# *Remarks on hyperfunctions with analytic parameters*

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## Introduction

In these notes we give some elementary results on hyperfunctions with analytic parameters. Among them we discuss the problem of unique continuation with respect to the analytic parameters. Let $f(x', x_n)$ be a hyperfunction which contains $x_n$ as a real analytic parameter. Then we can take the countably many initial data $D_n^k f(x', 0)$, $k=1, 2, \ldots$. M. Sato has shown by an example that these data do not determine $f(x', x_n)$, and at the same time conjectured that the continuously many data $J(D_n)f(x', 0)$ would be sufficient to determine $f$, where $J(D_n)$ runs over the local operators with constant coefficients of the normal derivative $D_n$. This conjecture is affirmatively solved in §3, but to my regret in somewhat weakened form.

Many elementary lemmas concerning hyperfunctions arise in the course of the proof, whose provision is also a purpose of this paper. Especially, in §2 we introduce some concept of limit in the theory of hyperfunctions by way of the general theory of boundary value problem for linear partial differential equations. This concept of limit, suggesting further possibility of applications, proves to be a very useful tool for our present purpose.

A part of these results was announced in [2] with a brief sketch of the proof. Some of these results will be applied in future to the theory of continuation of real analytic solutions of linear partial differential equations. (See [3].)

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## § 1. Provision on Fourier hyperfunctions and real analytic parameters

We shortly review here the definitions and the properties used in the sequel on these materials. For the details see the indicated references. First we review on Fourier hyperfunctions. These concepts were first introduced by Sato [10 bis] with the proof only for $n=1$. Later Kawai [4] has given detailed proofs for most part of Sato's assertion.

$\boldsymbol{D}^n$ denotes the directional compactification of $\boldsymbol{R}^n$ : $\boldsymbol{D}^n=\boldsymbol{R}^n\cup S_\infty^{n-1}$. $\tilde{\mathcal{O}}$ denotes the sheaf on $\boldsymbol{D}^n\times i\boldsymbol{R}^n$ defined in the following way: For an open set $U\subset\boldsymbol{D}^n\times i\boldsymbol{R}^n$, $\tilde{\mathcal{O}}(U)$ consists of functions holomorphic on $U\cap\boldsymbol{C}^n$ and satisfying

$$\sup_{z\in K\cap C^n}|f(z)|e^{-\varepsilon|z|}<\infty\,, \tag{1.1}$$

for any $\varepsilon>0$ and for any $K\Subset U$. $\underset{\sim}{\mathcal{O}}$ denotes the sheaf on $\boldsymbol{D}^n\times i\boldsymbol{R}^n$ defined in the following way: For an open set $U\subset\boldsymbol{D}^n\times i\boldsymbol{R}^n$, $\underset{\sim}{\mathcal{O}}(U)$ consists of functions holomorphic on $U\cap\boldsymbol{C}^n$ such that for any $K\Subset U$, there exists some $\varepsilon_K>0$ satisfying

$$\sup_{z\in K\cap C^n}|f(z)|e^{\varepsilon_K|z|}<\infty\,. \tag{1.2}$$

Further, for a fixed $\varepsilon>0$ we define $\tilde{\mathcal{O}}^{-\varepsilon}$ by replacing in (1.2) $\varepsilon_K$ by $\varepsilon'$ which runs independently of $K$ satisfying $\varepsilon'<\varepsilon$.

For a real open set $U\subset\boldsymbol{D}^n$, we employ the notation $\mathcal{P}(U)=\tilde{\mathcal{O}}(U)$ and $\mathcal{P}_*(U)=\underset{\sim}{\mathcal{O}}(U)$. Note that the elements of $\mathcal{P}(\boldsymbol{D}^n)$ or $\mathcal{P}_*(\boldsymbol{D}^n)$ are holomorphic on a strip with a fixed breadth around the real axis and satisfy (1.1) or (1.2) there. In the sequel we call the elements $\mathcal{P}(\boldsymbol{D}^n)$ or $\mathcal{P}_*(\boldsymbol{D}^n)$ the slowly increasing real analytic functions or the rapidly decreasing real analytic functions respectively. It is evident that by the natural inductive limit topology $\mathcal{P}_*(\boldsymbol{D}^n)$ is a (DFS) space, i.e., the inductive limit of a compact sequence. As for $\mathcal{P}(\boldsymbol{D}^n)$, it is the inductive limit of the sequence of (FS) spaces $\tilde{\mathcal{O}}(\{|\mathrm{Im}\,z|<1/k\})$. But this sequence is not compact. Hence $\mathcal{P}(\boldsymbol{D}^n)$ is not (DFS) and its topological property is not studied well. Since we need not much knowledge on the topology of this space, we will employ a less definitive expression on this point.

We define the sheaf of Fourier hyperfunctions $\mathcal{Q}$ on $\boldsymbol{D}^n$ by $\mathcal{Q}=\mathcal{H}^n_{\boldsymbol{D}^n}(\tilde{\mathcal{O}})$. Roughly speaking, an element $f(x)$ of $\mathcal{Q}(U\cap\boldsymbol{D}^n)$ is given as the boundary value of a function $F(z)$ in $\tilde{\mathcal{O}}(U\#\boldsymbol{D}^n)$ to $U\cap\boldsymbol{D}^n$ in the way $f(x)=\sum_\sigma \mathrm{sgn}\,\sigma F(x+i\sigma 0)$, where $\sigma=(\sigma_1,\dots,\sigma_n)$, $\sigma_j=\pm1$, $\mathrm{sgn}\,\sigma=\sigma_1\dots\sigma_n$, and

$$U\#\boldsymbol{D}^n=\{z\in U;\ \mathrm{Im}\,z_j\neq0,\quad j=1,\dots,n\}\,.$$

$F(z)$ is called a defining function of $f(x)$. We further define the sheaf of rapidly decreasing Fourier hyperfunctions $\mathcal{Q}_*$ on $\boldsymbol{D}^n$ by

$$\mathcal{Q}_* = \varinjlim_{\varepsilon} \mathcal{H}^n_{\boldsymbol{D}^n}(\tilde{\mathcal{O}}^{-\varepsilon}).$$

Though the latter concept is not discussed in [4], it is easily accepted because the cohomological property of $\tilde{\mathcal{O}}^{-\varepsilon}$ is connected with that of $\tilde{\mathcal{O}}$ by means of multiplication by $\exp(-\varepsilon\sqrt{z_1^2+\cdots+z_n^2+1})$ on a neighborhood of $\boldsymbol{D}^n$. In fact we can consider that

$$\mathcal{Q}_* = \varinjlim_{\varepsilon} \exp(-\varepsilon\sqrt{x_1^2+\cdots+x_n^2+1})\mathcal{Q}.$$

It is known that $\mathcal{Q}(\boldsymbol{D}^n)$ becomes an (FS) space as the dual of $\mathcal{P}_*(\boldsymbol{D}^n)$. As for $\mathcal{Q}_*(\boldsymbol{D}^n)$ we endow with the topology induced by the inductive limit.

As usual, $\mathcal{A}(\boldsymbol{R}^n)$ denotes the space of real analytic functions on $\boldsymbol{R}^n$. Note that as the spaces of global sections we have $\mathcal{A}(\boldsymbol{R}^n)=\mathcal{P}(\boldsymbol{R}^n)=\mathcal{P}_*(\boldsymbol{R}^n)$. Hence elements of this space are not necessarily holomorphic on any strip. $\mathcal{B}_*(\boldsymbol{R}^n)$ denotes the space of hyperfunctions with compact support in $\boldsymbol{R}^n$. For a fixed compact set $K\subset\boldsymbol{R}^n$, $\mathcal{B}[K]$ denotes the space of hyperfunctions with support in $K$. $\mathcal{B}(\boldsymbol{R}^n)$ denotes the space of hyperfunctions on $\boldsymbol{R}^n$. Then we have $\mathcal{B}(\boldsymbol{R}^n)=\mathcal{Q}(\boldsymbol{R}^n)=\mathcal{Q}_*(\boldsymbol{R}^n)$. The restriction $\mathcal{Q}(\boldsymbol{D}^n)\to\mathcal{B}(\boldsymbol{R}^n)$ or $\mathcal{Q}_*(\boldsymbol{D}^n)\to\mathcal{B}(\boldsymbol{R}^n)$ is defined by forgetting the growth condition in the defining functions.

We employ as the Fourier transform $\mathcal{F}[u](\xi)=\tilde{u}(\xi)=\int_{\boldsymbol{R}^n} e^{ix\xi}u(x)dx$, and as its inverse $\mathcal{F}^{-1}[v](x)=(2\pi)^{-n}\int_{\boldsymbol{R}^n} e^{-ix\xi}v(\xi)d\xi$, where $x\xi=x_1\xi_1+\cdots+x_n\xi_n$. As for the derivatives, we employ the notation $D=(D_1,\cdots,D_n)$ with $D_j=i\partial/\partial x_j$, and $D^\alpha=D_1^{\alpha_1}\ldots D_n^{\alpha_n}$ for the multi-index $\alpha=(\alpha_1,\cdots,\alpha_n)$.

LEMMA 1.1. 1) *The rows of the diagram*

$$\begin{array}{ccccc}
\mathcal{B}_*(\boldsymbol{R}^n) & \subset & \mathcal{Q}_*(\boldsymbol{D}^n) & \subset & \mathcal{Q}(\boldsymbol{D}^n) \\
\updownarrow & & \updownarrow & & \updownarrow \\
\mathcal{A}(\boldsymbol{R}^n) & \supset & \mathcal{P}(\boldsymbol{D}^n) & \supset & \mathcal{P}_*(\boldsymbol{D}^n)
\end{array}$$

*consist of continuous inclusions with dense range. Between the corresponding terms in each column, there is a separately continuous inner product which is the extension of* $\int_{\boldsymbol{R}^n} f(x)g(x)dx$ *for* $f(x)\in C_0^\infty(\boldsymbol{R}^n)$ *and* $g(x)\in\mathcal{P}_*(\boldsymbol{D}^n)$.

2) *Let $K$ be a compact subset of $\boldsymbol{D}^n$. Let $\mathcal{Q}[K]$ be the space of Fourier hyperfunctions with support in $K$. Let $\mathcal{P}_*(K)$ be the space of rapidly decreasing*

*real analytic functions defined on a neighborhood of $K$. Then $\mathcal{Q}[K]$ is* (FS), *$\mathcal{P}_*(K)$ is* (DFS) *and these are dual to each other. The inner product is the natural extension of the one given in* 1). *If $K\subset \boldsymbol{R}^n$, this relation reduces to the usual one between $\mathcal{B}[K]$ and $\mathcal{A}(K)$.*

3) *The Fourier transform maps $\mathcal{P}_*(\boldsymbol{D}^n)$, $\mathcal{Q}_*(\boldsymbol{D}^n)$ and $\mathcal{Q}(\boldsymbol{D}^n)$ isomorphically onto $\mathcal{P}_*(\boldsymbol{D}^n)$, $\mathcal{P}(\boldsymbol{D}^n)$ and $\mathcal{Q}(\boldsymbol{D}^n)$ respectively. The Parseval formula holds for $f\in\mathcal{P}_*(\boldsymbol{D}^n)$ and $g\in\mathcal{Q}(\boldsymbol{D}^n)$, or for $f\in\mathcal{Q}_*(\boldsymbol{D}^n)$ and $g\in\mathcal{P}(\boldsymbol{D}^n)$.*

4) *The convolution $f*g$ is defined as the inverse Fourier image of the product. It is a separately continuous bilinear mapping from $\mathcal{P}_*(\boldsymbol{D}^n)\times\mathcal{Q}(\boldsymbol{D}^n)$ to $\mathcal{P}(\boldsymbol{D}^n)$, from $\mathcal{Q}_*(\boldsymbol{D}^n)\times\mathcal{P}(\boldsymbol{D}^n)$ to $\mathcal{P}(\boldsymbol{D}^n)$ or from $\mathcal{Q}_*(\boldsymbol{D}^n)\times\mathcal{Q}(\boldsymbol{D}^n)$ to $\mathcal{Q}(\boldsymbol{D}^n)$. In the former two cases, the result is pointwise determined by $\langle f(x-t), g(t)\rangle_t$ or by $\langle f(t), g(x-t)\rangle_t$.*

5) *The sheaves $\mathcal{Q}$, $\mathcal{Q}_*$ are flabby. For any open set $U\subset\boldsymbol{D}^n$, we have $H^k(U,\mathcal{P})=H^k(U,\mathcal{P}_*)=0$ for $k\geq 1$.*

The proofs of these assertions are either contained in [4] or easily derived from the results there.

REMARK 1.2. The dual of $\mathcal{Q}_*(\boldsymbol{D}^n)$ is not $\mathcal{P}(\boldsymbol{D}^n)$ but a little larger space

$$\mathcal{P}^*(\boldsymbol{D}^n)=\varprojlim_{\varepsilon}\exp(\varepsilon\sqrt{x_1^2+\cdots+x_n^2+1})\mathcal{P}_*(\boldsymbol{D}^n)\,.$$

The latter agrees with the Fourier image of $H^n_{\boldsymbol{D}^n}(\boldsymbol{D}^n\times i\boldsymbol{R}^n, \underset{\sim}{\mathcal{O}})$, which is a little larger than $\mathcal{Q}_*(\boldsymbol{D}^n)$. As the rapidly decreasing Fourier hyperfunctions we preferred $\mathcal{Q}_*(\boldsymbol{D}^n)$ because of many profits. For example, in calculating the Fourier transform of $f(x)$ as the Laplace transform by dividing the support of $f(x)$ to cones, we can employ a more elementary partition (e.g., $f(x)=\sum_{\sigma}\varphi_{\sigma_1}(x_1)\cdots\varphi_{\sigma_n}(x_n)f(x)$, where $\varphi_+(x)=e^x/(e^x+1)$ and $\varphi_-(x)=1/(e^x+1)$) permitting the exudation of support in $\mathcal{Q}_*(\boldsymbol{D}^n)$. The ambiguity is transformed into zero cohomology class by the Fourier transform.

We will employ many other properties on Fourier hyperfunctions. We will also employ the local operators with constant coefficients, a class of differential operators of infinite order. These are briefly listed up at §1 in [1], so that we will not repeat them here.

Next we recall the concept of the real analytic parameters. For the general references see [11], [12] or [13]. We say that a hyperfunction $u(x)$ of $n$-variables $x=(x_1,\cdots,x_{n-1},x_n)$ contains $x_n$ as a real analytic parameter at the origin if the singular spectrum S.S.$u$ of $u$ does not contain the two points $(0,\pm idx_n\infty)\in i\boldsymbol{S}^*_\infty\boldsymbol{R}^n=\boldsymbol{R}^n\times i\boldsymbol{S}^{*n-1}_\infty$, namely, if on a neighborhood $U$ of the origin, $u(x)$ can be written as the sum of the boundary values

$$u(x)=\sum_{j=1}^{N}F_j(x+i\Gamma_j0),$$

where each $\Gamma_j\subset R_y^n$ is a convex open cone with the vertex at the origin and with a non-void intersection with the plane $\{y_n=0\}$; $F_j(z)$ is the defining function holomorphic in $U\times i\{\Gamma_j\cap\{|y|<\varepsilon\}\}$ for some $\varepsilon>0$. In terms of the standard covering, $u(x)$ contains $x_n$ as a real analytic parameter at the origin if for some complex neighborhood $V$ of the origin there exists a defining function $F(z)$ of $u$ on $V\sharp R^n$ such that each component $F_\sigma(z)$ on $V_\sigma$ can be holomorphically extended over the side $\{\operatorname{Im} z_n=0\}$ to another wedge. Here

$$V\sharp R^n=\{z\in V;\quad \operatorname{Im} z_j\neq 0,\quad j=1,\cdots,n\},$$

and

$$V_\sigma=\{z\in V;\quad \sigma_j\operatorname{Im} z_j>0,\quad j=1,\cdots,n\},$$

with a multi-signature $\sigma=(\sigma_1,\cdots,\sigma_n)$; $\sigma_j=\pm 1$. Hence $F_\sigma(z)$ is extended to an open set of the following type: with some $\lambda>0$,

$$\{z\in V;\quad \sigma_j\operatorname{Im} z_j>0,\quad \sigma_n\operatorname{Im} z_n>-\lambda\sigma_j\operatorname{Im} z_j,\quad j=1,\cdots,n-1\}. \tag{1.4}$$

Let $U\subset R^n$ be open. We say that a hyperfunction $u\in\mathcal{B}(U)$ contains $x_n$ as a real analytic parameter if it satisfies the above condition at every point of $U$.

If $u(x)$ contains $x_n$ as a real analytic parameter, then for any partial differential operator $p(x,D)$ with real analytic coefficients, also $p(x,D)u(x)$ contains $x_n$ as a real analytic parameter, as is easily shown by defining functions. Moreover we can apply a local operator $J(D)$ instead of $p(x,D)$. Sato's fundamental theorem (e.g., [13] Chapter III, Corollary 2.1.2) asserts that if $\pm i dx_n\infty$ is a non-characteristic direction with respect to a partial differential operator $p(x,D)$, then every hyperfunction solution $u(x)$ of $p(x,D)u(x)=0$ contains $x_n$ as a real analytic parameter.

If $u(x',x_n)$ is a hyperfunction on a cylindrical domain $U\times\{|x_n|<\delta\}$ and contains $x_n$ as a real analytic parameter, then we can take the specialization (restriction) $u(x',0)\in{}'\mathcal{B}(U)$ to the hyperplane $\{x_n=0\}$, where ${}'\mathcal{B}(U)$ denotes the hyperfunctions of the first $(n-1)$-variables $x'=(x_1,\cdots,x_{n-1})$ on the open set $U\subset R^{n-1}$. By what is mentioned above, we can also take the higher order specializations $(\partial/\partial x_n)^k u(x)|_{x_n=0}$, or more generally $J(D)u(x)|_{x_n=0}$. The specialization is given by way of the defining function which is obtained by restricting to $z_n=0$ the components of the original defining function $F(z)$ after extending them to the open sets of the form (1.4). By the localized version of Bochner's tube theorem, we can show that various defining functions give the same result. Hence the specialization is a locally determined sheaf homomorphism $\mathcal{B}|_{x_n=0}\to{}'\mathcal{B}$.

LEMMA 1.3. *Let $u(x)$ be a hyperfunction on $R^{n-1}\times\{|x_n|<\delta\}$ containing $x_n$ as a real analytic parameter. Suppose that* $\operatorname{supp} u\subset K\times\{|x_n|<\delta\}$, *where $K\subset R^{n-1}$ is compact. Let $U$ be a relatively compact open neighborhood of $K$ in $R^{n-1}$. Let $0<\delta'<\delta$. Then we can find $\varepsilon>0$ and open convex cones $\Gamma_j\subset R^n_y$, $j=1,\dots,N$, with vertex at the origin, each having a non-void intersection with $\{y_n=0\}$, and functions $F_j(z)$ each holomorphic on $U\times\{|x_n|<\delta'\}\times i\{\Gamma_j\cap\{|y|<\varepsilon\}\}$ and real analytic up to $(U\backslash K)\times\{|x_n|<\delta'\}$ such that*

$$u(x)=\sum_{j=1}^{N}F_j(x+i\Gamma_j0)\,. \tag{1.5}$$

PROOF. We can divide the compact set $S=\text{S.S.}(u(x)|_{R^{n-1}\times\{|x_n|\le\delta'\}})$ into the union of $2^n$ subsets $S_\sigma=S\cap(R^n\times\{i\xi dx\infty;\ \sigma_j\xi_j>0\})$, where $\sigma=(\sigma_1,\dots,\sigma_n)$ runs the $2^n$ sets of multi-signature. Each $S_\sigma$ has the convex hull $S'_\sigma$ which does not contain $R^{n-1}\times\{|x_n|\le\delta'\}\times\{\pm idx_n\infty\}$. Let $\operatorname{sp}(u)$ be the image of $u(x)$ by the canonical mapping of $\mathcal{B}$ to $\pi_*\mathcal{C}$, where $\mathcal{C}$ denotes the sheaf of microfunctions on $R^n\times iS^{*n-1}_\infty$ and $\pi$ denotes the projection of the latter space to $R^n$. By the flabbiness of $\mathcal{C}$ (see [13] Chapter III, Corollary 2.1.5), $\operatorname{sp}(u)$ is decomposed into the sum

$$\operatorname{sp}(u)=\sum_\sigma v_\sigma\,,$$

where $v_\sigma$ is a microfunction on $R^{n-1}\times\{|x_n|\le\delta'\}\times iS^{*n-1}_\infty$ satisfying $\operatorname{supp} v_\sigma\subset S'_\sigma$. By [9], Théorème (6.1), we can find hyperfunctions $u_\sigma$ on $\bar U\times\{|x_n|\le\delta'\}$ satisfying $\operatorname{sp}(u_\sigma)=v_\sigma$ such that $u_\sigma(x)=F_\sigma(x+i\Gamma_\sigma0)$, where each $\Gamma_\sigma\subset R^n_y$ is an open convex cone contained in the cone $\{y\in R^n;\ \langle\xi,y\rangle<0$ for any $i\xi dx\infty\in S'_\sigma\}$, but still having a non-void intersection with $\{y_n=0\}$, and $F_\sigma(z)$ is holomorphic in $\bar U\times\{|x_n|\le\delta'\}\times i\{\Gamma_\sigma\cap\{|y|<\varepsilon\}\}$ with some $\varepsilon>0$. Since $u_\sigma(x)$ are real analytic on $(\bar U\backslash K)\times\{|x_n|\le\delta'\}$, $F_\sigma(z)$ must obviously be real analytic up to there. The difference $u(x)-\sum u_\sigma(x)$ is real analytic on $\bar U\times\{|x_n|\le\delta'\}$. Thus modifying, e.g. $u_1$ by this difference, we have given the required decomposition. q.e.d.

LEMMA 1.4. *Let $u(x)$ be a hyperfunction on $U\times\{|x_n|<\delta\}$, where $U\subset R^{n-1}$ is open. Assume that $u(x)$ contains $x_n$ as a real analytic parameter and* $\operatorname{supp} u\subset K\times\{|x_n|<\delta\}$, *where $K\subset U$ is compact. Let $f(x,y)$ be a real analytic function of $(x,y)\in U\times\{|x_n|<\delta\}\times V$, where $V$ is an open set in the parameter space of $y$. Then*

1) *${}'\langle u(x',t),f(x',t,y)\rangle_{x'}$ is a real analytic function of $(t,y)\in\{|t|<\delta\}\times V$, where ${}'\langle\,,\rangle$ denotes the duality between ${}'\mathcal{B}[K]$ and ${}'\mathcal{A}(K)$.*

2) $\dfrac{\partial}{\partial t}{}'\langle u(x',t), f(x',t,y)\rangle_{x'} = {}'\Big\langle \dfrac{\partial}{\partial t}u(x',t), f(x',t,y)\Big\rangle_{x'} + {}'\Big\langle u(x',t), \dfrac{\partial}{\partial t}f(x',t)\Big\rangle_{x'}$

*or more generally, for a local operator $J(D_t)$, we have*

$$J(D_t)'\langle u(x',t), f(x',y)\rangle_{x'} = {}'\langle J(D_t)u(x',t), f(x',y)\rangle_{x'}\,.$$

3) *Let $\varepsilon$ be a constant satisfying $0<\varepsilon<\delta$. Then the product $Y(\varepsilon^2-x_n^2)u(x)$ is a well-defined element of $\mathcal{B}[K\times\{|x_n|\le\varepsilon\}]$ and*

$$\langle Y(\varepsilon^2-x_n^2)u(x), f(x)\rangle = \int_{-\varepsilon}^{\varepsilon} {}'\langle u(x',t), f(x',t)\rangle_{x'}dt\,, \tag{1.6}$$

*for every $f(x)\in\mathcal{A}(K\times\{|x_n|\le\delta\})$, where $Y(\varepsilon^2-x_n^2)$ denotes the characteristic function of the interval $[-\varepsilon,\varepsilon]$.*

PROOF. Let $L\subset U$ be a compact set which is the closure of an open neighborhood of $K$ in $R^{n-1}$. Take a decomposition of the form (1.5) in Lemma 1.3. Near the boundary $\partial L$ of the chain $L$, each $u_j(x',t)$ is real analytic in $x'$. Hence the integral

$$\int_L u_j(x',t)f(x',t,y)dx' \tag{1.7}$$

is well defined. For $t$ fixed, we have obviously $u(x',t)=\sum_{j=1}^{n}\chi_L(x')u_j(x',t)$ where $\chi_L$ is the characteristic function of $L$. Thus we have obviously

$${}'\langle u(x',t), f(x',t,y)\rangle_{x'} = \sum_{j=1}^{N}\int_L u_j(x',t)f(x',t,y)dx'\,.$$

On the other hand, each integral (1.7) is calculated by changing the chain $L$ by $\tilde{L}$, with the boundary $\partial\tilde{L}=\partial L$ fixed, into the complex region $U\times i\Gamma_j'$, where $\Gamma_j'=\Gamma_j\cap\{y_n=0\}$:

$$\int_{\tilde{L}_j} F_j(z',t)f(z',t,y)dz'\,. \tag{1.8}$$

The latter is a usual integration of a holomorphic function. By the assumption $F_j(z',t)$ is real analytic in $t$ when $z'\in\tilde{L}_j$, hence we can let $t$ (and of course $y$) run on a complex neighborhood. Thus (1.8), hence the original expression is real analytic in $t$ and $y$. The formula for the derivative is directly obtained from (1.8) by taking the derivative under the integral sign.

The products $Y(\varepsilon^2-x_n^2)u(x)$ and $Y(\varepsilon^2-x_n^2)u_j(x)$ are well-defined due to the general rule (see [13], Chapter I, Corollary 2.4.2). In this case, they are explicitly given by way of the defining functions $((-1/2\pi i)\log((z_n+\varepsilon)/(z_n-\varepsilon)))F_j(z)$. We have

$$(1.9)\qquad \langle Y(\varepsilon^2-x_n^2)u(x), f(x)\rangle=\int_{U\times\{|x_n|<\delta\}} Y(\varepsilon^2-x_n^2)u(x)f(x)dx$$
$$=\sum_{j=1}^{N}\int_{L\times\{|x_n|<\delta\}} Y(\varepsilon^2-x_n^2)u_j(x)f(x)dx$$
$$=\sum_{j=1}^{N}\int_{\gamma}\left(-\frac{1}{2\pi i}\log\frac{z_n+\varepsilon}{z_n-\varepsilon}\right)dz_n\int_{\tilde{L}_j}F_j(z',z_n)f(z',z_n)dz',$$

where $\gamma$ is a complex contour surrounding the interval $[-\varepsilon,\varepsilon]$. By the assumption, $F_j(z',z_n)$ is analytic up to real $z_n$ when $z'\in\tilde{L}_j$. Thus by the absolute convergence we can change the path $\gamma$ to the interval $[-\varepsilon,\varepsilon]$ itself. Then the last side of (1.9) converges to $\sum_{j=1}^{N}\int_{-\varepsilon}^{\varepsilon}dt\int_{\tilde{L}_j}F_j(z',t)f(z',t)dz'$. This is just the right hand side of (1.6). q.e.d.

The following theorem has long be known though unpublished. The method of proof here is due to [13] (cf. Chapter III, Proposition 2.1.3).

THEOREM 1.5. *Let $u(x)$ be a hyperfunction on $\boldsymbol{R}^{n-1}\times\{|x_n|<\delta\}$ containing $x_n$ as a real analytic parameter. Suppose that* $\operatorname{supp}u\subset K\times\{|x_n|<\delta\}$, *where $K\subset\boldsymbol{R}^{n-1}$ is compact. Assume that for every $k=0,1,2,\ldots$, $(\partial/\partial x_n)^k u|_{x_n=0}\equiv0$. Then $u(x)\equiv0$.*

PROOF. Take $\varepsilon$ satisfying $0<\varepsilon<\delta$. Take a test function $f(x)\in\mathcal{A}(K\times\{|x_n|\le\varepsilon\})$. Then, in the formula (1.6) in Lemma 1.4, the function $'\langle u(x',t), f(x',t)\rangle_{x'}$ is real analytic. We see that it has zero of infinite order at $t=0$ by the repeated use of the formula for the derivative given in Lemma 1.4 and the assumption. Thus it vanishes identically. Hence the integral on the right hand side of (1.6) has the value zero. Since $f(x)$ is arbitrary, this implies that $Y(\varepsilon^2-x_n^2)u(x)\equiv0$. This means that $u(x)\equiv0$ in $|x_n|<\varepsilon$. Since $\varepsilon$ is arbitrary, we conclude that $u(x)\equiv0$. q.e.d.

We remark that the assumption that $\operatorname{supp}u\subset K\times\{|x_n|<\delta\}$ is essential. See Example 4.10. A generalization of this unique continuation property is the main subject of the present article.

Finally we give two more elementary lemmas for later use.

LEMMA 1.6. *Let $u(x,t)$ be a hyperfunction on $\boldsymbol{R}^n\times U$ such that* $\operatorname{supp}u\subset K\times U$, *where $K\subset\boldsymbol{R}^n$ is a compact set of the product type $K_1\times\cdots\times K_n$. Let $f(x,s)$ be a real analytic function on $U\times V$. Here, $U\subset\boldsymbol{R}^m$, $V\subset\boldsymbol{R}^l$ are the open sets in the corresponding spaces of parameters. Then the integral $\int_K u(x,t)f(x,s)dx$ is well defind. It can be calculated, e.g., employing a defining function $F(z,\tau)$ holomorphic in $z$ in $(\boldsymbol{C}\backslash K_1)\times(\boldsymbol{C}\backslash K_2)\times\cdots\times(\boldsymbol{C}\backslash K_n)$ when* $\operatorname{Im}\tau_k\neq0$, *$k=1,\ldots,m$, in the following way:*

$$(1.10) \qquad (-1)^n \oint_{\gamma_1} \cdots \oint_{\gamma_n} F(z, \tau) f(z, s) dz_1 \cdots dz_n ,$$

*where $\gamma_j$ denotes a path surrounding $K_j$. If* S.S.$u$ *does not contain any point with the direction which is a linear combination of $\pm idt_k\infty$, $k=1, \dots, m$, then we can take the specialization of $u(x, t)$ with respect to the set of variables $t$. The above integral then agrees with the function calculated pointwise by the integral of the hyperfunction $u(x, t)|_{t=t^0} f(x, s^0)$ with compact support. The result is rel analytic in $(t, s)$. We can therefore transfer the action of a local operator $J(D_t)$ under the integral sign to $u$.*

The line of the proof is the same as that of Lemma 1.4, a lemma analogous to Lemma 1.3 employed. Note that $K$ need not be of the product type. For general $K$, only the integral representation becomes complicated.

LEMMA 1.7. *Let $u(x)$ be an infinitely differentiable function on $\boldsymbol{R}^{n-1} \times \{|x_n| < \delta\}$ such that* $\operatorname{supp} u \subset K \times \{|x_n| < \delta\}$, *where $K \subset \boldsymbol{R}^{n-1}$ is compact. Assume that $u$ contains $x_n$ as a real analytic parameter as a hyperfunction. Then the restriction $u(x)|_{x_n=0}$ in the sense of real analytic parameter agrees with the classical restriction $u(x', 0)$.*

PROOF. Take a positive constant $\varepsilon < \delta$, and consider $u_\varepsilon(x) = Y(\varepsilon^2 - x_n^2) u(x)$. The standard defining function $F_\varepsilon(z)$ for $u_\varepsilon(x)$ is given by

$$(1.11) \qquad F_\varepsilon(z) = \left(\frac{1}{2\pi i}\right)^n \int_{-\varepsilon}^{\varepsilon} \frac{dx_n}{x_n - z_n} \int_K \frac{u(x) dx'}{(x_1 - z_1) \cdots (x_{n-1} - z_{n-1})} ,$$

where the integral is the one in the classical sense. Without loss of generality we can assume that $K = K_1 \times \cdots \times K_{n-1}$. The assumption and Lemma 1.6 shows that the integral

$$G(z', x_n) = \left(\frac{1}{2\pi i}\right)^{n-1} \int_K \frac{u_\varepsilon(x)}{(x_1 - z_1) \cdots (x_{n-1} - z_{n-1})} dx'$$

taken in the sense of hyperfunction is holomorphic on a neighborhood of $(C \backslash K_1) \times \cdots \times (C \backslash K_{n-1}) \times \{|x_n| < \varepsilon\}$. The integral can be calculated employing the defining function (1.11) in the following way

$$(1.12) \qquad H(z', z_n) = \left(-\frac{1}{2\pi i}\right)^{n-1} \int_{\gamma_1 \times \cdots \times \gamma_{n-1}} \frac{F_\varepsilon(\zeta', z_n) d\zeta'}{(\zeta_1 - z_1) \cdots (\zeta_{n-1} - z_{n-1})}$$
$$= \left(\frac{1}{2\pi i}\right)^{2n-1} (-1)^{n-1} \int_{-\varepsilon}^{\varepsilon} \frac{dx_n}{x_n - z_n} \int_K \int_{\gamma_1 \times \cdots \times \gamma_{n-1}} \frac{u(x)}{(x_1 - \zeta_1) \cdots (x_{n-1} - \zeta_{n-1})}$$
$$\times \frac{d\zeta' dx'}{(\zeta_1 - z_1) \cdots (\zeta_{n-1} - z_{n-1})}$$

where $\gamma_j$ is a path surrounding $K_j$ and $G(z', x_n) = H(z', x_n + i0) - H(z', x_n - i0)$ as a hyperfunction. By the Cauchy integral formula we see that (1.12) reduces to (1.11). Thus we conclude that $F_\varepsilon(z)$ is holomorphically continued to a neighborhood of $(C \backslash K_1) \times \cdots \times (C \backslash K_{n-1}) \times \{|x_n| < \varepsilon\}$ from the both sides $\pm \operatorname{Im} z_n > 0$. Now $G(z', 0)$ gives the standard defining function of the restriction $u_\varepsilon(x)|_{x_n=0}$ in the sense of real analytic parameter. On the other hand, the same function $F_\varepsilon(z', i0) - F_\varepsilon(z', -i0)$ gives a defining function of $u_\varepsilon(x', 0)$. In fact, we have the local uniform convergence

$$F(z', \eta) - F(z', -\eta) = \left(\frac{1}{2\pi i}\right)^{n-1} \frac{\eta}{\pi} \int_{-\varepsilon}^{\varepsilon} \frac{dx_n}{x_n^2 + \eta^2} \int_K \frac{u(x) dx'}{(x_1 - z_1) \cdots (x_{n-1} - z_{n-1})}$$

$$\longrightarrow \left(\frac{1}{2\pi i}\right)^{n-1} \int_K \frac{u(x', 0) dx'}{(x_1 - z_1) \cdots (x_{n-1} - z_{n-1})}$$

when $\eta \to 0$. The result is just the standard defining function of $u_\varepsilon(x', 0)$. Thus we have shown $u(x)|_{x_n=0} = u_\varepsilon(x)|_{x_n=0} = u_\varepsilon(x', 0) = u(x', 0)$. q.e.d.

The assertion of Lemma 1.7 will hold without any condition on the support of $u(x)$. To prove this one must establish for a function of class $C^\infty$ the accordance of the usual singular spectrum with its analogue in the sense $C^\infty \bmod \mathcal{A}$.

## § 2. Boundary value problem and the concept of limit in the theory of hyperfunctions

One way of introducing generalized functions is to take the limit of a weakly converging sequence of regular functions. For a hyperfunction $f(x)$, as is remarked in Sato's original paper [10], it is also obtained as an ideal "limit" of a harmonic function $u(x, y)$ in the upper half space $\{z = x + iy;\ y > 0\}$, when $y$ tends to zero. Strictly speaking, this is the correspondence between the solutions of the Laplacian and their boundary values. Komatsu and Kawai [7] has generalized this in the following way. Let $p(x, D)$ be a linear partial differential operator with real analytic coefficients. Assume that the hyperplane $S = \{x_n = 0\}$ is non-characteristic with respect to the operator $p$. Then we can construct the operator $b_+^0$ of taking the boundary value: It attaches for every solution $u$ of the equation $p(x, D)u = 0$ in the upper half space $\{x_n > 0\}$, its boundary value $b_+^0(u)$.

Now assume that for every $\varepsilon > 0$, the hyperplane $S_\varepsilon = \{x_n = \varepsilon\}$ is non-characteristic with respect to $p$. Then, by Sato's fundamental theorem the solution $u(x', x_n)$ contains $x_n$ as a real analytic parameter. Hence we can take the restriction $u(x', \varepsilon)$. We intend to consider that the following is a limit process in the theory of hyperfunctions: $u(x', \varepsilon) \to b_+^0(u)$, when $\varepsilon \downarrow 0$. In the sequel we construct lemmas which support this intention.

First of all we shortly review the definition of the boundary value. Let $\omega$ be an open set in $S=\{x_n=0\}$ which is isomorphic to $\boldsymbol{R}^{n-1}$. Let $W$ be an open set in $\boldsymbol{R}^n$ such that $W\cap S=\omega$. Let $u$ be a solution of $p(x,D)u=0$ in $W\cap\{x_n>0\}$. By the flabbiness of $\mathcal{B}$ we can take an extension $\tilde{u}\in\mathcal{B}(W)$ of $u$ which vanishes in $W\cap\{x_n<0\}$. Then $p(x,D)\tilde{u}$ is a well defined element of the quotient space $\mathcal{B}_S(W)/p(x,D)\mathcal{B}_S(W)$, where $\mathcal{B}_S(W)$ denotes the space of hyperfunctions in $W$ with support contained in $S$. Let $'\mathcal{B}(\omega)$ be the hyperfunction of $n-1$ variables in $\omega$. Assume that $p$ is of order $m$. Then we can construct a mapping $({}^t\rho)^{-1}$: $\mathcal{B}_S(W)/p(x,D)\mathcal{B}_S(W)\to{}'\mathcal{B}(\omega)^m$ in the following way: Let $K\subset\omega$ be compact. Then the Cauchy-Kowalevsky theorem asserts that the mapping $\rho$: $\mathcal{A}_{{}^tp}(K)\to{}'\mathcal{A}(K)^m$, which assigns to every real analytic solution $f$ of ${}^tp(x,D)f=0$ near $K$ the initial values $C_j(x,D)f(x)|_{x_n=0}$, $j=0,1,\dots,m-1$, is a continuous isomorphism, where $C_j(x,D)$ denote the dual system of the boundary condition $D_n^j$ with respect to the operator $p$.[1] Hence the transposed mapping ${}^t\rho$: $'\mathcal{B}_K(\omega)^m\to\mathcal{B}_K(W)/p(x,D)\mathcal{B}_K(W)$ is also an isomorphism. Since it preserves the support it can be extended to a sheaf isomorphism ${}^t\rho$: $'\mathcal{B}(\omega)^m\to\mathcal{B}_s(W)/p(x,D)\mathcal{B}_s(W)$. The above quoted mapping is the inverse of this isomorphism. Finally we define $b_+^j(u)=({}^t\rho^{-1})_j(p(x,D)\tilde{u})$, $j=0,1,\dots,m-1$, where the suffix $j$ signifies the $j$-th component in $'\mathcal{B}(\omega)^m$. In the sequel we mainly employ the 0-th component $b_+^0(u)$.

$b_+^j(u)$ are characterized as the unique element of $'\mathcal{B}(\omega)^m$ which satisfies

$$p(x,D)\tilde{u}=\sum_{j=0}^{m-1}{}^tC_{m-1-j}(x,D)(b_+^j(u)\otimes\delta(x_n)) \tag{2.1}$$

with some (in fact unique) extension $\tilde{u}\in\mathcal{B}(W)$ of $u$ which vanishes on $W\cap\{x_n<0\}$. If $u$ can be extended as a solution to a neighborhood of $\omega$ in $W$, then the boundary values $b_+^j(u)$ agree with the restrictions $D_n^ju(x',0)$. For in this case we can take the product $\tilde{u}=Y(x_n)u$ with the Heaviside function $Y(x_n)$, because $u$ contains $x_n$ as a real analytic parameter even at $x_n=0$. Then we have

$$p(x,D)(Y(x_n)u)-Y(x_n)p(x,D)u=\sum_{j=0}^{m-1}{}^tC_{m-1-j}(x,D)(D_n^ju(x',0)\otimes\delta(x_n))\,, \tag{2.2}$$

by the definition of $C_j(x,D)$.

LEMMA 2.1. *$b_+^j$ have the local property. Namely, if $\omega_1\subset\omega$ is another open subset of $S$ and $W_1\subset W$ is an open set in $\boldsymbol{R}^n$ such that $W_1\cap S=\omega_1$, then the*

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1) We assume that $C_j$ is of order $j$, hence somewhat different of the notation in [7].

*boundary values $b_+^j(u|_{W_1})$ of the solution $u|_{W_1}$ of $p$ in $W_1$ agree with the restrictions of $b_+^j(u)$ to $\omega_1$.*

Though this is clear from the construction of $b_+^j$, we dare to list it up here as a lemma, since this local property is the main motivation to employ $b_+^0$ as a limit process. As the explanation we give here an example which shows that the topology of hyperfunctions introduced as that of analytic functionals does not have any local property.

EXAMPLE 2.2. We construct a sequence of hyperfunctions $u_k \in \mathcal{B}[\{0\}]$ of one variable with support at zero, which converges to zero in a larger space $\mathcal{B}[0,2]$ but does not converge to zero in the original space. (Cf. Remark 1.10 in [1].) Represent $\mathcal{B}[0,2]$ by the space $\mathcal{O}_0(\boldsymbol{P}^1\setminus[0,2])$ of functions holomorphic outside $[0,2]$ on the Riemann sphere and vanishing at infinity. Employ as the fundamental seminorms of the latter space the following:

$$\|f(z)\|_k = \sup\left\{|f(z)|;\ \operatorname{dis}(z,[0,2]) \geq \frac{1}{k}\right\}.$$

In the following we identify $\mathcal{B}[0,2]$ with $\mathcal{O}_0(\boldsymbol{P}^1\setminus[0,2])$ including the topology, which is permissible by Polya's theorem. Since $\mathcal{B}[\{0\}]$ is dense in $\mathcal{B}[0,2]$ we can choose $u_k \in \mathcal{B}[\{0\}]$ satisfying

$$\left\|\frac{1}{2^k}\delta(x-2) - u_k(x)\right\|_k \leq \frac{1}{3^k}, \qquad \text{in} \quad \mathcal{B}[0,2],$$

for $k=1,2,\ldots.$ Since $(1/2^k)\delta(x-2)$ converges to zero in $\mathcal{B}[0,2]$, we have $u_k \to 0$ in $\mathcal{B}[0,2]$. We claim that $u_k$ does not converge to zero in $\mathcal{B}[\{0\}]$. In fact, assume the contrary. Take $f_k(x)=x^k$, $k=1,2,\ldots.$ Then $f_k$ converges to zero in $\mathcal{A}(\{0\})$. Therefore we have

$$\langle u_k, f_k\rangle \longrightarrow 0, \quad \text{when} \quad k \longrightarrow \infty.$$

On the other hand we have

$$\begin{aligned}\langle u_k, f_k\rangle &= \left\langle u_k - \frac{1}{2^k}\delta(x-2), f_k\right\rangle + \left\langle \frac{1}{2^k}\delta(x-2), f_k\right\rangle \\ &= \left\langle u_k - \frac{1}{2^k}\delta(x-2), f_k\right\rangle + 1.\end{aligned}$$

Estimating the pairing of the first term by representing it by the contour integral along the path $\gamma=\{z;\ \operatorname{dis}(z,[0,2])=1/k\}$, we have

$$\left|\left\langle u_k - \frac{1}{2^k}\delta(x-2), f_k\right\rangle\right| \leq \frac{1}{3^k}\left(2+\frac{1}{k}\right)^k |\gamma|.$$

Thus the first term converges to zero, which is a contradiction.

LEMMA 2.3. *Let $u$ be a classical solution of $p(x, D)u=0$ which can be extended as a function of class $C^m$ to a neighborhood of $\omega$ in $W$. Then $b_+^j(u)$ agree with the classical data $D_n^j u(x', 0)$.*

PROOF. First we must emphasize that such a kind of assertion is not always apparent. This time we can make the product $Y(x_n)u$ in the classical sense. Thus formula (2.2) holds in the sense of distributions. It remains to verify that the tensor product $D_n^j u(x', 0)\otimes\delta(x_n)$ in the sense of distributions and that of hyperfunctions agree. Recall that the latter has been defined in [7] as a unique extension to the sheaf homomorphism of the tensor product of $'\mathcal{B}_*(S)$ and $\delta(x_n)$ which is defined by the duality with the real analytic functions. Thus, for a distribution $f$ with compact support the tensor product $f\otimes\delta(x_n)$ clearly agrees with that in the sense of hyperfunctions. Then, in the general case, dividing $f$ to a locally finite sum with compact support, we also obtain the same result in both sense of the tensor product. q.e.d.

LEMMA 2.4. *Assume that $b_+^j(u)$, $j=0, 1, \dots, m-1$, are real analytic. Then $u$ is a real analytic solution in a neighborhood of $\omega$ and can be extended analytically to a neighborhood of $\omega$ in $W$.*

PROOF. Let $v$ be the solution of the analytic Cauchy problem

$$\begin{cases} p(x, D)v=0 \\ D_n^j v(x', 0)=b_+^j(u)\,, \qquad j=0, 1, \dots, m-1\,. \end{cases}$$

Then for the difference $w=u-v$ we have $b_+^j(w)=0$, $j=0, 1, \dots, m-1$. Thus by Theorem 5 in [7], we have $w\equiv 0$ in a neighborhood of $\omega$. q.e.d.

In this lemma the assumption that $b_+^j(u)$ are analytic is superfluous. We will discuss the interesting problem of finding the necessary and sufficient condition elsewhere in the future.

THEOREM 2.5. *Let $p(x, D)$ be a partial differential operator with respect to which the hyperplane $S=\{x_n=0\}$ is non-characteristic. Let $K\subset S$ be compact, $\omega\supset K$ be open in $S$, and let $u(x', x_n)$ be a solution of $p(x, D)u=0$ defined in $\omega\times\{0<x_n<\delta\}$. Assume that* $\operatorname{supp} u\subset K\times\{0<x_n<\delta\}$. *Then $u(x', \varepsilon)$ converges to $b_+^0(u)$ in the topology of $'\mathcal{B}[K]$ when $\varepsilon\downarrow 0$.*

PROOF. Since $'\mathcal{B}[K]$ is (FS), it suffices to show the weak convergence. Take

$f(x') \in {}'\mathcal{A}(K)$ arbitrarily. We will show

$$'\langle u(x', \varepsilon), f(x')\rangle \longrightarrow {}'\langle b_+^0(u), f(x')\rangle$$

when $\varepsilon \downarrow 0$, where $'\langle\ ,\ \rangle$ denotes the pairing between $'\mathcal{B}[K]$ and $'\mathcal{A}(K)$. Consider the analytic Cauchy problem

$$\begin{cases} {}^t p(x, D) F_\varepsilon(x) = 0 , \\ C_j(x, D) F_\varepsilon(x)|_{x_n = \varepsilon} = 0 , \qquad j = 0, 1, \ldots, m-2 , \\ C_{m-1}(x, D) F_\varepsilon(x)|_{x_n = \varepsilon} = f(x') . \end{cases} \tag{2.3}$$

Due to a precise form of the Cauchy-Kowalevsky theorem (see, e.g., Leray [8]), we can assume that the solutions $F_\varepsilon(x)$ exist up to $K \times \{0\}$ for $0 < \varepsilon \leq \varepsilon_0$ with sufficiently small $\varepsilon_0$. Let $\tilde{u}$ be an extension satisfying (2.1). Then (2.1) and the Green's formula (2.2) give

$$\begin{aligned} &'\langle b_+^0(u), C_{m-1}(x, D) F_\varepsilon(x)|_{x_n=0}\rangle + \cdots + {}'\langle b_+^{m-1}(u), C_0(x, D) F_\varepsilon(x)|_{x_n=0}\rangle \\ &\quad = \langle p(x, D) u(x), F_\varepsilon(x)\rangle \\ &\quad = \langle Y(\varepsilon - x_n) u(x), {}^t p(x, D) F_\varepsilon(x)\rangle + {}'\langle u(x', \varepsilon), f(x')\rangle \\ &\quad = {}'\langle u(x', \varepsilon), f(x')\rangle . \end{aligned}$$

Here $\langle\ ,\ \rangle$ denotes the pairing between $\mathcal{B}[K \times [0, \varepsilon]]$ and $\mathcal{A}(K \times [0, \varepsilon])$. The product $Y(\varepsilon - x_n) u(x)$ is well defined for sufficiently small $\varepsilon$, since by Sato's fundamental theorem $u$ contains $x_n$ as a real analytic parameter in a neighborhood of $K \times \{0\}$. Now when we let $\varepsilon \downarrow 0$, then we have obviously $F_\varepsilon(x', 0) \to f(x')$, and $D_n^j F_\varepsilon(x', 0) \to 0$ for $j = 1, \ldots, m-1$ in $'\mathcal{A}(K)$. Thus the left hand side tends to $'\langle b_+^0(u), f(x')\rangle$. q.e.d.

COROLLARY 2.6. *Let $S$ be an $(n-1)$-dimensional oriented real analytic manifold and let $p(x, D)$ be a partial differential operator on $S \times \{-\delta < x_n < \delta\}$. Let $K \subset S$ be compact and assume that $S \times \{0\}$ is non-characteristic with respect to $p$ at every point of $K \times \{0\}$. Let $\omega \supset K$ be an open set in $S$ and let $u$ be a solution of $p(x, D) u = 0$ defined in $\omega \times \{0 < x_n < \delta\}$. Assume that* supp $u \subset K \times \{0 < x_n < \delta\}$. *Then $u(x', \varepsilon)$ converges to $b_+^0(u)$ in $'\mathcal{B}[K]$ when $\varepsilon \downarrow 0$.*

In fact, by the unique continuation property, we can solve the analytic Cauchy problem globally with respect to the coordinates on $S$ on a neighborhood of $K \times \{0\}$. Thus the proof goes without modification as above.

## § 3. Generalized unique continuation property

We begin with the simplest case.

DEFINITION 3.1. Let $u(x)$ be a hyperfunction defined on a neighborhood of the origin. Let $v(x)$ be an element of $\mathcal{B}_*(\boldsymbol{R}^n)$ which agrees with $u(x)$ on a neighborhood of the origin. If the finite limit

$$\lim_{\varepsilon\downarrow 0}\frac{1}{(2\pi)^n}\int_{\boldsymbol{R}^n} e^{-\varepsilon|\xi|}\tilde{v}(\xi)d\xi \tag{3.1}$$

exists, we define it to be the value of $u(x)$ at the origin.

LEMMA 3.2. *Whether the limit* (3.1) *exists or not, or the value itself, does not depend on the choice of* $v(x)\in\mathcal{B}_*(\boldsymbol{R}^n)$.

PROOF. Let $v(x)\in\mathcal{B}_*(\boldsymbol{R}^n)$ be identically equal to zero on a neighborhood of the origin. It sufficies to prove

$$\lim_{\varepsilon\downarrow 0}\frac{1}{(2\pi)^n}\int_{\boldsymbol{R}^n} e^{-\varepsilon|\xi|}\tilde{v}(\xi)d\xi=0\ . \tag{3.2}$$

Put

$$E(x,\varepsilon)=\mathcal{F}^{-1}[e^{-\varepsilon|\xi|}]=\frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}\ \frac{\varepsilon}{(|x|^2+\varepsilon^2)^{(n+1)/2}}\ . \tag{3.3}$$

Thus $E(x,\varepsilon)$ is a slowly increasing real analytic function of $x$ for each fixed $\varepsilon>0$. When we let $\varepsilon\downarrow 0$, $E(x,\varepsilon)$ converges to zero uniformly on a complex neighborhood of any real compact set $K\subset\boldsymbol{R}^n$ which does not contain the origin. Choose $K=\operatorname{supp} v$. Thus $E(x,\varepsilon)\to 0$ in $\mathcal{A}(K)$. Considering the inner product $\langle v(x), E(x,\varepsilon)\rangle$ as the one between $\mathcal{B}[K]$ and $\mathcal{A}(K)$, we thus obtain

$$\lim_{\varepsilon\downarrow 0}\langle v(x), E(x,\varepsilon)\rangle=0\ . \tag{3.4}$$

On the other hand, when $\varepsilon>0$ is fixed we can consider the same inner product as the one between $\mathcal{Q}_*(\boldsymbol{D}^n)$ and $\mathcal{P}(\boldsymbol{D}^n)$. Then we apply the Parseval formula

$$\langle v(x), E(x,\varepsilon)\rangle=\frac{1}{(2\pi)^n}\int_{\boldsymbol{R}^n} e^{-\varepsilon|\xi|}\tilde{v}(\xi)d\xi\ . \tag{3.5}$$

(3.4) and (3.5) imply (3.2). q.e.d.

LEMMA 3.3. *Assume that a hyperfunction $u(x)$ is of class $C^{2n}$ on a neighborhood of the origin. Then it has a finite value at the origin in the sense of Definition* 3.1. *It agrees with the usual data* $u(0)$.

PROOF. Let $v(x)\in C_0^{2n}(\boldsymbol{R}^n)$ be a cutting off of $u(x)$. Then in the classical sense we have $u(0)=v(0)$. On the other hand, when $v(x)$ belongs to $C_0^{2n}(\boldsymbol{R}^n)$, the limit

(3.1) exists and agrees with

$$\frac{1}{(2\pi)^n}\int_{R^n}\tilde{v}(\xi)d\xi=v(0)\ ,$$

as is easily seen by the Lebesgue convergence theorem and the classical Fourier inversion formula. q.e.d.

REMARK 3.4. Employing a result on harmonic analysis (see [14], Chapter II, Theorem 1.10), we can strengthen Lemma 3.3. For example, $u(x)$ has a finite value in our sense if it is continuous on a neighborhood of the origin. Since Lemma 3.3 is sufficient for our later application, we will go without quoting so much from the harmonic analysis.

THEOREM 3.5. *A hyperfunction $u(x)$ is real analytic at the origin if and only if for any local operator $J(D)$ with constant coefficients, the derived function $J(D)u(x)$ has a finite value at the origin in the sense of Definition* 3.1.

PROOF. First suppose that $u(x)$ is real analytic at the origin. Then for any local operator $J(D)$ with constant coefficients, $J(D)u(x)$ is infinitely differentiable on a neighborhood of the origin. Thus by Lemma 3.3 it has a finite value.

Conversely assume that for any $J(D)$, $J(D)u(x)$ has a finite value at the origin. We can assume without loss of generality that $u(x)\in\mathcal{B}_*(\boldsymbol{R}^n)$. Thus the assumption implies that

$$\frac{1}{(2\pi)^n}\int_{R^n}e^{-\varepsilon|\xi|}J(\xi)\tilde{u}(\xi)d\xi=\mathcal{F}^{-1}[e^{-\varepsilon|\xi|}J(\xi)\tilde{u}(\xi)](0)=[J(D)(u(x)*E(x,\varepsilon))]|_{x=0} \tag{3.6}$$

has a finite limit when $\varepsilon\downarrow 0$. Now put $u_k(x)=u(x)*E(x,\varepsilon_k)$, where $\varepsilon_k\downarrow 0$. The above assumption implies that $u_k(x)$ is a converging sequence in $\mathcal{A}_{\mathcal{J}}(\{0\})$. (Here, $\mathcal{A}_{\mathcal{J}}(K)$ denotes the space of real analytic functions $f(x)$ on a compact set $K\subset\boldsymbol{R}^n$ endowed with the seminorms $\|f\|_J=\sup_{x\in K}|J(D)f(x)|$, $J(D)$ running over all the local operators with constant coefficients; see [1], Definition 2.1). Thus by Proposition 2.4 in [1], $u_k(x)$ converges uniformly on some complex neighborhood of the origin. Thus the limit function is real analytic on a neighborhood of the origin. Since the sequence $\varepsilon_k$ is arbitrary, we conclude that the function $u(x,\varepsilon)=u(x)*E(x,\varepsilon)$ converges to the unique real analytic limit function in $\mathcal{A}(K)$, where $K$ is a real compact neighborhood of the origin.

We claim that the limit function agrees with the original $u(x)$ on a neighborhood of the origin. If this is established, the proof will be completed. To establish the claim we need the result in §2. As we have already remarked there, we must pay special attention in deducing the local accordance of the limit function in the theory of hyperfunctions. Now the function $u(x, \varepsilon)$ is real analytic and satisfies the equation

$$\left(\Delta_x + \frac{\partial^2}{\partial\varepsilon^2}\right)u(x, \varepsilon) = 0 \tag{3.7}$$

on $\varepsilon > 0$ as a function of $n+1$ variables, where $\Delta_x$ denotes the Laplacian on $x$. Hence

$$\frac{\partial^2}{\partial\varepsilon^2} u(x, \varepsilon) = -\Delta_x u(x, \varepsilon)$$

also converges in $\mathcal{A}(K)$ when $\varepsilon \downarrow 0$. Further

$$\frac{\partial}{\partial\varepsilon} u(x, \varepsilon) = \int_{\varepsilon_0}^{\varepsilon} \frac{\partial^2}{\partial\varepsilon^2} u(x, \varepsilon) d\varepsilon + u(x, \varepsilon^0)$$

also converges in $\mathcal{A}(K)$. Thus we conclude that $u(x, \varepsilon)$ can be extended as a classical function of class $C^2$ across the boundary $\varepsilon = 0$ on a neighborhood of the origin in $\boldsymbol{R}^{n+1}$. Therefore by Lemma 2.3 the boundary value $b_+^0(u(x, \varepsilon))$ of $u(x, \varepsilon)$ to the non-characteristic plane $\{\varepsilon = 0\}$ with respect to the operator in (3.7), agrees with the uniform limit $\lim_{\varepsilon \downarrow 0} u(x, \varepsilon)$, which we have seen to be real analytic on $K$.

Next we consider the limit in a global sense. Let $\boldsymbol{S}^n$ be the one point compactification of $\boldsymbol{R}^n$. It is a compact real analytic manifold with the coordinates

$$y_j = \frac{x_j}{x_1^2 + \cdots + x_n^2}, \quad j = 1, \cdots, n-1, \quad y_n = -\frac{x_n}{x_1^2 + \cdots + x_n^2}, \tag{3.8}$$

at infinity. As the volume element we can employ

$$\frac{dx_1 \cdots dx_n}{(1 + x_1^2 + \cdots + x_n^2)^n} = \frac{dy_1 \cdots dy_n}{(1 + y_1^2 + \cdots + y_n^2)^n} \tag{3.9}$$

which is induced by the stereographic projection. Then, the hyperfunction $u(x) \in \mathcal{B}_*(\boldsymbol{R}^n)$ can be considered as an element of $\mathcal{B}[\boldsymbol{S}^n]$. Moreover consider the real analytic manifold $\boldsymbol{S}^n \times \boldsymbol{R}^1$. Then the kernel $E(x, \varepsilon)$ is a real analytic function on the open subset $\{\varepsilon > 0\}$ of the manifold. Hence the function

$$\begin{aligned} u(x, \varepsilon) &= \int_{\boldsymbol{R}^n} u(y)(1 + y_1^2 + \cdots + y_n^2)^n E(x - y, \varepsilon) \frac{dy}{(1 + y_1^2 + \cdots + y_n^2)^n} \\ &= \langle u(y)(1 + y_1^2 + \cdots + y_n^2)^n, \ E(x - y, \varepsilon) \rangle_y \end{aligned} \tag{3.10}$$

is also real analytic there, where $\langle \, , \rangle$ denotes the inner product between $\mathcal{B}[\boldsymbol{S}^n]$

and $\mathcal{A}(\boldsymbol{S}^n)$. The equation (3.7) can be written by the local coordinates (3.8) at infinity in the form

$$\left[(y_1^2+\cdots+y_n^2)^2\Delta_y+(4-2n)(y_1^2+\cdots+y_n^2)\sum_j y_j\frac{\partial}{\partial y_j}+\frac{\partial^2}{\partial\varepsilon^2}\right]u(y,\varepsilon)=0\,. \tag{3.11}$$

Thus the hypersurface $\{\varepsilon=0\}$ is everywhere non-characteristic with respect to the operator.

Finally we verify that

$$u(x,\varepsilon)\to u(x)\quad\text{in}\quad\mathcal{B}[\boldsymbol{S}^n]\quad\text{when}\quad\varepsilon\downarrow 0\,. \tag{3.12}$$

If this is established, we can conclude by Corollary 2.6 that $u(x)$ agrees with the boundary value $b_+^0(u(x,\varepsilon))$ of $u(x,\varepsilon)$ with respect to the operator in (3.7)-(3.11), because the topology of $\mathcal{B}[\boldsymbol{S}^n]$ is Hausdorff. The boundary value is locally determined by Lemma 2.1, and we have already seen that it is real analytic on a neighborhood of the origin. Thus we can conclude that $u(x)$ is real analytic on the same neighborhood.

Now we verify (3.12). Take a test function $f(x)\in\mathcal{A}(\boldsymbol{S}^n)$. Then we have

$$\begin{aligned}{}_{\mathcal{B}[S^n]}\langle u(x,\varepsilon),\ f(x)\rangle_{\mathcal{A}(S^n)}&=\int_{R^n}u(x,\varepsilon)f(x)\frac{dx}{(1+x_1^2+\cdots+x_n^2)^n}\\&=\int_{R^n}\langle u(y),\ E(x-y,\varepsilon)\rangle_y f(x)\frac{dx}{(1+x_1^2+\cdots+x_n^2)^n}\,.\end{aligned} \tag{3.13}$$

The inner product under the integral sign denotes the one between $\mathcal{B}[K]$ and $\mathcal{A}(K)$, where $K=\operatorname{supp}u$. For fixed $\varepsilon>0$, $E(x-y,\varepsilon)$ is an $\mathcal{A}(K)$-valued continuous function of $x$ and the integral

$$I_\varepsilon(y)=\int_{R^n}E(x-y,\varepsilon)f(x)\frac{dx}{(1+x_1^2+\cdots+x_n^2)^n} \tag{3.14}$$

converges in $\mathcal{A}(K)$. Thus we can take the integral into the inner product and (3.13) becomes $\langle u(y),I_\varepsilon(y)\rangle$. Thus it suffices to show that the integral (3.14) converges to $f(y)/(1+y_1^2+\cdots+y_n^2)^n$ in $\mathcal{A}(K)$ when we let $\varepsilon\downarrow 0$. Since $E(x-y,\varepsilon)$ and $f(x)/(1+x_1^2+\cdots+x_n^2)^n$ are functions of class $L_2$, we can apply the classical Fourier transform. The Parseval formula gives

$$I_\varepsilon(y)=\frac{1}{(2\pi)^n}\int_{R^n}e^{-\varepsilon|\xi|+iy\xi}\mathcal{F}\left[\frac{f(x)}{(1+x_1^2+\cdots+x_n^2)^n}\right](\xi)d\xi\,. \tag{3.15}$$

The function $F[f(x)/(1+x_1^2+\cdots+x_n^2)^n]$ is rapidly decreasing as a Fourier transform of the real analytic function. Thus when we let $\varepsilon\downarrow 0$, (3.15) converges to

$$I(y)=\frac{1}{(2\pi)^n}\int_{R^n} e^{iy\xi}\mathcal{F}\left[\frac{f(x)}{(1+x_1^2+\cdots+x_n^2)^n}\right](\xi)d\xi=\frac{f(y)}{(1+y_1^2+\cdots+y_n^2)^n}$$

uniformly for $y$, when $y$ runs on a complex neighborhood of $K$. Thus we have verified the weak convergence, hence the strong convergence of (3.12). q.e.d.

COROLLARY 3.6. *Let $u(x)$ be a hyperfunction defined on a neighborhood of the origin. Then it is identically equal to zero on a neighborhood of the origin if and only if for each local operator $J(D)$ with constant coefficients $J(D)u(x)$ has value zero at the origin.*

PROOF. We only have to prove the sufficiency. By Theorem 3.5 we see that $u(x)$ is real analytic on a neighborhood of the origin. Therefore employing the discrete data $D^\alpha u(0)=0$, we conclude that $u(x)\equiv 0$ there by the unique continuation property. q.e.d.

It is well known in elementary analysis that the vanishing of every finite derivative $D^\alpha u(0)$ is not sufficient for $u(x)\equiv 0$.

The following result strengthens Theorem 3.3 in [1] (see Remark 3.4 there).

COROLLARY 3.7. *Let $u(x)$ be a germ of hyperfunction at the origin. Assume that for every local operator $J(D)$ with constant coefficients, $J(D)u(x)$ defines a germ of continuous function at the origin. Then $u(x)$ is indeed a germ of real analytic function.*

PROOF. From the assumption we see easily that every $J(D)u(x)$ defines a germ of class $C^{2n}$. Thus by Lemma 3.3, each $J(D)u(x)$ has a finite value at the origin. Hence by Theorem 3.5, $u(x)$ is real analytic on a neighborhood of the origin. q.e.d.

Note that even if every finite derivative $D^\alpha u(x)$ defines a germ of continuous function we cannot conclude that $u(x)$ defines a germ of class $C^\infty$.

For a Fourier hyperfunction $u(x)$, the following criterion is sometimes useful.

THEOREM 3.8. *A Fourier hyperfunction $u(x)$ is real analytic at the origin if and only if for every infra-exponential entire function $J(\xi)$ the finite limit*

$$\lim_{\varepsilon\downarrow 0}\frac{1}{(2\pi)^n}\langle J(\xi)\tilde{u}(\xi), e^{-\varepsilon\sqrt{|\xi|^2+1}}\rangle=\lim_{\varepsilon\downarrow 0}\frac{1}{(2\pi)^n}\int_{R^n} e^{-\varepsilon\sqrt{|\xi|^2+1}}J(\xi)\tilde{u}(\xi)d\xi \tag{3.16}$$

*exists.*

PROOF. Instead of the Poisson kernel $E(x,\varepsilon)$, we employ the Yukawa kernel

$$K(x, \varepsilon)=\mathcal{F}^{-1}[e^{-\varepsilon\sqrt{|\xi|^2+1}}] \tag{3.17}$$
$$=\frac{2}{(2\pi)^{n/2+1}}\frac{1}{|x|^n}\int_0^\infty e^{-(\varepsilon/|x|)\sqrt{r^2+|x|^2}}r^{n/2}J_{n/2-1}(r)dr\,.$$

When $n=1$, we have

$$\int_0^\infty e^{-(\varepsilon/|x|)\sqrt{r^2+|x|^2}}r^{1/2}J_{-1/2}(\alpha r)dr=|x|\frac{2}{\sqrt{\alpha\pi}}\frac{\varepsilon}{\sqrt{\varepsilon^2+\alpha^2|x|^2}}K_1(\varepsilon^2+\alpha^2|x|^2)\,,$$

where $K_1(z)$ denotes the modified Bessel function with the asymptotic $K_1(z)\sim\sqrt{\pi/2z}\,e^{-z}$ for $|\arg z|<3\pi/2$ and $|z|\gg1$. When $n=2$, we have

$$\int_0^\infty e^{-(\varepsilon/|x|)\sqrt{r^2+|x|^2}}rJ_0(\alpha r)dr=|x|^2\frac{e^{-\sqrt{\varepsilon^2+\alpha^2|x|^2}}}{(\varepsilon^2+\alpha^2|x|^2)^{3/2}}(1+\sqrt{\varepsilon^2+\alpha^2|x|^2})\,.$$

Therefore the integral (3.17) for general $n$ can be calculated by the following recurrence formula

$$\frac{\partial}{\partial\alpha}[\alpha^{-\nu}J_\nu(\alpha r)]=-\alpha^{-\nu}rJ_{\nu+1}(\alpha r)\,.$$

The power of $|x|$ in the coefficients cancels. Thus if we take $|x|^2=x_1^2+\cdots+x_n^2$ for complex $x$, we see that $K(x, \varepsilon)$ is a rapidly decreasing real analytic function for fixed $\varepsilon>0$, and when $\varepsilon\downarrow0$, $e^{|z|/2}K(z, \varepsilon)$ converges uniformly to zero on $\{z\in\boldsymbol{C}^n;$ $|\mathrm{Im}\,z|\leq\delta,\ |\mathrm{Re}\,z|\geq2\delta$ for any $\delta$ satisfying $0<\delta<1$. Thus $K(x, \varepsilon)$ converges to zero in $\mathcal{P}_*(K)$ for any compact subset $K\subset\boldsymbol{D}^n$ which does not contain the origin. Hence, if $K=\mathrm{supp}\,u$ does not contain the origin, we have

$$\lim_{\varepsilon\downarrow0}\frac{1}{(2\pi)^n}\langle\tilde{u}(\xi), e^{-\varepsilon\sqrt{|\xi|^2+1}}\rangle=\lim_{\varepsilon\downarrow0}\langle u(x), K(x, \varepsilon)\rangle=0\,,$$

where the inner product is considered at first between $\mathcal{Q}(\boldsymbol{D}^n)$ and $\mathcal{P}_*(\boldsymbol{D}^n)$, and later between $\mathcal{Q}[K]$ and $\mathcal{P}_*(K)$. Therefore the limit (3.16) does not change if we replace $u(x)$ by a hyperfunction $v(x)\in\mathcal{B}_*(\boldsymbol{R}^n)$ which agrees with $u(x)$ on a neighborhood of the origin. Thus we only have to prove the assertion for those $u(x)\in\mathcal{B}_*(\boldsymbol{R}^n)$. The analogy of Lemma 3.3 for our present situation clearly holds. On the other hand, the real analytic function $u(x, \varepsilon)=u(x)*K(x, \varepsilon)$ satisfies

$$\left[\Delta_x-1+\frac{\partial^2}{\partial\varepsilon^2}\right]u(x, \varepsilon)=0 \tag{3.18}$$

on $\varepsilon>0$. As in the proof of Theorem 3.5, we introduce the real analytic manifold $\boldsymbol{S}^n\times\boldsymbol{R}^1$. Then $u(x, \varepsilon)$ can be extended as an infinitely differentiable solution of (3.18) on $\boldsymbol{S}^n\times\{\varepsilon>0\}$. The hypersurface $\{\varepsilon=0\}$ is non characteristic with respect to the operator in (3.18) even at infinity. Thus the remaining part of the proof goes just in the same way as that in Theorem 3.5. q.e.d.

REMARK 3.9. We can define the value of a Fourier hyperfunction $u(x)$ at the origin as

$$\lim_{\varepsilon \downarrow 0} \langle \tilde{u}(\xi), e^{-\varepsilon\sqrt{|\xi|^2+1}} \rangle . \tag{3.19}$$

Since the kernel $K(x, \varepsilon)$ belongs to $\mathcal{P}_*(\boldsymbol{D}^n)$, it is theoretically convenient. But we preferred (3.1) as the definition of the value because of the elementary look of the Poisson kernel $E(x, \varepsilon)$. I do not know whether the two values (3.1) and (3.19) formally agree for $u(x) \in \mathcal{B}_*(\boldsymbol{R}^n)$.

Next we treat real analytic parameters.

THEOREM 3.10. *Let $u(x)$ be a hyperfunction defined on a cylindrical domain $U \times \{|x_n| < \delta\}$, where $U \subset \boldsymbol{R}^{n-1}$ is open. Assume that $u(x)$ contains $x_n$ as a real analytic parameter, and that for every local operator $J(D)$ with constant coefficients we have*

$$J(D)u(x)|_{x_n=0} \in {}'\mathcal{A}(U) . \tag{3.20}$$

*Then $u(x)$ is real analytic on a neighborhood of $U \times \{0\}$.*

PROOF. We are going to show that $u(x)$ is real analytic on a neighborhood of every point $x^0$ of $U \times \{0\}$. Without loss of generality we can assume that $x^0$ is the origin of $\boldsymbol{R}^n$. Let $V \Subset U$ be another neighborhood of the origin in $\boldsymbol{R}^{n-1}$. Then we can find a hyperfunction $v_1(x) \in \mathcal{B}(\boldsymbol{R}^{n-1} \times \{|x_n| < \delta\})$ such that 1) $v_1(x)$ contains $x_n$ as a real analytic parameter, 2) $v_1(x)$ is real analytic outside $\bar{V} \times \{|x_n| < \delta\}$, where $\bar{V}$ denotes the closure of $V$ in $\boldsymbol{R}^{n-1}$, 3) the difference $h(x) = u(x) - v_1(x)$ is real analytic in $V \times \{|x_n| < \delta\}$. In fact, let $\mathrm{sp}\,(u|_{V \times \{|x_n| < \delta\}})$ be the image of $u|_{V \times \{|x_n| < \delta\}}$ by the canonical mapping to the global section of the sheaf $\mathcal{C}$ of microfunctions on $V \times \{|x_n| < \delta\} \times i\boldsymbol{S}_\infty^{*n-1}$. By the flabbiness of the sheaf $\mathcal{C}$ ([13], Chapter III, Corollary 2.1.5), we can find a hyperfunction $v_1(x)$ on $\boldsymbol{R}^{n-1} \times \{|x_n| < \delta\}$ such that $\mathrm{sp}\,(v_1) = \mathrm{sp}\,(u)$ in $V \times \{|x_n| < \delta\} \times i\boldsymbol{S}_\infty^{*n-1}$ and $\mathrm{supp}\,\mathrm{sp}\,(v_1) \subset \overline{\mathrm{supp}\,\mathrm{sp}\,(u|_{V \times \{|x_n| < \delta\}})}$, where the closure is taken in $\boldsymbol{R}^{n-1} \times \{|x_n| < \delta\} \times i\boldsymbol{S}_\infty^{*n-1}$. Thus we see obviously that $v_1(x)$ satisfies all the required properties.

Next let $K$ be a ball in $\boldsymbol{R}^{n-1}$ containing $\bar{V}$ in its interior. Let $\chi_K(x')$ be the characteristic function of $K$. Since $v_1(x)$ is real analytic on $\partial K \times \{|x_n| < \delta\}$, the product $v(x) = \chi_K(x')v_1(x)$ is locally well defined, and contains $x_n$ as a real analytic parameter. The condition (3.20) holds for $v(x)$ with ${}'\mathcal{A}(U)$ replaced by ${}'\mathcal{A}(V)$.

Thus it suffices to prove that under this condition $v(x)$ is real analytic on a neighborhood of the origin.

Put

$$v(x,\varepsilon)={}'\langle v(y',x_n),{}'E(x'-y',\varepsilon)\rangle_{y'}, \tag{3.21}$$

where ${}'E(x',\varepsilon)$ denotes the $(n-1)$-dimensional Poisson kernel, and ${}'\langle,\rangle$ denotes the inner product between ${}'\mathcal{B}[K]$ and ${}'\mathcal{A}(K)$. Let $\boldsymbol{S}^{n-1}$ be the one point compactification of $\boldsymbol{R}^{n-1}$. Then by Lemma 1.4, $v(x,\varepsilon)$ is a real analytic function of $x$ and $\varepsilon$ on the open subset $\boldsymbol{S}^{n-1}\times\{|x_n|<\delta\}\times\{\varepsilon>0\}$ of the real analytic manifold $\boldsymbol{S}^{n-1}\times\boldsymbol{R}^1\times\boldsymbol{R}^1$. Put

$$w(x,\varepsilon)=Y\Big(\frac{\delta^2}{4}-x_n^2\Big)v(x,\varepsilon)\,. \tag{3.22}$$

For each fixed $\varepsilon>0$, this is a hyperfunction with compact support $L=\boldsymbol{S}^{n-1}\times\{|x_n|\le\delta/2\}$ in $\boldsymbol{S}^{n-1}\times\boldsymbol{R}^1$. We claim that

$$w(x,\varepsilon)\longrightarrow w(x)=Y\Big(\frac{\delta^2}{4}-x_n^2\Big)v(x)\qquad\text{in}\quad \mathcal{B}[L]\,, \tag{3.23}$$

where the product in the right hand side is well defined because $v(x)$ contains $x_n$ as a real analytic parameter. Take a test function $f(x)\in\mathcal{A}(L)$. Then we have

$$\begin{aligned}
&{}_{\mathcal{B}[L]}\langle w(x,\varepsilon),f(x)\rangle_{\mathcal{A}(L)} \\
&=\int_{-\delta/2}^{\delta/2}\Big\{\int_{R^{n-1}}v(x,\varepsilon)f(x',x_n)\frac{dx'}{(1+|x'|^2)^{n-1}}\Big\}dx_n \\
&=\int_{-\delta/2}^{\delta/2}{}'\Big\langle v(y',x_n),\int_{R^{n-1}}{}'E(x'-y',\varepsilon)f(x',x_n)\frac{dx'}{(1+|x'|^2)^{n-1}}\Big\rangle_{y'}dx_n\,,
\end{aligned} \tag{3.24}$$

where we have employed $dx'dx_n/(1+|x'|^2)^{n-1}$ as the volume element on $\boldsymbol{S}^{n-1}\times\boldsymbol{R}^1$. The last deformation is legal because the integral

$$I_\varepsilon(y',x_n)=\int_{R^{n-1}}{}'E(x'-y',\varepsilon)f(x',x_n)\frac{dx'}{(1+|x'|^2)^{n-1}} \tag{3.25}$$

converges in ${}'\mathcal{A}(K)$ for fixed $x_n$ and $\varepsilon$. Since $I_\varepsilon(y',x_n)$ belongs to $\mathcal{A}(K\times\{|x_n|\le\delta/2\})$, we can rewrite (3.24) by Lemma 1.4 as follows: $\langle Y(\delta^2/4-x_n^2)v(x),I_\varepsilon(x',x_n)\rangle$, where the inner product is used between $\mathcal{B}[K\times\{|x_n|\le\delta/2\}]$ and $\mathcal{A}(K\times\{|x_n|\le\delta/2\})$. When we let $\varepsilon\downarrow 0$, $I_\varepsilon(x',x_n)$ converges to $f(x)/(1+|x'|^2)^{n-1}$ in $\mathcal{A}(K\times\{|x_n|\le\delta/2\})$. The proof is similar to that in the last part of the proof of Theorem 3.5. Thus we conclude that (3.24) converges to $\langle Y(\delta^2/4-x_n^2)v(x),f(x)/(1+|x'|^2)^{n-1}\rangle$, which is obviously equal to ${}_{\mathcal{B}[L]}\langle Y(\delta^2/4-x_n^2)v(x),f(x)\rangle_{\mathcal{A}(L)}$. Thus (3.23) was proved.

Now $w(x, \varepsilon)$, as a function of $n+1$ variables on an open subset $\boldsymbol{S}^{n-1}\times\boldsymbol{R}^1\times\{\varepsilon>0\}$ of the real analytic manifold $\boldsymbol{S}^{n-1}\times\boldsymbol{R}^1\times\boldsymbol{R}^1$, satisfies

$$\text{(3.26)}\qquad \left(\Delta_{x'}+\frac{\partial^2}{\partial\varepsilon^2}\right)w(x, \varepsilon)=Y\left(\frac{\delta^2}{4}-x_n^2\right)\left(\Delta_{x'}+\frac{\partial^2}{\partial\varepsilon^2}\right)v(x, \varepsilon)$$
$$={}'\left\langle v(y', x_n), \left(\Delta_{x'}+\frac{\partial^2}{\partial\varepsilon^2}\right)'E(x'-y', \varepsilon)\right\rangle_{y'}=0\,,$$

because the differentiation $(\Delta_{x'}+\partial^2/\partial\varepsilon^2)'E(x'-y', \varepsilon)$ converges in $'\mathcal{A}(K)$. A calculation similar to (3.11) shows that the hypersurface $\{\varepsilon=0\}$ is everywhere non-characteristic with respect to the operator in (3.26) even at infinity. Further, $\operatorname{supp} w(x, \varepsilon)$ is contained in $L\times\{\varepsilon>0\}$, where $L$ is the compact subset of $\boldsymbol{S}^{n-1}\times\boldsymbol{R}^1$ defined above. Thus by Corollary 2.6, we conclude that the limit function $w(x)$ in (3.23) agrees with the boundary value $b_+^0(w(x, \varepsilon))$ with respect to the operator.

Now we examine the convergence in fuller detail at the origin. Near the origin, $w(x, \varepsilon)=v(x, \varepsilon)$ is a real analytic function of $x$ for fixed $\varepsilon>0$. By Lemma 1.1, (3.21) can be rewritten as follows

$$v(x, \varepsilon)={}'\langle v(x'-y', x_n), {}'E(y', \varepsilon)\rangle_{y'}\,.$$

Hence by Lemma 1.5, for any local operator $J(D)$ with constant coefficients we have near the origin,

$$J(D)w(x, \varepsilon)=J(D)v(x, \varepsilon)={}'\langle J(D_x)v(x'-y', x_n), {}'E(y', \varepsilon)\rangle_{y'}$$
$$={}'\langle (J(D)v)(x'-y', x_n), {}'E(y', \varepsilon)\rangle_{y'}\,.$$

Now take the value at the origin.

$$\text{(3.27)}\qquad J(D)w(x, \varepsilon)(0)={}'\langle (J(D)v)(-y', 0), {}'E(y', \varepsilon)\rangle_{y'}\,.$$

By the condition on $v(x)$, the function $(J(D)v)(-y', 0)$ is real analytic on a neighborhood of the origin (in fact in $-V$), and has compact support $K$. Thus if we let $\varepsilon\downarrow 0$, (3.27) converges to the finite value $(J(D)v)(-y', 0)|_{y'=0}$ by Lemma 3.3. Thus in the same way as in the proof of Theorem 3.5, we conclude that when $\varepsilon\downarrow 0$, $w(x, \varepsilon)$ converges to a real analytic function of $x$ on a neighborhood of the origin in $\boldsymbol{R}^n$. By the same technique used there, employing the equation (3.26) we can show that $w(x, \varepsilon)$ can be extended as a classical function of class $C^2$ on a neighborhood of the origin in $\boldsymbol{R}^{n+1}$ beyond the boundary $\{\varepsilon=0\}$. Thus by Lemma 2.3, we conclude that the boundary value $w(x)$ is real analytic on a neighborhood of the origin in $\boldsymbol{R}^n$. Since $w(x)$ is equal to $v(x)$ near the origin, we have completed the proof. q.e.d.

COROLLARY 3.11. *Let $u(x)$ be a hyperfunction defined on a cylindrical domain $U\times\{|x_n|<\delta\}$, containing $x_n$ as a real analytic parameter. Assume that for every local operator $J(D)$ with constant coefficients we have*

$$J(D)u(x)|_{x_n=0}=0 \quad in \quad '\mathcal{B}(U). \tag{3.28}$$

*Then we have $u(x)\equiv 0$ on a neighborhood of $U\times\{0\}$.*

In fact, by Theorem 3.10 we conclude that $u(x)$ is real analytic on a neighborhood of $U\times\{0\}$. Thus employing the discrete data $D_n^k u(x)|_{x_n'=0}=0$, $k=0,1,\dots$, we conclude that $u(x)\equiv 0$ there. Note that we cannot assert that $u(x)\equiv 0$ in $U\times\{|x_n|<\delta\}$. For example, $\delta(t-x^\alpha)$ is a hyperfunction of two variables on $\{x>0\}\times R_t^1$ containing $t$ as a real analytic parameter. Its support contains $\{t=x^\alpha\}$.

REMARK 3.12. In Sato's original conjecture, the assumption (3.28) contained only the normal derivatives $J(D_n)$. If we prove the following assertion, we can prove the conjecture in the original form without adding any further technique. "Let $\{f_k(x)\}$ be a sequence of germs of real analytic functions at the origin. Assume that for every local operator with constant coefficients of the product type, the value $J_1(D_1)\cdots J_n(D_n)f_k(0)$ converges to a finite limit when $k\to\infty$. Then $f_k(x)$ converges in $\mathcal{A}(\{0\})$." Note that the topology defined by the seminorms $J_1(D_1)\cdots J_n(D_n)f(0)$ of the product type is really weaker than the usual weak topology of $\mathcal{A}(\{0\})$. Thus the above conjecture makes sense only for the sequences.

## §4. Complex holomorphic parameters and examples

In this section we first give the definition of complex holomorphic parameters, and later give critical examples concerning analytic parameters, mainly for the results in this paper.

We say that a hyperfunction $u(x)$ of $n$-variables $x=(x',x_n)$ contains $x_n$ as a complex holomorphic parameter at the origin if it admits the following defining function $F(z)$: There exists a complex neighborhood $V$ of the origin such that each component $F_\sigma(z)$ on $V_\sigma$ can be holomorphically extended to

$$V_{\sigma'}=\{z\in V;\ \sigma_j\operatorname{Im} z_j>0\,,\quad j=1,\dots,n-1\}\,. \tag{4.1}$$

We say that a hyperfunction $u(x)\in\mathcal{B}(U)$ contains $x_n$ as a complex holomorphic parameter if it satisfies the above condition at every point of $U$. The sheaf of hyperfunctions with the complex holomorphic parameter $x_n$ is defined by the relative cohomology $\mathcal{H}^{n-1}_{R^{n-1}\times C_{z_n}}(\mathcal{O})|_{R^n}$. For such argument see [11], [12] or [13]. A real analytic function contains $x_n$ as a complex holomorphic parameter. The complex holomorphic parameter is preserved by the operation of differential operators, or even of local operators. These are readily seen from the definition. If $u(x) \in \mathcal{B}(U)$ contains $x_n$ as a complex holomorphic parameter, we can choose a global defining function $F(z)$ satisfying the above definition at every point of $U$. The complex holomorphic parameter is a special case of the real analytic parameter. Therefore we can take the restriction $u(x', 0)$ or the product $Y(x^2 - x_n^2)u(x)$.

By Sato's fundamental theorem, the real analytic parameter is preserved by solving an elliptic equation. As for the complex holomorphic parameter we have

LEMMA 4.1. *Let $u$ be a hyperfunction which contains $x_n$ as a complex holomorphic parameter. Let $J(D)$ be an elliptic local operator with constant coefficients. Then every solution $v$ of $J(D)V=u$ also contains $x_n$ as a complex holomorphic parameter.*

PROOF. The problem is local and the solutions of the homogeneous equation $J(D)v=0$ are real analytic, hence contain $x_n$ as complex holomorphic parameters. Thus it suffices to prove the assertion locally for a special solution $v$. We construct $v$ solving the equation $J(D)G_\sigma(z) = F_\sigma(z)$. We can assume that $F_\sigma(z)$ is holomorphic on $V_{\sigma'}$ of the form (4.1), where $V$ is convex. Thus by the existence theorem of the holomorphic solution on convex domains for the local operators (see, e.g., [1], Theorem 4.1) we can find holomorphic solutions $G_\sigma(z)$ on $V_{\sigma'}$. Let $v$ be the hyperfunction defined by the defining functions $G_\sigma(z)|_{V_\sigma}$. Then $v$ contains $x_n$ as a complex holomorphic parameter and satisfies $J(D)v=u$. q.e.d.

LEMMA 4.2. *A hyperfunction $u(x) \in \mathcal{B}(U)$ contains $x_n$ as a complex holomorphic parameter if and only if it is the restriction of a solution $v(x, y_n) \in \mathcal{B}(W)$, of the equation $\bar{\partial}_n v=0$ to the non-characteristic plane $y_n=0$, where*

$$\bar{\partial}_n = \frac{1}{2}\frac{\partial}{\partial x_n} + \frac{i}{2}\frac{\partial}{\partial y_n},$$

*and $W \subset \boldsymbol{R}^{n+1}$ is an open set such that $W \cap \{y_n=0\} = U$.*

PROOF. Let $u(x) \in \mathcal{B}(U)$ be a hyperfunction containing $x_n$ as a complex holomorphic parameter. Let $F(z)$ be one of the global defining function on a complex neighborhood $V$ satisfying the definition. Let $v(x, y_n)$ be the hyperfunction of $(n+1)$-variables defined by the defining functions

$$G_{(\sigma,+1)}(z, w) = F_\sigma(z', z_n + iw)$$

on

$$V_{(\sigma,+1)}=\{(z,w)\in \boldsymbol{C}^{n+1};\ (z',z_n+iw)\in V,\ \sigma_j\operatorname{Im} z_j>0\,,\quad j=1,\dots,n,\ \operatorname{Im} w>0\}\,,$$

$$G_{(\sigma,-1)}(z,w)=0$$

on

$$V_{(\sigma,-1)}=\{(z,w)\in \boldsymbol{C}^{n+1};\ (z',z_n+iw)\in V,\ \sigma_j\operatorname{Im} z_j>0\,,\quad j=1,\dots,n,\ \operatorname{Im} w<0\}\,.$$

Put

$$W=\{(x,y_n)\in \boldsymbol{R}^{n+1};\ (x',x_n+iy_n)\in V\}\,.$$

Then by way of the defining function we see easily that $v(x,y_n)$ is a solution of $\bar{\partial}_n v=0$ in $\mathcal{B}(W)$ satisfying $v|_{y_n=0}=u$.

Conversely let $v(x,y_n)$ be a solution of $\bar{\partial}_n v=0$ on an open set $W\subset \boldsymbol{R}^{n+1}$ satisfying $W\cap\{y_n=0\}=U$. Let $G(z,w)$ be a defining function of $v$ on a complex neighborhood $X$ of $W$. Since the assertion is local, we can assume that $U$ is convex, $W=U\times\{|y_n|<\delta\}$ and $X=W\times\{|\operatorname{Im} z_j|<\delta,\ |\operatorname{Im} w|<\delta\}$, where $\delta>0$ is sufficiently small. Then $\bar{\partial}_n v=0$ implies that

$$\frac{1}{2}\left(\frac{\partial}{\partial z_n}+i\frac{\partial}{\partial w}\right)G(z,w)=\sum_{k=1}^{n+1} H_k(z,w)\,,$$

where $H_k(z,w)$ is holomorphic on

$$X_k=\{(z,w)\in X;\ \operatorname{Im} z_j\neq 0\quad \text{for}\quad j\neq k\}\,,$$

where we understand $w=z_{n+1}$. Since each connected component of $X_k$ is convex, we can easily find holomorphic solutions $G_k(z,w)$ of $\bar{\partial}_n G_k=H_k$ on $X_k$. Then $G(z,w)-\sum_{k=1}^{n+1} G_k(z,w)$ is another defining function of $v(x,y_n)$. Thus we can assume without loss of generality that

$$\frac{1}{2}\left(\frac{\partial}{\partial z_n}+i\frac{\partial}{\partial w}\right)G(z,w)=0\,.$$

Put $F_\sigma(z)=G_{(\sigma,\pm1)}(z,w^0)$ for a fixed $w^0$ satisfying $\pm\operatorname{Im} w^0>0$ with the same sign. Then we have

$$\frac{1}{2}\left(\frac{\partial}{\partial z_n}+i\frac{\partial}{\partial w}\right)F_\sigma(z',z_n+iw)=0\,.$$

Thus by the uniqueness of the Cauchy problem we conclude that $G_{(\sigma,\pm1)}(z,w)=F_\sigma(z',z_n+i(w-w^0))=G_{(\sigma,\pm1)}(z',z_n+i(w-w^0),w^0)$. Choose $w^0=-(\delta/2)\sigma_n\pm(\delta/2)i$. Then we have

$$G_{(\sigma,\pm1)}(z,w)=G_{(\sigma,\pm1)}\left(z', z_n+iw+\frac{\delta}{2}\sigma_n i\pm\frac{\delta}{2},\ -\frac{\delta}{2}\sigma_n\pm\frac{\delta}{2}i\right),$$

where the double signs are taken in the same way. Since

$$\sigma_n \operatorname{Im}\left(z_n+iw+\frac{\delta}{2}\sigma_n i\pm\frac{\delta}{2}\right)=\sigma_n \operatorname{Im} z_n+\sigma_n \operatorname{Re} w+\frac{\delta}{2},$$

$$\operatorname{Re}\left(z_n+iw+\frac{\delta}{2}\sigma_n i\pm\frac{\delta}{2}\right)=\operatorname{Re} z_n-\operatorname{Im} w\pm\frac{\delta}{2},$$

the above formula implies that $G_{(\sigma,\pm1)}(z,w)$ can be continued holomorphically to

$$\left[U_\delta\times\bigcap_{j=1}^{n-1}\{0<\sigma_j \operatorname{Im} z_j<\delta\}\cap\left\{|\operatorname{Im} z_n|<\frac{\partial}{4}\right\}\right]\times\left\{|w|<\frac{\delta}{4}\right\},$$

where $U_\delta=\{x\in U;\ \operatorname{dis}(x,\partial U)>\delta\}$. Thus the restriction $u(x)=v(x,0)$ defined by the defining functions

$$F_\sigma(z)=G_{(\sigma,+1)}(z,0)-G_{(\sigma,-1)}(z,0)$$

contains $x_n$ as a complex holomorphic parameter on $U_\delta$. Thus in the general case we see that $u(x)$ contains $x_n$ as a complex holomorphic parameter at every point of the given domain. q.e.d.

LEMMA 4.3. *Let $v(x',x_n,y_n)$ be a function of class $C^\infty$ ($\mathcal{D}'$, or ultradistribution) on $V'\times V_n$, where $V'\subset \boldsymbol{R}^{n-1}$, $V_n\subset\boldsymbol{R}^2$ are open. Then $v$ satisfies the partial Cauchy-Riemann equation $\bar\partial_n v=0$ if and only if $v$ is naturally regarded as a $C^\infty(V')$-valued (respectively $\mathcal{D}'(V')$-valued, or ultradistribution-valued) holomorphic function of $z_n=x_n+iy_n$ on $V_n$.*

PROOF. Since the proofs are similar, we give one for $C^\infty$ case. Let $v(x',x_n,y_n)$ be a solution of $\bar\partial_n v=0$ of class $C^\infty$. Take a test function $\varphi(x')\in\mathcal{E}'(V')$. Then the product $\varphi(x')v(x',x_n,y_n)$ belongs to $\mathcal{D}'(V'\times V_n)$. We define the integral along the compact fibre

$$\langle\varphi, v\rangle_{x'}\equiv\int_{R^{n-1}}\varphi(x')v(x',x_n,y_n)dx'\in\mathcal{D}'(V_n)$$

by the following formula

$$\langle\psi(x_n,y_n),\langle\varphi,v\rangle_{x'}\rangle=\int_{R^{n+1}}\psi(x_n,y_n)\varphi(x')v(x',x_n,y_n)dx'dx_ndy_n,$$

where $\psi\in C_0^\infty(V_n)$ is a test function. We have

$$\begin{aligned}\langle\psi(x_n, y_n), \bar{\partial}_n\langle\varphi, v\rangle_{x'}\rangle &= \langle -\bar{\partial}_n\psi(x_n, y_n), \langle\varphi, v\rangle_{x'}\rangle \\ &= -\int_{R^{n+1}}(\bar{\partial}_n\psi(x_n, y_n))\varphi(x')v(x', x_n, y_n)dx'dx_ndy_n \\ &= \int_{R^{n+1}}\psi(x_n, y_n)\varphi(x')\bar{\partial}_nv(x', x_n, y_n)dx'dx_ndy_n \\ &= 0\,.\end{aligned}$$

Hence $\langle\varphi, v\rangle_{x'}$ is in fact a holomorphic function of $x_n+iy_n$. Thus $v(x', x_n, y_n)$ is a $C^\infty(V')$-valued weakly (hence strongly) holomorphic function.

Conversely, let $v(x'; z_n)$ be a $C^\infty(V')$-valued holomorphic function of $z_n=x_n+iy_n$. Then $v$ and its derivatives on $z_n$ up to a fixed order form a bounded subset of $C^\infty(V')$ when $(x_n, y_n)$ runs a compact subset of $V_n$. For a test function of the type $\varphi(x')\psi(x_n, y_n)\in\mathcal{E}'(V'\times V_n)$, we define the functional $v(x', x_n, y_n)$ by

$$\langle\varphi(x')\psi(x_n, y_n), v(x', x_n, y_n)\rangle=\int_{R^2}\langle\varphi(x'), v(x';\ z_n)\rangle_{x'}\psi(x_n, y_n)dx_ndy_n\,.$$

Then we see that $v(x', x_n, y_n)$ is a continuous functional, hence $v(x', x_n, y_n)\in C^\infty(V'\times V_n)$. From the same formula we have obviously $\bar{\partial}_nv(x', x_n, y_n)=0$.

The two correspondences given above are clearly inverse to each other. q.e.d.

Since $C^\infty$- ($\mathcal{D}'$-, or ultradistribution-) solutions of $\bar{\partial}_nv=0$ are of course hyperfunction solutions of the same equation, we see that those functions in Lemma 4.3 contain $x_n$ as a complex holomorphic parameter due to Lemma 4.2. Conversely, based on these functions we can locally characterize general hyperfunctions containing $x_n$ as a complex holomorphic parameter.

THEOREM 4.4. *Let $u(x)\in\mathcal{B}(V)$ be a hyperfunction containing $x_n$ as a complex holomorphic parameter. Then on every relatively compact subdomain of $V$ of the type $V'\times\{a<x_n<b\}$, where $V'\subset\boldsymbol{R}^{n-1}$ is convex, we can find an elliptic local operator $J(D')$ of $n-1$ variables $x'$, and a function $v(x)$ which is a $C^\infty(V')$-valued holomorphic function of $x_n$, such that $u=J(D')v$.*

PROOF. By Lemma 4.2 we can take a hyperfunction solution $u_1(x, y_n)$ of $\bar{\partial}_nu_1=0$ such that $u_1(x, 0)=u(x)$, where $V_n$ is a convex open set in $\boldsymbol{R}^2$ with the system of coordinates $(x_n, y_n)$ such that $V_n\cap\{y_n=0\}=\{a<x_n<b\}$. By [1], Theorem 1.3, we can find an elliptic local operator $Q(D, D_{y_n})$ such that every hyperfunction solution $u_2(x, y_n)$ of $Q(D, D_{y_n})u_2=u_1$ is of class $C^\infty$. In fact we can take one of the type

$$J\left(\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_{n-1}^2}+\frac{\partial^2}{\partial x_n^2}+\frac{\partial^2}{\partial y_n^2}\right),$$

where $J$ is an entire function of one variable of order 1/2. (See the proof of the quoted theorem.) Consider the system of equations

$$\begin{cases} Q(D, D_{y_n})w = u_1 , \\ \bar{\partial}_n w = 0 . \end{cases} \tag{4.2}$$

If the polynomial corresponding to the operator $\bar{\partial}_n$ does not divide the entire function corresponding to $Q(D, D_{y_n})$, we can solve (4.2) with respect to $w$, because the compatibility condition is satisfied ([1], Theorem 4.1). Thus $w(x, y_n)$ is a solution of $\bar{\partial}_n w = 0$ of class $C^\infty$. Due to the second equation of (4.2), the first equation can be rewritten as $Q(D, iD_n)w = u_1$, where $D_n = i\partial/\partial x_n$. Now put $v(x) = w(x, 0)$. By Lemma 4.3, $v$ is a $C^\infty(V')$-valued holomorphic function of $x_n$ and satisfies $Q(D, iD_n)v = u$. For our choice of $Q$, we have

$$Q(D, iD_n) = J\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{n-1}^2}\right),$$

which is an elliptic local operator of $x'$. q.e.d.

REMARK 4.5. We can also take

$$J\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{n-1}^2} + 2\frac{\partial^2}{\partial x_n^2} + \frac{\partial^2}{\partial y_n^2}\right)$$

as the operator $Q(D, D_{y_n})$. Then we can finally get

$$J(D) = J\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)$$

as the regularizing operator. Since the latter is elliptic in all variables $x$, every solution $v$ of $J(D)v = u$ satisfies the property required in Theorem 4.4.

REMARK 4.6. If $u(x)$ is a $C^\infty(V')$-valued (or $\mathcal{D}'$- or ultradistribution-valued) holomorphic function of $x_n$, we can decompose the support of $u$ parallel to the $x_n$-axis in the way $u(x) = \sum \varphi_\alpha(x')u(x)$, where $\varphi_\alpha(x') \in C_0^\infty(V')$ are appropriately regular functions and each term $\varphi_\alpha(x')u(x)$ is a $C^\infty(V')$-valued (or $\mathcal{D}'$- or ultradistribution-valued) holomorphic function of $x_n$. Hence in general if $u(x)$ is a hyperfunction containing $x_n$ as a complex holomorphic parameter, then regularizing $u(x)$ by Theorem 4.4 as $u(x) = J(D)v(x)$, we have the decomposition $u(x) = \sum J(D)[\varphi_\alpha(x')v(x)]$, where each term in the right hand side contains $x_n$ as a complex holomorphic parameter, and has a support whose intersection with $\{x_n = \text{const.}\}$ is compact. Thus for the complex holomorphic parameter, we have a way of decomposition more elementary than that used in the proof of Theorem 3.10 employing the flabbiness of $\mathcal{C}$.

Now we give miscellaneous examples. For the sake of simplicity we put $n=2$ and write $x$, $t$ for $x_1$, $x_n$.

EXAMPLE 4.7. Put

$$u(x,t)=\begin{cases}\exp\left(-\dfrac{it}{x^2}-\dfrac{1}{|x|}\right) & \text{for } x\neq 0,\\ 0 & \text{for } x=0.\end{cases}$$

Then $u$ is an infinitely differentiable function of $x$, $t$, and for each fixed $x$, $u$ can be continued holomorphically in $t$ to the whole complex line $\boldsymbol{C}$. But $u$ does not contain $t$ as a real analytic parameter (much less as a complex holomorphic parameter). In fact, suppose that $u(x,t)$ contains $t$ as a real analytic parameter. Let $\varphi(x)$ be an infinitely differentiable non-negative even function such that $\mathrm{supp}\,\varphi\subset[-1,1]$ and $\varphi\equiv 1$ on $[-1/2,1/2]$. Then the product $\varphi(x)u(x,t)$ must also contain $t$ as a real analytic parameter, because $u(x,t)$ is real analytic outside $\{x=0\}$. Then by Lemma 1.7 and Lemma 1.4, the integral

$$I(t)=\int_{-\infty}^{\infty}\varphi(x)u(x,t)dx$$

must be a real analytic function of $t$. Putting $1/x^2=y$, we have

$$I(t)=\int_1^{\infty}e^{-ity-\sqrt{y}}\varphi\left(\frac{1}{\sqrt{y}}\right)\frac{1}{\sqrt{y}^3}dy\,.$$

The result cannot be real analytic at $t=0$, which is a contradiction. For, choosing $J(y)=y^2(e^{\sqrt{y}}+e^{-\sqrt{y}})$, we have

$$\begin{aligned}J(D_t)I(0)&=\lim_{\varepsilon\downarrow 0}\int_1^{\infty}e^{-\sqrt{y}}\varphi\left(\frac{1}{\sqrt{y}}\right)\frac{1}{\sqrt{y}^3}y^2(e^{\sqrt{y}}+e^{-\sqrt{y}})e^{-\varepsilon\sqrt{y^2+1}}dy\\ &\geq\lim_{\varepsilon\downarrow 0}\int_2^{\infty}\sqrt{y}\,e^{-\varepsilon\sqrt{y^2+1}}dy=\infty\,.\end{aligned}$$

Thus by Theorem 3.8 we reach the conclusion.

EXAMPLE 4.8. Consider the function

$$F(z,\tau)=\exp\left(-\frac{ie^{-\tau}}{\sqrt[4]{z}}-\frac{1}{\sqrt[8]{z}}\right),$$

where $\sqrt[4]{z}$ etc. denote those branches which take the positive value on the positive real axis. Clearly $F(z,\tau)$ is holomorphic on $\{\operatorname{Im} z>0\}\times\{\tau\in\boldsymbol{C}\}$, hence defines a hyperfunction $u(x,t)=F(x+i0,t+i0)$ which contains $t$ as a complex holomorphic parameter. Since for $\operatorname{Im} z>0$

$$\frac{5}{4}\pi-\operatorname{Im}\tau<\arg\left(-\frac{ie^{-\tau}}{\sqrt[4]{z}}\right)<\frac{3}{2}\pi-\operatorname{Im}\tau\,,$$

when we let $\operatorname{Im}\tau\downarrow 0$, $\operatorname{Im} z\downarrow 0$ from the side $\operatorname{Im}\tau\geq 0$, $F(z,\tau)$ converges in $C^\infty(\boldsymbol{R}^2_{x,t})$ to the infinitely differentiable function

$$f(x,t)=\begin{cases}\exp\left(-\dfrac{ie^{-t}}{\sqrt[4]{x+i0}}-\dfrac{1}{\sqrt[8]{x+i0}}\right) & \text{for } x\neq 0\,,\\ 0 & \text{for } x=0\,.\end{cases}$$

We have $u(x,t)=f(x,t)$. We can check this, e.g., by showing that $F(z,\tau)$ defines the Fourier hyperfunction $f(x,t)$ employing the rapidly decreasing real analytic test functions. Thus $u(x,t)$ is an infinitely differentiable function containing $t$ as a complex holomorphic parameter as a hyperfunction. But it is not a $C^\infty(\boldsymbol{R}_x)$-valued holomorphic function of $t$. For, assume that at least $u(x,t)$ is a $\mathcal{D}'(\boldsymbol{R}_x)$-valued holomorphic function of $t$. Then for every $\varphi(x)\in C_0^\infty(R_x)$, $\varphi(x)u(x,t)$ must also be a $\mathcal{D}'(\boldsymbol{R}_x)$-valued holomorphic function of $t$, hence

$$I(t)=\int_{-\infty}^{\infty}\varphi(x)u(x,t)dx$$

must be analytic in $t$. Choose $\varphi(x)\in C_0^\infty[0,2]$ such that $\varphi(x)=\exp(i/\sqrt[4]{x})\psi(x)$, where $\psi(x)\in C_0^\infty[0,2]$ is a non-negative function satisfying $\psi(x)\geq\exp(-1/\sqrt[8]{x})$ on $0\leq x\leq 1$. Then we have, putting $1/\sqrt[4]{x}=y$,

$$I(t)=\int_{1/\sqrt[4]{2}}^{\infty}e^{-ie^{-t}y-\sqrt{y}}e^{iy}\psi\left(\frac{1}{y^4}\right)\frac{4}{y^5}dy\,.$$

Thus taking $J(y)=y^5(e^{2\sqrt{y}}+e^{-2\sqrt{y}})$ we have

$$\begin{aligned}J(D_t)I(0)&=\lim_{\varepsilon\downarrow 0}\int_{1/\sqrt[4]{2}}^{\infty}e^{-\sqrt{y}}\psi\left(\frac{1}{y^4}\right)\frac{4}{y^5}y^5(e^{2\sqrt{y}}+e^{-2\sqrt{y}})e^{-\varepsilon\sqrt{y^2+1}}dy\\&\geq\lim_{\varepsilon\downarrow 0}\int_1^{\infty}4e^{-\varepsilon\sqrt{y^2+1}}dy=\infty\,.\end{aligned}$$

Due to Theorem 3.8 this shows that $I(t)$ is not analytic at $t=0$.

The above argument simultaneously shows that the product $\varphi(x)u(x,t)$ does not contain $t$ even as a real analytic parameter. Since $\text{S.S.}\varphi(x)\subset\boldsymbol{R}^2\times\{\pm idx\infty\}$, $\text{S.S.}u(x,t)\subset\boldsymbol{R}^2\times\{\pm idx\infty\}$, this shows that the rule of estimating the singular spectrum of the product (see [13], Chapter I, Corollary 2.4.2) does not hold for such product.

EXAMPLE 4.9. Put

$$v(x, t)=\begin{cases}\exp\left(\dfrac{it}{\sqrt[4]{x+i0}}-\dfrac{1}{\sqrt[8]{x+i0}}\right) & \text{for } x\neq 0\,,\\ 0 & \text{for } x=0\,.\end{cases}$$

Thus on a neighborhood of $\{t=-1\}$, we have

$$v(x, t)=u(x, -\log(-t)),$$

where $u(x, t)$ is the one defined in Example 4.8. Since $-\log(-t)$ is a holomorphic transformation on a neighborhood of $t=-1$, $v(x, t)$ is a function of class $C^\infty$ containing $t$ as a complex holomorphic parameter on a neighborhood of $\{t=-1\}$. Take the local operator $J(D_t)=e^{\sqrt{D_t}}+e^{-\sqrt{D_t}}$. Then we have

$$J(D_t)v(x, t)=\exp\left(\frac{it}{\sqrt[4]{x+i0}}-\frac{1}{\sqrt[8]{x+i0}}\right)\left\{\exp\left(-\frac{1}{\sqrt[4]{x+i0}}\right)+\exp\left(\frac{1}{\sqrt[4]{x+i0}}\right)\right\}.$$

Thus the result goes beyond the class $C^\infty$, because it is not bounded at $x=0$.

EXAMPLE 4.10. Finally we give a famous counter-example by M. Sato that Theorem 1.5 does not hold in general when the condition for $\operatorname{supp} u(x)$ is dropped. This example has long be known though unpublished. Let $P_n(z)$ be a sequence of polynomials in one variable which approximate $1/z$ locally uniformly outside the negative real axis. Namely, there exists a sequence of compact subsets $K_1\subset K_2\subset\cdots\subset K_n\subset\cdots$, such that $\cup K_n=\boldsymbol{C}\setminus\,]-\infty, 0]$, and a decreasing sequence of positive numbers $\varepsilon_n$ such that

$$\left|\frac{1}{z}-P_n(z)\right|\leq\varepsilon_n\,,\qquad \text{if}\quad z\in K_n\,.$$

Further, writing $\delta_n=\operatorname{dis}(\{0\}, K_n)$, we can assume that $\sqrt[n]{\delta_n}$ tends to zero when $n$ tends to infinity. Then the power series $F(z, \tau)=\sum_{n=0}^{\infty}P_n(z)\tau^n$ defines a holomorphic function in $(\boldsymbol{C}\setminus\,]-\infty, 0])\times\{\tau\in\boldsymbol{C};\ |\tau|<1\}$, since for $z\in K_m$

$$|P_n(z)|\leq\left|\frac{1}{z}\right|+\varepsilon_n\leq\frac{1}{\delta_m}+\varepsilon_0\qquad \text{for}\quad n\geq m\,.$$

The associated hyperfunction $f(x, t)=F(x+i0, t+i0)-F(x-i0, t+i0)$ contains $t$ as a complex holomorphic parameter. Clearly every derivative of the finite order vanishes at $t=0$. But $\operatorname{supp} f(x, t)$ contains the origin. In fact if $f(x, t)$ vanishes identically on a neighborhood of the origin, then the above series must obviously converge uniformly on a neighborhood of the origin. This implies

$$\sqrt[n]{\max_{|z|\leq r}|P_n(z)|}\leq C$$

for some $r>0$ and $C<\infty$. Since by the assumption

$$|P_n(\delta_n)|\geq\frac{1}{\delta_n}-\varepsilon_n\geq\frac{1}{\delta_n}-\varepsilon_0,$$

hence

$$\sqrt[n]{|P_n(\delta_n)|}\to\infty \text{ when } n\to\infty,$$

we have a contradiction.

Note that if $F(z,\tau)$ in the above example would have an estimate of the form

$$|F(z,\tau)|\leq C\exp\left(\frac{1}{|\mathrm{Im}\, z|^q}\right) \tag{4.3}$$

near the real axis, then $f(x,t)$ would become an ultradistribution by Komatsu's criterion (see [6] Theorem 11.5). Then $f(x,t)$ would be an ultradistribution-valued holomorphic function of $t$. Hence, choosing a sufficiently regular function $\varphi(x)\in C_0^\infty(\boldsymbol{R}_x)$, we could make the product $\varphi(x)f(x,t)$, thus cutting off the support without losing the assumption that the restriction of every finite order derivative to $t=0$ was zero. Thus by Theorem 1.5 we could conclude that $\varphi(x)f(x,t)\equiv 0$. Since $\varphi(x)$ was arbitrary, this would show $f(x,t)\equiv 0$. Thus we conclude that any $F(z,\tau)$ appearing in such counter-example cannot satisfy the estimate of type (4.3).

REMARK 4.11. Kolm and Nagel [5] has made an interesting work concerning the problem on the "edge of the wedge". When $n=2$, their result can be rephrased as follows: Let $u(x,t)$ be a distribution containing $t$ as a real analytic parameter in the sense of hyperfunction. Assume that every derivative of the finite order vanish at $t=0$. Then $u(x,t)$ vanishes identically on a neighborhood of $t=0$. When $n\geq 3$, their situation is different from ours. Thus the following problem occurs: If in addition we assume in Theorem 3.10 that $u(x)$ is a distribution, then are only the derivatives of the finite order sufficient to give the same conclusion?

**Supplement**

Professor Komatsu has kindly pointed out that the proof of Theorem 4.1 in [1] is incomplete in showing that the operator $J(\zeta)$ in (4.3) there has closed range.

Since we quote it in the proof of Theorem 4.4 in this article, we give here a least necessary complement. We assume in addition that $J(\zeta)$ has the form

$$J(\zeta)=\prod_{m=1}^{\infty}\left(1+\frac{\zeta_1^2+\cdots+\zeta_n^2}{(m\varphi(m))^2}\right)$$

with a monotone increasing function $\varphi(m)\nearrow\infty$, and show that

$$J(\zeta):\ [\mathcal{O}'(V)]^t/\,q'(\zeta)[\mathcal{O}'(V)]^r \longrightarrow [\mathcal{O}'(V)]^t/q'(\zeta)[\mathcal{O}'(V)]^r \tag{s.1}$$

has closed range. This is sufficient for most of our application, especially for the present article and for the proof of Theorem 4.2 in [1].

Applying the Fundamental Principle, we obtain the following commutative exact diagram

$$\begin{array}{ccccccc}
[\mathcal{O}'(V)]^r & \xrightarrow{q'(\zeta)} & [\mathcal{O}'(V)]^t & \xrightarrow{d} & \widetilde{\mathcal{O}'(V)}\{q',d\} & \longrightarrow & 0\\
{\scriptstyle J(\zeta)}\downarrow & & {\scriptstyle J(\zeta)}\downarrow & & {\scriptstyle I(\zeta)}\downarrow & & \\
[\mathcal{O}'(V)]^r & \xrightarrow{q'(\zeta)} & [\mathcal{O}'(V)]^t & \xrightarrow{d} & \widetilde{\mathcal{O}'(V)}\{q',d\} & \longrightarrow & 0\,.
\end{array}$$

Here $\widetilde{\mathcal{O}'(V)}\{q',d\}$ denotes the space of holomorphic $q'$-functions with respect to a normal noetherian operator $d$ satisfying the growth condition of the type $\widetilde{\mathcal{O}'(V)}$. By [15], Lemma 2.2, the induced operator $I(\zeta)$ has the form $\partial'(z,D)J(\zeta)$, where

$$\partial'(z,D)=\begin{bmatrix}1 & & 0\\ & \ddots & \\ * & & 1\end{bmatrix}$$

with rational coefficients whose denominators do not vanish identically on every irreducible component of the family of algebraic varieties $N(q')$ associated with $q'(\zeta)$. By the assumption also $J(\zeta)$ does not vanish identically on every irreducible component. Thus the matrix is invertible there except on a proper algebraic subvariety of each component and the zeros of $J(\zeta)$. The inverse matrix can be written as the product of a negative power of $J(\zeta)$ and a matrix with coefficients of infra-exponential growth. By the assumption on $J(\zeta)$ we can show below that given $\varepsilon>0$ arbitrarily

$$J(\zeta)\geq C_\varepsilon e^{-\varepsilon|\zeta|}\,, \tag{s.2}$$

outside

$$|\zeta_1^2+\cdots+\zeta_n^2+(m\varphi(m))^2|<1\,,\qquad m=1,2,\cdots. \tag{s.3}$$

These exceptional sets are mutually disjoint for large $m$. Since the family of algebraic varieties $N(q')$ can contain only a finite number of components contained in $\zeta_1^2+\cdots+\zeta_n^2=0$ at infinity, all but a finite number of irreducible components of $N(q')$ intersect with the exceptional sets (s.3) in a neighborhood of breadth one of some corresponding proper algebraic subvarieties. Due to the usual Fundamental Principle, we can remove the finite exceptional factors. Thus by (s.2) and by the maximum modulus principle we can prove: for any $\varepsilon>0$ there exists $C_\varepsilon>0$ such that

$$\sup_{\zeta\in N(q')}|F(\zeta)e^{-H_K(\zeta)}|\le C_\varepsilon \sup_{\zeta\in N(q')}|I(\zeta)F(\zeta)e^{-H_K(\zeta)+\varepsilon|\zeta|}|\,,$$

for an arbitrary holomorphic $q'$-function $F(\zeta)$. This inequality obviously implies that $I(\zeta)$, hence (s.1) has a closed range.

Now we prove (s.2) on (s.3). We have for $m\varphi(m)\ge\sqrt{2}\,|\zeta|$,

$$\begin{aligned}\left|1+\frac{\zeta_1^2+\cdots+\zeta_n^2}{(m\varphi(m))^2}\right|^2&=\left(1-\frac{|\zeta|^2}{(m\varphi(m))^2}\right)^2+\frac{4\{(\mathrm{Re}\,\zeta_1)^2+\cdots+(\mathrm{Re}\,\zeta_n)^2\}}{(m\varphi(m))^2}\\&\ge 1\Big/\left(1+\frac{2|\zeta|^2}{(m\varphi(m))^2}\right)^2.\end{aligned}$$

Thus

$$\begin{aligned}\prod_{m\varphi(m)\ge\sqrt{2}|\zeta|}\left|1+\frac{\zeta_1^2+\cdots+\zeta_n^2}{(m\varphi(m))^2}\right|&\ge 1\Big/\prod_{m=1}^{\infty}\left(1+\frac{2|\zeta|^2}{(m\varphi(m))^2}\right)\\&\ge C_\varepsilon\Big/\prod_{m=1}^{\infty}\left(1+\frac{\varepsilon^2|\zeta|^2}{\pi^2m^2}\right)\\&=C_\varepsilon\frac{\sinh\varepsilon|\zeta|}{|\zeta|}\\&\ge C'_\varepsilon e^{-\varepsilon|\zeta|}\,.\end{aligned}$$

Next assuming $N\varphi(N)\le|\zeta|\le(N+1)\varphi(N+1)$ we have

$$\left|1+\frac{\zeta_1^2+\cdots+\zeta_n^2}{(N\varphi(N))^2}\right|\ge\frac{1}{(N\varphi(N))^2}\ge\frac{1}{|\zeta|^2}\,,$$

and, if $(N+1)\varphi(N+1)\le\sqrt{2}\,|\zeta|$,

$$\left|1+\frac{\zeta_1^2+\cdots+\zeta_n^2}{((N+1)\varphi(N+1))^2}\right|\ge\frac{1}{((N+1)\varphi(N+1))^2}\ge\frac{1}{2|\zeta|^2}\,.$$

Further

$$\begin{aligned}\prod_{m=1}^{N-1}\left|1+\frac{\zeta_1^2+\cdots+\zeta_n^2}{(m\varphi(m))^2}\right|&\ge\prod_{m=1}^{N-1}\left(\frac{|\zeta|^2}{(m\varphi(m))^2}-1\right)\\&\ge\prod_{m=1}^{N-1}\frac{(N\varphi(N))^2-(m\varphi(m))^2}{(m\varphi(m))^2}\end{aligned}$$

$$\begin{aligned}&\geq \frac{(N-1)!^2\varphi(1)^2}{(N-1)!^2(\varphi(N-1))^{2(N-1)}}\\&=\varphi(1)^2e^{-2(N-1)\log\varphi(N-1)}\\&\geq\varphi(1)^2C_\varepsilon e^{-\varepsilon(N-1)\varphi(N-1)}\\&\geq C'_\varepsilon e^{-\varepsilon|\zeta|}\,.\end{aligned}$$

Finally assuming $M\varphi(M)\leq\sqrt{2}|\zeta|\leq(M+1)\varphi(M+1)$,

$$\begin{aligned}\prod_{m=N+2}^{M}\left|1+\frac{\zeta_1^2+\cdots+\zeta_n^2}{(m\varphi(m))^2}\right|&\geq\prod_{m=N+2}^{M}\left(1-\frac{|\zeta|^2}{(m\varphi(m))^2}\right)\\&\geq\prod_{m=N+2}^{M}\frac{(m\varphi(m))^2-((N+1)\varphi(N+1))^2}{(m\varphi(m))^2}\\&\geq\frac{(\varphi(N+1))^2}{(\varphi(M))^{2(M-N-1)}}\,.\end{aligned}$$

We have

$$\sqrt{2}|\zeta|-|\zeta|\geq M\varphi(M)-(N+1)\varphi(N+1)\geq(M-N-1)\varphi(N+1)\,,$$

hence

$$M-N-1\leq\frac{(\sqrt{2}-1)|\zeta|}{\varphi(N+1)}\leq(\sqrt{2}-1)(N+1)\,.$$

Therefore

$$\begin{aligned}\frac{(\varphi(N+1))^2}{(\varphi(M))^{2(M-N-1)}}&\geq\frac{\varphi(1)^2}{(\varphi(N+1))^{2(M-N-1)}}\\&\geq\varphi(1)^2e^{-2(\sqrt{2}-1)(N+1)\log\varphi(N+1)}\\&\geq C_\varepsilon e^{-\varepsilon(N+1)\varphi(N+1)/\sqrt{2}}\\&\geq C_\varepsilon e^{-\varepsilon M\varphi(M)/\sqrt{2}}\\&\geq C_\varepsilon e^{-\varepsilon|\zeta|}\,.\end{aligned}$$

q.e.d.

Note that the above proof applies even if we replace $\zeta_1^2+\cdots+\zeta_n^2$ for any quadratic form.

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