*On potential good reduction of abelian varieties**

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Introduction

Let k be a field, $\mathfrak{p}$ a discrete valuation of K, and R the valuation ring of $\mathfrak{p}$. Let A be an abelian variety defined over k. We say that A *has good reduction at* $\mathfrak{p}$ if there exists an abelian scheme A_R over Spec (R) such that $A \cong A_R \times_R k$. We say also that A *has potential good reduction at* $\mathfrak{p}$ if there exist a finite extension k' of k and a prolongation $\mathfrak{p}'$ of $\mathfrak{p}$ to k' such that $A \times_k k'$ has good reduction at $\mathfrak{p}'$. For example, an elliptic curve E has potential good reduction at $\mathfrak{p}$ if and only if its modular invariant j is integral at $\mathfrak{p}$.

Let Ω be a PEL-type, $\mathcal{Q}$ a PEL-structure of type Ω in the sense of Shimura [8]. In particular, $\mathcal{Q}$ is a structure consisting of an abelian variety A, its polarization, an injection of a ring into the endomorphism ring of A and a finite set of points of finite order on A, all of the prescribed type Ω. Let V_Ω be the moduli variety of PEL-structures of type Ω. It is a quotient space of a bounded symmetric domain by an arithmetic discontinuous group.

Now we conjecture that the underlying abelian variety A of the PEL-structure $\mathcal{Q}$ has potential good reduction at any discrete place of the field of definition of $\mathcal{Q}$ if the moduli variety V_Ω is compact. For example, an abelian variety with complex multiplication, which has potential good reduction everywhere, gives a PEL-structure of type Ω such that V_Ω consists of one point. We shall study potential good reduction of abelian varieties from such a point of view and, using the stable reduction theorem of Grothendieck, prove that our conjecture is true in some cases. We shall prove in particular that the PEL-structures which Shimura used in [9] to construct the canonical model of an arithmetic quotient of a product of several copies of the Siegel upper half plane has potential good reduction if the corresponding arithmetic quotient is compact.

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Notation. We denote by $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{Z}_p$, $\boldsymbol{Q}_p$, $\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{F}_p$, respectively, the ring of rational integers, the rational number field, the ring of p-adic integers, the p-adic number field, the real number field, the complex number field, and the finite field with p elements. For any field k, $\bar{k}$ denotes the algebraic closure of k in a universal domain.

§1. The first method

1-1. Let A be an abelian variety defined over a field k and let X be an ample divisor defined over k. Let $\hat{A}$ be the Picard variety (i.e., the dual abelian variety) of A defined over k. Let $\varphi_X : A \to \hat{A}$ be the isogeny obtained from X (cf. Lang [3]). It is known that φ_X is also defined over k.

Let n be a natural number which is prime to the characteristic of k. Let a be an n-section point on A and let x be an n-section point on $\hat{A}$. Let $e_n(a, x)$ be the n-th root of unity that corresponds to a and x as in Lang [3], p. 189, Proposition 3. Let k_s be the separable closure of k and let G be the Galois group of k_s over k. Then we see from the definition of $e_n(a, x)$ that $e_n(a^\sigma, x^\sigma) = e_n(a, x)^\sigma$ holds for any $\sigma \in G$.

Let l be a prime number different from the characteristic of k. Let $T_l(A)$ (resp. $T_l(\hat{A})$) be the Tate module of A (resp. $\hat{A}$). Then we define a $\boldsymbol{Z}_l$-valued $\boldsymbol{Z}_l$-bilinear form $(,)$ on $T_l(A) \times T_l(\hat{A})$ as in Lang [3], p. 192. Then we see from the above mentioned equality that $(x^\sigma, \hat{y}^\sigma) = (x, \hat{y})^\sigma$ holds for any $x \in T_l(A)$, $\hat{y} \in T_l(\hat{A})$ and $\sigma \in G$, where σ acts on x (resp. $\hat{y}$) by the natural action on $T_l(A)$ (resp. $T_l(\hat{A})$) and σ acts on $(x, y) \in \boldsymbol{Z}_l$ by the natural action on the group of l^ν-th roots of unity. Therefore we have

$$(x, \varphi_X y)^\sigma = (x^\sigma, (\varphi_X y)^\sigma) = (x^\sigma, \varphi_X y^\sigma)$$

for any $x, y \in T_l(A)$ and $\sigma \in G$. Here we have used the fact that φ_X is defined over k.

Let $\varphi_{X,l}$ be the l-adic representation of φ_X with respect to the bilinear form $(,)$. Let $\sigma_{A,l}$ (resp. $\sigma_{m,l}$) be the l-adic representation of $\sigma \in G$ corresponding to the Galois module $T_l(A)$ (resp. the Galois module $T_l(G_m)$ of the Tate module of the multiplicative group G_m). Then we obtain from the above formula that

$$\sigma_{m,l} = {}^t\varphi_{X,l}^{-1}\, {}^t\sigma_{A,l}\, {}^t\varphi_{X,l}\, \sigma_{A,l}\,.$$

Hence we have

$${}^t\sigma_{A,l}\, \varphi_{X,l}\, \sigma_{A,l} = \sigma_{m,l}\, \varphi_{X,l}\,.$$

This fact can be expressed by saying that the l-adic representation ρ_l corresponding to the Galois module $T_l(A)$ maps G into the group of all similitudes of the alternating form $\varphi_{X,l}$.

1-2. Let A, k, G, etc. be as in 1-1. Let B be a simple algebra over the rational number field Q. Let $\mathcal{O}$ be an order of B and let θ be an injection of $\mathcal{O}$ into $\mathrm{End}_k(A)$ satisfying $\theta(1_B)=1_A$. Let λ_l be the l-adic representation of $\mathrm{End}_k(A)$ on $T_l(A)$. We denote the composition $\lambda_l\circ\theta$ by θ_l. Since every element of $\theta(\mathcal{O})$ is defined over k, $\theta_l(\mathcal{O})$ is contained in the commuting algebra of $\rho_l(G)$ in $\mathrm{End}_{Z_l}(T_l(A))$. In other words, ρ_l maps the Galois group G into the commuting algebra of $\theta_l(\mathcal{O})$ in $\mathrm{End}(T_l(A))$.

1-3. Let F be a field contained in the center of B. Put $\mathfrak{o}=\mathcal{O}\cap F$. We assume hereafter that $\mathfrak{o}$ is the maximal order of the finite algebraic number field F. Let $l=\mathfrak{l}_1^{e_1}\cdots\mathfrak{l}_s^{e_s}$ be the prime ideal decomposition of l in $\mathfrak{o}$. Then we have $\mathfrak{o}\otimes_Z Z_l=\mathfrak{o}_{\mathfrak{l}_1}\oplus\cdots\oplus\mathfrak{o}_{\mathfrak{l}_s}$, where $\mathfrak{o}_{\mathfrak{l}_i}$ is the $\mathfrak{l}_i$-adic completion of $\mathfrak{o}$. Since θ_l induces an injection of $\mathfrak{o}\otimes_Z Z_l$ into $\mathrm{End}(T_l(A))$, $\mathfrak{o}_{\mathfrak{l}_1}\oplus\cdots\oplus\mathfrak{o}_{\mathfrak{l}_s}$ acts on $T_l(A)$. Hence we can decompose $T_l(A)$ as a direct sum $T_{\mathfrak{l}_1}(A)\oplus\cdots\oplus T_{\mathfrak{l}_s}(A)$, where $T_{\mathfrak{l}_i}(A)$ is defined by $T_{\mathfrak{l}_i}(A)=\theta_l(1_{\mathfrak{o}_{\mathfrak{l}_i}})T_l(A)$ $(i=1,\cdots,s)$. Let α be any element of $\mathrm{End}_k(A)$ which is contained in the commuting algebra of $\mathfrak{o}$ in $\mathrm{End}_k(A)$. Then we see that the l-adic representation of α on $T_l(A)$ induces representations of α on $T_{\mathfrak{l}}(A)$ $(\mathfrak{l}=\mathfrak{l}_1,\cdots,\mathfrak{l}_s)$ and that the l-adic representation of α is equivalent to the direct sum of these s representations. In this way, we have the $\mathfrak{l}$-adic representation $\lambda_{\mathfrak{l}}$ of the commuting algebra of $\mathfrak{o}$ in $\mathrm{End}_k(A)$.

Let φ_X, $\varphi_{X,l}$ be as before. We denote by $*$ the positive involution

$$\mathrm{End}_k(A)\ni\alpha\longmapsto\alpha^*=\varphi_X^{-1}\alpha\varphi_X\in\mathrm{End}_k(A)$$

of $\mathrm{End}_k(A)$. We assume hereafter that $*$ induces the identity map on $\mathfrak{o}$. Then we see that $T_l(\hat{A})$ is the direct sum of $T_{\mathfrak{l}}(\hat{A})=\theta(1_{\mathfrak{o}_{\mathfrak{l}}})T_l(\hat{A})$ $(\mathfrak{l}=\mathfrak{l}_1,\cdots,\mathfrak{l}_s)$ and that $\varphi_{X,l}$ maps $T_{\mathfrak{l}}(A)$ into $T_{\mathfrak{l}}(\hat{A})$. Let $\varphi_{X,\mathfrak{l}}$ be the restriction of $\varphi_{X,l}$ to $T_{\mathfrak{l}}(A)$. Then we see that $\varphi_{X,l}$ is the direct sum of the $\varphi_{X,\mathfrak{l}_i}$ and that the group of all similitudes of $\varphi_{X,l}$ is the direct product of the groups of all similitudes of $\varphi_{X,\mathfrak{l}_i}$.

Let $\mathfrak{g}_{\mathfrak{l}}(\varphi_X,\mathcal{O})$ be the set of all elements of $\mathrm{End}_{\mathfrak{o}_{\mathfrak{l}}}(T_{\mathfrak{l}}(A))$ that belong to the commuting algebra of $\theta_{\mathfrak{l}}(\mathcal{O})$ in $\mathrm{End}(T_{\mathfrak{l}}(A))$ and that belong to the group of all similitudes of $\varphi_{X,\mathfrak{l}}$. Since A, $\hat{A}$, φ_X and all elements of $\theta(\mathcal{O})$ are defined over k, the l-adic representation ρ_l of the Galois group G induces a map $\rho_{\mathfrak{l}}$ of G into $\mathfrak{g}_{\mathfrak{l}}(\varphi_X,\mathcal{O})$. We call this map $\rho_{\mathfrak{l}}$ the $\mathfrak{l}$-adic representation of the Galois group.

Now we have the following

THEOREM 1. *Suppose that* (i) k *is a* $\mathfrak{p}$*-adic field* (*i.e., a finite extension of* $\boldsymbol{Q}_p$) *and* (ii) *there exists a prime ideal* $\mathfrak{l}$ *of* F *such that the characteristic of the residue field of* $\mathfrak{p}$ *is prime to* l $(l=\mathfrak{l}\cap \boldsymbol{Z})$, *and* $\mathfrak{g}_{\mathfrak{l}}(\varphi_X, \mathcal{O})$ *contains no unipotent element other than* 1. *Then* A *has potential good reduction at* $\mathfrak{p}$.

PROOF. Let χ be a homomorphism of $\mathfrak{o}$ into the endomorphism ring of an abelian variety C (resp. a torus S) satisfying $\chi(1_{\mathfrak{o}})=1_C$ (resp. 1_S). Let $T_l(C)$ (resp. $T_l(S)$) be the Tate module of C (resp. S). Let χ_l be the l-adic representation of $\mathfrak{o}$ induced from χ on $T_l(C)$ (resp. $T_l(S)$). Then, for any $\alpha \in \mathfrak{o}$, the characteristic polynomial of $\chi_l(\alpha)$ has rational coefficients. Hence, by Shimura-Taniyama [10], p. 38, Lemma 1, χ_l is equivalent to the sum of a multiple of a regular representation of F over $\boldsymbol{Q}$ and a 0-representation. Since χ_l maps the identity element of F to the identity element of the endomorphism ring of the Tate module, χ_l is equivalent to a multiple of a regular representation of F over $\boldsymbol{Q}$.

Let I be the inertia group of $\mathfrak{p}$. Then, by using the above-mentioned fact, we can prove, following step by step the proof of Serre-Tate [5], Theorem 1, that A has potential good reduction if and only if $\rho_{\mathfrak{l}}$ maps the inertia group I into a finite subgroup of $\mathfrak{g}_{\mathfrak{l}}(\varphi_X, \mathcal{O})$. So we shall prove that $\rho_{\mathfrak{l}}(I)$ is finite.

By the result of Grothendieck (cf. Serre-Tate [5], p. 515), there exists an open subgroup H of I such that $\rho_{\mathfrak{l}}(h)$ is unipotent for any $h\in H$. In particular, $\rho_{\mathfrak{l}}(h)$ is a unipotent element of $\mathfrak{g}_{\mathfrak{l}}(\varphi_X, \mathcal{O})$ for any $h\in H$. Since $\mathfrak{g}_{\mathfrak{l}}(\varphi_X, \mathcal{O})$ contains no unipotent element other than 1, this implies that $\rho_{\mathfrak{l}}(h)=1$ for any $h\in H$. Since H is an open subgroup of the compact group I, $\rho_{\mathfrak{l}}$ maps I into a finite group.

Q.E.D.

REMARK. Let $\mathfrak{p}$ be a discrete valuation of k. Assume that the residue field $\tilde{k}$ of $\mathfrak{p}$ has the following property:

(C_l) No finite extension of $\tilde{k}$ contains all the roots of unity of order power of l.

Then the result of Grothendieck holds. Hence the condition (i) of Theorem 1 may be replaced by this condition (C_l).

§2. The second method

2-1. Let A be an abelian variety defined over a field k. Let $\mathcal{O}$ be an order of a simple algebra B over the rational number field. Let θ be an injection of $\mathcal{O}$ into $\mathrm{End}_k(A)$ satisfying $\theta(1_B)=1_A$. Let $\mathfrak{p}$ be a discrete place of k and let R (resp. $\tilde{k}$) be the valuation ring (resp. the residue field) of $\mathfrak{p}$.

Let A_R be the Néron minimal model relative to $\mathfrak{p}$: A_R is a smooth group scheme of finite type over R, together with an isomorphism $A_R \times_R k \cong A$, which represents the functor

$$Y \longmapsto \mathrm{Hom}_k(Y \times_R k, A)$$

on the category of schemes Y smooth over R.

Let $\tilde{A}$ be the special fibre of A_R and $\tilde{A}^0$ the connected component of $\tilde{A}$. Then $\tilde{A}$ is a commutative algebraic group defined over $\tilde{k}$. Hence $\tilde{A}^0$ is an extension of an abelian variety C by a linear group H. By Lemma 3 of Serre-Tate [5], it is known that $\tilde{A}$ is an abelian variety if and only if H is reduced to $\{1\}$.

Now the stable reduction theorem of Grothendieck (cf. Grothendieck [14] or Serre-Tate [5], p. 499) states that there is a finite extension K of k such that the connected component of the special fibre of the Néron model of $A \times_k K$ over K is an extension of an abelian variety by a torus. Therefore, changing k by some finite extension if necessary, we assume that $\tilde{A}^0$ is an extension of an abelian variety C by the (trivial) torus $S = (G_m)^r$, where G_m is the multiplicative group, and that all of them are defined over $\tilde{k}$.

Now, by the universal mapping property of the Néron model, every element g of $\mathrm{End}_k(A)$ induces an element of $\mathrm{End}_R(A_R)$. Hence it induces an endomorphism of the special fibre. Since every endomorphism is continuous in the Zariski topology, g induces an endomorphism $\tilde{g}$ of the connected component of the special fibre. Moreover, since S is the maximal linear subgroup of $\tilde{A}^0$ and since the homomorphic image of a linear group is linear, $\tilde{g}$ maps S into S. Hence g induces an endomorphism g_1 of the torus $S = (G_m)^r$ and an endomorphism g_2 of the abelian variety C.

Now we see from the construction that

$$\chi : \mathrm{End}_k(A) \ni g \longmapsto g_1 \in \mathrm{End}_{\tilde{k}}(S)$$

is a ring homomorphism satisfying $\chi(1_A) = 1_S$. Since $\mathrm{End}_{\tilde{k}}(S) = \mathrm{End}_{\tilde{k}}(G_m^r) \cong M_r(\boldsymbol{Z})$, χ induces a homomorphism of $\mathrm{End}_k(A) \otimes_Z \boldsymbol{Q}$ into $M_r(\boldsymbol{Z}) \otimes_Z \boldsymbol{Q} = M_r(\boldsymbol{Q})$. Therefore we have a homomorphism $\chi \circ \theta$ of B into $M_r(\boldsymbol{Q})$ satisfying $(\chi \circ \theta)(1_B) = 1_{M_r(\boldsymbol{Q})}$. Since B is a simple algebra over $\boldsymbol{Q}$, $(\chi \circ \theta)(B)$ is either isomorphic to B or equal to $\{0\}$. Hence, if r is not zero, $(\chi \circ \theta)(B)$ is isomorphic to B since it contains the identity element of $M_r(\boldsymbol{Q}) \neq \{0\}$. Hence we have the following

THEOREM 2. *Let A be an abelian variety. Let B be a simple algebra over the rational number field. Let θ be an injection of B into* $\mathrm{End}(A) \otimes_Z \boldsymbol{Q}$ *satisfying $\theta(1_B) = 1_A$. Suppose that B is not isomorphic to any subalgebra of $M_r(\boldsymbol{Q})$* $(r \leq \dim A)$

containing the unit element of $M_r(Q)$. *Then A has potential good reduction at any discrete place of the field of definition of A.*

2-2. Let the notation and assumptions be as in 2-1. In particular, $\chi\circ\theta$ is an injection of $\mathcal{O}$ into $\mathrm{End}_{\tilde{k}}(S)$ if $r=\dim S$ is not zero. Let ι be the representation of $\mathrm{End}_k(A)$ on the tangent space at the origin of A. Let $\tilde{\iota}$ (resp. ι_1) be the representation of $\mathrm{End}_{\tilde{k}}(\tilde{A}^0)$ (resp. $\mathrm{End}_{\tilde{k}}(S)$) on the tangent space at the origin of $\tilde{A}^0$ (resp. S). We see that ι_1 is equivalent to $M_r(\boldsymbol{Z})\ni\gamma\longmapsto\gamma\in M_r(\boldsymbol{Z})$ (resp. $M_r(\boldsymbol{Z})\ni\gamma\longmapsto\gamma \bmod p\in M_r(\boldsymbol{F}_p)$) if the characteristic of the residue field $\tilde{k}$ is zero (resp. a prime number p).

Since A_R is a smooth scheme over R, we see that the tangent space at the origin of $\tilde{A}^0$ is obtained by reduction modulo $\mathfrak{p}$ of the tangent space at the origin of A (for a more detailed proof, see Shimura-Taniyama [10], pp. 84-92). Moreover, since $\tilde{A}^0$ is the extension of the abelian variety C by the torus S, the tangent space at the origin of S is a subspace of the tangent space at the origin of $\tilde{A}^0$.

Now we restrict ι (resp. ι_1) to the subring $\theta(\mathcal{O})$ (resp. $(\chi\circ\theta(\mathcal{O}))$. Then we see from the above consideration that the representation $\iota_1\circ\chi\circ\theta$ of $\mathcal{O}$ is a subrepresentation of the representation of $\mathcal{O}$ obtained from $\iota\circ\theta$ by reduction modulo $\mathfrak{p}$ of the representation space.

Now we have the following

THEOREM 3. *Let A be an abelian variety defined over a field k. Let* $\mathfrak{o}$ *be the maximal order of a finite algebraic number field F. Let* θ *be an injection of* $\mathfrak{o}$ *into* $\mathrm{End}_k(A)$ *satisfying* $\theta(1_F)=1_A$. *Let* ι *be the representation of* $\mathrm{End}_k(A)$ *on the tangent space at the origin of A. Suppose that* $(\iota\circ\theta)(\mathfrak{o})$ *does not generate a* $\bar{k}$*-subalgebra of dimension* $[F:\boldsymbol{Q}]$ *of the endomorphism ring of the tangent space at the origin of A. Then A has potential good reduction at any discrete place of the field of definition of A.*

PROOF. Let the notation be as before. Suppose that A does not have potential good reduction at $\mathfrak{p}$. Then $r=\dim S$ is not zero and χ induces an injection of $\mathfrak{o}$ into $M_r(\boldsymbol{Q})$ satisfying $\chi(1_F)=1_{M_r(\boldsymbol{Q})}$. Hence χ is equivalent (as a representation of F) to a multiple of a regular representation of F over $\boldsymbol{Q}$ (cf. the argument in the first part of the proof of Theorem 1). In particular, the image of $\mathfrak{o}$ generates a $\bar{\boldsymbol{Q}}$-subalgebra of $M_r(\bar{\boldsymbol{Q}})$ of rank $[F:\boldsymbol{Q}]$. Since $\mathfrak{o}$ is the maximal order and since ι_1 is given by

$$M_r(\boldsymbol{Z})\ni\gamma\longmapsto\gamma\in M_r(\boldsymbol{Z}) \quad \text{or} \quad M_r(\boldsymbol{Z})\ni\gamma\longmapsto\gamma \bmod p\in M_r(\boldsymbol{F}_p)\,,$$

the image of $\mathfrak{o}$ generates a $\bar{\boldsymbol{Q}}$ or $\bar{\boldsymbol{F}}_p$-subalgebra of rank $[F:\boldsymbol{Q}]$ of the endomorphism ring of the tangent space at the origin of S. Hence $(\bar{\iota}\circ\chi\circ\theta)(\mathfrak{o})$ generates a $\bar{Q}$ or $\bar{F}_p$-subalgebra of rank $[F:Q]$ of the endomorphism ring of the tangent space at the origin of $\tilde{A}^0$. Since $\bar{\iota}\circ\chi\circ\theta$ is obtained from $\iota\circ\theta$ by reduction modulo $\mathfrak{p}$ of the representation space, this contradicts our assumption that $(\iota\circ\theta)(\mathfrak{o})$ generates a $\bar{Q}$ or $\bar{F}_p$-algebra of rank less than $[F:Q]$. Hence we have completed the proof of Theorem 3.

§ 3. Applications

3-1. Let B be a totally indefinite division quaternion algebra over a totally real algebraic number field F of degree g. Let A be a $2g$-dimensional abelian variety defined over a field k. Let θ be an injection of B into $\mathrm{End}_k(A)\otimes_Z Q$ satisfying $\theta(1_B)=1_A$. Then we see that the conditions of Theorem 2 are satisfied. Hence A has potential good reduction at any discrete place of the field of definition of (A, θ).

3-2. Let F be a totally real algebraic number field of degree g. Let K be a totally imaginary quadratic extension of F. Let $\tau_{0,1}, \cdots, \tau_{0,g}$ be all isomorphisms of F into R, and let $\tau_1, \cdots, \tau_g$ be extensions of $\tau_{0,1}, \cdots, \tau_{0,g}$ to K. Let $\bar{\tau}_\nu$ be the complex conjugate of τ_ν ($\nu=1, \cdots, g$). Let A be an abelian variety defined over a subfield k of C. Let θ be an injection of the maximal order $\mathfrak{o}_K$ of K into $\mathrm{End}_k(A)$ satisfying $\theta(1_K)=1_A$. Suppose that the representation of K on the tangent space at the origin of A is equivalent to

$$\sum_{\nu=1}^{g}(r_\nu\tau_\nu+s_\nu\bar{\tau}_\nu) \quad (r_\nu, s_\nu \in Z) .$$

Then we see that the condition in Theorem 3 is satisfied with K in place of F if at least one of the r_ν or s_ν is zero. Hence A has potential good reduction at any discrete place of k if at least one of the r_ν or s_ν is zero.

3-3. Let $\Omega=(L, \Phi, \rho; T, V, \mathfrak{M}; x_1, \cdots, x_s)$ be a PEL-type in the sense of Shimura [8]: L is a division algebra over Q, ρ is a positive involution of L, Φ is a representation of L into $M_n(C)$ such that $\Phi+\bar{\Phi}$ is equivalent to a rational representation, V is a left L-module of rank m, where $m=2n/[L:Q]$, T is a non-degenerate L-valued ρ-antihermitian form on V, $\mathfrak{M}$ is a free Z-submodule of V of rank $2n$, and the x_i are elements of V. Let $\mathcal{Q}=(A, \mathcal{C}, \theta; t_1, \cdots, t_s)$ be a PEL-structure in the sense of Shimura [8]. That is, A is an abelian variety defined over C, $\mathcal{C}$ is a polarization of A, θ is an isomorphism of L into $\mathrm{End}(A)\otimes_Z Q$ and the t_i are points of finite order on A. We say that $\mathcal{Q}$ is of type Ω if there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{M} & \longrightarrow & V\otimes_Q \boldsymbol{R} & \longrightarrow & (V\otimes_Q \boldsymbol{R})/\mathfrak{M} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow f & & \downarrow & & \\ 0 & \longrightarrow & D & \longrightarrow & \boldsymbol{C}^n & \xrightarrow{\xi} & A & \longrightarrow & 0 \end{array}$$

such that the following five conditions are satisfied:

(i) ξ gives a holomorphic isomorphism of $\boldsymbol{C}^n/D$ to A;

(ii) f is an $\boldsymbol{R}$-linear isomorphism satisfying $f(\mathfrak{M})=D$;

(iii) $f(\alpha x)=\Phi(\alpha)f(x)$, and $\Phi(\alpha)$ defines $\theta(\alpha)$ for every $\alpha \in L$;

(iv) $\mathcal{C}$ contains a basic polar divisor X which determines a Riemann form E on $\boldsymbol{C}^n/D$ such that $E(f(x), f(y))=\mathrm{Tr}_{L/Q}(T(x, y))$ for every $(x, y) \in V\times V$;

(v) $t_i=\xi(f(x_i))$ for every i.

Let G be the algebraic group defined over $\boldsymbol{Q}$ such that $G_{\boldsymbol{Q}}$ can be identified with the group of all L-linear automorphisms α of V satisfying $T(x\alpha, y\alpha)=T(x, y)$. Let S be the quotient space of $G_{\boldsymbol{R}}$ by a maximal compact subgroup. Put

$$\Gamma=\{\alpha \in G_{\boldsymbol{Q}} \mid \mathfrak{M}\alpha=\mathfrak{M},\ x_i\alpha\equiv x_i \bmod \mathfrak{M}\ (i=1, \cdots, s)\}\ .$$

Then Γ is commensurable with $G_{\boldsymbol{Z}}$ and acts naturally on S. Moreover, it is known that $V=\Gamma\backslash S$ has a structure of an algebraic variety defined over an algebraic number field and V is in one-to-one correspondence with the set of all isomorphism classes of PEL-structure of type Ω, if $\dim S>1$ or $\Gamma\backslash S$ is compact (cf. Shimura [8], for more detailed properties of the moduli space V). By Borel-Harish-Chandra [1], $V=\Gamma\backslash S$ is compact if and only if the reductive group G is anisotropic over $\boldsymbol{Q}$.

Let F be the set of all elements x in the center of L such that $x^\rho=x$. Then it is known that F is a totally real algebraic number field. We see from the definition of G that there is an algebraic group G' defined over F such that $G'_F=G_{\boldsymbol{Q}}$. Therefore $V=\Gamma\backslash S$ is compact if and only if G' is anisotropic over F. Hence, up to the Hasse principle of the ρ-antihermitian form T (cf. Weil [13], p. 80), $V=\Gamma\backslash S$ is compact if and only if there is a prime v of F such that G' is anisotropic over F_v.

Now we have the following

THEOREM 4. *Let the notation and assumptions be as above. Let k be a subfield of $\boldsymbol{C}$ and let $\mathcal{Q}=(A, \mathcal{C}, \theta; t_1, \cdots, t_s)$ be a PEL-structure of type Ω defined over k. Let $\mathfrak{p}$ be a discrete place of k. Then the following assertions hold:*

(i) *If there is an archimedean prime v of F such that G' is anisotropic* *over F_v, A has potential good reduction at $\mathfrak{p}$.*

(ii) *Suppose that there is a finite prime v of F such that the characteristic of the residue field of $\mathfrak{p}$ is prime to v and that G' is anisotropic over F_v. Then, if k is a $\mathfrak{p}$-adic field (i.e., a finite extension of $\boldsymbol{Q}_p$), A has potential good reduction at $\mathfrak{p}$.*

(iii) *Suppose that $m=1$ (i.e., $V \cong L$). Then A has potential good reduction at $\mathfrak{p}$.*

PROOF. If $m=1$, then $2\dim A=[L:\boldsymbol{Q}]$. Since L is a division algebra, L cannot be imbedded in $M_r(\boldsymbol{Q})$ ($r \leqq \dim A$). Hence we obtain the third assertion from Theorem 2.

We note that the algebraic group G' depends only on L, Φ, ρ, V and T, and does not depend on $\mathfrak{M}$ and the x_i. Now there exists a free $\boldsymbol{Z}$-submodule $\mathfrak{M}'$ of V of rank $2n$ such that

$$\mathcal{O}=\{a \in L \mid a\mathfrak{M}' \subseteq \mathfrak{M}'\}$$

is a maximal order of L. Moreover we see that, for any PEL-structure $\mathcal{Q}=(A, \mathcal{C}, \theta; t_1, \cdots, t_s)$, there is a PEL-structure $\mathcal{Q}'=(A', \mathcal{C}', \theta')$ of type $\Omega'=(L, \Phi, \rho; V, cT, \mathfrak{M}')$ with $c \in \boldsymbol{Q}$ that is isogenous to $(A, \mathcal{C}, \theta)$. Since θ' maps $\mathcal{O}$ into $\mathrm{End}(A')$ and since two isogenous abelian varieties have potential good reduction if one of them has potential good reduction, we may assume that θ maps a maximal order of L into $\mathrm{End}(A)$ and $\mathcal{Q}$ has no level structure. Moreover, it is clear that θ maps the identity element of L to the identity element of $\mathrm{End}(A)$.

Now suppose that there is an archimedean prime v of F such that G' is anisotropic over F_v. Then we see from the results of Shimura [7] that $m=1$ or L is a central simple algebra over a totally imaginary quadratic extension K of F. Since we have already proved Theorem 4 in the case $m=1$, we assume the latter case. Let τ and $\bar{\tau}$ be extensions of v to isomorphisms of K into $\boldsymbol{C}$. Let r (resp. s) be the multiplicity of τ (resp. $\bar{\tau}$) in the restriction of Φ to K. Then we see from the results of Shimura [7] that G'_{F_v} is isomorphic to the unitary group of type (r, s). Hence G' is anisotropic over F_v if and only if r or s is zero. Hence A and $\theta \mid K$ satisfies the conditions of 3-2. Hence A has potential good reduction at any discrete place of k.

Next suppose that the conditions in (ii) are satisfied. Let X be the basic polar divisor of A that corresponds to the Riemann form $E(f(x), f(y))=\mathrm{Tr}_{L/\boldsymbol{Q}}(T(x, y))$ (see [10], p. 25, (7)). Let $\mathfrak{l}$ be the prime ideal of F corresponding to v. Then we see that all conditions of §1 become satisfied if we replace k by some finite extension of k. Moreover we see that the alternating form $\varphi_{X,\mathfrak{l}}$ corresponds to $\mathrm{Tr}_{L/F_v}(T(x, y))$. Hence $\mathfrak{g}_{\mathfrak{l}}(\varphi_X, \mathcal{O})$ is isomorphic to a subgroup of the group H of all $L\otimes_F F_v$-linear automorphisms α of $V\otimes_F F_v$ satisfying $T(x\alpha, y\alpha)=\mu(\alpha)T(x, y)$ with $\mu(\alpha)\in F_v$. Since $H\ni\alpha\to\mu(\alpha)\in F_v$ gives a rational character of the algebraic group H defined over F_v, μ maps unipotent elements of H to the identity element of the multiplicative group of F_v. Hence any unipotent element α of H satisfies $T(x\alpha, y\alpha)=T(x, y)$. Hence α is a unipotent element of G'_{F_v}. Since G' is anisotropic over F_v, this implies that $\alpha=1$. Hence H has no unipotent element. Therefore $\mathfrak{g}_{\mathfrak{l}}(\varphi_X, \mathcal{O})$ has no unipotent element. Therefore our assertion (ii) follows from Theorem 1. Q.E.D.

REMARK. We can prove the second assertion of Theorem 4 without assuming that k is a $\mathfrak{p}$-adic field by considering the imbedding of the moduli variety $V=\Gamma\backslash S$ into the moduli scheme of Mumford. But we shall not discuss this here.

REMARK. W. Casselman studied the same problem and proved the second assertion and the third assertion of Theorem 4 independently. The author thinks that parts of the results of this paper may have been known to some other specialists.

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