i}\left|\int_{\Delta}((1+Q(\lambda+i\varepsilon))^{-1}D^{-1}GR_0(\lambda+i\varepsilon)f, DR_0(\lambda-i\varepsilon)g)_{\mathfrak{X}_0}d\lambda\right|\\&\leqq c\|f\|_{\mathfrak{X}_0}\|g\|_{\mathfrak{X}_0}\,.\end{aligned}$$

Here we used the concrete expression of $D^{-1}GR_0(z)$;

$$\begin{aligned}D^{-1}GR_0(z)=&\begin{pmatrix}d^{-1} & 0\\ 0 & d^{-1}\end{pmatrix}\left\{\frac{1}{2}Vr_0(0)\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\right.\\&\left.+\frac{1}{2}K\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\right\}\begin{pmatrix}r_0(z) & 0\\ 0 & -r_0(-z)\end{pmatrix},\end{aligned}$$

the Schwarz inequality and the smoothness condition. Summarizing these facts, we obtain the following theorem.

THEOREM 4.7. *The sesqui-linear forms* $\{\mathfrak{E}(\Delta;\varepsilon)[f, g]\,;\,\varepsilon\in(0,\delta)\}$ *are uniformly bounded in* ε *for a fixed compact* Δ *which does not intersect with* Γ *and have a limit* $\mathfrak{E}(\Delta)[f, g]$ *when* $\varepsilon\downarrow 0$, *which is also bounded.*

PROOF. The existence of the limit is proved for $f, g\in\mathfrak{R}(D)$ by Assumption 4.5 together with Proposition 4.6. In the general case we use the estimate (4.2) and obtain the desired result. q.e.d.

REMARK. The compactness of Δ is assumed in Theorem 4.7. But if $(1+Q(\lambda\pm i0))^{-1}$ are uniformly bounded in Δ, we can drop this condition.

This theorem shows that $\mathfrak{E}(\Delta)[f, g]$ defines a bounded operator $E(\Delta)$ which is given as

$$(\mathfrak{E}(\Delta)f, g)_{\mathfrak{X}_0}=E(\Delta)[f, g]\,.$$

We call this family of operators $E(\Delta)$ the perturbed spectral measure and shall investigate its properties in the next subsection.

4.3. Properties of the perturbed spectral measure

We shall in this subsection show that the family of operators $E(\Delta)$ has most of the properties of a usual spectral measure. Namely we can prove the next theorem. We owe the proof of (5) of the theorem to Lax-Phillips [17].

THEOREM 4.8. *Let Δ and Δ_j $(j=1,2)$ be bounded open sets whose closures do not intersect with Γ in Proposition 4.6. Then we have*: (1) $E(\Delta_1)E(\Delta_2)=E(\Delta_1\cap\Delta_2)$; (2) $(E(\Delta)f,g)_{\mathfrak{X}}=(f,E(\Delta)g)_{\mathfrak{X}}$; (3) $(E(\Delta)f,f)_{\mathfrak{X}}\geqq 0$, *and if* $\Delta\not\ni 0$, $(E(\Delta)f,f)_{\mathfrak{X}}=0\Leftrightarrow E(\Delta)f=0$; (4) *if Δ contains a point of the spectrum of A_0, $\Re(E(\Delta))$ is non-trivial*; (5) *if* $\Delta\not\ni 0$,

$$c_1\|E(\Delta)f\|_{\mathfrak{X}}\leqq\|E(\Delta)f\|_{\mathfrak{X}_0}\leqq c_2\|E(\Delta)f\|_{\mathfrak{X}},\ c_1,c_2>0\,.$$

PROOF OF (1). We need only to show the equality in the weak form for the elements in $\Re(D)$. Now, we have that

$$\begin{aligned}(E(\Delta_1)&E(\Delta_2)Df,Dg)_{\mathfrak{X}_0}\\&=\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\Delta_1}(\{R(\lambda+i\varepsilon)-R(\lambda-i\varepsilon)\}E(\Delta_2)Df,Dg)_{\mathfrak{X}_0}d\lambda\\&=\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\Delta_1}\left\{\lim_{\tau\downarrow 0}\frac{1}{2\pi i}\int_{\Delta_2}(\{R(\lambda+i\varepsilon)-R(\lambda-i\varepsilon)\}\right.\\&\qquad\left.\times\{R(\eta+i\tau)-R(\eta-i\tau)\}Df,Dg)_{\mathfrak{X}_0}d\eta\right\}d\lambda\,.\end{aligned}$$

Then, using the resolvent equations for $R(z)$ and the fact that $DR(\eta\pm i\tau)D$ have boundary values, we have

$$\begin{aligned}(E(\Delta_1)&E(\Delta_2)Df,Dg)_{\mathfrak{X}_0}\\&=\lim_{\varepsilon\downarrow 0}\int_{\Delta_1}d\lambda\int_{\Delta_2}d\eta\left[\frac{\varepsilon}{\pi\{(\lambda-\eta)^2+\varepsilon^2\}}\frac{1}{2\pi i}(D\{R(\eta+i0)-R(\eta-i0)\}Df,g)_{\mathfrak{X}_0}\right]\\&=\frac{1}{2\pi i}\int_{\Delta_1\cap\Delta_2}(D\{R(\eta+i0)-R(\eta-i0)\}Df,g)_{\mathfrak{X}_0}d\eta=(E(\Delta_1\cap\Delta_2)Df,Dg)_{\mathfrak{X}_0}\,,\end{aligned}$$

here we used the notations $DR(\eta\pm i0)D$ to denote the limits of $DR(\eta\pm i\tau)D$ when τ tends to zero, which do exist by Theorem 4.7.

PROOF OF (2). Since the form $(f,g)_{\mathfrak{X}}$ is bounded, it has the representation

$$(f,g)_{\mathfrak{X}}=(f,Pg)_{\mathfrak{X}_0}=(Pf,g)_{\mathfrak{X}_0}$$

with some bounded selfadjoint operator P. Thus we have

$$
\begin{aligned}
(E(\varDelta)f, g)_{\mathfrak{X}} &= (E(\varDelta)f, Pg)_{\mathfrak{X}_0} \\
&= \lim_{\varepsilon\downarrow 0} \frac{1}{2\pi i} \int_{\varDelta} (\{R(\lambda+i\varepsilon)-R(\lambda-i\varepsilon)\}f, Pg)_{\mathfrak{X}_0} d\lambda \\
&= \lim_{\varepsilon\downarrow 0} \frac{1}{2\pi i} \int_{\varDelta} (\{R(\lambda+i\varepsilon)-R(\lambda-i\varepsilon)\}f, g)_{\mathfrak{X}} d\lambda \\
&= \lim_{\varepsilon\downarrow 0} \frac{1}{2\pi i} \int_{\varDelta} (f, \{R(\lambda-i\varepsilon)-R(\lambda+i\varepsilon)\}g)_{\mathfrak{X}} d\lambda \quad \text{(by Proposition 3.6)} \\
&= \lim_{\varepsilon\downarrow 0} \frac{1}{2\pi i} \int_{\varDelta} (Pf, \{R(\lambda-i\varepsilon)-R(\lambda+i\varepsilon)\}g)_{\mathfrak{X}_0} d\lambda = (Pf, E(\varDelta)g)_{\mathfrak{X}_0} = (f, E(\varDelta)g)_{\mathfrak{X}} .
\end{aligned}
$$

PROOF OF (3). To prove the nonnegativity, we first observe that

$$
\begin{aligned}
(4.3)\qquad & (E(\varDelta)Df, Df)_{\mathfrak{X}} \\
&= \lim_{\varepsilon\downarrow 0} \frac{1}{2\pi i} \int_{\varDelta} (\{R(\lambda+i\varepsilon)-R(\lambda-i\varepsilon)\}Df, Df)_{\mathfrak{X}} d\lambda \\
&= \lim_{\varepsilon\downarrow 0} \frac{\varepsilon}{\pi} \int_{\varDelta} (R(\lambda\pm i\varepsilon)Df, R(\lambda\pm i\varepsilon)Df)_{\mathfrak{X}} d\lambda \\
&= \lim_{\varepsilon\downarrow 0} \left\{ \frac{\varepsilon}{\pi} \int_{\varDelta} \|R(\lambda\pm i\varepsilon)Df\|^2_{\mathfrak{X}_0} d\lambda + \frac{\varepsilon}{\pi} \int_{\varDelta} V[R(\lambda\pm i\varepsilon)Df, R(\lambda\pm i\varepsilon)Df] d\lambda \right\} .
\end{aligned}
$$

Now, using the concrete expression of $V[f, g]$, the formula (3.1) and the fact that $R(\lambda+i\varepsilon)Df = R_0(\lambda+i\varepsilon)D(1+Q(\lambda+i\varepsilon))^{-1}f$, we have that

$$
\begin{aligned}
V[R(\lambda+i\varepsilon)Df, R(\lambda+i\varepsilon)Df] = \frac{1}{2}(&d^{-1}Vr_0(0)\{r_0(\lambda+i\varepsilon)df(z)^1 - r_0(-\lambda-i\varepsilon)df(z)^2\}, \\
&dr_0(0)\{r_0(\lambda+i\varepsilon)df(z)^1 - r_0(-\lambda-i\varepsilon)df(z)^2\}) ,
\end{aligned}
$$

with $f(z) = (1+Q(\lambda+i\varepsilon))^{-1}f = \begin{pmatrix} f(z)^1 \\ f(z)^2 \end{pmatrix}$

which is uniformly bounded in ε by Assumption 4.5 together with Proposition 4.6. Therefore, the second term of (4.3) tends to zero as $\varepsilon\downarrow 0$. This proves the nonnegativity. Furthermore, if $(E(\varDelta)f, f)_{\mathfrak{X}}=0$, then $(E(\varDelta)f, g)_{\mathfrak{X}} = (E(\varDelta)f, E(\varDelta)g)_{\mathfrak{X}} = 0$ for all $g \in \mathfrak{X}_0$ by (1) and (2) and the Schwarz inequality. This implies that

$$
T^{-1}E(\varDelta)f = \begin{pmatrix} k \\ 0 \end{pmatrix}, \ k \in \mathfrak{N}(H) .
$$

Thus $AE(\varDelta)f=0$ and

$$
\begin{aligned}
(E(\varDelta)f, g)_{\mathfrak{X}_0} &= \lim_{\varepsilon\downarrow 0} \frac{1}{2\pi i} \int_{\varDelta} (\{R(\lambda+i\varepsilon)-R(\lambda-i\varepsilon)\}E(\varDelta)f, g)_{\mathfrak{X}_0} d\lambda \\
&= \lim_{\varepsilon\downarrow 0} \frac{1}{2\pi i} \int_{\varDelta} \left\{ \frac{1}{\lambda-i\varepsilon} - \frac{1}{\lambda+i\varepsilon} \right\} (E(\varDelta)f, g)_{\mathfrak{X}_0} d\lambda = 0 .
\end{aligned}
$$

This shows that $E(\varDelta)f=0$, which concludes the proof.

PROOF OF (4). Since $(E(\Delta)Df, Df)_{\mathfrak{X}}$ is expressed as

$$(E(\Delta)Df, Df)_{\mathfrak{X}}=\lim_{\varepsilon\downarrow 0}\frac{\varepsilon}{\pi}\int_{\Delta}\|R(\lambda\pm i\varepsilon)Df\|^2_{\mathfrak{X}_0}d\lambda\ ,$$

we have, using the resolvent equation,

$$(E(\Delta)Df, Df)_{\mathfrak{X}}$$
$$=\frac{1}{2\pi i}\int_{\Delta}(D\{R_0(\lambda+i0)-R_0(\lambda-i0)\}D(1+Q(\lambda+i0))^{-1}f, (1+Q(\lambda+i0))^{-1}f)_{\mathfrak{X}_0}d\lambda\ .$$

Now, let λ belong to the spectrum of A_0, then the integrand of the above equation does not vanish identically in some neighbourhood of λ for some element f. This implies that $E(\Delta)Df\neq 0$, because the form $(f, g)_{\mathfrak{X}}$ does not degenerate on $\mathfrak{R}(E(\Delta))$ by (3) and (1).

PROOF OF (5). We shall prove this fact by contradiction. Let $\{f_n\}$ be a sequence of elements in $\mathfrak{R}(E(\Delta))$ such that

$$\|f_n\|_{\mathfrak{X}_0}=1 \text{ and } \|f_n\|_{\mathfrak{X}}\to 0 \text{ as } n\to\infty\ .$$

Then, since $\mathfrak{R}(E(\Delta))$ is closed, we can assume without loss of generality that $\{f_n\}$ converges weakly to some element $f\in\mathfrak{R}(E(\Delta))$. On the other hand, by the assumption we have

$$\|f_n\|^2_{\mathfrak{X}}=\|f_n\|^2_{\mathfrak{X}_0}+\frac{1}{2}((h_0^{-1}Vh_0^{-1})^a(f_n^1+f_n^2), (f_n^1+f_n^2))\to 0,\ \ n\to\infty\ .$$

Thus, since $(h_0^{-1}Vh_0^{-1})^a$ is compact by Lemma 4.2, the second term of the above formula converges to $V[f, f]$. Therefore, since $\|f_n\|^2_{\mathfrak{X}_0}=1$, we have that $V[f, f]=-1$. Then, using the well-known fact that $\|f\|^2_{\mathfrak{X}_0}\leqq\varliminf_{n\to 0}\|f_n\|^2_{\mathfrak{X}_0}=1$, we obtain

$$\|f\|^2_{\mathfrak{X}}=\|f\|^2_{\mathfrak{X}_0}+V[f, f]\leqq 0\ .$$

This implies that $\|f\|_{\mathfrak{X}}=0$ and consequently that $f=0$, which contradicts the fact that $V[f, f]=-1$. q.e.d.

Next, we shall examine the relation between the inner product $(f, g)_{\mathfrak{X}}$ and A on $\mathfrak{R}(E(\Delta))$.

THEOREM 4.9. *Let Δ satisfy the same condition as in Theorem 4.7, then we have*: (1) $E(\Delta)A\subset AE(\Delta)$; (2) *$A$ is bounded on $\mathfrak{R}(E(\Delta))$.*

PROOF. Let $f\in\mathfrak{D}(A)$ and $g\in\mathfrak{D}(A^*)$, where A^* is the adjoint of A in $\mathfrak{X}_0$. Then we have the following equality

$$(E(\varDelta)Af, g)_{\mathfrak{X}_0}=\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\varDelta}(\{R(\lambda+i\varepsilon)-R(\lambda-i\varepsilon)\}Af, g)_{\mathfrak{X}_0}d\lambda$$
$$=(E(\varDelta)f, A^*g)_{\mathfrak{X}_0}\,.$$

This implies that $E(\varDelta)f\in\mathfrak{D}(A)$ and also that $AE(\varDelta)f=E(\varDelta)Af$. Now, since $R(z)Af=AR(z)f=(1+zR(z))f$ for $f\in\mathfrak{D}(A)$, we have for $f\in\mathfrak{D}(A)$, that

$$(AE(\varDelta)f, g)_{\mathfrak{X}_0}=\lim_{\varepsilon\downarrow 0}\left\{\frac{1}{2\pi i}\int_{\varDelta}\lambda(\{R(\lambda+i\varepsilon)-R(\lambda-i\varepsilon)\}f, g)_{\mathfrak{X}_0}d\lambda\right.$$
$$\left.+\frac{\varepsilon}{2\pi}\int_{\varDelta}(\{R(\lambda+i\varepsilon)+R(\lambda-i\varepsilon)\}f, g)_{\mathfrak{X}_0}d\lambda\right\}$$
$$=\lim_{\varepsilon\downarrow 0}\left\{\frac{1}{2\pi i}\int_{\varDelta}\lambda(\{R_0(\lambda+i\varepsilon)-R_0(\lambda-i\varepsilon)\}f, g)_{\mathfrak{X}_0}d\lambda\right.$$
$$\left.+\frac{\varepsilon}{2\pi}\int_{\varDelta}(\{R_0(\lambda+i\varepsilon)+R_0(\lambda-i\varepsilon)\}f, g)_{\mathfrak{X}_0}d\lambda+\text{the remainder}\right\}$$
$$=(A_0E_0(\varDelta)f, g)_{\mathfrak{X}_0}+\text{the remainder}\,.$$

The remainder term is estimated in the same way as in the proof of Theorem 4.7 together with the fact that $\varepsilon\|R_0(\lambda+i\varepsilon)\|_{\mathfrak{X}_0}\leqq 1$, and we obtain the estimate

$$|(AE(\varDelta)f, g)_{\mathfrak{X}_0}|\leqq c\|f\|_{\mathfrak{X}_0}\cdot\|g\|_{\mathfrak{X}_0}\,.$$

This shows that A is bounded on $\mathfrak{R}(E(\varDelta))$ since A is densely defined and closed. q.e.d.

Now, we define the Hilbert space $\mathfrak{X}(\varDelta)$ as $\mathfrak{X}(\varDelta)=\mathfrak{R}(E(\varDelta))$ with an inner product $(f, g)_{\mathfrak{X}}$ and the operator $A|_{\mathfrak{X}(\varDelta)}$ as the restriction of A onto $\mathfrak{X}(\varDelta)$. Then we can summarize the essential part of the foregoing results as follows.

THEOREM 4.10. *Let $\varDelta$ satisfy the same condition as in Theorem 4.7 and $0\notin\varDelta$, then the operator $A|_{\mathfrak{X}(\varDelta)}$ is bounded and selfadjoint. The family of operators $\{E(\Lambda)\,;\,\Lambda\subset\varDelta\}$ is a spectral measure of $A|_{\mathfrak{X}(\varDelta)}$.*

PROOF. We have only to prove the last statement. The equality

$$(E(\varDelta)f, g)_{\mathfrak{X}}=\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\varDelta}(\{R(\lambda+i\varepsilon)-R(\lambda-i\varepsilon)\}f, g)_{\mathfrak{X}}d\lambda$$

and the fact that $R(\lambda+i\varepsilon)E(\varDelta)f=(A|_{\mathfrak{X}(\varDelta)}-(\lambda+i\varepsilon))^{-1}E(\varDelta)f$ imply the desired result. q.e.d.

§5. Unitary equivalence and scattering theory

5.1. Construction of wave operators

We denote by $\mathfrak{X}_0(\varDelta)$ the Hilbert space $E_0(\varDelta)\mathfrak{X}_0$. Then we shall establish the unitary equivalence between the operators $A_0|_{\mathfrak{X}_0(\varDelta)}$, the restriction of A_0 onto $\mathfrak{X}_0(\varDelta)$, and $A|_{\mathfrak{X}(\varDelta)}$. We prove this fact using the abstract stationary method of the scattering theory (see Kato-Kuroda [13], [14]). We first prepare a representation space $\mathfrak{L}^2(\varDelta\,;\mathfrak{X}_0)$ in which $A_0|_{\mathfrak{X}_0(\varDelta)}$ is diagonal. Namely, we introduce the following space $\mathfrak{C}(\varDelta\,;\mathfrak{X}_0)$ for a bounded $\varDelta$:

$$\mathfrak{C}(\varDelta\,;\mathfrak{X}_0)=\{f(\lambda)\,;\,\lambda\in\varDelta,\ f(\lambda)=\mathfrak{M}_0(\lambda)h(\lambda)\}\,,$$

where $\mathfrak{M}_0(\lambda)=\sqrt{\dfrac{1}{2\pi i}D\{R_0(\lambda+i0)-R_0(\lambda-i0)\}D}$ and $h(\lambda)$ is any strongly continuous function of λ with its values in $\mathfrak{X}_0$. Then $\mathfrak{C}(\varDelta\,;\mathfrak{X}_0)$ is a pre-Hilbert space with an inner product

$$(f(\lambda),g(\lambda))_{\mathfrak{L}^2(\varDelta;\mathfrak{X}_0)}=\int_\varDelta (f(\lambda),g(\lambda))_{\mathfrak{X}_0}d\lambda\,.$$

We complete this space and get a Hilbert space $\mathfrak{L}^2(\varDelta\,;\mathfrak{X}_0)$. Now we have the following proposition.

PROPOSITION 5.1. *There exists a unitary operator $J_0(\varDelta)$ from $\mathfrak{X}_0(\varDelta)$ to $\mathfrak{L}^2(\varDelta\,;\mathfrak{X}_0)$ such that*: (1) $J_0(\varDelta)E_0(\varDelta)Df=\chi_\varDelta(\lambda)\mathfrak{M}_0(\lambda)f$, *where $\chi_\varDelta(\lambda)$ is a characteristic function of $\varDelta$*; (2) $J_0(\varDelta)A_0|_{\mathfrak{X}_0(\varDelta)}J_0(\varDelta)^*=\lambda$, *the multiplication operator by λ.*

This proposition is an easy consequence of the spectral representation of the selfadjoint operator $A_0|_{\mathfrak{X}_0(\varDelta)}$, and we omit the proof. Then we shall proceed to the investigation of the perturbed operator $A|_{\mathfrak{X}(\varDelta)}$. Since we have the formula:

$$\begin{aligned}\|E(\varDelta)Df\|_{\mathfrak{X}}^2&=\lim_{\varepsilon\downarrow 0}\frac{\varepsilon}{\pi}\int_\varDelta \|R_0(\lambda+i\varepsilon)D(1+Q(\lambda\pm i\varepsilon))^{-1}f\|_{\mathfrak{X}_0}^2 d\lambda\\&=\int_\varDelta \|\mathfrak{M}_0(\lambda)(1+Q(\lambda\pm i0))^{-1}f\|_{\mathfrak{X}_0}^2 d\lambda\,,\end{aligned}$$

we can define the unitary operator $J_\pm(\varDelta)$ from $\mathfrak{X}(\varDelta)$ to $\mathfrak{L}^2(\varDelta\,;\mathfrak{X}_0)$ as

$$J_\pm(\varDelta)E(\varDelta)Df=\chi_\varDelta(\lambda)\mathfrak{M}_0(\lambda)(1+Q(\lambda\pm i0))^{-1}f\,.$$

Here the fact that $J_\pm(\varDelta)$ are onto is derived from the facts that $1+Q(\lambda\pm i\lambda)$, which depend continuously on λ, $\lambda\in\varDelta$, are one-to-one and onto, and that $J_0(\varDelta)$ is unitary. Now, we have the next theorem.

THEOREM 5.2. *The operators $J_\pm(\varDelta)$ give two diagonal representations of $A|_{\mathfrak{X}(\varDelta)}$ in the sense that*

$$J_\pm(\varDelta)A|_{\mathfrak{X}(\varDelta)}J_\pm(\varDelta)^*=\lambda\,.$$

The wave operators $W_\pm(\varDelta)$, which are defined as

$$W_{\pm}(\Delta)=J_{\pm}(\Delta)^{*}J_{0}(\Delta) ,$$

are unitary from $\mathfrak{X}_0(\Delta)$ *to* $\mathfrak{X}(\Delta)$ *and have the intertwining properties*:

$$W_{\pm}(\Delta)A_{0}|_{\mathfrak{X}_{0}(\Delta)}=A|_{\mathfrak{X}(\Delta)}W_{\pm}(\Delta) .$$

PROOF. The proof of the first part of this theorem is the same as that for $J_0(\Delta)$, if we remember the proof of Theorem 4.8, (3) and Theorem 4.10. The unitarity of $W_{\pm}(\Delta)$ is a direct consequence of the properties of $J_0(\Delta)$ and $J_{\pm}(\Delta)$. Furthermore, we have

$$\begin{aligned} &W_{\pm}(\Delta)A_{0}|_{\mathfrak{X}_{0}(\Delta)}E_{0}(\Delta)Df \\ &\quad =J_{\pm}(\Delta)^{*}J_{0}(\Delta)A_{0}J_{0}(\Delta)^{*}J_{0}(\Delta)E_{0}(\Delta)Df \\ &\quad =J_{\pm}(\Delta)^{*}\lambda J_{0}(\Delta)E_{0}(\Delta)Df=J_{\pm}(\Delta)^{*}J_{\pm}(\Delta)AJ_{\pm}(\Delta)^{*}J_{0}(\Delta)E_{0}(\Delta)Df \\ &\quad =A|_{\mathfrak{X}(\Delta)}W_{\pm}(\Delta)E_{0}(\Delta)Df . \end{aligned}$$

This concludes the proof. q.e.d.

5.2. Time-dependent formulation

We shall give in this subsection another expression of $W_{\pm}(\Delta)$, which represents the asymptotic behavior of $U(t)|_{\mathfrak{X}(\Delta)}$. Namely, we have the next theorem.

THEOREM 5.3. *The wave operators* $W_{\pm}(\Delta)$ *have the expression*:

$$W_{\pm}(\Delta)f=\lim_{t\to\pm\infty}U(-t)E(\Delta)U_{0}(t)f,\ f\in\mathfrak{X}_{0}(\Delta) ,$$

where the limits are taken with respect to $\|f\|_{\mathfrak{X}}$ *or* $\|f\|_{\mathfrak{X}_0}$. *Furthermore, we have the following asymptotic behavior for* $U(t)$:

$$\lim_{t\to\pm\infty}\|U(t)W_{\pm}(\Delta)f-U_{0}(t)f\|_{\mathfrak{X}_{0}}=0 ,\quad f\in\mathfrak{X}_{0}(\Delta) .$$

PROOF. Let us define the following function

$$Z_{\pm}(t\,;f)=\|\{W_{\pm}(\Delta)-U(-t)E(\Delta)U_{0}(t)\}E_{0}(\Delta)f\|_{\mathfrak{X}}^{2} .$$

Since $U(t)$ is unitary, using the intertwining properties of $W_{\pm}(\Delta)$, we have

$$Z_{\pm}(t\,;f)=\|\{W_{\pm}(\Delta)-E(\Delta)\}U_{0}(t)E_{0}(\Delta)f\|_{\mathfrak{X}}^{2} .$$

Now we fix t and approximate $U_0(t)E_0(\Delta)f$ by a sequence of elements $\{Df_j(t)\}$, then

$$\begin{aligned}
Z_{\pm}(t\,;f) &= \|\{J_0(\Delta)-J_{\pm}(\Delta)E(\Delta)\}U_0(t)E_0(\Delta)f\|^2_{\mathfrak{L}^2(\Delta;\mathfrak{X}_0)} \\
&= \lim_{j\to\infty} \|\{J_0(\Delta)-J_{\pm}(\Delta)E(\Delta)\}Df_j(t)\|^2_{\mathfrak{L}^2(\Delta;\mathfrak{X}_0)} \\
&= \lim_{j\to\infty}\lim_{\varepsilon\downarrow 0} \|\mathfrak{M}_0(\lambda)\{1-(1+Q(\lambda\pm i\varepsilon))^{-1}\}f_j(t)\|^2_{\mathfrak{L}^2(\Delta;\mathfrak{X}_0)} \\
&= \lim_{j\to\infty}\lim_{\varepsilon\downarrow 0} \|\mathfrak{M}_0(\lambda)(1+Q(\lambda\pm i\varepsilon))^{-1}D^{-1}GR_0(\lambda\pm i\varepsilon)Df_j(t)\|^2_{\mathfrak{L}^2(\Delta;\mathfrak{X}_0)}\,,
\end{aligned}$$

and we have the fact that $\mathfrak{M}_0(\lambda)(1+Q(\lambda\pm i\varepsilon))^{-1}$ is uniformly bounded in $\lambda\in\Delta$ and ε. Therefore, expressing the resolvent $R_0(\lambda\pm i\varepsilon)$ by $U_0(t)$ we obtain (see Kato [10] Lemma 3.5) that

$$\begin{aligned}
Z_{\pm}(t\,;f) &\leqq c\lim_{j\to\infty}\lim_{\varepsilon\downarrow 0}\int_{\Delta}\|D^{-1}GR_0(\lambda\pm i\varepsilon)Df_j(t)\|^2_{\mathfrak{X}_0}d\lambda \\
&\leqq c\lim_{j\to\infty}\lim_{\varepsilon\downarrow 0}\int_0^{\pm\infty}\pm e^{\mp\varepsilon s}\|D^{-1}GU_0(s)Df_j(t)\|^2_{\mathfrak{X}_0}ds \\
&= c\lim_{j\to\infty}\int_0^{\pm\infty}\pm\|D^{-1}GU_0(s)Df_j(t)\|^2_{\mathfrak{X}_0}ds\,.
\end{aligned}$$

Here, we remark that $U_0(s)Df_j(t)\in\mathfrak{D}(D^{-1}G)$ for almost every s by the smoothness of $D^{-1}G$ with respect to A_0 (see Kato [10] Remark 3.8). On the other hand, the last term equals to

$$\begin{aligned}
&c\int_0^{\pm\infty}\pm\|D^{-1}GU_0(s+t)E_0(\Delta)f\|^2_{\mathfrak{X}_0}ds \\
&\qquad = c\int_t^{\pm\infty}\pm\|D^{-1}GU_0(s)E_0(\Delta)f\|^2_{\mathfrak{X}_0}ds\,,
\end{aligned}$$

which is finite by the smoothness condition, so that it tends to zero as $t\to\pm\infty$. This concludes the first part of the theorem. Since, for $f\in\mathfrak{X}_0(\Delta)$,

$$\begin{aligned}
&\|U(t)W_{\pm}(\Delta)f-U_0(t)f\|_{\mathfrak{X}_0} \\
&\qquad\leqq c\|U(t)W_{\pm}(\Delta)f-E(\Delta)U_0(t)f\|_{\mathfrak{X}}+\|(1-E(\Delta))U_0(t)f\|_{\mathfrak{X}_0}
\end{aligned}$$

by Theorem 4.8, (5), we can prove the second part by showing that

$$\lim_{t\to\pm\infty}\|(1-E(\Delta))U_0(t)E_0(\Delta)f\|_{\mathfrak{X}_0}=0,\ f\in\mathfrak{X}_0\,.$$

This is shown by the following calculation :

$$\begin{aligned}
&\lim_{t\to\pm\infty}\|(1-E(\Delta))U_0(t)E_0(\Delta)f\|^2_{\mathfrak{X}_0} \\
&\qquad=\lim_{t\to\pm\infty}\|(1-E(\Delta))U_0(t)E_0(\Delta)f\|^2_{\mathfrak{X}} \\
&\qquad=\lim_{t\to\pm\infty}\|U_0(t)E_0(\Delta)f\|^2_{\mathfrak{X}}-\lim_{t\to\pm\infty}\|E(\Delta)U_0(t)E_0(\Delta)f\|^2_{\mathfrak{X}} \\
&\qquad=\|E_0(\Delta)f\|^2_{\mathfrak{X}_0}-\lim_{t\to\pm\infty}\|U(-t)E(\Delta)U_0(t)E_0(\Delta)f\|^2_{\mathfrak{X}} \\
&\qquad=\|E_0(\Delta)f\|^2_{\mathfrak{X}_0}-\|W_{\pm}(\Delta)E_0(\Delta)f\|^2_{\mathfrak{X}}=0
\end{aligned}$$

where we used the abbreviation $(f,f)_{\mathfrak{X}}=\|f\|_{\mathfrak{X}}^2$ and the facts that (1) $U_0(t)E_0(\Delta)f$ tends weakly to zero; (2) $\|f\|_{\mathfrak{X}}^2=\|f\|_{\mathfrak{X}_0}^2+V[f,f]$ and $V[f,f]=\frac{1}{2}((h_0^{-1}Vh_0^{-1})^a f,f)$, where $(h_0^{-1}Vh_0^{-1})^a$ is compact, so that $V[f(t),f(t)]$ tends to zero when $f(t)$ $(=U_0(t)E_0(\Delta)f)$ tends weakly to zero. q.e.d.

§ 6. Applications to the Klein-Gordon equations

6.1. Applications

We shall now apply the preceding results to the Klein-Gordon equation (1.1). We consider the equation in $X=L^2(R^3)$ and define the operators H_0, H, V and K as follows:

$$\mathfrak{D}(H_0)=\mathfrak{D}(H)=\mathfrak{D}(V)=H^2(R^3);$$

$$H_0f=-\sum_{j=1}^{3}\frac{\partial^2}{\partial x_j^2}f(x)+m^2f(x),\quad m>0;$$

$$Hf=-\sum_{j=1}^{3}\left(\frac{\partial}{\partial x_j}-ib_j(x)\right)^2f(x)+m^2f(x)+(q(x)-b_0(x)^2)f(x);$$

$$Vf=Hf-H_0f=2i\sum_{j=1}^{3}b_j(x)\left(\frac{\partial}{\partial x_j}f(x)\right)$$
$$+\left\{\sum_{j=1}^{3}i\left(\frac{\partial}{\partial x_j}b_j(x)\right)+\sum_{j=1}^{3}b_j(x)^2+q(x)-b_0(x)^2\right\}f(x);$$

$$\mathfrak{D}(K)=\mathfrak{D}(H_0^{1/2})=H^1(R^3);$$

$$Kf=2b_0(x)f(x),$$

where $H^s(R^3)$ $(s=1,2)$ are the Sobolev spaces of order s[4], and $b_j(x)$ $(j=0,1,2,3)$ and $q(x)$ are bounded real functions with bounded derivatives $\frac{\partial}{\partial x_j}b_j(x)$ $(j=1,2,3)$. Now, we impose the following conditions on these functions.

ASSUMPTION 6.1. The functions $b_j(x)$ $(j=0,1,2,3)$ and $q(x)$ are all real bounded measurable functions in R^3 which satisfy: (1) $|b_j(x)|\leqq c|x|^{-2-\varepsilon}$ $(j=0,1,2,3)$, $\varepsilon>0$; (2) $b_j(x)$ $(j=1,2,3)$ are differentiable and $\left|\frac{\partial}{\partial x_j}b_j(x)\right|\leqq c|x|^{-2-\varepsilon}$; (3) $|q(x)|\leqq c|x|^{-2-\varepsilon}$.

Then we have the next theorem.

THEOREM 6.2. *Under Assumption 6.1 all the preceding results in the sections 4 and 5 are valid.*

PROOF. We may check Assumptions 3.1, 4.1 and 4.5. First, the compactness of V and K are derived directly from Assumption 6.1 and Rellich's com-

[4] The Sobolev space $H^s(R^3)$ is the set of all square integrable functions in R^3 which have square integrable distributional derivatives up to the order s.

pactness theorem. This implies Assumption 4.1 and also 3.1. Next, we put $d=(1+|x|^2)^{-(1+\varepsilon)/2}$. Then $d^{-1}V$ and $d^{-1}K$ are well defined operators from $\mathfrak{D}(H_0)$ to X and from $\mathfrak{D}(r_0)$ to X respectively, and the compactness of $dr_0(z)d$, $dr_0(0)r_0(z)d$, $d^{-1}Kr_0(z)d$ and $d^{-1}Vr_0(0)r_0(z)d$ can be proved by Rellich's theorem. Now we shall proceed to the investigation of the behavior of these operators near the real axis. For this purpose we introduce the following operators $T(\lambda\,;\tilde{d})$, $\lambda>m^2$, from X to $L^2(\Omega)$, here Ω is the unit sphere in R^3 with its center 0. We define (see Kuroda [15] and [16])

$$(T(\lambda\,;\tilde{d})f)(\omega)={}^{1/4}\sqrt{\frac{\lambda-m^2}{2}}[\mathfrak{F}(\tilde{d}f)](\sqrt{\lambda-m^2},\omega),\ f\in X\,,$$

where $\mathfrak{F}$ denotes the Fourier transformation and we use the spherical coordinates (r,ω), $r\in(0,\infty)$ and $\omega\in\Omega$, in the dual space. We denote one of d, $dr_0(0)$, $d^{-1}K$ and $d^{-1}Vr_0(0)$ by $\tilde{d}$. Then $T(\lambda\,;\tilde{d})$ belongs to

$$L^2((m^2,\infty)\,;L^2(\Omega))=\left\{f(\lambda)\,;\ f(\lambda)\in L^2(\Omega),\int_{m^2}^{\infty}\|f(\lambda)\|_{L^2(\Omega)}^2<\infty\right\},$$

and

$$\|T(\lambda\,;\tilde{d})f\|_{L^2((m^2,\infty);L^2(\Omega))}=\|\tilde{d}f\|\,.$$

Furthermore, $T(\lambda\,;\tilde{d})$ gives a diagonal representation of H_0 and $\tilde{d}r_0(z)d$ is represented as

$$\tilde{d}r_0(z)df=\int_{m^2}^{\infty}\frac{1}{\xi^{1/2}-z}T(\xi\,;\tilde{d})^*T(\xi\,;d)fd\xi\,.$$

Here $T(\xi\,;\tilde{d})$ and $T(\xi\,;d)$ are Hölder continuous (see Lions-Magenes [18]), so that for $z=\lambda+i\varepsilon$ with $\lambda\neq m^2$, the above integral converges to some bounded operator as $\varepsilon\downarrow 0$ in the operator norm topology (Privalov's theorem; see Kato-Kuroda [14]). The smoothness conditions are shown as follows. First, as is well-known, d is H_0-smooth, which is expressed as (see Kato [10])

$$\sup_{\substack{\lambda\in R^1,\varepsilon\in(0,\infty)\\ \|f\|=1}}\frac{1}{2\pi i}(d\{R(\lambda+i\varepsilon\,;H_0)-R(\lambda-i\varepsilon\,;H_0)\}d^*f,f)<\infty\,.$$

Furthermore, by Agmon's weighted-L^2 estimate (see Agmon [1] Appendix A and also Ginibre-Moulin [6]), we have

$$\|dR(z\,;H_0)d\|\leqq c(1+|z|)^{-1/2}\,.$$

This shows that $\|T(\xi\,;d)\|\leqq c(1+|\xi|)^{-1/4}$, since

$$T(\xi; d)^*T(\xi\,;\,d)=\frac{1}{2\pi i}d\{R(\xi+i0\,;\,H_0)-R(\xi-i0\,;\,H_0)\}d\ .$$

Then, using the facts that

$$\|T(\xi\,;\,(dr_0(0))^*)\|,\ \|T(\xi\,;\,(d^{-1}K)^*)\|,\ \|T(\xi\,;\,(d^{-1}Vr_0(0))^*)\|\leqq c\|T(\xi\,;\,d)\|\ ,$$

we have, for $\tilde{d}$,

$$\begin{aligned}&\sup_{\substack{\lambda\in R^1,\varepsilon\in(0,\infty)\\ \|f\|=1}}\left|\frac{1}{2\pi i}(\tilde{d}\{r_0(\lambda+i\varepsilon)-r_0(\lambda-i\varepsilon)\}\tilde{d}^*f,f)\right|\\ &\leqq\sup_{\lambda,\varepsilon,\|f\|=1}\int_{-\infty}^{\infty}\frac{\varepsilon}{\pi\{(\xi^{1/2}-\lambda)^2+\varepsilon^2\}}\|T(\xi\,;\,\tilde{d}^*)f\|^2d\xi\\ &\leqq\sup_{\lambda,\varepsilon}\int_{m^2}^{\infty}\frac{\varepsilon}{\pi\{(\xi^{1/2}-\lambda)^2+\varepsilon^2\}}\,\frac{1}{(1+\xi)^{1/2}}\,d\xi\\ &\leqq\sup_{\lambda,\varepsilon}\int_{m}^{\infty}\frac{\varepsilon}{\pi\{(\xi-\lambda)^2+\varepsilon^2\}}\,\frac{2\xi}{(1+\xi^2)^{1/2}}\,d\xi\leqq c\ .\end{aligned}$$

This proves the smoothness of the operators d, $dr_0(0)$, $d^{-1}K$ and $d^{-1}Vr_0(0)$ with respect to h_0. q.e.d.

6.2. Nonexistence of an exceptional set

Under the same condition as Assumption 6.1, we can prove that the exceptional set Γ is included in the union of the point spectrum of A and $\pm m$. Therefore, if no point spectrum is included in the continuous spectrum of A, there is no exceptional set in the continuous spectrum. We thus obtain the next proposition.

PROPOSITION 6.3. *If λ belongs to the exceptional set Γ and $\lambda\neq\pm m$, there exists a non-zero vector $g\in\mathfrak{X}_0$ such that $Ag=\lambda g$.*

PROOF. Let λ satisfy the condition, then there is a non-zero vector $f\in\mathfrak{X}_0$ such that

$$(1+Q(\lambda+i0))f=0\ \text{ or }\ (1+Q(\lambda-i0))f=0\ .$$

We can assume without loss of generality that the first equation holds. Then we have

$$(1+Q(\lambda+i\varepsilon))f=D^{-1}(1+GR_0(\lambda+i\varepsilon))Df\to0,\ \text{as}\ \varepsilon\downarrow0\ . \tag{6.1}$$

On the other hand, since $(f,g)_{\mathfrak{X}}=(f,g)_{\mathfrak{X}_0}+V[f,g]$, we have

$$(R_0(\lambda+i\varepsilon)Df, D(1+Q(\lambda+i\varepsilon))f)_{\mathfrak{X}}$$
$$=(DR_0(\lambda+i\varepsilon)Df, (1+Q(\lambda+i\varepsilon))f)_{\mathfrak{X}_0}$$
$$+\frac{1}{2}(h_0^{-1}Vh_0^{-1}\{r_0(\lambda+i\varepsilon)df^1-r_0(-\lambda-i\varepsilon)df^2\}, \tilde{f}(\varepsilon)) ,$$

where $\tilde{f}(\varepsilon)$ tends to zero as $\varepsilon\downarrow 0$. Thus, using Assumption 4.5 and (6.1), we have

$$(R_0(\lambda+i\varepsilon)Df, D(1+Q(\lambda+i\varepsilon))f)_{\mathfrak{X}}\to 0, \text{ as } \varepsilon\downarrow 0 .$$

This implies, since $D(1+Q(z))=(A-z)R_0(z)D$, that

$$(R_0(\lambda+i\varepsilon)Df, (A-(\lambda+i\varepsilon))R_0(\lambda+i\varepsilon)Df)_{\mathfrak{X}}\to 0, \text{ as } \varepsilon\downarrow 0 .$$

Taking the imaginary part of the above formula, we have

$$\varepsilon(R_0(\lambda+i\varepsilon)Df, R_0(\lambda+i\varepsilon)Df)_{\mathfrak{X}}\to 0 \text{ as } \varepsilon\downarrow 0 .$$

Then, since $V[R_0(\lambda+i\varepsilon)Df, R_0(\lambda+i\varepsilon)Df]$ is bounded (see the proof of Theorem 4.8, (3)), we have

$$\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}(\{R_0(\lambda+i\varepsilon)-R_0(\lambda-i\varepsilon)\}Df, Df)_{\mathfrak{X}_0}$$
$$=\lim_{\varepsilon\downarrow 0}\frac{\varepsilon}{\pi}(R_0(\lambda+i\varepsilon)Df, R_0(\lambda+i\varepsilon)Df)_{\mathfrak{X}_0}$$
$$=\lim_{\varepsilon\downarrow 0}\frac{\varepsilon}{\pi}(R_0(\lambda+i\varepsilon)Df, R_0(\lambda+i\varepsilon)Df)_{\mathfrak{X}}=0 .$$

We can apply Agmon's trace lemma (see Agmon [1] Appendix B) and obtain that $\lim_{\varepsilon\downarrow 0} R_0(\lambda+i\varepsilon)Df=R_0(\lambda+i0)Df\in X_0$. We set $g=R_0(\lambda+i0)Df$. Then, using (6.1), we have easily that $\lim_{\varepsilon\downarrow 0} AR_0(\lambda+i\varepsilon)Df=\lambda g$ and also that $g\neq 0$. Since A is closed, this implies that $Ag=\lambda g$, $g\neq 0$. q.e.d.

The equation $Ag=\lambda g$ implies that there exists a non-zero vector $h\in\mathfrak{Y}_0$ such that

$$\lambda h=Bh=\begin{pmatrix}0 & -i\\ iH & K\end{pmatrix}h,\ h=T^{-1}g .$$

Thus h^1 satisfies the equation:

$$(H+\lambda K)h^1=\lambda^2 h^1 .$$

Using the results of Ikebe-Uchiyama [8] for the non-existence of a positive eigenvalue of the second-order elliptic operators, we have the following proposition.

PROPOSITION 6.4. *Under Assumption 6.1 and the unique continuation property of the operators* $H+\lambda K$, $\lambda\in(\pm m, \pm\infty)$, *there is no eigenvalue of* A

which is embedded in the continuous spectrum of A, and the continuous part of the spectrum of A absolutely continuous.

Note added in proof; After this paper was accepted, K. Yajima and the present author ([29]) extended the applications of the smooth perturbation to include the so-called short-range perturbations. Using those results, we can relax the conditions in Assumption 6.1 such that $b_j(x)$ $(j=0, 1, 2, 3)$, $\frac{\partial}{\partial x_j} b_j(x)$ $(j=1, 2, 3)$ and $q(x)$ behave like $0(|x|^{-1-\epsilon})$ at infinity. In that case, we have to make some slight modifications in Assumption 4.5, that is, the range of integration in (4) has to be replaced by some subset of R, and then we have also to make the corresponding modifications in the subsequent lemmas and theorems. The details will be published elsewhere.

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