A note on reduction of positive operators, II

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(Communicated by S. Itô)

§ 1. Introduction and preliminaries

Extending the methods of I. Sawashima and F. Niiro [6], the author gave in the preceding note [4] a reduction theory for positive ergodic operators in the space of all continuous functions on a compact Hausdorff space. But the result is somewhat incomplete in the sense that the irreducible component obtained by the reduction is generally a positive operator in the space of continuous functions vanishing at infinity on a locally compact Hausdorff space. In this note the incompleteness is eliminated by giving the reduction theory for positive ergodic operators in an arbitrary (AM) space.

It seems to be useful to recall some basic facts about (AM) spaces before the exposition of the results.

An (AM) space E is a Banach lattice having the following property; $\|x\vee y\|=\mathrm{Max}\{\|x\|, \|y\|\}$ for $x, y\in E$, $x, y\geqq 0$. By the well-known representation theory of S. Kakutani [2], any (AM) space is isometric and lattice isomorphic to a closed sublattice of the space of all continuous functions on a compact Hausdorff space with its usual order and norm. Precisely speaking, for any (AM) space E, there exists a compact Hausdorff space X and a system of pairs of points x_α, x'_α ($x_\alpha, x'_\alpha\in X, \alpha\in A$) and real numbers c_α ($0\leqq c_\alpha\leqq 1, \alpha\in A$) such that E is isometric and lattice isomorphic to the space $C(X; x_\alpha, x'_\alpha, c_\alpha; \alpha\in A)$ of all bounded continuous real-valued functions $f(x)$ which are defined on X and satisfy the following relations;

$$f(x_\alpha)=c_\alpha f(x'_\alpha)\ , \qquad \alpha\in A\ ,$$

where A is a set of indices α whose power may be arbitrarily large. Without loss of generality, we may assume that the family $\{(x_\alpha, x'_\alpha, c_\alpha)\}_{\alpha\in A}$ is "saturated", i.e., it covers all the triple $(x, x', c)\in X\times X\times[0,1]$ which satisfies $f(x)=c\cdot f(x')$ for any $f\in C(X; x_\alpha, x_\alpha', c_\alpha; \alpha\in A)$.

From now on we fix an (AM) space E and identify it with its representation $C(X; x_\alpha, x_\alpha', c_\alpha; \alpha\in A)$, where the family $\{(x_\alpha, x_\alpha', c_\alpha)\}_{\alpha\in A}$ is saturated. For convenience we make the following

DEFINITION. *A closed set $Y\subset X$ is called an E-closed set if it satisfies the* *following two conditions.*

(i) $E_Y \equiv \{f \in C(Y)\,;\, f=g|_Y$ *for some* $g \in E\}$ *is closed in* $C(Y)$.

(ii) *For any* $\alpha \in A$ *such that* $c_\alpha \neq 0$, $x_\alpha \in Y$ *if and only if* $x_\alpha' \in Y$, *and* Y *contains any* x_α *if* $c_\alpha = 0$.

The following proposition shows that any closed order ideal of E is characterized by an E-closed set. (A linear subspace I of E is called an ideal if $x \in I$, $|y| \leqq |x|$ imply $y \in I$.)

PROPOSITION 1. *Let* I *be a closed order ideal of* E. *Then there exists an* E*-closed set* Y *for which* $I=\{f \in E\,;\, f=0$ *on* $Y\}$. *Moreover,* E/I *is isometrically isomorphic to* $E_Y \equiv \{f \in C(Y)\,;\, f=g|_Y$ *for some* $g \in E\}$.

PROOF. Let $Y=\{x \in X\,;\, f(x)=0$ for any $f \in I\}$, and $I_Y=\{f \in E\,;\, f=0$ on $Y\}$. We will show that $I=I_Y$. It is clear that $I \subset I_Y$. To show the converse inclusion it suffices to show that any $f \in I_Y$, $f \geqq 0$ belongs to I since I_Y is also a closed order ideal of E. Let $f \in I_Y$, $f \geqq 0$, and ε be an arbitrary positive number. Then the set $A_\varepsilon=\{x \in X\,;\, f(x) \geqq \varepsilon\}$ is a compact set disjoint from Y. Therefore for any $x \in A_\varepsilon$ there exists an $f_x \geqq 0$, $f_x \in I$, for which $f \leqq f_x$ holds in a neighbourhood of x. By the compactness A_ε, there exists a finite set of points of A_ε, $\{x_i\}_{i=1,2,\dots,k}$, and positive elements of I, $\{f_{x_i}\}_{i=1,\dots,k}$, for which $f \leqq \bigvee_{i=1}^{k} f_{x_i}$ holds on A_ε. Let $g=f \wedge \left(\bigvee_{i=1}^{k} f_{x_i}\right)$. Then $g \in I$ and $\|f-g\| \leqq \varepsilon$. This shows that $f \in I$, hence $I=I_Y$. Next we prove that E/I is isometrically isomorphic to E_Y. To show this it is sufficient to verify that for any positive $f \in E_Y$ and any positive number ε, there exists a positive extension $g \in E$ of f such that $\|g\| \leqq \|f\|+\varepsilon$. Let h be an arbitrary positive extension of f and $B_\varepsilon=\{x \in X\,;\, h(x) \geqq \|f\|+\varepsilon\}$. Then B_ε is a compact set disjoint from Y. Therefore for any $x \in B_\varepsilon$ there exists $f_x \in I$, $f_x \geqq 0$ for which $f_x \geqq h$ holds in a neighbourhood of x. By the same argument used in the former part of the proof, we get an element $k \in I$, $k \geqq 0$ such that $k \geqq h$ on B_ε. Let g be the positive part of $h-k$. Then g is an extension of f since $k=0$ on Y, and $\|g\| \leqq \|f\|+\varepsilon$. Thus E/I is isometrically isomorphic to E_Y, and hence Y is an E-closed set.//

The set Y defined in the proof of the above proposition is called the support of the closed order ideal I. The following proposition shows that such a set is nothing but an E-closed set.

PROPOSITION 2. *A closed set* Y *of* X *is the support of a closed order ideal if and only if it is an* E*-closed set.*

PROOF. It suffices to show that an E-closed set is the support of a closed order ideal, since the converse is proved in Proposition 1. Let Y be an E-closed set and $I_Y=\{f\in E\,;\ f=0 \text{ on } Y\}$, and Z be the support of I_Y. Then obviously $Y\subset Z$. On the other hand any point x of Z defines a lattice homomorphic functional on $E_Y=\{f\in C(Y)\,;\ f=g|_Y \text{ for some } g\in E\}$. Namely, the mapping

$$f=g|_Y\in E_Y\to g(x)$$

is well defined and lattice homomorphic by the definition of Z, and we denote it by ε_x. Since E_Y is closed in $C(Y)$, ε_x is a bounded linear functional on E_Y. If $\varepsilon_x=0$, $f(x)=0$ for any $f\in E$, hence $x\in Y$. So we may assume that ε_x is not zero. In this case, for a suitable positive constant c, $c\cdot\varepsilon_x$ is a nonzero extreme point of the positive portion of the unit ball in E'_Y (cf. [7] Chap. 5, 1.7). Next we show that there exists a point y of Y for which $\varepsilon_y=c\cdot\varepsilon_x$ holds. If not, there exists an element $f\in E_Y$ which satisfies

$$c\cdot\varepsilon_x(f)>\sup_{y\in Y}f(y)$$

(cf. [1] Chap. 2, §7 n°1, Proposition 2). If the positive part f_+ of f is nonzero, we have

$$c\cdot\varepsilon_x(f_+)>\sup_{y\in Y}f_+(y)=\|f_+\|\ ,$$

which contradicts $\|c\cdot\varepsilon_x\|=1$. On the other hand, if the positive part of f is zero, $0\leqq c\cdot\varepsilon_x(f_-)<\inf_{y\in Y}f_-(y)$, where f_- is the negative part of f. Then for any $g\in E_Y$, $g\geqq 0$,

$$g\leqq\frac{\|g\|}{\inf_{y\in Y}f_-(y)}f_-\ .$$

This implies $c\cdot\varepsilon_x(g)\leqq\dfrac{c\cdot\varepsilon_x(f_-)}{\inf_{y\in Y}f_-(y)}\|g\|$, which means $\|c\cdot\varepsilon_x\|<1$ since $|c\cdot\varepsilon_x(h)|=\operatorname{Max}\{c\cdot\varepsilon_x(h_+),\ c\cdot\varepsilon_x(h_-)\}$ holds for any $h\in E_Y$. Thus there exists a point $y\in Y$ for which $f(y)=c\cdot f(x)$ holds for any $f\in E$. Since Y is an E-closed set, this implies $x\in Y$, and hence $Z\subset Y$. Thus we have $Z=Y$.//

The following is an easy corollary of Proposition 1.

COROLLARY. *Let E be an* (AM) *space and I be an closed order ideal of E. Then E/I is also an* (AM) *space.*

§2. Reduction theory

Let E be as in §1 and T be a bounded operator in E satisfying the following conditions:

I) $T \geqq 0$,

II) $M_n \equiv \frac{1}{n}(I+T+\cdots+T^{n-1})$ converges strongly as $n \to \infty$.

(These conditions imply that the spectral radius of T is less than or equal to 1.)

Let P denote the limit operator of $\{M_n\}$. Then P is a positive projection whose range PE is the eigenspace of T for the eigenvalue 1, since $PT=TP=P$. A function p, which is important for the investigation of the structure of PE, is defined as follows.

DEFINITION. *p denotes the function on X whose value at $x \in X$ is* $\sup\{Pf(x) ; f \in S_+\}$, *where S_+ is the positive portion of the unit ball in E.*

It is clear that p is a positive lower semi-continuous function satisfying $p(x_\alpha)=c_\alpha p(x'_\alpha)$ for any $\alpha \in A$ and $p(x) \leqq \|P\|$ for any $x \in X$. Moreover p has the following property.

LEMMA. *Let $f \in E$ and $f \leqq p$. Then $Pf \leqq p$.*

PROOF. Let ε be an arbitrary positive number. Then for every $x \in X$ there exists a function $f_x \in S_+$ for which $f \leqq Pf_x+\varepsilon$ holds in a neighbourhood U_x of x. Since X is compact, it is covered by finite union of such U_x's, say $X=\bigcup_{i=1}^n U_{x_i}$. Then $g=\bigvee_{i=1}^n f_{x_i}$ belongs to S_+ and $f \leqq Pg+\varepsilon$ holds on X. This shows $\|(f-Pg)_+\| \leqq \varepsilon$. Since $f \leqq Pg+(f-Pg)_+$, $Pf \leqq Pg+P(f-Pg)_+ \leqq p+\varepsilon\|P\|$. By the arbitrariness of ε, this proves the lemma.//

A new norm on PE can be defined through the function p. That is, if we denote the number $\inf\{c ; -c \cdot p \leqq f \leqq c \cdot p\}$ by $\|f\|_0$, it is easy to see that $\|f\|_0$ really defines a norm on PE which is equivalent to the original norm induced by that on E. Hereafter whenever the space PE is concerned, its norm should be considered to be $\|\cdot\|_0$. The following proposition is easily proved by the lemma.

PROPOSITION 3. *Equipped with norm $\|\cdot\|_0$ and the order induced by that in E, PE is an* (AM) *space.*
(The supremum of $f, g \in PE$ in PE is $P(f \vee g)$ which will be denoted by $f \curlyvee g$ as in [4].)

PROPOSITION 4. *$(PE)'$ is isometrically isomorphic to $P'E'$ as a Banach lattice. In more detail, the mapping*

$$\psi \in (PE)' \to \phi=\psi \circ P \in P'E'$$

is an order preserving isometric linear mapping of $(PE)'$ onto $P'E'$, and the inverse mapping is the restriction of ϕ on the subspace PE.

PROOF. Since P is positive, the two mappings in the proposition are both positive, and are inverse to each other. By Proposition 3, $\||\psi|\|=\|\psi\|$ for $\psi\in(PE)'$. Therefore if we prove that $\|\phi\|=\|\psi\|$ holds for positive ψ, the proposition is completely proved. If $\psi\in(PE)'$ is positive, $\|\psi\|=\sup\{\psi(f)\,;\, f\in PE,\ f\geqq 0,\ \|f\|_0\leqq 1\}\geqq\sup\{\psi(Pf)\,;\, f\in S_+\}=\|\phi\|$, since $\|Pf\|_0\leqq 1$ for $f\in S_+$. On the other hand, for any $f\in PE$ and arbitrary positive number ε such that $f\geqq 0$ and $\|f\|_0\leqq 1$, there exists an element $g\in S_+$ which satisfies $\|(f-Pg)_+\|<\varepsilon$ as was shown in the proof of the lemma. Since ψ is the restriction of ϕ, this implies

$$\psi(f)\leqq\phi(Pg)+\phi((f-Pg)_+)\leqq(1+\varepsilon)\cdot\|\phi\|\ . \quad \text{Hence} \quad \|\psi\|=\|\phi\|\ .//$$

Let Φ be the set $\{\phi\,;\,\phi\in E',\ \phi\geqq 0,\ \|\phi\|\leqq 1,\ T'\phi=\phi\}$, and let Λ be the set of all nonzero extreme points of Φ. Then Φ is identified with the positive portion of the unit ball in $(PE)'$ by Proposition 4, since $T'\phi=\phi$ is equivalent to $P'\phi=\phi$. So an element $\phi\in\Phi$ belongs to Λ if and only if $\|\phi\|=1$ and ϕ is lattice homomorphic on PE, i.e., $\phi(f\veebar g)=\mathrm{Max}\,\{\phi(f),\ \phi(g)\}$ for any $f, g\in PE$ (cf. [7] Chap. 5, 1.7). For any $\lambda\in\Lambda$, let $I_\lambda=\{f\in E\,;\,\lambda(|f|)=0\}$ and $S_\lambda=\{x\in X\,;\, f(x)=0$ for any $f\in I_\lambda\}$ (=the support of I_λ), and let S be the closure of $\bigcup_{\lambda\in\Lambda}S_\lambda$. Then we have

PROPOSITION 5. *$P'\varepsilon_x=p(x)\cdot\lambda$ holds for any $\lambda\in\Lambda$ and $x\in S_\lambda$.*

PROOF. First we remark that $f\veebar g-f\vee g\in I_\lambda$ since $f\veebar g\geqq f\vee g$ and $\lambda(f\veebar g)=\lambda(f\vee g)$. Therefore $f\veebar g=f\vee g$ on S_λ, hence $P'\varepsilon_x$ is lattice homomorphic on PE and $\|P'\varepsilon_x\|=p(x)$ for $x\in S_\lambda$. This implies the existence of $\mu\in\Lambda$ for which $P'\varepsilon_x=p(x)\cdot\mu$. If we show that $p(x)\neq 0$ implies $\mu=\lambda$, the proposition is proved. Suppose that $p(x)\neq 0$ and $\mu\neq\lambda$. Then there exists a function $f\in PE$ such that $\lambda(f)=0$ and $\mu(f)\neq 0$, since if $\lambda(f)=0$ implies $\mu(f)=0$ for any $f\in PE$, μ is proportional to λ. As was remarked before the proposition, λ and μ are lattice homomorphic on PE. Hence replacing f by $f\veebar(-f)$, we have an element $f\in PE$ such that $f\geqq 0$, $\lambda(f)=0$ and $\mu(f)>0$. Then

$$f(x)=P'\varepsilon_x(f)=p(x)\mu(f)>0\ .$$

This contradicts $\lambda(f)=0$.//

For any $\lambda\in\Lambda$, let

$$X_\lambda=\{x\in X\,;\ P'\varepsilon_x=p(x)\cdot\lambda\}\ ,$$

and let

$$X_0=\{x\in X\,;\ f\veebar g(x)=\mathrm{Max}\,\{f(x),\, g(x)\}\quad\text{for any } f, g\in PE\}\ ,$$
$$N=\{x\in X\,;\ p(x)=0\}\ .$$

Then we have the following

THEOREM 1. i) *For any* $\lambda \in \Lambda$, S_λ *is a compact set and* $S_\lambda \cap S_\mu \cap N^c = \emptyset$ *if* $\mu \in \Lambda$ *and* $\mu \neq \lambda$. ii) *For any* $\lambda \in \Lambda$, X_λ *is a compact set and* $X_\lambda \cap X_\mu \cap N^c = \emptyset$ *if* $\mu \in \Lambda$ *and* $\mu \neq \lambda$. iii) X_0 *is compact and* $X_0 = \bigcup_{\lambda \in \Lambda} X_\lambda$. iv) *For any* $\lambda \in \Lambda$, *the following closed order ideals are all T-invariant.*

a) $\{f \in E\,;\ f=0$ *on* $S_\lambda\}$

b) $\{f \in E\,;\ f=0$ *on* $S\}$

c) $\{f \in E\,;\ f=0$ *on* $X_\lambda\}$

d) $\{f \in E\,;\ f=0$ *on* $X_0\}$

e) $\{f \in E\,;\ f=0$ *on* $N\}$.

PROOF. We prove only ii) and iv) c); the proof of the remaining part is not too difficult.

Let $\{x_\beta\}_{\beta \in B}$ be a net of points of X_λ converging to x_0. Then the net $\{P'\varepsilon_{x_\beta}\}_{\beta \in B}$ converges weakly* to $P'\varepsilon_{x_0}$. This implies that $P'\varepsilon_{x_0}$ is proportional to λ, since $P'\varepsilon_{x_\beta} = p(x_\beta)\lambda$. Considering the norm of $P'\varepsilon_{x_0}$, we have $P'\varepsilon_{x_0} = p(x_0)\lambda$, hence $x_0 \in X_\lambda$. This proves the first half of ii). The second half of ii) is clear from the definition of X_λ. To prove iv) c), it is sufficient to show that if $f \in E$, $f \geqq 0$ and $f=0$ on X_λ, then $Tf=0$ on X_λ is implied. Let f be such an element and ε be an arbitrary positive number. Since $N \subset X_\lambda$, the set $A_\varepsilon = \{x \in X\,;\ f(x) \geqq \varepsilon\}$ is a compact set disjoint from N, and the mapping $\tau : x \to P'\varepsilon_x / p(x)$ is defined on A_ε. $\tau(A_\varepsilon)$ is contained in Φ and its closure does not contain λ. This is proved as follows. If λ belongs to the closure of $\tau(A_\varepsilon)$, there exists a net $\{x_\beta\}_{\beta \in B}$ of points of A_ε for which the net $\{\tau(x_\beta)\}_{\beta \in B}$ converges weakly* to λ. Since $p(x)$ is uniformly bounded, this implies that $\{P'\varepsilon_{x_\beta} - p(x_\beta)\lambda\}_{\beta \in B}$ converges weakly* to 0. On the other hand, there exists a subnet $\{x_\gamma\}_{\gamma \in C}$ of $\{x_\beta\}_{\beta \in B}$ converging to some point $x \in A_\varepsilon$ by the compactness of A_ε. Then $\{P'\varepsilon_{x_\gamma}\}_{\gamma \in C}$ converges to $P'\varepsilon_x$, hence $\{p(x_\gamma)\}_{\gamma \in C}$ also converges. These imply that $P'\varepsilon_x$ is proportional to λ, and considering the norm of $P'\varepsilon_x$, we have $P'\varepsilon_x = p(x)\lambda$, which implies $x \in X_\lambda$ and contradicts $x \in A_\varepsilon$. Therefore there exists a $g \in PE$ such that $\lambda(g)=0$ and $\inf\{\phi(g)\,;\ \phi \in \tau(A_\varepsilon)\} > 0$ (cf. [1] Chap. 2, §7 n°1, Proposition 2). We may assume $g \geqq 0$ since λ is lattice homomorphic on PE. On the other hand, p has positive minimum on A_ε because it is lower semi-continuous. Consequently g has a positive minimum c on A_ε. Let $h = \frac{\|f\|}{c} g$ and $k = (f-h)_+$. Then $\|k\| \leqq \varepsilon$ and $f \leqq h+k$. Hence

$$0 \leqq Tf \leqq Th + Tk \leqq h + Tk\,.$$

Therefore $0 \leqq Tf(x) \leqq \varepsilon \|T\|$ on X_λ since $Th=h=0$ on X_λ, this implies c).//

By the above theorem, $I_\lambda = \{f \in E\,;\ f=0$ on $S_\lambda\}$ (cf. Proposition 1) is a T-invariant closed order ideal, hence it is also P-invariant. Consequently, T and P

naturally induce operators U_λ and Q_λ in E/I_λ respectively. Namely U_λ (resp. Q_λ) is defined as follows: $U_\lambda(\pi(f))=\pi(T(f))$ (resp. $Q_\lambda(\pi(f))=\pi(P(f))$) where $f\in E$ and π is the natural mapping of E onto E/I_λ. Then U_λ is a positive operator in E/I_λ, and it is clear that U_λ is also strongly ergodic with limit operator Q_λ. The eigenspace for the eigenvalue 1 of U_λ (resp. Q_λ) is 1 dimensional with the base $p(x)|_{S_\lambda}$ by Proposition 5. By Proposition 1, E/I_λ is identified with $E_{S_\lambda}=\{f\in C(S_\lambda)\,;\ f=g|_{S_\lambda}$ for some $g\in E\}$. Under this identification we can easily prove the following

PROPOSITION 6. *Let $K_\lambda=\{f\in E/I_\lambda\,;\ f=0$ on $S_\lambda\cap N\}$. Then K_λ is the smallest nonzero U_λ-invariant closed order ideal in E/I_λ.*

Proposition 6 shows that we have operators in K_λ restricting U_λ and Q_λ to K_λ. We denote the restriction of U_λ and Q_λ by T_λ and P_λ respectively. Then we have

THEOREM 2. *T_λ is an irreducible (i.e., having no nonzero proper closed invariant ideal), positive, strongly ergodic operator with limit operator P_λ.*

The following proposition will be useful in the next section.

PROPOSITION 7. *Let $I_0=\{f\,;\ P|f|=0\}$ and S_0 be the support of I_0. Then $S\subset S_0\subset X_0$.*

PROOF. If $P|f|=0$, $\lambda(P|f|)=\lambda(|f|)=0$ for any $\lambda\in\Lambda$, hence $f=0$ on S. This shows $S\subset S_0$. On the other hand, for any $f, g\in PE$, $f\,\dot\vee\, g-f\vee g\in I_0$, since $f\,\dot\vee\, g-f\vee g\geqq 0$ and $P(f\,\dot\vee\, g-f\vee g)=0$. This implies $S_0\subset X_0$.//

COROLLARY. *If P is strictly positive (i.e., $f\in E$, $P|f|=0$ imply $f=0$), then $X_0=X$.*

PROOF. The strict positivity of P implies $I_0=\{0\}$ hence $S_0=X$, thus we get $X_0=X$.//

REMARK. Contrary to the case of $E=C(X)$, the strict positivity of P is not equivalent to $S=X$ as the following example shows. (This example is due to Professor F. Niiro.)

EXAMPLE. Let X be the compact subset of $\boldsymbol{R}\times\boldsymbol{R}$ defined as $[0,1]\times[0,1]\cup\{(-1,0)\}$, and $E=\{f\in C(X)\,;\ f(-1,0)=(1/2)\cdot f(0,0)\}$. We define a positive projection operator P in E by the following formula;

$$Pf(x,y)=\int_0^1 g(z,y)f(z,y)dz\cdot h(x,y)\,,\quad (x,y)\in[0,1]\times[0,1]\,,$$

$$Pf(-1,0)=(1/2)Pf(0,0)\,,$$

where

$$g(x, y)=\begin{cases} y & 0\leqq x\leqq 1/2 \\ 8(1-y)(x-1/2)+y & 1/2\leqq x\leqq 1 \end{cases}$$

and $h(x, y)=1$ identically (or we may choose any continuous positive function such that $\int_0^1 g(x, y)h(x, y)dx=1$ for any $y\in[0,1]$). Then P satisfies the condition I) and II) in the first paragraph of this section. And the set Λ is identified with the set $[0,1]$ by the correspondence $y\in[0,1]\rightarrow\lambda_y$, where λ_y is the functional $f\in E\rightarrow\int_0^1 g(x, y)f(x, y)dx$. Then $S_{\lambda_y}=\{(x, y)\,;\ 0\leqq x\leqq 1\}$ for $y\in(0,1]$, and $S_{\lambda_0}=\{(x,0)\,;\ 1/2\leqq x\leqq 1\}$, hence $S=(\bigcup_{\lambda\in\Lambda} S_\lambda)^-=[0,1]\times[0,1]\neq X$ although P is strictly positive.

§ 3. Spectral properties on the unit circle

In this section we consider the relation between the spectrum of T and those of $\{T_\lambda\}_{\lambda\in\Lambda}$ or $\{U_\lambda\}_{\lambda\in\Lambda}$ obtained in § 2. To do this we introduce new operators $\{V_\lambda\}_{\lambda\in\Lambda}$ and T_0 which is induced by T in the same way as U_λ's. Precisely speaking, V_λ is the operator in the space E/J_λ where $J_\lambda=\{f\in E\,;\ f=0$ on $X_\lambda\}$ and $V_\lambda(\pi(f))=\pi(Tf)$ where $f\in E$ and π is the natural mapping of E onto E/J_λ; T_0 is the operator in E/J_0 where $J_0=\{f\in E\,;\ f=0$ on $X_0\}$ and T_0 is defined in the same way as V_λ. (These operators are defined unambiguously by the T-invariance of J_λ and J_0, which is proved in Theorem 1.) Then V_λ and T_0 satisfy the conditions I) and II) in the first paragraph of § 2, and the limit of $\frac{1}{n}(I+V_\lambda+\cdots+V_\lambda^{n-1})$ or $\frac{1}{n}(I+T_0+\cdots+T_0^{n-1})$ is induced by P in a similar way. Let ρ_∞ denote the unbounded connected component of the resolvent set. Then we have

PROPOSITION 8. i) *For any* $\lambda\in\Lambda$, $\rho_\infty(T_\lambda)\supset\rho_\infty(U_\lambda)\supset\rho_\infty(V_\lambda)\supset\rho_\infty(T)$, *and, if* $\alpha\in\rho_\infty(T)$, *then* $\|R(\alpha, T_\lambda)\|\leqq\|R(\alpha, U_\lambda)\|\leqq\|R(\alpha, V_\lambda)\|\leqq\|R(\alpha, T)\|$. ii) *If* $\alpha\in\rho_\infty(T)$ *and P is strictly positive, then* $\sup_{\lambda\in\Lambda}\|R(\alpha, V_\lambda)\|=\|R(\alpha, T)\|$.

PROOF. i) is the direct consequence of Lemma 2 and Corollary 1 in [5]. Under the assumption of ii), $X_0=X$ by Proposition 7. Let $c=\sup_{\lambda\in}\|R(\alpha, V_\lambda)\|$. Then $c\leqq\|R(\alpha, T)\|$ by i). Conversely for any $f\in E$, $\|f|_{X_\lambda}\|\leqq c\|(\alpha-T)f|_{X_\lambda}\|$ holds for any $\lambda\in\Lambda$ by the definition of c. This implies $\|R(\alpha, T)\|\leqq c$.//

Using the above proposition, we can easily prove the following

PROPOSITION 9. *Let Γ denote the unit circle in* $\boldsymbol{C}$. *Then*

$$\sigma(T)\cap\Gamma\supset(\bigcup_{\lambda\in\Lambda}\sigma(V_\lambda))^-\cap\Gamma\supset(\bigcup_{\lambda\in\Lambda}\sigma(U_\lambda))^-\cap\Gamma\supset(\bigcup_{\lambda\in\Lambda}\sigma(T_\lambda))^-\cap\Gamma\,.$$

In the rest of this section we show that the inclusion in Proposition 9 is replaced by equality if we replace the condition II) for T by the following stronger condition;

II)′ T is uniformly ergodic, i.e., $\frac{1}{n}(I+T+\cdots+T^{n-1})$ converges uniformly as $n\to\infty$.

This condition implies the following II)″ under the condition I). (Cf. [3] Theorem 4.)

II)″ $R(\alpha, T)$ has a pole of order at most 1 at $\alpha=1$.

And the limit operator P of $\frac{1}{n}(I+T+\cdots+T^{n-1})$ is equal to the residual operator of $R(\alpha, T)$, which is defined by $\frac{1}{2\pi i}\int_{|\alpha-1|=r} R(\alpha, T)d\alpha$ where r is a positive number such that $\{\alpha\,;\ 0<|\alpha-1|\leqq r\}\subset\rho(T)$.

Hereafter we assume that the operator T satisfies the conditions I) and II)′. Then we can apply the results in §2 for T, and we still use the same notations defined in the various stage of reduction, such as Λ, V_λ, T_λ, T_0. Then combining Proposition 7 in §2 and Lemma 2 in [4], we can easily prove the following

PROPOSITION 10. *Let T be a uniformly ergodic positive operator in an* (AM) *space E. Then the following relations hold*:

$$\sigma(T)\cap\Gamma=\sigma(T_0)\cap\Gamma\ ,$$
$$R_\sigma(T)\cap\Gamma=R_\sigma(T_0)\cap\Gamma\ ,$$
$$P_\sigma(T)\cap\Gamma=P_\sigma(T_0)\cap\Gamma\ ,$$
$$C_\sigma(T)\cap\Gamma=C_\sigma(T_0)\cap\Gamma\ ,$$

where T_0 is the operator defined in the first paragraph of this section.

We also have

PROPOSITION 11. *Let $\alpha_0\in\Gamma$ satisfy the following condition*;

$\alpha_0\in\rho(V_\lambda)$ *for any* $\lambda\in\Lambda$ *and* $\sup_{\lambda\in\Lambda}\|R(\alpha, V_\lambda)\|<\infty$.

Then $\alpha_0\in\rho(T)$.

PROOF. By the assumption and Lemma 3 in [6], there exists a positive number d such that $\sup_{\lambda\in\Lambda}\|R(\alpha, V_\lambda)\|$ is bounded in the set $\{\alpha\,;\ |\alpha-\alpha_0|<d\}$. Since $\|R(\alpha, T_0)\|=\sup_{\lambda\in\Lambda}\|R(\alpha, V_\lambda)\|$, $\|R(\alpha, T_0)\|$ is bounded by the same upper bound in the set $\{\alpha\,;\ |\alpha-\alpha_0|<d, |\alpha|>1\}$, hence $\alpha_0\in\rho(T_0)$. Together with Proposition 10, this implies $\alpha_0\in\rho(T)$.//

Although the following proposition is the key to the main result of this section, we give only the outline of the proof since it is almost parallel to that of Theorem 8 in [4].

PROPOSITION 12. *Let T be a uniformly ergodic positive operator in an* (AM) *space E. If Λ is the set of all nonzero extreme points of the set of positive T'-invariant functionals in the unit ball of E', then*

$$\sigma(T)\cap\Gamma=(\bigcup_{\lambda\in\Lambda}\sigma(V_\lambda))^-\cap\Gamma$$

where V_λ is the operator defined in the first paragraph of this section.

PROOF. Since the inclusion

$$\sigma(T)\cap\Gamma\supset(\bigcup_{\lambda\in\Lambda}\sigma(V_\lambda))^-\cap\Gamma$$

is proved in Proposition 9, it suffices to show the inverse inclusion which is equivalent to

$$\rho(T)\supset(\bigcap_{\lambda\in\Lambda}\rho(V_\lambda))^\circ\cap\Gamma .$$

Let α_0 be in $(\bigcap_{\lambda\in\Lambda}\rho(V_\lambda))^\circ\cap\Gamma$. By Proposition 11 it is enough to show that the assumption of unboundedness of the set $\{\|R(\alpha, V_\lambda)\|\,;\,\lambda\in\Lambda\}$ yields a contradiction. We shall show this in the following four steps.

The first step: Let r and b be positive numbers satisfying

$$\{\alpha\,;\,|\alpha-\alpha_0|<r\}\subset\bigcap_{\lambda\in\Lambda}\rho(V_\lambda)\,,\qquad\{\alpha\,;\,0<|\alpha-1|<r\}\subset\rho(T)$$

and

$$\sup_{\alpha>1}\|R(\alpha, T)(I-P)\|\leqq b\,.$$

Let s be a positive number less than r and $1/2b$. Then by the argument as in the first step of the proof of Theorem 8 in [4], there exists an α_1 and a sequence $\{\lambda_n\}$ of elements of Λ such that $|\alpha_1-\alpha_0|<s$ and $\|R(\alpha_0, V_{\lambda_n})\|>n$, $\|R(\alpha_1, V_{\lambda_n})\|>n$ hold for any n.

The second step: From the sequence $\{V_{\lambda_n}\}$ obtained in the first step, we construct a new (AM) space $\tilde{E}$ and a positive operator $\tilde{T}$ in $\tilde{E}$ following the method described before Lemma 4 in [4]. Let $\tilde{J}$ denote the closed order ideal in $\tilde{E}$ generated by the eigenspace of $\tilde{T}$ for the eigenvalue 1. Then $\tilde{J}$ is the minimal $\tilde{T}$-invariant closed order ideal, hence the restriction of $\tilde{T}$ to $\tilde{J}$, which is denoted by $\tilde{T}|_{\tilde{J}}$, is an irreducible positive operator having the following properties; $R(\alpha, \tilde{T}|_{\tilde{J}})$ has a simple pole at $\alpha=1$, and $\{\alpha\,;\,0<|\alpha-1|<r\}\subset\rho(\tilde{T}|_{\tilde{J}})$. (To

prove that $\tilde{J}$ is the smallest nonzero closed invariant ideal we use the fact that the eigenspace for the eigenvalue 1 of V_λ is 1-dimensional with the base $p(x)|_{X_\lambda}$, where $p(x)$ is the function defined in § 2.)

The third step: α_0 and α_1 are shown to belong to $P_\sigma(\tilde{T})$ by the same way as in the third step of the proof of Theorem 8 in [4].

The fourth step: Applying Lemma 4 in [4] to the results of the third step, we have $\alpha_0, \alpha_1 \in \sigma(\tilde{T}|_{\tilde{J}})$. This contradicts the fundamental property of irreducible positive operators (cf. [6] Lemma 6) since $\rho(\tilde{T}|_{\tilde{J}}) \supset \{\alpha\,;\ 0<|\alpha-1|<r\}$ and $s<r$.//

Using this proposition we have the following

THEOREM 3. *Let T be a uniformly ergodic positive operator in an* (AM) *space E, and let Λ be the set of all nonzero extreme points of the set of all positive T'-invariant functionals in the unit ball of E'. Then for any $\lambda \in \Lambda$, there exists a positive irreducible operator T_λ which is induced by T in the sense of the definition in* § 3 *in* [4], *and the following relation holds*;

$$\sigma(T) \cap \Gamma = \Big(\bigcup_{\lambda \in \Lambda} \sigma(T_\lambda)\Big)^- \cap \Gamma\ .$$

PROOF. Let T_λ be the operator defined before Theorem 2 in § 2, and r be a positive number such that $\{\alpha\,;\ 0<|\alpha-1|<r\} \subset \rho(T)$. Then if we denote the set $\{\alpha\,;\ |\alpha|>1-r\}$ by B, we have $\sigma(V_\lambda) \cap B = \sigma(T_\lambda) \cap B$. This is proved as follows. By Proposition 8, $R(\alpha, V_\lambda)$ has a simple pole at $\alpha=1$ and $\{\alpha\,;\ 0<|\alpha-1|<r\} \subset \rho(V_\lambda)$. As described in the first paragraph of this section, V_λ is defined in the space $E_\lambda = E/J_\lambda$ where $J_\lambda = \{f \in E\,;\ f=0 \text{ on } X_\lambda\}$, and the residual operator of V_λ, which we denote by W_λ, is induced by P. Let $L_\lambda = \{f \in E\,;\ W_\lambda|f|=0\}$. Then using Proposition 1 and the definition of X_λ, we can easily prove that E_λ/L_λ is isometric and lattice isomorphic to $E_{S_\lambda} = \{f \in C(S_\lambda)\,;\ f=g|_{S_\lambda} \text{ for some } g \in E\}$. Hence the operator U_λ defined before Proposition 6 is identified with the operator in E_λ/L_λ induced by V_λ. Together with Lemma 2 in [4], this shows that $\sigma(U_\lambda) \cap B = \sigma(V_\lambda) \cap B$. Since T_λ is the restriction of U_λ to the closed U_λ-invariant order ideal generated by the eigenspace of U_λ for the eigenvalue 1 (this is clear from the definition of K_λ in Proposition 6), we may show $\sigma(T_\lambda) \cap B = \sigma(U_\lambda) \cap B$ using Lemma 3 in [4]. Hence we have $\sigma(T_\lambda) \cap B = \sigma(V_\lambda) \cap B$; accordingly $\big(\bigcup_{\lambda \in \Lambda} \sigma(T_\lambda)\big)^- \cap \Gamma = \big(\bigcup_{\lambda \in \Lambda} \sigma(V_\lambda)\big)^- \cap \Gamma = \sigma(T) \cap \Gamma$ by Proposition 12.//

Using Theorem 3, we get the following theorem in a similar way as in the case of Theorem 7 in [6].

THEOREM 4. *Let T be a uniformly ergodic positive operator in an* (AM) *space E. Then $\sigma(T) \cap \Gamma$ is a finite set. If $\alpha_0 \in \Gamma$ is an isolated point of $\sigma(T)$, α_0 is a pole $R(\alpha, T)$ of order* 1.

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(Received September 11, 1975)

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