A Poisson formula and exponential sums[1)]

By Jun-ichi IGUSA

Introduction. We shall first recall our Poisson formula in [4] with precise definitions: let k denote a global field; we shall denote by the subscripts A and k the adelization relative to k and the taking of k-rational points, respectively. We shall denote by k_A and $k_A^\times$ the adele and idele groups of k; we shall fix a non-trivial character ψ of k_A/k. Let X denote an affine n-space over k with x_1, $x_2, \cdots, x_n$ as its affine coordinates; let dx denote the exterior product of their differentials and $|dx|_A$ the Haar measure on X_A such that X_A/X_k is of measure 1. We shall fix a form $f(x)$ of degree $m \geqq 2$ in $x_1, x_2, \cdots, x_n$ with coefficients in k; we shall assume that char(k) does not divide m, $f(x)$ is "strongly non-degenerate", i.e., the projective hypersurface defined by $f(x)=0$ is non-singular, and $n>2m$. Let $U(i)$ denote the set of simple points of the hypersurface $f(x)=i$ and $\theta_i(x)$ the residue of $(f(x)-i)^{-1}dx$ along $U(i)$; then if i is in k, the gauge-form θ_i on $U(i)$ gives rise to a measure $|\theta_i|_A$ on $U(i)_A$. Finally let $\mathscr{S}(X_A)$ denote the Schwartz-Bruhat space of X_A and take Φ from $\mathscr{S}(X_A)$. Then we have the following identity:

$$\sum_{i \in k} \int_{U(i)_A} \Phi|\theta_i|_A = \sum_{i^* \in k} \int_{X_A} \psi(i^* f(x))\Phi(x)|dx|_A ;$$

both sides have dominant series if Φ is restricted to a compact subset of $\mathscr{S}(X_A)$. We shall add the Dirac measure δ_0 to the so-defined tempered measure on X_A and denote the new measure by E; then $E(\Phi)-\Phi(0)$ is given by either side of the above identity.

On the other hand

$$E'(\Phi) = \sum_{\xi \in X_k} \Phi(\xi)$$

defines a tempered measure E' on X_A independently of $f(x)$ for every n. We observe that Supp(E'), the support of E', is the discrete subset X_k of X_A while Supp(E) is in general the union of $f^{-1}(i)_A$ for all i in k; Supp(E) becomes the union of $\{0\}$ and $f^{-1}(i)_A = U(i)_A$ for all i in $k^\times$ if $U(0)_A$ is empty. The *main objective* of this paper is to prove the following remarkable relation between E and E':

1) This work was partially supported by the National Science Foundation.

We convert $G_A=k_A\times k_A^\times$ into a (locally compact) group as

$$(u, t)\cdot(u', t')=(u+t^m u', tt')$$

and for every $g=(u, t)$ in G_A we put

$$(U(g)\Phi)(x)=|t|_A^{(1/2)n}\psi(uf(x))\Phi(tx)\ ,$$

in which $|t|_A$ is the modulus of t. *Then* $(E'-E)(U(g)\Phi)$ *becomes a* G_k-*invariant continuous function on* G_A *and it vanishes at every k-rational boundary point of* G_A *to the order* $s-1$ *where* $s=n/2m$. This is a generalization of a classical theorem stating that the theta series defined by a quadratic form behaves up to lower order terms like the corresponding Eisenstein series at every cusp. We recall that if $f(x)$ is a quadratic form, then the above G_A can be embedded in the "metaplectic group" which is at most a two-sheeted covering group of $(SL_2)_A$ and which contains $(SL_2)_k$ as a subgroup and that $E'-E$ is $(SL_2)_k$-invariant; in view of the reduction theory for $(SL_2)_A$ the above-recalled theorem then implies the boundedness of $(E'-E)(U(g)\Phi)$ on G_A. We refer to Weil [12], Chap. V for the details.

The theory of metaplectic groups for higher degree forms is not yet available. However our result seems to indicate that a boundedness as above holds in some form. We have included a remark that a weak form of the boundedness implies a generalization of the Hasse-Minkowski theorem. We have also included as an appendix an observation showing that a naive generalization of the metaplectic group might give something different from SL_2. It is hoped that this paper serves as a guide to the future arithmetic theory of higher degree forms.

1. Reduction theory for G_A. We shall denote by p a (not necessarily non-archimedean) valuation on the global field k, by k_p the corresponding completion of k, and by $|\ |_p$ the usual absolute value on k_p. If t_p is an element of $k_p^\times$, then $|t_p|_p$ is the modulus of t_p, i.e., the rate of measure change in k_p under the multiplication by t_p. Moreover the modulus $|t|_A$ of an element $t=(t_p)_p$ of $k_A^\times$ becomes the product of all $|t_p|_p$. We recall that $|t|_A=1$ if t is in $k^\times$; we shall use this "product-formula" all the time.

Since $G_A=k_A\times k_A^\times$, every element g of G_A is a pair (u, t) of an element u of k_A and an element t of $k_A^\times$; g is in G_k if u is in k and t in $k^\times$. We put

$$\boldsymbol{t}(u)=U(u, 1)\ ,\qquad \boldsymbol{d}(t)=U(0, t)\ ;$$

then we get

$$(\boldsymbol{t}(u)\Phi)(x)=\psi(uf(x))\Phi(x)\ ,\qquad (\boldsymbol{d}(t)\Phi)(x)=|t|_A^{(1/2)n}\Phi(tx)$$

for every Φ in $\mathscr{S}(X_A)$ and

$$U(u, t)=\boldsymbol{t}(u)\boldsymbol{d}(t)=\boldsymbol{d}(t)\boldsymbol{t}(t^{-m}u) .$$

We recall that the correspondence $(g, \Phi)\to U(g)\Phi$ does define a mapping $G_A\times \mathscr{S}(X_A)\to\mathscr{S}(X_A)$ and that it is continuous; such a continuity problem was examined quite generally by Levin [7]. Moreover $g\to U(g)$ uniquely extends to a unitary representation of G_A in $L^2(X_A)$. Since E, E' are tempered distributions on X_A, both $E(U(g)\Phi)$ and $E'(U(g)\Phi)$ become complex-valued continuous functions on $G_A\times\mathscr{S}(X_A)$. We observe that E, E' are G_k-invariant, i.e., $E(U(g)\Phi)=E(\Phi)$, $E'(U(g)\Phi)=E'(\Phi)$ for every g in G_k and Φ in $\mathscr{S}(X_A)$; the verification is straightforward.

We shall construct a convenient subset of G_A whose left translates by elements of G_k cover the whole G_A; such a procedure is called a reduction theory for G_A. We shall first recall the reduction theories for k_A and $k_A^\times$: if k is a number field, we shall denote by S_∞ the set of all archimedean valuations on k; then k_p is an $\boldsymbol{R}$-field, i.e., $k_p=\boldsymbol{R}$ or $\boldsymbol{C}$, for every p in S_∞. We shall denote by $\mathfrak{o}$ the ring of integers of k, i.e., the integral closure of $\boldsymbol{Z}$ in k. If k is a function field, we shall denote by $\boldsymbol{F}_{q_0}$ the algebraic closure in k of the prime field; we then choose a valuation p_∞ such that the corresponding residue class field $\boldsymbol{F}_q$ is of degree at least twice the genus of k over $\boldsymbol{F}_{q_0}$, and we take $\{p_\infty\}$ as S_∞. According to the Riemann-Roch theorem there exists an element T of k with p_∞ as its polar divisor, and we have

$$k_{p_\infty}=\boldsymbol{F}_q((T^{-1})) .$$

This time we shall denote by $\mathfrak{o}$ the integral closure of $\boldsymbol{F}_{q_0}[T]$ in k.

In either case if k_p is a p-field, i.e., if k_p is not an $\boldsymbol{R}$-field, we shall denote by X_p^0 the compact open subgroup $(k_p^0)^n$ of $X_p=k_p^n$, in which k_p^0 is the maximal compact subring of k_p, and by X_0 the restricted product of X_p relative to X_p^0 for all p not in S_∞; then

$$X_0^0=\prod_{p\notin S_\infty} X_p^0$$

becomes a compact open subgroup of X_0. We shall denote by X_∞ the product of X_p for all p in S_∞; then we have

$$X_A=X_0\times X_\infty ;$$

accordingly we shall express an element x of X_A as (x_0, x_∞). We shall denote by $X_\mathfrak{o}$ the subgroup of X_k consisting of $\mathfrak{o}$-rational points of X. If we diagonally embed X_k in X_A, then we have

$$X_A = X_k + (X_0^0 \times X_\infty), \qquad X_0 = X_k \cap (X_0^0 \times X_\infty);$$

moreover the projection of X_0 to X_∞ becomes a lattice in X_∞; we shall denote X_0, X_0^0, X_∞ for $n=1$ by k_0, k_0^0, k_∞. This is the reduction theory for X_A and in particular for k_A.

If k is a number field and τ is in $\boldsymbol{R}_+^\times$, the multiplicative group of positive real numbers, then we shall denote by a_τ the element of $k_A^\times$ defined by $a_\tau = (1, \tau)$, i.e., by the condition that non-archimedean components are 1 and archimedean components are τ. If k is a function field, we replace $\boldsymbol{R}_+^\times$ by the cyclic subgroup of $(k_{p_\infty})^\times$ generated by T and define a_τ similarly as above. Then in either case we have

$$k_A^\times = k^\times \cdot \{a_\tau\}_\tau \cdot \text{compact};$$

this is the reduction theory for $k_A^\times$ and it implies the finiteness of the ideal class group and the theorem of units of $\mathfrak{o}$. All these are well known as the Iwasawa-Tate theory; we refer to [11], [13] for the details.

The reduction theory for G_A follows from what we have recalled; we have

$$G_A = G_k \cdot \{(u, a_\tau);\ u_0 = 0,\ u_\infty \in \text{compact}\} \cdot \text{compact}.$$

The proof is as follows: let (u, t) denote an arbitrary element of G_A; then we can write

$$t = i a_\tau c, \qquad i^{-m} u = i^{-m} i^* + (v_0, c^*)$$

with i in $k^\times$, c in a fixed compact subset of $k_A^\times$, i^* in k, v_0 in k_0^0, and c^* in a fixed compact subset of k_∞; and we get

$$(u, t) = (i^*, i)((0, c^*), a_\tau)((v_0, 0), c).$$

This proves the assertion.

We observe that the middle space can be injectively mapped to $k_\infty \times \boldsymbol{R}_+^\times$. We take the one point compactification of $k_\infty \times (\boldsymbol{R}_+^\times \cup \{0\})$ and regard the boundary of $k_\infty \times \boldsymbol{R}_+^\times$ as the boundary of G_A. In particular points of $k \times \{0\}$ and the point at infinity will be considered as k-rational boundary points of G_A. We shall examine the behavior of $(E' - E)(U(g)\Phi)$ as g approaches such boundary points; for the exact statement of our results we refer to § 4, Th. 3 and Th. 4.

2. A dominant series. If $f(x)$ is a quadratic form and k is a number field, the series

$$E(\Phi) = \Phi(0) + \sum_{i^* \in k} \int_{X_A} \psi(i^* f(x)) \Phi(x) |dx|_A$$

appeared (as a special case) in Weil [12], Chap. IV; he called it the Eisenstein-Siegel series. We recall that if $k=\boldsymbol{Q}$ and $f(x_\infty)$ is a positive-definite quadratic form, then $E(U(g)\Phi)$ becomes the classical holomorphic Eisenstein series for a suitable Φ. We shall show in the general case where $f(x)$ is a higher degree form that $E(U(g)\Phi)$ has a certain Eisenstein series as a dominant series; we shall first construct an equivalent dominant series.

If $g=(u,t)$ is in G_A, we get

$$\int_{X_A}\psi(i^*f(x))(U(g)\Phi)(x)|dx|_A$$
$$=|t|_A^{(1/2)n-n}\cdot\int_{X_A}\psi((i^*+u)t^{-m}f(x))\Phi(x)|dx|_A ,$$

hence its absolute value is at most equal to

$$c(\Phi)|t|_A^{(1/2)n}\cdot\prod_p \max(|u_p+i^*|_p, |t_p|_p^m)^{-n/m} ,$$

in which $c(\Phi)$ is independent of i^*, t, and u; cf. [4]. Actually if we restrict Φ to a compact subset of $\mathcal{S}(X_A)$, then we can replace $c(\Phi)$ by a constant independent of Φ; cf. 2 bis. We shall show that $E(U(g)\Phi)$ has

$$\text{const.}\ |t|_A^{(1/2)n}(1+\sum_{i^*\in k}\prod_p \max(|u_p+i^*|_p, |t_p|_p^m)^{-n/m})$$

as a dominant series; we have only to show that this series is convergent.

For the sake of simplicity we shall write $|u|_p$, etc. instead of $|u_p|_p$, etc. We shall also use the following notation: if ϕ and ϕ' are complex-valued functions on a set X such that

$$|\phi(x)|\leqq c\cdot\phi'(x)$$

for every x in X, in which c is a constant, we write $\phi(x)\prec\phi'(x)$ or $\phi'(x)\succ\phi(x)$ on X; if we have both $\phi(x)\prec\phi'(x)$ and $\phi(x)\succ\phi'(x)$ on X, we write $\phi(x)\succ\prec\phi'(x)$ on X. If there is no ambiguity about the set X, we sometimes omit it. Finally we shall use this notation to define a family of subsets of a given set. The following lemma will settle the convergence problem:

LEMMA 1. *Suppose that $s>1$ and (u,v) is in $k_A\times k_A^\times$; then we have*

$$\sum_{i^*\in k}\prod_p \max(|u+i^*|_p, |v|_p)^{-2s}\prec|v|_A^{1-2s}$$

on the subset of $k_A\times k_A^\times$ defined by $|v|_A\succ 1$.

PROOF. By the reduction theory for $k_A^\times$ we can write v as $ia_\tau c$, in which i is in $k^\times$ and c is in a fixed compact subset of $k_A^\times$; then we get $|v|_A\succ\prec|a_\tau|_A$ and

$$\prod_p \max(|u+i^*|_p, |ia_\tau c|_p)$$
$$\succ\prec \prod_p \max(|(i^{-1}u)+(i^{-1}i^*)|_p, |a_\tau|_p) .$$

Therefore for our purpose we may assume that $v=a_\tau$ and $|v|_A \succ 1$; this implies $|v|_p \succ 1$ for every p in S_∞ and $|v|_p=1$ for every other p. By the reduction theory for k_A we may assume that $|u|_p \prec 1$ for every p in S_∞ and $|u|_p \leqq 1$ for every other p. Then we get

$$\max(|u+i^*|_p, |v|_p) \succ\prec \max(|i^*|_p, |v|_p)$$

for every p in S_∞ and

$$\max(|u+i^*|_p, |v|_p) = \max(|i^*|_p, |v|_p)$$

for every other p. Therefore we get

$$\sum_{i^* \in k} \prod_p \max(|u+i^*|_p, |v|_p)^{-2s}$$
$$\succ\prec \sum_{i^* \in k} \prod_p \max(|i^*|_p, |v|_p)^{-2s} ;$$

the rest of the proof is as follows:

Since the class number of k or rather of $\mathfrak{o}$ is finite, we can express i^* as ab^{-1} with a, b in $\mathfrak{o}$, $b \neq 0$ such that

$$\prod_{p \notin S_\infty} \max(|a|_p, |b|_p) \succ\prec 1 .$$

Furthermore, in view of the theorem of units, we may assume that b is among a fixed set, say R, of representatives of orbits in $\mathfrak{o}-\{0\}$ by the group of units of $\mathfrak{o}$ with the property that $|b|_p \succ 1$ for every p in S_∞. Then we get

$$\sum_{i^* \in k} \prod_p \max(|i^*|_p, |v|_p)^{-2s}$$
$$\prec \sum_{b \in R} \sum_{a \in \mathfrak{o}} \prod_{p \in S_\infty} \max(|a|_p, |bv|_p)^{-2s} ;$$

and we have

$$\sum_{a \in \mathfrak{o}} \prod_{p \in S_\infty} \max(|a|_p, |bv|_p)^{-2s} \prec \prod_{p \in S_\infty} |bv|_p^{1-2s} .$$

This is the crucial point of our proof: if k is a number field, it follows from Lemma 12 in [3]; and if k is a function field, it follows from a counterpart of "Lemma 12" proved in [4], p. 227; cf. also Mars [8], pp. 133-136. At any rate, since $|v|_p=1$ for every p not in S_∞, we get

$$\prod_{p \in S_\infty} |v|_p = |v|_A .$$

Moreover we have

$$\sum_{b\in R}\Big(\prod_{p\in S_\infty}|b|_p\Big)^{1-2s}<\infty$$

because this series is a partial sum of the Dedekind zeta series of $\mathfrak{o}$ evaluated at $2s-1>1$. q.e.d.

2 bis. We shall outline a method to obtain a dominant series for any series such as $E(\Phi)$; we shall first make some of our notations precise: k is a global field, ψ is a non-trivial character of k_A/k, and ψ_p is the product of the canonical injection $k_p\to k_A$ and ψ; X is an affine n-space over k, $[x, y]$ is a symmetric non-degenerate bilinear form on $X\times X$ defined over k, and $|dx|_p$ is the Haar measure on X_p which is autodual relative to the bicharacter $\psi_p([x, y])$ of $X_p\times X_p$; the restricted product measure $|dx|_A$ of all $|dx|_p$ then becomes the Haar measure on X_A such that X_A/X_k is of measure 1.

Let $f(x)$ denote, for a moment, an arbitrary polynomial in the affine coordinates $x_1, x_2, \cdots, x_n$ of X with coefficients in k and suppose that

$$(*)\qquad \left|\int_{X_p}\psi_p(i^*f(x))\Phi(x)|dx|_p\right|\leqq c(\Phi)\cdot\max(|i^*|_p, 1)^{-\sigma}$$

for every i^* in k_p and Φ in $\mathscr{S}(X_p)$, the Schwartz-Bruhat space of X_p; it is understood that $c(\Phi)$ is independent of i^* and σ is independent of i^* and Φ. Then for any compact subset C of $\mathscr{S}(X_p)$ we can replace $c(\Phi)$ by a constant which works for all Φ in C. The proof depends on the following fact: "Let B denote a subset of the space $\mathscr{S}(X_p)'$ of tempered distributions on X_p which is bounded in the sense that

$$\sup_{T\in B}|T(\Phi)|<\infty$$

for each Φ in $\mathscr{S}(X_p)$; then we have

$$\sup_{T\in B,\Phi\in C}|T(\Phi)|<\infty\ ."$$

(If k_p is a p-field, this follows from the fact that C is contained in a finite dimensional subspace of $\mathscr{S}(X_p)$. If k_p is an $\boldsymbol{R}$-field, then $\mathscr{S}(X_p)$ is a complete metric space; and we have only to apply the Baire category theorem.) If we put

$$T_{i^*}(\Phi)=\max(|i^*|_p, 1)^{\sigma}\cdot\int_{X_p}\psi_p(i^*f(x))\Phi(x)|dx|_p$$

for every Φ in $\mathscr{S}(X_p)$, we get a subset $\{T_{i^*}\}_{i^*}$ of $\mathscr{S}(X_p)'$ parametrized by k_p.

The assumption (*) shows that this subset is bounded; the rest follows from the fact that we have recalled.

We can go one step further: "Let B, B' denote bounded subsets of $\mathscr{S}(X_p)'$, $\mathscr{S}(X_{p'})'$, respectively; then the set of $T\otimes T'$ for all T in B and T' in B' is bounded in $\mathscr{S}(X_p\times X_{p'})'$." (This becomes trivial if either k_p or $k_{p'}$ is a p-field; and if both k_p and $k_{p'}$ are $\boldsymbol{R}$-fields, then it can be proved by applying twice the above-recalled fact.) Therefore if we have (*) for every p, we can take a finite product. More precisely suppose that S is a finite set of valuations on k and let X_S, k_S denote the products of X_p, k_p for all p in S; then we have

$$\left|\int_{X_A}\Big(\prod_{p\in S}\psi_p(i_p^*f(x_p))\Big)\Phi(x)\Big(\bigotimes_{p\in S}|dx_p|_p\Big)\right| \leqq c\cdot\prod_{p\in S}\max(|i^*|_p, 1)^{-\sigma_p}$$

for every $i^*=(i_p^*)_p$ in k_S and Φ in a compact subset of $\mathscr{S}(X_S)$.

Finally for any non-archimedean valuation p let k_p^0 denote the maximal compact subring of k_p and Φ_p the characteristic function of $X_p^0=(k_p^0)^n$; suppose that we can choose $c(\Phi_p)$'s so that their product is convergent; then we have

$$\left|\int_{X_A}\psi(i^*f(x))\Phi(x)|dx|_A\right|\leqq c'\cdot\prod_p\max(|i^*|_p, 1)^{-\sigma_p}$$

for every i^* in k_A and Φ in a compact subset of $\mathscr{S}(X_A)$. Therefore if the series

$$\sum_{i^*\in k}\prod_p\max(|i^*|_p, 1)^{-\sigma_p}$$

is convergent, by adding 1 and multiplying a suitable constant we will get a dominant series for $E(\Phi)$. We have shown in [4] that this is the case with $\sigma_p=n/m$ if $f(x)$ is a form of degree $m\geqq 2$ such that char(k) does not divide m, the projective hypersurface defined by $f(x)=0$ is non-singular, and $n>2m$.

3. Eisenstein series. We shall introduce an Eisenstein series which will serve as a dominant series for $E(U(g)\Phi)$ and examine some of its properties; we shall first make preliminary observations.

Let K denote a local field; K is either an $\boldsymbol{R}$-field or a p-field. We embed K in a division ring $L=K+Kw$ as follows: if K is an $\boldsymbol{R}$-field, we have a unique choice for L; it is $\boldsymbol{C}$ or $\boldsymbol{H}$, the Hamilton quaternion algebra, according as K is $\boldsymbol{R}$ or $\boldsymbol{C}$. We shall normalize w as $w^2=-1$. If K is a p-field, we shall take as L an unramified quadratic extension of K; it exists and is unique. Let K^0, L^0 denote the maximal compact subrings of K, L, respectively; then we shall take as w an element of L such that $L^0=K^0+K^0w$. In either case we shall denote by Nz the norm of an element z of L; N gives rise to a continuous homomorphism of $L^\times$ to $K^\times$.

Let

$$\sigma=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, \qquad z=u+vw$$

denote arbitrary elements of $GL_2(K)$, $Z=K+K^\times w$, respectively; then we have $\gamma z+\delta\neq 0$. Therefore

$$\sigma\cdot z=(\alpha z+\beta)(\gamma z+\delta)^{-1}=u'+v'w$$

is defined and we get

$$v'=v(\sigma\cdot z)=\det(\sigma)v\cdot N(\gamma z+\delta)^{-1}\neq 0\ ;$$

hence $\sigma\cdot z$ is also in Z. Moreover the mapping $GL_2(K)\times Z\to Z$ defined by $(\sigma\cdot z)\to\sigma\cdot z$ is continuous and it gives a group action of $GL_2(K)$ on Z; the action is transitive because

$$\begin{pmatrix}v & u\\ 0 & 1\end{pmatrix}\cdot w=u+vw$$

for every $z=u+vw$ in Z.

We shall go back to the global field k and take k_p as K; and we shall denote the above w by w_p. For the sake of completeness (and clarity) we shall prove the following lemma:

LEMMA 2. *Let u_p, v_p denote arbitrary elements of k_p; then we have*

$$|N(u_p+v_pw_p)|_p \succ\!\prec \max(|u_p|_p, |v_p|_p)^2$$

*if k_p is an **R**-field and*

$$|N(u_p+v_pw_p)|_p=\max(|u_p|_p, |v_p|_p)^2$$

if k_p is a p-field.

PROOF. If a, b are non-negative real numbers, we certainly have

$$\max(a, b)\leqq a+b\leqq 2\cdot\max(a, b)\ ;$$

this takes care of the **R**-field case. In the p-field case, if a, b are elements of $k_p(w_p)$ satisfying $|a|_p\neq|b|_p$, we have

$$|a+b|_p=\max(|a|_p, |b|_p)\ .$$

Therefore if $|u_p|_p\neq|v_p|_p$, we have the equality in question. We shall assume that $|u_p|_p=|v_p|_p$; then by homogeneity we may further assume that $|u_p|_p=|v_p|_p=1$. And we have only to show that $|N(u_p+v_pw_p)|_p=1$. By our choice of w_p the residue classes of 1, w_p are linearly independent over the residue class field of k_p. Since u_p, v_p are units, the residue class of $u_p+v_pw_p$ is different from 0; hence $u_p+v_pw_p$ is a unit, and hence $|N(u_p+v_pw_p)|_p=1$. q.e.d.

We choose w_p for each p and put

$$w=(w_p)_p\,, \qquad \mathscr{Z}=k_A+k_A^{\times}w\,;$$

with the obvious topology $\mathscr{Z}$ become a locally compact space. We shall denote an element of $\mathscr{Z}$ by $z=u+vw=(u_p+v_pw_p)_p$ and define Nz componentwise, i.e., as $(Nz)_p=Nz_p$ for every p; then we see by Lemma 2 that N gives a continuous mapping of $\mathscr{Z}$ to $k_A^{\times}$. Furthermore we have

$$|N(z+i^*)|_A \succ\!\prec \max(|u+i^*|_p, |v|_p)^2$$

on $\mathscr{Z}\times k$. Therefore if s is any complex number satisfying $\mathrm{Re}(s)>1$, we see by Lemma 1 that the series

$$E(z, s)=|v|_A^s(1+\sum_{i^*\in k}|N(z+i^*)|_A^{-s})$$

is absolutely convergent; this is the Eisenstein series useful for our purpose. The following theorem follows from what we have observed:

THEOREM 1. *Consider the mapping $G_A\to\mathscr{Z}$ defined by $g=(u, t)\to z=u+vw$ where $v=t^m$; then we have*

$$\int_{X_A}\phi(i^*f(x))(U(g)\Phi)(x)|dx|_A \prec (|v|_A|N(z+i^*)|_A^{-1})^{n/2m}$$

on $G_A\times k$ if Φ is restricted to a compact subset of $\mathscr{S}(X_A)$. Therefore a constant multiple of $E(u+t^mw, n/2m)$ serves as a dominant series for $E(U(g)\Phi)$ on G_A if Φ is restricted as above. Moreover for any $s>1$ we have

$$E(z, s)-|v|_A^s \prec |v|_A^{1-s}$$

on the subset of $\mathscr{Z}$ defined by $|v|_A>1$.

We shall examine the invariance property of $E(z,s)$ under the action of $GL_2(k)=(GL_2)_k$ on $\mathscr{Z}$; we shall first define the action of $(GL_2)_A$ on $\mathscr{Z}$: let $\sigma=(\sigma_p)_p$, $z=u+vw$ denote arbitrary elements of $(GL_2)_A$, $\mathscr{Z}$, respectively; we define $\sigma\cdot z=u'+v'w$ componentwise, i.e., as

$$(\sigma\cdot z)_p=u'_p+v'_pw_p=\sigma_p\cdot z_p$$

for every p. We shall show that $\sigma\cdot z$ is also in $\mathscr{Z}$: we have only to show that if k_p is a p-field, σ_p is in $GL_2(k_p^0)$, and $|u|_p \leqq 1$, $|v|_p = 1$, then $|u'|_p \leqq 1$, $|v'|_p = 1$. Let $(\gamma\ \delta)$ denote the second row of σ; then $(\sigma_p \cdot z_p) N(\gamma_p z_p + \delta_p)$ is certainly in $k_p^0 + k_p^0 w_p$. Moreover, since either γ_p or δ_p is a unit of k_p^0, we get

$$|N(\gamma_p z_p + \delta_p)|_p = \max(|\gamma u + \delta|_p, |\gamma v|_p)^2 = 1 ;$$

this implies $|u'|_p \leqq 1$ and

$$|v'|_p = |\det(\sigma)|_p |v|_p |N(\gamma z + \delta)|_p^{-1} = 1 .$$

We observe that the action of $(GL_2)_A$ on $\mathscr{X}$ thus defined is continuous and transitive.

THEOREM 2. *The Eisenstein series* $E(z, s)$ *is* $GL_2(k)$*-invariant, i.e.,* $E(\sigma \cdot z, s) = E(z, s)$ *for every* σ *in* $GL_2(k)$.

PROOF. We let GL_2 act on the projective line P_1 as

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot i^* = (\alpha i^* + \beta)(\gamma i^* + \delta)^{-1}$$

with the obvious understanding in the case where $i^* = \infty$; then we get a transitive action of $GL_2(k)$ on $P_1(k) = k \cup \{\infty\}$. Let σ denote an arbitrary element of $GL_2(k)$ with coefficients α, β, γ, δ and z an element of $\mathscr{X}$; then for every i^* in k such that $-\gamma i^* + \alpha \neq 0$, i.e., $\sigma^{-1} \cdot i^* \neq \infty$, we get

$$\sigma \cdot z - i^* = (-\gamma i^* + \alpha)(z - \sigma^{-1} \cdot i^*)(\gamma z + \delta)^{-1} .$$

Moreover if $-\gamma i^* + \alpha = 0$, then $\gamma \neq 0$ and we get

$$\sigma \cdot z - i^* = -\det(\sigma) \gamma^{-1} (\gamma z + \delta)^{-1} .$$

Therefore we get

$$\begin{aligned} E(\sigma \cdot z, s) &= |v(\sigma \cdot z)|_A^s \Big(1 + \sum_{i^* \in k} |N(\sigma \cdot z - i^*)|_A^{-s}\Big) \\ &= |v|_A^s \Big(|N(\gamma z + \delta)|_A^{-s} + \sum_{i^* \neq \infty, \sigma \cdot \infty} |N(z - \sigma^{-1} \cdot i^*)|_A^{-s} + 1\Big) \\ &= |v|_A^s \Big(1 + \sum_{j^* \in k} |N(z - j^*)|_A^{-s}\Big) \end{aligned}$$

if $\sigma \cdot \infty \neq \infty$; and similarly $E(\sigma \cdot z, s) = E(z, s)$ if $\sigma \cdot \infty = \infty$. q.e.d.

COROLLARY. *For any given* i^* *in* k *we have*

$$E(z, s) - |v|_A^s |N(z + i^*)|_A^{-s} < |v|_A^{s-1}$$

on the subset of $\mathscr{X}$ *defined by*

$$(*) \qquad |N(z + i^*)|_A < |v|_A^2 < 1 ;$$

we also have

$$E(z,s)-|v|_A^s|N(z+i^*)|_A^{-s}<1$$

on the subset of $\mathscr{Z}$ *defined by*

$$(**) \qquad |N(z+i^*)|_A<|v|_A .$$

The proof is quite simple: choose an element σ of $GL_2(k)$ satisfying $\sigma^{-1}\cdot\infty=-i^*$; then by Th. 2 we get

$$E(z,s)-|v|_A^s|N(z+i^*)|_A^{-s}=E(\sigma\cdot z,s)-|v(\sigma\cdot z)|_A^s .$$

And $(*)$ implies

$$|v(\sigma\cdot z)|_A=|v|_A|N(z+i^*)|_A^{-1}\succ|v|_A^{-1}\succ 1$$

while $(**)$ implies $|v(\sigma\cdot z)|_A\succ 1$. The rest follows from Th. 1.

One way to construct a subset of $\mathscr{Z}$ satisfying $(*)$ is as follows: we restrict v as $|v|_A<1$ and u by the condition that $(u+i^*)v^{-1}$ remains in a compact subset of k_A, i.e., as $|u+i^*|_p<|v|_p$ for every p in a fixed finite set and $|u+i^*|_p\leqq|v|_p$ for every other p. In fact by Lemma 2 we then get

$$|N(z+i^*)|_A\succ\prec\prod_p\max(|u+i^*|_p,|v|_p)^2$$
$$\succ\prec\prod_p|v|_p^2=|v|_A^2<1 .$$

We shall examine our definitions and results in the special case where $k=\boldsymbol{Q}$. In order to make the objects visible, we shall assume that $u_0=0$ and $v_0=1$ in $z=u+vw=(z_0,z_\infty)$; then we get

$$E(z,s)=|\mathrm{Im}(z_\infty)|^s(1+\sum_{\gamma,\delta}|\gamma z_\infty+\delta|^{-2s}) ,$$

in which γ, δ are relatively prime integers with $\gamma\geqq 1$. This is a classical Eisenstein series; cf., e.g., Kubota [5]. Moreover the condition $(*)$ and the more restrictive condition explained after the corollary both become

$$|z_\infty+i^*|<|\mathrm{Im}(z_\infty)|<1 ;$$

if we restrict z_∞ to the upper-half plane, this defines a V-shaped region which touches the real axis at $-i^*$. On the other hand the condition $(**)$ becomes

$$|z_\infty+i^*|^2<|\mathrm{Im}(z_\infty)| ;$$

this defines a closed circle which is tangent to the real axis at $-i^*$. In the first case the height and the angle between the two lines through $-i^*$ are arbitrary and in the second case the radius is arbitrary. We have thus shown that the second part of Th. 1 and the corollary become the well-known theorem describing the behavior of the classical Eisenstein series at cusps; cf. Kubota, op. cit.

4. **Main results.** We shall go back to our $(E'-E)(U(g)\Phi)$: in view of Th. 1 we put

$$|g|_A=|(u,t)|_A=|t|_A^m\,, \qquad s=n/2m\,;$$

then $s>1$ and $g\to|g|_A$ gives a continuous homomorphism of G_A to $\boldsymbol{R}_+^\times$.

THEOREM 3. *We restrict g as $|g|_A>1$ and Φ to a compact subset of $\mathscr{S}(X_A)$; then we have*

$$(E'-E)(U(g)\Phi)\prec|g|_A^{1-s}\,.$$

The proof of this theorem does not depend on the degree m of $f(x)$; it can be extracted from the proof of Prop. 7 in Weil [12], Chap. V. Since we have included the function field case, we shall give an outline:

LEMMA 3. *Let F denote a closed subset of $X_A=X_0\times X_\infty$ such that $F\cap(X_0\times\{0\})=\varnothing$; then for any $N\geqq 0$ and any compact subset C of $\mathscr{S}(S_A)$ there exists an element ϕ of $\mathscr{S}(X_A)$ satisfying*

$$|a_\tau|_A^N|\Phi(a_\tau x)|\leqq\phi(x)$$

for every x in F, every a_τ with $|a_\tau|_A\geqq 1$, and every Φ in C.

PROOF. This lemma is known if k is a number field; cf. Weil [12], Chap. I, Lemma 6 and also Mars [8], Lemma 7. The proof in the function field case is simpler: by using coordinates of x_∞ we define $|x_\infty|_\infty$ as

$$|x_\infty|_\infty=\max_{1\leqq i\leqq n}\{|x_{\infty,i}|_{p_\infty}\}\,.$$

Since C is compact, there exist compact open subgroups L_0, L_∞ of X_0, X_∞, respectively, such that $\mathrm{Supp}(\Phi)$ is contained in $L_0\times L_\infty$ for every Φ in C; we may assume that L_∞ is defined by $|x_\infty|_\infty\leqq c$ for some $c>0$. Let ϕ_0 denote a constant multiple of the characteristic function of L_0 and ϕ_∞ the characteristic function of L_∞; then if we take the constant large enough, we will have

$$|\Phi(x)|\leqq\phi_0(x_0)\phi_\infty(x_\infty)$$

for every $x=(x_0,x_\infty)$ in X_A and Φ in C. Since $F\cap(L_0\times L_\infty)$ is compact and has an empty intersection with $X_0\times\{0\}$, we have $|x_\infty|_\infty\geqq d$ for some $d>0$ and for every x in $F\cap(L_0\times L_\infty)$. We shall show that

$$\phi=(cd^{-1})^N(\phi_0\otimes\phi_\infty)$$

has the required property.

If $\Phi(a_\tau x)\neq 0$ for some x in X_A and Φ in C, we get $\phi_0(x_0)\phi_\infty(\tau x_\infty)\neq 0$; hence x_0 is in L_0 and τx_∞ is in L_∞. If $|a_\tau|_A=|\tau|_\infty\geqq 1$, then x_∞ itself is in L_∞. And if x is also in F, then $|\tau|_\infty\leqq c|x_\infty|_\infty^{-1}\leqq cd^{-1}$; hence

$$\begin{aligned}|a_\tau|_A^N|\Phi(a_\tau x)|&\leqq(cd^{-1})^N\phi_0(x_0)\phi_\infty(\tau x_\infty)\\&=(cd^{-1})^N\phi_0(x_0)\phi_\infty(x_\infty)\,.\end{aligned}$$ q.e.d.

The proof of Th. 3 is as follows: we have shown in §1 that

$$G_A=G_k\cdot\{(u,a_\tau);\ u_0=0,u_\infty\in\text{compact}\}\cdot\text{compact}\,.$$

If we express an element g of G_A as the product of an element of G_k, (u,a_τ), and an element of the compact set, then we have $|g|_A\succ\prec|a_\tau|_A^m$. Therefore we may assume that $g=(u,a_\tau)$, in which $u_0=0$, u_∞ is in a fixed compact subset of k_∞, and $|a_\tau|_A\succ 1$; then $(a_\tau^{-m}u,1)$ in

$$(u,a_\tau)=(0,a_\tau)(a_\tau^{-m}u,1)$$

remains in a compact subset of G_A. Therefore we may assume that $g=(0,a_\tau)$ where $|a_\tau|_A\geqq 1$. We recall that for $g=(0,t)$ we have

$$(E'-E)(U(g)\Phi)=|t|_A^{(1/2)n}\Big(\sum_{\xi\in X_k-\{0\}}\Phi(t\xi)-\sum_{i\in k}\int_{U(i)_A}\Phi(tx)|\theta_i(x)|_A\Big)\,.$$

We shall take $t=a_\tau$ where $|a_\tau|_A\geqq 1$. Then by applying Lemma 3 to $F=X_k-\{0\}$ we get

$$|t|_A^{(1/2)n}\cdot\sum_{\xi\in X_k-\{0\}}\Phi(t\xi)\prec|g|_A^{-N}$$

for any $N\geqq 0$; by applying Lemma 3 to $F=f^{-1}(k^\times)$ we get

$$|t|_A^{(1/2)n}\cdot\sum_{i\in k^\times}\int_{U(i)_A}\Phi(tx)|\theta_i(x)|_A\prec|g|_A^{-N}$$

for any $N\geqq 0$; and finally we have

$$|t|_A^{(1/2)n}\cdot\int_{U(0)_A}\Phi(tx)|\theta_0(x)|_A=|g|_A^{1-s}\cdot\int_{U(0)_A}\Phi|\theta_0|_A\,.$$

We have only to put the above three relations together.

THEOREM 4. *Let i^* denote any given element of k; we restrict $g=(u,t)$ by $|t|_A<1$ and by the additional condition that $(u+i^*)t^{-m}$ remains in a compact* *subset of* k_A; *we also restrict* Φ *to a compact subset of* $\mathscr{S}(X_A)$. *Then we have*

$$(E'-E)(U(g)\Phi) \prec |g|_A^{s-1}.$$

PROOF. We define the singular term at $-i^*$ as

$$\text{s.t. at } -i^* = |t|_A^{(1/2)n} \cdot \int_{X_A} \phi((u+i^*)f(x))\Phi(tx)|dx|_A ;$$

then by Th. 1, its corollary, and a remark after the corollary we get

$$E(U(g)\Phi) - \text{s.t. at } -i^* \prec |g|_A^{s-1}$$

on the subset under consideration. Therefore we have only to show that

$$E'(U(g)\Phi) - \text{s.t. at } -i^* \prec |g|_A^N$$

for any $N \geqq 0$. Since $\phi(uf(\xi)) = \phi((u+i^*)f(\xi))$ for every ξ in X_k, by applying the usual Poisson formula we get

$$\begin{aligned} E'(U(g)\Phi) &= |t|_A^{(1/2)n} \cdot \sum_{\xi \in X_k} \phi((u+i^*)f(\xi))\Phi(t\xi) \\ &= |t|_A^{(1/2)n} \cdot \sum_{\xi \in X_k} \int_{X_A} \phi([\xi, y] + (u+i^*)f(y))\Phi(ty)|dy|_A . \end{aligned}$$

Therefore if we put

$$\Psi(x) = \phi((u+i^*)t^{-m}f(x))\Phi(x)$$

and denote its Fourier transform by $\Psi^*(x)$, we get

$$E'(U(g)\Phi) - \text{s.t. at } -i^* = |t|_A^{-(1/2)n} \cdot \sum_{\xi \in X_k - \{0\}} \Psi^*(t^{-1}\xi) .$$

We recall that the mapping $G_A \times \mathscr{S}(X_A) \to \mathscr{S}(X_A)$ defined by $(g, \Phi) \to U(g)\Phi$ is continuous. Since $(u+i^*)t^{-m}$ remains in a compact subset of k_A, therefore, the set $\{\Psi\}$ is relatively compact in $\mathscr{S}(X_A)$. Since the Fourier transformation is bicontinuous, the set $\{\Psi^*\}$ is also relatively compact in $\mathscr{S}(X_A)$. The remaining part of the proof is as follows:

We write $t = i a_\tau c$ with i in $k^\times$ and c in a fixed compact subset of $k_A^\times$; then by assumption we have $|t|_A \succ\prec |a_\tau|_A \prec 1$ and

$$|t|_A^{-(1/2)n} \cdot \sum_{\xi \in X_k - \{0\}} \Psi^*(t^{-1}\xi) \succ\prec |a_\tau^{-1}|_A^{(1/2)n} \cdot \sum_{\xi \in X_k - \{0\}} \Psi^*(c^{-1}a_\tau^{-1}\xi) .$$

Since the set $\{\Psi^*(c^{-1}x)\}$ with c in the compact subset is still relatively compact and $|a_\tau^{-1}|_A \succ 1$, we can apply Lemma 3 to $F = X_k - \{0\}$. In this way we get

$$|a_\tau^{-1}|_A^{(1/2)n} \cdot \sum_{\xi \in X_k - \{0\}} \Psi^*(c^{-1}a_\tau^{-1}\xi) \prec |a_\tau^{-1}|_A^{-mN} \succ\prec |g|_A^N$$

for any $N \geqq 0$. q.e.d.

We shall examine Th. 4 in the special case where $k=\boldsymbol{Q}$; for the sake of simplicity we shall assume that $f(x)$ has integer coefficients: we shall use the particular ψ defined by the condition that $\psi_\infty(u_\infty)=\boldsymbol{e}(u_\infty)$ for every u_∞ in $\boldsymbol{Q}_\infty=\boldsymbol{R}$ and $\psi_p(u_p)=\psi_\infty(-\langle u_p\rangle)$ for every u_p in the Hensel p-adic field $\boldsymbol{Q}_p$, where $\langle u_p\rangle$ is the fractional part of u_p. We shall use as Φ a function of the form $\Phi_0\otimes\Phi_\infty$, in which Φ_∞ is a Schwartz function on $X_\infty=\boldsymbol{R}^n$; for the sake of simplicity we shall assume that Φ_0 is the characteristic function of X_0^0. And we shall take as g a special pair (u, a_τ) where $u_0=0$. Then we get

$$(U(g)\Phi)(x)=\tau^{(1/2)n}\boldsymbol{e}(u_\infty f(x_\infty))\Phi_\infty(\tau x_\infty)$$

if x_0 is in X_0^0 and $(U(g)\Phi)(x)=0$ otherwise; hence

$$\tau^{-(1/2)n}E'(U(g)\Phi)=\sum_{\xi\in Z^n}\boldsymbol{e}(u_\infty f(\xi))\Phi_\infty(\tau\xi)\ . \tag{1}$$

One way to visualize this series is to consider the limit case where Φ_∞ becomes the characteristic function of a relatively compact, say open, subset J of $\boldsymbol{R}^n$; then we get the following finite sum:

$$\sum_{\xi\in Z^n\cap\tau^{-1}J}\boldsymbol{e}(u_\infty f(\xi))\ .$$

This is classically known as an exponential sum; it behaves quite delicately as a function of u_∞ and τ; cf., e.g., Birch [2].

We shall also make the two series for $E(U(g)\Phi)$ explicit: suppose that $i^*=\gamma^{-1}\delta$, where γ, δ are relatively prime integers with $\gamma\geqq 1$, and put

$$G(i^*)=\gamma^{-n}\cdot\sum_{\xi \bmod \gamma}\boldsymbol{e}(-i^*f(\xi))\,;$$

then one expression for $\tau^{-(1/2)n}E(U(g)\Phi)$ is

$$1+\sum_{i^*\in Q}G(i^*)\cdot\int_{R^n}\boldsymbol{e}((u_\infty+i^*)f(x_\infty))\Phi_\infty(\tau x_\infty)dx_\infty\ , \tag{2}$$

in which dx_∞ is the usual measure on $\boldsymbol{R}^n$. On the other hand, for every integer i and a positive integer Q let $N_Q(i)$ denote the number of $\xi \bmod Q$ satisfying $f(\xi)\equiv i \bmod Q$; we let Q tend to ∞ so that Q becomes divisible by any positive integer; then the following limit:

$$S(i)=\lim_{Q\to\infty}Q^{-(n-1)}N_Q(i)$$

exists. In fact the condition of convergence is $n\geqq 4$ if $i\neq 0$ and $n>\max(m+1, 4)$ if $i=0$; and this is weaker than our assumption that $n>2m$ (and $m\geqq 2$). We might also mention that in the fourth and fifth lines of [4], p. 224 the condition

$n > \max(m+1, 4)$ was incorrectly stated as $n > \max(m, 4)$. At any rate another expression for $\tau^{-(1/2)n}E(U(g)\Phi)$ is

$$(3) \qquad 1+\sum_{i \in Z} S(i)\cdot\int_{f^{-1}(i)} \Phi_\infty(\tau x_\infty)|\theta_i(x_\infty)|_\infty\cdot e(iu_\infty)\ .$$

And Th. 4 shows that as the point

$$z_\infty = u_\infty + (-1)^{1/2}\tau^m$$

approaches any rational number $-i^*$ in the ∇-shaped region explained in the previous section, (1) is given by (2) and (3) with a remainder term of order τ^{-m}; we observe that (1) itself can be of order τ^{-n}.

We shall also explain the classical case where $f(x)$ is a quadratic form: after Hermite and Siegel we choose a "majorant" $h(x_\infty)$ of $f(x_\infty)$ and put

$$\Phi_\infty(x_\infty) = \exp(-2\pi h(x_\infty))\ ;$$

we can define $h(x_\infty)$ as a positive-definite quadratic form (with real coefficients) such that the above Φ_∞ is equal to its Fourier transform relative to the bicharacter $\psi_\infty(f(x_\infty, y_\infty))$ of $X_\infty \times X_\infty$, in which

$$f(x, y) = f(x+y) - f(x) - f(y)\ .$$

If $f(x_\infty)$ is positive-definite, then $h(x_\infty) = f(x_\infty)$ is the only choice. At any rate under the above specialization (1) becomes the following theta series:

$$\sum_{\xi \in Z^n} e(\mathrm{Re}\,(z_\infty)f(\xi) + (-1)^{1/2}\,\mathrm{Im}\,(z_\infty)h(\xi))$$

and (2) becomes the following Eisenstein series:

$$1 + e((1/8)(p-q))|d|^{-(1/2)}\cdot\sum_{i^* \in Q} G(i^*)(z_\infty+i^*)^{-(1/2)p}(\bar{z}_\infty+i^*)^{-(1/2)q}\ ,$$

in which p, q are the numbers of positive and negative eigenvalues of the coefficient matrix of $f(x, y)$ and d is its determinant. Furthermore in the special case where $q=0$ (3) becomes

$$1 + (2\pi)^{(1/2)n}\Gamma\left(\frac{1}{2}n\right)^{-1} d^{-(1/2)}\cdot\sum_{i=1}^{\infty} S(i)i^{(1/2)n-1}e(iz_\infty)\ .$$

We recall that such series and their generalizations appeared in many works of Siegel.

5. A conjecture. In the special case where $f(x)$ is a quadratic form we know that $(E'-E)(U(g)\Phi)$ is bounded on G_A if Φ is restricted to a compact subset of $\mathscr{S}(X_A)$. We refer to Weil [12], Prop. 7 for the proof; we also refer to Siegel

[9] and Ariturk [1]. We have shown in the general case that $(E'-E)(U(g)\Phi)$ vanishes to the order $s-1$ at every k-rational boundary point of G_A and hence is bounded around every such point. Therefore it is reasonable to think that the boundedness is not restricted to the case of a quadratic form. We in fact propose the following weaker statement as a conjecture:

CONJECTURE. *For any given Φ in $\mathscr{S}(X_A)$ there exists a positive real number ε such that*

$$(E'-E)(U(g)\Phi) \prec |g|_A^{1-s+\varepsilon}$$

or at least

$$\int_{k_A/k} (E'-E)(U(g)\Phi)|du|_A \prec |g|_A^{1-s+\varepsilon}$$

on the subset of G_A defined by $|g|_A \leqq 1$.

We observe that $(E'-E)(U(u,t)\Phi)$ is a continuous function on $(k_A/k)\times k_A^\times$ and that the first hypothesis implies the second. We can prove this conjecture if n is sufficiently large compared to m. The point is that it may be true under the assumption $n>2m$ and it is very likely to be true if $n>m^2$ and that it implies the following generalization of the Hasse-Minkowski theorem:

"If the non-singular projective hypersurface defined by $f(x)=0$ has a k_p-rational point for every p, then it has a k-rational point".

(In this way we can reproduce Birch's result in [2] at least for a non-singular projective hypersurface.) The above implication can be proved as follows: we have

$$(E'-E)(\boldsymbol{t}(u)\Phi)=\sum_{i\in k} c_i(\Phi)\psi(iu)\ ,$$

in which

$$c_i(\Phi)=\sum_{\xi\in U(i)_k}\Phi(\xi)-\int_{U(i)_A}\Phi|\theta_i|_A$$

for every i in k. Since we also have

$$c_i(\Phi)=\int_{k_A/k}(E'-E)(\boldsymbol{t}(u)\Phi)\psi(-iu)|du|_A\ ,$$

if the conjecture is true, we will have

$$c_0(\boldsymbol{d}(t)\Phi)\prec |t|_A^{m(1-s+\varepsilon)}$$

on the subset of $k_A^\times$ defined by $|t|_A\leqq 1$. By using the O-symbol as $|t|_A\to 0$, this can be rewritten as

$$\sum_{\xi\in U(0)_k}\Phi(t\xi)=\left(\int_{U(0)_A}\Phi|\theta_0|_A+O(|t|_A^{m\varepsilon})\right)|t|_A^{m-n}\ .$$

We recall that for every i in k the support of the measure $|\theta_i|_A$ is the whole space $U(i)_A$. Therefore if $U(0)_A$ is not empty, for any $\Phi\geqq 0$ which is not the constant 0 on $U(0)_A$ its integral over $U(0)_A$ is positive; then the right hand side, hence also the left hand side, is positive for all small $|t|_A$. Therefore $U(0)_k$ can not possibly be empty; this completes the proof.

Appendix

One of the points we have made in this paper is that we can do something significant by using only such a small group as G_A. However the last section clearly indicates that a good generalization of the metaplectic group will be extremely useful. For that purpose we must start examining, as in Kubota [6], possible generalizations of the metaplectic group over a local field. Since the problem is still quite difficult, it was suggested to us by Shalika to examine the same problem over a finite field. In this appendix we shall explain our fragmental results in this simplest case.

We shall introduce a naive generalization of the metaplectic group over $K=\boldsymbol{F}_q$: let X denote an n-dimensional vector space over K and $L(X)$ the Hilbert space of (complex-valued) functions on X with the following norm:

$$\|\Phi\|^2=\text{card}\,(X)^{-(1/2)}\cdot\sum_{x\in X}|\Phi(x)|^2\ ;$$

then for every t in $K^\times$ the scalar multiplication by t in X defines a unitary operator in $L(X)$. Let ψ denote a non-trivial character of K and $f(x)$ a form of degree m on X, i.e., a homogeneous element of degree m of the symmetric algebra of the dual of X; then for every u in K the multiplication by $\psi(uf(x))$ in $L(X)$ also defines a unitary operator in $L(X)$. Finally let $[x, y]$ denote a symmetric non-degenerate K-bilinear form on $X\times X$; then

$$\Phi^*(x)=\text{card}\,(X)^{-(1/2)}\cdot\sum_{y\in X}\psi([x,y])\Phi(y)$$

defines a unitary operator in $L(X)$. We shall assume that $f(x)$ is non-degenerate, i.e., $f(x)$ is not a form on the quotient of X by a proper subspace; and we shall denote by Mp the subgroup of the full unitary group $\text{Aut}\,(L(X))$ generated by the above three types of unitary operators.

The structure of Mp is known for $m\leqq 2$: if $m=1$, we necessarily get $n=1$; and Mp becomes a semidirect product of a Heisenberg group of order q^3 by a generalized quaternion group of order $2(q-1)$. If $m=2$, q is odd, and $[x, y]=f(x, y)$, then Mp becomes a semidirect product of a cyclic group of order 2 or 4 (according as $q\equiv\pm1 \bmod 4$) by $SL_2(K)$. In the general case we have the following theorem:

THEOREM 5. *Suppose that* $n=1$ *and let* e *denote the G.C.D. of* m *and* $q-1$. *Then if* e *is even,* Mp *decomposes into* $\frac{1}{2}e+1$ *inequivalent irreducible representations of degrees*

$$\frac{q-1}{e}+1,\ \frac{q-1}{e},\ \frac{2(q-1)}{e},\ \ldots,\ \frac{2(q-1)}{e};$$

and if e *is odd,* Mp *decomposes into* $\frac{1}{2}(e+1)$ *inequivalent irreducible representations of degrees*

$$\frac{q-1}{e}+1,\ \frac{2(q-1)}{e},\ \ldots,\ \frac{2(q-1)}{e}.$$

PROOF. For a moment we shall drop the assumption that $n=1$; let T denote an element of $\mathrm{End}(L(X))$, i.e., a K-linear transformation of $L(X)$ to itself; let δ_y denote the Dirac function on X satisfying $\delta_y(y)=1$ and put $k(x, y)=(T\delta_y)(x)$; then we get

$$(T\Phi)(x)=\sum_{y\in X} k(x, y)\Phi(y)$$

for every Φ in $L(X)$. The set A of all T's which elementwise commute with Mp forms a subalgebra of $\mathrm{End}(L(X))$. Let $k(x, y)$ denote the kernel of T defined as above; then T is in A if and only if

($k1$) $k(tx, ty)=k(x, y)$ for every t in $K^\times$;

($k2$) $k(x, y)\neq 0$ only if $f(x)=f(y)$;

($k3$) $\sum_{z\in X} k(x, z)\phi([z, y])=\sum_{z\in X}\phi([x, z])k(z, y)$.

We shall closely examine these conditions resuming the assumption that $n=1$: if we identify X with K, we get $f(x)=cx^m$, $[x, y]=dxy$ for some c, d in $K^\times$. We define a function ϕ on $K^\times$ as $\phi(t)=k(1, t)$; then we get

($\phi1$) $\mathrm{Supp}(\phi)\subset(K^\times)_e$, i.e., $\phi(t)\neq 0$ only if $t^e=1$;

($\phi2$) $\phi(t)=\phi(t^{-1})$.

Conversely for every such ϕ if we define $k(x, y)$ as

$$k(x, y)=\begin{cases}\phi(x^{-1}y) & xy\neq 0\\ 0 & xy=0\ , \quad (x, y)\neq(0, 0)\\ \sum_t \phi(t) & (x, y)=(0, 0)\ ,\end{cases}$$

then k satisfies (k1)-(k3). And the correspondence $k \rightarrow \phi$ gives rise to a $\boldsymbol{C}$-linear bijection.

Consider the group ring $\boldsymbol{C}(K^{\times})$ of $K^{\times}$; then the correspondence

$$T \rightarrow k \rightarrow a = \sum_{t} \phi(t)t$$

gives rise to an injective algebra homomorphism of A to $\boldsymbol{C}(K^{\times})$. In particular A is commutative. Since Mp is completely reducible, therefore, every irreducible subrepresentation of Mp has multiplicity one. As for its degree, it can be determined as follows:

We observe that the $\boldsymbol{C}$-linear extension, say ω, of a character χ of $K^{\times}$ is an algebra homomorphism $\boldsymbol{C}(K^{\times}) \rightarrow \boldsymbol{C}$ mapping 1 to 1 and the correspondence $\chi \rightarrow \omega$ gives rise to a bijection. In particular there are $q-1$ such algebra homomorphisms. If we arrange them in some order, we get a representation of $\boldsymbol{C}(K^{\times})$ as the algebra of all diagonal matrices of degree $q-1$; and by restricting this representation to A we get a diagonalization of A.

After this remark suppose that ω, ω' correspond to χ, χ', respectively; then we get $\omega' = \omega$ on A if and only if $\chi' = \chi^{\pm 1}$ on $(K^{\times})_e$. Therefore the restriction of ω to A has multiplicity $(q-1)/e$ or $2(q-1)/e$ according as $\chi^2 = 1$ or $\chi^2 \neq 1$ on $(K^{\times})_e$. Moreover if ω_0 corresponds to the character 1 of $K^{\times}$, then we get

$$k(0, 0) = \sum_{t} \phi(t) = \omega_0(a) .$$

We have thus obtained a diagonalization of A as an algebra of q-by-q matrices from which we can read off the degrees of irreducible subrepresentations of Mp. q.e.d.

COROLLARY. *If q is odd, $e \geqq 3$, and $(e, q) \neq (3, 7)$, $(4, 5)$, then Mp is not a central extension of $SL_2(K)$; if q is odd and $e \geqq 5$, then every homomorphism of $SL_2(K)$ to Mp is trivial.*

The proof is as follows: we recall that the degrees of irreducible representations of $SL_2(K)$ are

$$1 , \quad \frac{1}{2}(q \pm 1) , \quad q \pm 1 , \quad q ;$$

cf. Tanaka [10]. And we have

$$\frac{1}{2}(q-1) > 2(q-1)/e$$

if (and only if) $e \geqq 5$. Therefore if $e \geqq 5$, the degree of any non-trivial irreducible representation of $SL_2(K)$ is larger than the degrees of irreducible subrepresentations of Mp. After this observation suppose that π is a homomorphism of $SL_2(K)$ to Mp; then π followed by any irreducible subrepresentation of Mp is necessarily a sum of the trivial representation of $SL_2(K)$. Therefore π is trivial; this proves the second part and also the first part for $e \geqq 5$ (because any central extension of $SL_2(K)$ splits). The proof of the remaining cases is similar.

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(Received September 5, 1975)

Department of Mathematics
The Johns Hopkins University
Baltimore, Md. 21218
U.S.A.