Eigenfunction expansions associated with uniformly propagative systems and their applications to scattering theory

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Introduction

The present paper is a direct continuation of writer's previous paper [12] and is concerned with the spectral and scattering theory, especially eigenfunction expansions, for linear partial differential systems describing the wave propagation phenomena of classical physics in inhomogeneous anisotropic media filling the whole space. It was shown by C. H. Wilcox [11] that these systems can be written in the form

$$\frac{1}{i}\frac{\partial u}{\partial t}=L(D)u(x,t)+f(x,t)\ , \tag{0.1}$$

$$L(D)=M(x)^{-1}\sum_{j=1}^{n}A_jD_j\ , \tag{0.2}$$

where $t\in\boldsymbol{R}^1$ (time), $x={}^t(x_1,x_2,\cdots,x_n)\in\boldsymbol{R}^n$ (space), $D_j=\dfrac{1}{i}\dfrac{\partial}{\partial x_j}$, $u(x,t)$ is C^m-valued function which describes the state of the media at time t and position x, $M(x)$ is an $m\times m$ positive definite hermitian matrix depending on x and A_j's are $m\times m$ constant hermitian matrices. These systems are named uniformly propagative systems.

The study of the spectral and scattering theory for uniformly propagative systems was initiated by Wilcox [11] himself and has been developed by many mathematicians from then on ([1], [5], [8], [9]). Especially, eigenfunction expansions associated with the systems was proved by Schulenberger-Wilcox [9] recently. These works, however, dealt with the systems under rather strong assumptions on the regularity and asymptotic relations at infinity of $M(x)$. For example, Schulenberger-Wilcox [9] assumes $M(x)\in C^1(\boldsymbol{R}^n)$ and $M(x)-I$ vanishes outside a compact subset.

On the other hand the limiting absorption principle for the system (0.2) under weaker conditions on $M(x)$ was proved recently by T. Ikebe, T. Suzuki [10] and the writer [12]. Using the results derived in these papers, we shall prove in this paper eigenfunction expansions associated with such systems and then give representation formulas for the wave and scattering operators associated with the pair $L(D)$ and $L_0(D)$ in terms of the quantities related to the eigenfunctions. Here $L_0(D)$ is the operator defined by

$$L_0(D)=\sum_{j=1}^{n} A_j D_j\,, \tag{0.3}$$

which will be considered as an unperturbed operator for $L(D)$.

The method used here for proving the formulas is the perturbation method developed by Kato-Kuroda [6] in an abstract form and the formulas derived here are somewhat different from the ones derived by Schulenberger-Wilcox [9].

The present paper is divided into six sections. In §1 we shall review some results and notations which are derived and used in [12], and then present two lemmas which are necessary in §3 and §4. In §2 eigenfunction expansions for the unperturbed operator $L_0(D)$ will be discussed in two ways; but the first one will not be discussed in detail in the following sections and only remarks on it will be made in §5. §3 and §4 are main parts of this paper. In §3 expansion formulas for $L(D)$ will be obtained by the perturbation method; and in §4 their applications to the scattering theory will be made, that is, representation formulas for the wave and scattering operators associated with the pair $L(D)$ and $L_0(D)$ will be obtained in terms of the quantities related to the eigenfunctions. In §5 we shall remark that if $M(x)-I$ decreases sufficiently rapidly we can construct the distorted plane waves in terms of the eigenfunctions obtained in §3. In §6 an example associated with Maxwell's equation will be treated.

§1. Preliminaries.

Since this paper is a direct continuation of [12], we use the same notations as in [12]. Theorems, formulas etc. appearing in [12] are referred to as Theorem 1.1.I for theorems and as (2.3.I) for formulas etc. However, we shall review shortly some of the important notations, assumptions and theorems appearing in [12] for the sake of reader's conveniences.

Throughout this paper we assume as in [12] that $L(D)$ and $L_0(D)$ satisfy (A.1.I) and (A.2.I). Namely $L_0(D)$ is a uniformly propagative system in the sense of Wilcox and $M(x)$ satisfies

$$C_1|\xi|^2\leqq(\xi, M(x)\xi)_{C^m}\leqq C_1^{-1}|\xi|^2 \quad \text{for all } x\in \boldsymbol{R}^n \text{ and all } \xi\in \Xi^n; \tag{1.1}$$

$$\sup_{1\leqq i,j\leqq n} |m_{ij}(x)-\delta_{ij}|\leqq C_2(1+|x|^2)^{-\delta/2} \quad \text{for all } x\in \boldsymbol{R}^n\,, \tag{1.2}$$

where δ is a constant satisfying the condition $\delta>1$, C_1 and C_2 are some positive constants and $m_{ij}(x)$ is the (i, j)-component of the matrix $M(x)$.

Two Hilbert spaces H_0 and H_1 in which operators $L_0(D)$ and $L(D)$ are considered, respectively, and some auxiliary spaces $H^s_{0,\sigma}$ and $H^s_{1,\sigma}$ $(s \in \boldsymbol{R}^1, \sigma \in \boldsymbol{R}^1)$ are defined in the following way:

$$H^s_{0,\sigma}=\{u \in \mathscr{S}'(\boldsymbol{R}^n, \boldsymbol{C}^m) : \|u\|^2_{H^s_{0,\sigma}} = \int_{\boldsymbol{R}^n} |\mathscr{F}^{-1}((1+|\xi|^2)^{s/2}\mathscr{F}u)(x)|^2 \times (1+|x|^2)^\sigma dx < \infty\};$$

$$H^s_{1,\sigma}=\{u \in \mathscr{S}'(\boldsymbol{R}^n, \boldsymbol{C}^m) : \|u\|^2_{H^s_{1,\sigma}} = \int_{\boldsymbol{R}^n} (\mathscr{F}^{-1}((1+|\xi|^2)^{s/2}\mathscr{F}u)(x), M(x)\cdot\mathscr{F}^{-1}((1+|\xi|^2)^{s/2}\mathscr{F}u)(x))_{\boldsymbol{C}^m}(1+|x|^2)^\sigma dx < \infty\};$$

$$H_0=H^0_{0,0} \quad \text{and} \quad H_1=H^0_{1,0}\,,$$

where $\mathscr{F}$ is the Fourier transform

$$(\mathscr{F}u)(\xi)=(2\pi)^{-n/2}\int_{\boldsymbol{R}^n} e^{-ix\cdot\xi}u(x)dx\,.$$

Then by Assumption (1.1), $H^s_{0,\sigma}=H^s_{1,\sigma}$ as a set. J is an identification operator from H_1 to H_0 defined as $Ju(x)=u(x)$. Then its adjoint operator $J^* : H_0 \to H_1$ can be written as $J^*v(x)=M(x)^{-1}v(x)$.

Under these conditions natural selfadjoint realizations of $L_0(D)$ and $L(D)$ in H_0 and H_1 are determined as follows. Let $L_0(\xi)$ be the maximal operator determined by the multiplication by the matrix $L_0(\xi)=\sum_{j=1}^n A_j\xi_j$. Then

$$L_0=\mathscr{F}^{-1}L_0(\xi)\mathscr{F}\,, \quad L=J^*L_0J$$

are natural selfadjoint realizations of $L_0(D)$ and $L(D)$ in H_0 and H_1, respectively.

The roots $\lambda_j(\xi)$ of the characteristic equation $p(\lambda, \xi)\equiv\det(\lambda I-L_0(\xi))=0$ can be enumerated as $\lambda_{-\mu}(\xi)<\lambda_{-\mu+1}(\xi)<\cdots<\lambda_{-1}(\xi)<\lambda_0(\xi)<\lambda_1(\xi)<\cdots<\lambda_{\mu-1}(\xi)<\lambda_\mu(\xi)$, where $\lambda_0(\xi)\equiv 0$ if it exists and will be omitted otherwise. Let us take $\delta_j(\xi)>0$ so small that $\Gamma_j(\xi)=\{\zeta \in \boldsymbol{C}^1 : |\zeta-\lambda_j(\xi)|=\delta_j(\xi)\}$ does not enclose any root of $p(\lambda, \xi)=0$ except $\lambda_j(\xi)$ and put

$$\hat{P}_j(\xi)=-\frac{1}{2\pi i}\int_{\Gamma_j(\xi)} (L_0(\xi)-\zeta I)^{-1}d\zeta\,. \tag{1.3}$$

Then $\hat{P}_j(\xi)$ are the projection matrices in $\boldsymbol{C}^m$ onto the eigenspace of $L_0(\xi)$ corresponding to the eigenvalue $\lambda_j(\xi)$. Let $\hat{P}_j$ $(j=-\mu, \cdots, \mu)$ be the operator in H_0 determined by the multiplication by the matrix $\hat{P}_j(\xi)$. Put $P_j=\mathscr{F}^{-1}\hat{P}_j\mathscr{F}$. Then

(2.1.I) $\{P_j\}_{j=-\mu,\dots,\mu}$ is a complete system of projectors in H_0 reducing the operator L_0;

(2.2.I) $L_0P_j=\mathscr{F}^{-1}\lambda_j(\xi)\mathscr{F}P_j$, where $\lambda_j(\xi)$ is the operator determined by the multiplication by $\lambda_j(\xi)$.

As in [12] the following notations and conventions will be used throughout this paper.

$\boldsymbol{R}_\pm=\{\lambda\gtrless 0:\lambda\in\boldsymbol{R}^1\}$. By definition $\operatorname{sign} j=\pm 1$, but it will also be used as a substitute for $+$ or $-$, for example, $\boldsymbol{R}_{\operatorname{sign} j}$ is $\boldsymbol{R}_+$ or $\boldsymbol{R}_-$ according as $j>0$ or $j<0$. The subscript j is reserved to enumerate the eigenvalues λ_j, projectors P_j etc. and varies over (a subset of) $\{-\mu,\cdots,\mu\}$. The notation $(j\neq 0)$ means that $j=-\mu,\cdots,-1,1,\cdots,\mu$. Furthermore $\sum_{j\neq 0}=\sum_{j=-\mu}^{-1}+\sum_{j=1}^{\mu}$, $\sum_{\operatorname{sign} j=\operatorname{sign}\lambda}=\sum_{j=1}^{\mu}$ if $\operatorname{sign}\lambda=1$ and $\sum_{\operatorname{sign} j=\operatorname{sign}\lambda}=\sum_{j=-\mu}^{-1}$ if $\operatorname{sign}\lambda=-1$.

I is the $m\times m$ identity matrix.

$L^2(\Omega,K,d\rho)$ is the L^2-space of all strongly measurable functions defined on a measure space $(\Omega,d\rho)$ taking the values in a Hilbert space K.

$R_{L_0}(\zeta)=(L_0-\zeta)^{-1}$ and $R_L(\zeta)=(L-\zeta)^{-1}$ $(\operatorname{Im}\zeta\neq 0)$.

For $f\in H_\sigma$ and $g\in H_{-\sigma}$ we put

$$\langle f,g\rangle_{H_\sigma,H_{-\sigma}}=\int_{R^n}(f(x),g(x))_{C^m}dx\,.$$

Our starting point in [12] was the following spectral representation for $L_0(D)$. Let $S_j=\{\xi\in\Xi^n:\lambda_j(\xi)=\operatorname{sign} j\}$ $(j\neq 0)$ be the slowness surface which is a C^∞-manifold without boundary and let $F_j:\Xi^n\backslash\{0\}\to\boldsymbol{R}_{\operatorname{sign} j}\times S_j$ be the mapping defined by

$$F_j(\xi)=(\lambda_j(\xi),(\operatorname{sign} j)\xi/\lambda_j(\xi))\,.$$

Let ds_j be the surface element of S_j, and let $d\sigma_j(\omega_j)=\dfrac{ds_j}{\sqrt{|\operatorname{grad}\lambda_j(\omega_j)|}}$. Let $d\rho_\pm(\lambda)=|\lambda|^{\frac{n-1}{2}}d\lambda$. (Although $d\rho_+$ and $d\rho_-$ have the same form we regard ρ_+ (or ρ_-) as a measure on $\boldsymbol{R}_+$ (or $\boldsymbol{R}_-$).) Then the operator $\hat{\Gamma}_j:L^2(\Xi^n,\boldsymbol{C}^m)\to L^2(\boldsymbol{R}_{\operatorname{sign} j},L^2(S_j,\boldsymbol{C}^m,d\sigma_j),d\rho_{\operatorname{sign} j})$ defined by the equation $(\hat{\Gamma}_jf)(\lambda,\omega_j)=f(F_j^{-1}(\lambda,\omega_j))$ for any $f\in L^2(\Xi^n,\boldsymbol{C}^m)$ is a unitary operator. The operator $\Gamma_j=\hat{\Gamma}_j\mathscr{F}P_j$ $(j\neq 0)$ has the following properties:

(2.4.I) $(\Gamma_jL_0u)(\lambda)=\lambda(\Gamma_ju)(\lambda)$ a.e. $\lambda\in\boldsymbol{R}_{\operatorname{sign} j}$, for all $u\in D(L_0)$;

(2.5.I) $\Gamma_jP_i=0$ when $i\neq j$;

(2.6.I) If $u\in H_{0,\sigma}$ $\left(\sigma>\dfrac{1}{2}\right)$ then $(\Gamma_ju)(\lambda):\boldsymbol{R}_{\operatorname{sign} j}\to L^2(S_j,\boldsymbol{C}^m,d\sigma_j)$ is locally Hölder continuous;

(2.7.I) If we define the operator $\Gamma_j(\lambda):H_{0,\sigma}\to L^2(S_j,\boldsymbol{C}^m,d\sigma_j)$ $\left(\sigma>\dfrac{1}{2}\right)$ by $\Gamma_j(\lambda)u$

$=(\Gamma_j u)(\lambda)$ for $\lambda\in \boldsymbol{R}_{\operatorname{sign} j}$, then the operator $\Gamma_j(\lambda)$ is a bounded operator and the mapping $\Gamma_j(\cdot): \boldsymbol{R}_{\operatorname{sign} j}\to B(H_{0,\sigma}, L^2(S_j, \boldsymbol{C}^m, d\sigma_j))$ is locally Hölder continuous.

Main theorems derived in [12] can be stated as follows:

THEOREM 1.1.I. *Let Assumption (A.1.I) be satisfied. Let $I_0=\boldsymbol{R}^1\setminus\{0\}$ and let $\Pi^\pm=\{\zeta\in \boldsymbol{C}^1 : \operatorname{Im}\zeta\gtrless 0\}$. Let ε be any positive constant. Then the following statements hold:*

(1) *$R_{L_0}(\zeta)(1-P_0)$ $(\operatorname{Im}\zeta\neq 0)$ can be extended to $\Pi^\pm\cup I_0$ as a $B(H_{0,(1+\varepsilon)/2}, H^1_{0,-(1+\varepsilon)/2})$-valued locally Hölder continuous function. Moreover $R_{L_0}(\zeta)$ itself can be extended to $\Pi^\pm\cup I_0$ as a $B(H_{0,(1+\varepsilon)/2}, H_{0,-(1+\varepsilon)/2})$-valued locally Hölder continuous function. Their boundary values on I_0 are denoted as $R'_{L_0}(\lambda\pm i0)$ and $R_{L_0}(\lambda\pm i0)$, respectively.*

(2) *For any $u\in H_{0,(1+\varepsilon)/2}$ and $\lambda\in I_0$, $(L_0-\lambda)R_{L_0}(\lambda\pm i0)u=u$, where the differentiation is in the sense of distributions.*

THEOREM 1.3.I. *Let Assumptions (A.1.I) and (A.2.I) be satisfied. Let $I_1=\boldsymbol{R}^1\setminus(\sigma_p(L)\cup\{0\})$, where $\sigma_p(L)$ is the point spectrum of L. Then the following statements hold:*

(1) *$R_L(\zeta)$ $(\operatorname{Im}\zeta\neq 0)$ can be extended to $\Pi^\pm\cup I_1$ as a $B(H_{1,\delta/2}, H_{1,-\delta/2})$-valued locally Hölder continuous function. We put $R_L(\lambda\pm i0)=\lim_{\eta\downarrow 0} R_L(\lambda\pm i\eta)$ for $\lambda\in I_1$.*

(2) *For any $u\in H_{1,\delta/2}$ and $\lambda\in I_1$, $(L-\lambda)R_L(\lambda\pm i0)u=u$, where the differentiation is in the sense of distributions.*

THEOREM 1.7.I. *Let Assumptions (A.1.I) and (A.2.I) be satisfied. Then $\sigma_p(L)\setminus\{0\}$ is discrete in $\boldsymbol{R}^1\setminus\{0\}$ and the only possible accumulation point is the origin.*

We finish our review of [12] with this theorem. We next give two lemmas which will play the important role in the following sections. The first one is proved implicitly in the proof of Theorem 1.1.I.

LEMMA 1.1. *Let $\lambda\in \boldsymbol{R}^1\setminus\{0\}$ and let $\phi_\lambda(\xi)$ be a C_0^∞-function such that $\phi_\lambda(\xi)=1$ in a neighbourhood of the origin and such that $(\sum_{j=1}^n A_j\xi_j-\lambda I)$ is nonsingular on the support of $\phi_\lambda(\xi)$; put $\tilde\phi_\lambda=1-\phi_\lambda$. Then for any $u\in H_{0,\sigma}$ $\left(\sigma>\frac{1}{2}\right)$ the relation*

$$\mathscr{F}(R_{L_0}(\lambda\pm i0)u)(\xi)=\phi_\lambda(\xi)(\sum_{j=1}^n A_j\xi_j-\lambda I)^{-1}\mathscr{F}u(\xi) + \lim_{\eta\downarrow 0}\sum_{j=-\mu}^{\mu}\frac{\tilde\phi_\lambda(\xi)\hat P_j(\xi)\mathscr{F}u(\xi)}{\lambda_j(\xi)-(\lambda\pm i\eta)}, \tag{1.3}$$

holds, where the limit is considered in $H_0^{-(1+\varepsilon)/2}$ $(\varepsilon>0)$.

LEMMA 1.2. *Let K be an arbitrary Hilbert space and let $u \in H^s(\boldsymbol{R}^1, K)$ (=K-valued Sobolev space of order s in the usual sense) with $s > \frac{1}{2}$. For $\lambda_0 \in \boldsymbol{R}^1$ put*

$$u_1(\lambda, \lambda_0) = \frac{u(\lambda) - u(\lambda_0)}{\lambda - \lambda_0} \quad \textit{and} \quad u_2^{\pm}(\lambda, \lambda_0) = \lim_{\eta \downarrow 0} \frac{u(\lambda_0)}{\lambda - (\lambda_0 \pm i\eta)} .$$

We fix λ_0 and regard u_1 and $u_2^{\pm}$ as K-valued distributions of the variable λ. Then $u_1 \in H^{s-1}(\boldsymbol{R}^1, K)$ and $u_2^{\pm} \in H^{-(1+\varepsilon)/2}(\boldsymbol{R}^1, K)$ for any $\varepsilon > 0$, and there exist constants C_1 and C_2 such that

$$\|u_1(\cdot, \lambda_0)\|_{H^{s-1}} \leqq C_1 \|u\|_{H^s}, \tag{1.4}$$

$$\|u_2^{\pm}(\cdot, \lambda_0)\|_{H^{-(1+\varepsilon)/2}} \leqq C_2 \|u\|_{H^s}. \tag{1.5}$$

Moreover, $u_1(\cdot, \lambda_0)$ (or $u_2^{\pm}(\cdot, \lambda_0)$) are $H^{s-1}(\boldsymbol{R}^1, K)$-valued (or $H^{-(1+\varepsilon)/2}(\boldsymbol{R}^1, K)$-valued) continuous function of λ_0.

Furthermore the following equation holds in $H^{-(1+\varepsilon)/2}$ ($\varepsilon > 0$) for any $u \in H^s$ ($s > 1/2$):

$$\lim_{\eta \downarrow 0} \frac{u(\lambda)}{\lambda - (\lambda_0 \pm i\eta)} = \frac{u(\lambda) - u(\lambda_0)}{\lambda - \lambda_0} + \lim_{\eta \downarrow 0} \frac{u(\lambda_0)}{\lambda - (\lambda_0 \pm i\eta)} . \tag{1.6}$$

PROOF. Since the other case can be proved similarly we shall give the proof only for "+" case and omit the sign "+" in what follows. Let $\eta > 0$. By Cauchy's integral formula we get

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-i\lambda x}}{\lambda - \lambda_0 - i\eta} d\lambda = \begin{cases} 0 & \text{for } x > 0 \\ e^{-i\lambda_0 x + \eta x} & \text{for } x < 0 . \end{cases} \tag{1.7}$$

Therefore

$$\begin{aligned} \mathscr{F}\left(\frac{u(\lambda)}{\lambda - \lambda_0 - i\eta}\right)(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathscr{F} u(y) \mathscr{F}\left(\frac{1}{\lambda - \lambda_0 - i\eta}\right)(x - y) dy \\ &= i \int_x^{\infty} (\mathscr{F} u)(y) e^{-i\lambda_0(x-y) + \eta(x-y)} dy . \end{aligned}$$

Since by assumption $u \in H^s(\boldsymbol{R}^1, K)$, Lebesgue's dominated convergence theorem shows that in $L^\infty(\boldsymbol{R}^1, K)$

$$\lim_{\eta \downarrow 0} \mathscr{F}\left(\frac{u(\lambda)}{\lambda - \lambda_0 - i\eta}\right)(x) = i \int_x^{\infty} (\mathscr{F} u)(y) e^{-i\lambda_0(x-y)} dy \tag{1.8}$$

exists. Hence in $H_{-(1+\varepsilon)/2}$ the limit also exists and equation (1.8) holds. Let $Y(x)$ be Heaviside function

$$Y(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x < 0 , \end{cases}$$

then

$$ie^{-i\lambda_0 x}\int_x^\infty \mathscr{F}u(y)e^{i\lambda_0 y}dy = ie^{-i\lambda_0 x}(1-Y(x))\int_{-\infty}^\infty \mathscr{F}u(y)e^{i\lambda_0 y}dy$$
$$+ie^{-i\lambda_0 x}\left(-\int_{-\infty}^x \mathscr{F}u(y)e^{i\lambda_0 y}dy + Y(x)\int_{-\infty}^\infty \mathscr{F}u(y)e^{i\lambda_0 y}dy\right)$$

holds in $H_{-(1+\varepsilon)/2}$. On the other hand

$$\lim_{\eta\downarrow 0}\frac{u(\lambda)}{\lambda-(\lambda_0+i\eta)} = \lim_{\eta\downarrow 0}\left[\frac{u(\lambda)-u(\lambda_0)}{\lambda-(\lambda_0+i\eta)}+\frac{u(\lambda_0)}{\lambda-(\lambda_0+i\eta)}\right]$$
$$=\frac{u(\lambda)-u(\lambda_0)}{\lambda-\lambda_0}+\lim_{\eta\downarrow 0}\frac{u(\lambda_0)}{\lambda-(\lambda_0+i\eta)}$$

in $\mathscr{S}'$, because $u(\lambda)$ is Hölder continuous and

$$\lim_{\eta\downarrow 0}\frac{u(\lambda)-u(\lambda_0)}{\lambda-(\lambda_0+i\eta)} = \frac{u(\lambda)-u(\lambda_0)}{\lambda-\lambda_0}\in L^1_{\mathrm{loc}}\,.$$

It is obvious by (1.7) that

$$\mathscr{F}\left(\lim_{\eta\downarrow 0}\frac{u(\lambda_0)}{\lambda-(\lambda_0+i\eta)}\right)(x) = \sqrt{2\pi}\, ie^{-i\lambda_0 x}(1-Y(x))u(\lambda_0)\,. \tag{1.9}$$

Hence

$$\mathscr{F}\left(\frac{u(\lambda)-u(\lambda_0)}{\lambda-\lambda_0}\right)(x) \tag{1.10}$$
$$= ie^{-i\lambda_0 x}\left(-\int_{-\infty}^x \mathscr{F}u(y)e^{i\lambda_0 y}dy + Y(x)\int_{-\infty}^\infty \mathscr{F}u(y)e^{i\lambda_0 y}dy\right).$$

Now we shall prove $u_1(\cdot,\lambda_0)\in H^{s-1}$. By assumption $u(\lambda)\in H^s$ and Schwarz inequality we get

$$x^{2s-1}\left|\int_x^\infty e^{i\lambda_0 y}\mathscr{F}u(y)dy\right|^2 \tag{1.11}$$
$$\leqq x^{2s-1}\left\{\int_x^\infty |\mathscr{F}u(y)|^2(1+|y|^2)^s dy\right\}\left\{\int_x^\infty y^{-2s}dy\right\}$$
$$=\frac{1}{2s-1}\int_x^\infty |\mathscr{F}u(y)|^2(1+|y|^2)^s dy \to 0 \quad \text{as} \quad x\to\infty\,.$$

Hence using the partial integration, we get

$$\int_0^\infty x^{2s-2}\left|\int_x^\infty e^{i\lambda_0 y}\mathscr{F}u(y)dy\right|^2 dx$$
$$=\lim_{R\to\infty}\frac{x^{2s-1}}{2s-1}\left|\int_x^\infty e^{i\lambda_0 y}\mathscr{F}u(y)dy\right|^2\Bigg|_{x=0}^{x=R}$$

$$+\int_0^\infty \frac{x^{2s-1}}{2s-1}\cdot 2\mathrm{Re}\left(e^{i\lambda_0 x}(\mathscr{F}u)(x),\ \int_x^\infty e^{i\lambda_0 y}\mathscr{F}u(y)\,dy\right)dx$$
$$\leqq \frac{2}{2s-1}\int_0^\infty \left\{x^s\cdot|\mathscr{F}u(x)|\cdot x^{s-1}\left|\int_x^\infty e^{i\lambda_0 y}\mathscr{F}u(y)dy\right|\right\}dx$$
$$\leqq \frac{2}{2s-1}\left\{\int_0^\infty |x^s\cdot\mathscr{F}u(x)|^2dx\right\}^{1/2}\left\{\int_0^\infty x^{2s-2}\left|\int_x^\infty e^{i\lambda_0 y}\mathscr{F}u(y)dy\right|^2dx\right\}^{1/2}.$$

Therefore we get

$$\int_x^\infty x^{2s-2}\left|\int_x^\infty e^{i\lambda_0 y}\mathscr{F}u(y)dy\right|^2dx \leqq \left(\frac{2}{2s-1}\right)^2\int_0^\infty |x^s\mathscr{F}u(x)|^2dx\,. \tag{1.12}$$

Similarly we can get

$$\int_{-\infty}^0 |x|^{2s-2}\left|\int_{-\infty}^x e^{i\lambda_0 y}\mathscr{F}u(y)dy\right|^2dx \leqq \left(\frac{2}{2s-1}\right)^2\int_{-\infty}^0 |x^s\mathscr{F}u(x)|^2dx\,. \tag{1.13}$$

Let $\chi_{[-1,1]}$ be the characteristic function of the interval $[-1, 1]$. Then using (1.10), (1.12) and (1.13), we get after a simple calculation

$$\begin{aligned}\|u_1\|^2_{H^{s-1}} &= \|\chi_{[-1,1]}(x)\mathscr{F}u_1(x)\|^2_{H_{s-1}}+\|(1-\chi_{[-1,1]})\mathscr{F}u_1(x)\|^2_{H_{s-1}}\\ &\leqq 2^{\max(s,1)}\|\mathscr{F}u_1(x)\|_{L^\infty}+\left\{\int_{-\infty}^0 |x|^{2s-2}\left|\int_{-\infty}^x e^{i\lambda_0 y}\mathscr{F}u(y)dy\right|^2dx\right.\\ &\qquad\left.+\int_0^\infty x^{2s-2}\left|\int_x^\infty e^{i\lambda_0 y}\mathscr{F}u(y)dy\right|^2dx\right\}\\ &\leqq 2^{\max(s,1)}\|\mathscr{F}u\|^2_{L^1}+\left(\frac{2}{2s-1}\right)^2\int_{-\infty}^\infty |x^s\mathscr{F}u(x)|^2dx\\ &\leqq C_s\|\mathscr{F}u\|^2_{H_s}=C_s\|u\|^2_{H^s}\,,\end{aligned} \tag{1.14}$$

where the constant C_s is determined only by s.

As for u_2 we get by (1.9) and the assumption $s>1/2$,

$$\|u_2\|^2_{H^{-(1+\varepsilon)/2}}=\|\mathscr{F}u_2\|^2_{H_{-(1+\varepsilon)/2}}=(2\pi)\int_{-\infty}^0 (1+x^2)^{-(1+\varepsilon)/2}dx|u(\lambda_0)|^2\leqq C_\varepsilon\|u\|_{H^s}\,, \tag{1.15}$$

where the constant C_ε is dependent only on ε.

Next we prove the continuity of $u_1(\cdot, \lambda_0)$ and $u_2(\cdot, \lambda_0)$ as stated in the lemma. A similar calculation used to prove inequality (1.14) shows

$$\begin{aligned}\|u_1(\cdot,\lambda_0)-u_1(\cdot,\lambda_0')\|^2_{H^{s-1}} &= \|\mathscr{F}u_1(x,\lambda_0)-\mathscr{F}u(x,\lambda_0')\|^2_{H_{s-1}}\\ &\leqq C_s\left\{\|(e^{i\lambda_0 x}-e^{i\lambda_0' x})\mathscr{F}u)\|_{L^1}\right.\\ &\qquad+\int_0^\infty x^{2s-2}\left|\int_x^\infty (e^{-i\lambda_0(x-y)}-e^{-i\lambda_0'(x-y)})\mathscr{F}u(y)dy\right|^2dx\\ &\qquad\left.+\int_{-\infty}^0 |x|^{2s-2}\left|\int\left\{\int_{-\infty}^x (e^{-i\lambda_0(x-y)}-e^{-i\lambda_0'(x-y)})\mathscr{F}u(y)dy\right|^2dx\right\}\,.\end{aligned}$$

Since $u \in H_s(R^1, K) \subset L^1(R^1, K)$, Lebesgue's dominated convergence theorem implies

$$\|(e^{i\lambda_0 x}-e^{i\lambda_0' x})\mathscr{F}u\|_{L^1} \to 0 \quad \text{as} \quad \lambda_0' \to \lambda_0 .$$

As for the second term we get

$$\begin{aligned}
&\int_0^\infty x^{2s-2}\left|\int_x^\infty (e^{-i\lambda_0(x-y)}-e^{-i\lambda_0'(x-y)})\mathscr{F}u(y)dy\right|^2 dx \\
&\leqq \int_0^\infty x^{2s-2}\left|\int_x^\infty (e^{i\lambda_0 y}-e^{i\lambda_0' y})\mathscr{F}u(y)dy\right|^2 dx \\
&\quad + \int_0^\infty x^{2s-2}|e^{-i\lambda_0 x}-e^{-i\lambda_0' x}|^2\left|\int_x^\infty e^{i\lambda_0' y}\mathscr{F}u(y)dy\right|^2 dx \\
&\leqq \left(\frac{2}{2s-1}\right)^2\int_0^\infty x^{2s}|e^{i\lambda_0 x}-e^{i\lambda_0' x}|^2|\mathscr{F}u(x)|^2 dx \\
&\quad + \int_0^\infty x^{2s-2}|e^{-i\lambda_0 x}-e^{-i\lambda_0' x}|^2\left|\int_x^\infty e^{i\lambda_0' y}u\mathscr{F}(y)dy\right|^2 dx .
\end{aligned}$$

The assumption and (1.12) imply that $x^{2s}|\mathscr{F}u(x)|^2$ and $x^{2s-2}\left|\int_x^\infty e^{i\lambda_0 y}\mathscr{F}u(y)dy\right|^2$ are absolutely integrable functions. Therefore we get, using Lebesgue's dominated convergence theorem

$$\int_0^\infty x^{2s-2}\left|\int_x^\infty (e^{-i\lambda_0(x-y)}-e^{-i\lambda_0'(x-y)})\mathscr{F}u(y)dy\right|^2 dx \to 0 \quad \text{as} \quad \lambda_0' \to \lambda_0 .$$

Similarly we get

$$\int_{-\infty}^0 |x|^{2s-2}\left|\int_{-\infty}^x (e^{-i\lambda_0(x-y)}-e^{-i\lambda_0'(x-y)})\mathscr{F}u(y)dy\right|^2 dx \to 0 \quad \text{as} \quad \lambda_0' \to \lambda_0 .$$

Hence $\|u_1(\cdot, \lambda_0)-u(\cdot, \lambda_0')\|_{H^{s-1}} \to 0$ as $\lambda_0' \to \lambda_0$. We can prove the continuity of the function $u_2(\cdot, \lambda_0)$ easily. (Q.E.D.)

§2. Eigenfunction expansions associated with L_0.

In this section we shall give eigenfunction expansion formulas associated with the unperturbed operator L_0 in two ways. The first one is the expansion by plane waves, which is a simple modification of the Fourier transformation. The second one is a modified form of the first one and can be considered to correspond to the partial wave expansion formula which has been used in the analysis of quantum mechanical systems. They can be stated as follows.

THEOREM 2.1. *For any $K>0$, any $f \in H_0$ and any $\hat{f} \in L^2(\Xi^n, C^m)$ we have*

$$(2\pi)^{-n/2}\int_{|x|<K} e^{-ix\cdot\xi}\hat{P}_j(\xi)f(x)dx \in L^2(\Xi^n, C^m) \quad \textit{and}$$

$$(2\pi)^{-n/2}\int_{|\xi|<K} e^{ix\cdot\xi}\hat{P}_j(\xi)\hat{f}(\xi)d\xi \in H_0 .$$

Furthermore

$$(T_{0,j}f)(\xi)=\underset{K\to\infty}{\mathrm{l.i.m.}}\ (2\pi)^{-n/2}\int_{|x|<K} e^{-ix\cdot\xi}\hat{P}_j(\xi)f(x)dx$$

and

$$(T'_{0,j}\hat{f})(x)=\underset{K\to\infty}{\mathrm{l.i.m.}}\ (2\pi)^{-n/2}\int_{|\xi|<K} e^{ix\cdot\xi}\hat{P}_j(\xi)\hat{f}(\xi)d\xi$$

exist, where l.i.m. *denotes the convergence in* $L^2(\Xi^n, \boldsymbol{C}^m)$ *and* H_0, *respectively. Let* $T_0: H_0 \to L^2(\Xi^n, \boldsymbol{C}^m)$ *and* $T'_0: L^2(\Xi^n, \boldsymbol{C}^m) \to H_0$ *be the operator defined by*

$$T_0 f=\sum_j T_{0,j}f , \qquad f\in H_0 ,$$

$$T'_0\hat{f}=\sum_j T'_{0,j}\hat{f} , \qquad \hat{f}\in L^2(\Xi^n, \boldsymbol{C}^m) ,$$

respectively. Then the following statements hold:

(1) $(L_0-\lambda_j(\xi)I)(e^{ix\cdot\xi}\hat{P}_j(\xi))=0$ *for* $j=-\mu, \cdots, \mu$ *and* $\xi\in\Xi^n$;

(2) T_0 *is a unitary operator from* H_0 *to* $L^2(\Xi^n, \boldsymbol{C}^m)$ *and* T'_0 *is its adjoint operator*;

(3) *For* $f\in H_0$, $f\in D(L_0)$ *if and only if* $\lambda_j(\xi)(T_{0,j}f)(\xi)\in L^2(\Xi^n, \boldsymbol{C}^m)$ *for all* $j=-\mu, \cdots, \mu$. *For* $f\in D(L_0)$, $T_0L_0f(\xi)=\lambda_j(\xi)(T_{0,j}f)(\xi)$.

Here in statement (1) the operator $(L_0-\lambda_j(\xi)I)$ *is applied to* $e^{ix\xi}\hat{P}_j(\xi)$ *by matrix multiplication rule.*

REMARK. By statement (1) we may say that $e^{ix\cdot\xi}\hat{P}_j(\xi)$ is a (matrix-valued) eigenfunction of L_0 corresponding to the eigenvalue $\lambda_j(\xi)$.

PROOF. See (2.1.I) and (2.2.I).

PROPOSITION 2.2. *For each* $j\neq 0$, *let* $\phi_j^{(k)}(\xi)$, $k=1, 2, 3, \cdots$ *be a* C^∞*-class complete orthonormal system of* $L^2(S_j, \boldsymbol{C}^1, d\sigma_j)$ *and let* $h_j(x, \lambda, k)$ *be the matrix depending on* $x\in\boldsymbol{R}^n$, $\lambda\in\boldsymbol{R}_{\mathrm{sign}\,j}$ *and* $k\in\boldsymbol{N}$ (*=the set of all natural numbers*) *defined by*

$$h_j(x, \lambda, k)=(2\pi)^{-n/2}\int_{S_j} e^{i|\lambda|\omega_j x}\phi_j^{(k)}(\omega_j)\hat{P}_j(\omega_j)d\sigma_j(\omega_j) . \tag{2.1}$$

Then $h_j(x, \lambda, k)$ *is a bounded function of all variables. Furthermore for each fixed* $\lambda\in\boldsymbol{R}_{\mathrm{sign}\,j}$, $k\in\boldsymbol{N}$ *and for any* $\varepsilon>0$ *each column* $h_j^{(i)}(x, \lambda, k)$ *of* $h_j(x, \lambda, k)$ *satisfies the following relations*:

(2.2) *For any* n*-tuple of nonnegative integers* $(\alpha_1, \alpha_2, \cdots, \alpha_n)$,

$$D_1^{\alpha_1}D_2^{\alpha_2}\cdots D_n^{\alpha_n}h_j^{(i)}(x, \lambda, k)\in H_{0,-(1+\varepsilon)/2} ;$$

$$(2.3)\qquad (L_0-\lambda I)h_j^{(l)}(x, \lambda, k)=0 .$$

REMARK. By (2.3) we can consider that $h_j(x, \lambda, k)$ is a (matrix-valued) eigenfunction of L_0 corresponding to the eigenvalue λ. In contrast to $e^{ix\cdot\xi}\hat{P}_j(\xi)$ which is also an eigenfunction of L_0 and describes the plane wave, $h_j(x, \lambda, k)$ describes a sort of the spherical wave (see §6).

PROOF. It is obvious that for every $\lambda\in \boldsymbol{R}_{\mathrm{sign}\, j}$ and $k\in N$, $h_j(x, \lambda, k)$ is a C^∞-function as a function of x and for any n-tuple of non-negative integers $\alpha_1, \alpha_2, \cdots, \alpha_n\geqq 0$,

$$(2.4)\qquad \begin{aligned} &D_1^{\alpha_1}D_2^{\alpha_2}\cdots D_n^{\alpha_n}h_j(x, \lambda, k)\\ &=(2\pi)^{-n/2}\int_{S_j}(|\lambda|\omega_1)^{\alpha_1}\cdots(|\lambda|\omega_n)^{\alpha_n}e^{ix\cdot|\lambda|\omega_j}\phi_j^{(k)}(\omega_j)\hat{P}_j(\omega_j)d\sigma_j(\omega_j) .\end{aligned}$$

Since $L_0(\omega_j)\hat{P}_j(\omega_j)=\lambda_j(\omega_j)\hat{P}_j(\omega_j)$ and $\lambda_j(\omega_j)=\mathrm{sign}\, j$ on S_j, we get for any $\lambda\in \boldsymbol{R}_{\mathrm{sign}\, j}$ and $k\in N$,

$$(2.5)\qquad \begin{aligned} &(L_0-\lambda I)h_j(x, \lambda, k)\\ &=(2\pi)^{-n/2}\int_{S_j}(|\lambda|L_0(\omega_j)-\lambda I)(e^{ix|\lambda|\omega_j}\phi_j^{(k)}(\omega_j)\hat{P}_j(\omega_j))d\sigma_j(\omega_j)\\ &=(2\pi)^{-n/2}\int_{S_j}[(\mathrm{sign}\, j)|\lambda|-\lambda](e^{ix|\lambda|\omega_j}\phi_j^{(k)}(\omega_j)\hat{P}_j(\omega_j))d\sigma_j(\omega_j)\\ &=0 .\end{aligned}$$

Finally we shall prove the relation

$$D_1^{\alpha_1}D_2^{\alpha_2}\cdots D_n^{\alpha_n}h_j(x, \lambda, k)\in H_{0,-(1+\varepsilon)/2} .$$

A simple consideration shows that the operator

$$f=\{f_\nu(\omega_j)\}_{\nu=1}^m\to\left\{(2\pi)^{-n/2}\int_{S_j}e^{i|\lambda|\omega_j\cdot x}f_\nu(\omega_j)d\sigma_j(\omega_j)\right\}_{\nu=1}^m, \qquad f\in L^2(S_j, \boldsymbol{C}^m, d\sigma_j)$$

is the adjoint operator of the operator $\hat{\Gamma}_j(\lambda): H_{0,(1+\varepsilon)/2}\to L^2(S_j, \boldsymbol{C}^m, d\sigma_j)$. Therefore (2.4) shows that $D_1^{\alpha_1}D_2^{\alpha_2}\cdots D_n^{\alpha_n}h_j(x, \lambda, k)$ is an element of $H_{0,-(1+\varepsilon)/2}$. (Q.E.D.)

THEOREM 2.3. *Let $h_j(x, \lambda, k)$ $(j\neq 0)$ be the function defined in Proposition 2.2. Then for any $f\in H_0$ and $K>0$, $\int_{|x|<K}h_j(x, \lambda, k)^*f(x)dx$ belongs to $L^2(\boldsymbol{R}_{\mathrm{sign}\, j}, l^2(\boldsymbol{C}^m), d\rho_{\mathrm{sign}\, j})$ and*

$$(T_jf)(\lambda, k)=\underset{K\to\infty}{\mathrm{l.i.m.}}\int_{|x|<K}h_j(x, \lambda, k)^*f(x)dx$$

exists, where, as usual, l.i.m. *denotes the convergence in $L^2(\boldsymbol{R}_{\mathrm{sign}\, j}, l^2(\boldsymbol{C}^m), d\rho_{\mathrm{sign}\, j})$. Let*

$$T: H_0 \to \sum_{j\neq 0} \oplus L^2(\boldsymbol{R}_{\mathrm{sign}\, j},\ l^2(\boldsymbol{C}^m), d\rho_{\mathrm{sign}\, j})$$

be the operator defined by

$$Tf = \sum_{j\neq 0} \oplus T_j f\ , \quad f\in H_0\ .$$

The operator

$$T'\ ; \sum_{j\neq 0} \oplus L^2(\boldsymbol{R}_{\mathrm{sign}\, j}, l^2(\boldsymbol{C}^m), d\rho_{\mathrm{sign}\, j}) \to H_0$$

can be defined analogously by $H_{0,-(1+\varepsilon)/2}$-valued Bochner integral as

$$T'(\sum_{j\neq 0} \oplus \hat{f}_j)(x) = \sum_{j\neq 0} \underset{K\to\infty}{\mathrm{l.i.m.}} \int_{R_{\mathrm{sign}\, j}\cap\{|\lambda|<K\}} \sum_{k=1}^{K} h_j(x, \lambda, k)\hat{f}_j(\lambda, k) d\rho_{\mathrm{sign}\, j}(\lambda)\ .$$

Furthermore the following statements hold:

(1) *T is a partially isometric operator from H_0 into $\sum_{j\neq 0} \oplus L^2(\boldsymbol{R}_{\mathrm{sign}\, j}, l^2(\boldsymbol{C}^m), d\rho_{\mathrm{sign}\, j})$ and T' is its adjoint operator:*

(2) *(Expansion formula)*

$$(I-P_0)f = T'Tf \quad \textit{for all} \quad f\in H_0:$$

(3) *(Diagonal representation formula) $f\in D(L_0)$ if and only if $\lambda(T_j f)(\lambda, k) \in L^2(\boldsymbol{R}_{\mathrm{sign}\, j}, l^2(\boldsymbol{C}^m), d\rho_{\mathrm{sign}\, j})$ for all $j\neq 0$. If $f\in D(L_0)$, $(T_j L_0 f)(\lambda, k) = \lambda(T_j f)(\lambda, k)$ for any $\lambda\in \boldsymbol{R}_{\mathrm{sign}\, j}$ and $k\in \boldsymbol{N}$.*

PROOF. By the definition of Γ_j, $\Gamma_j^*\Gamma_j = \mathscr{F}^*\hat{P}_j\hat{\Gamma}_j^*\hat{\Gamma}_j\hat{P}_j\mathscr{F} = P_j$. Hence $I-P_0 = \sum_{j\neq 0} \Gamma_j^*\Gamma_j$. We next define the operator $U_j: L^2(S_j, \boldsymbol{C}^m, d\sigma_j) \to l^2(\boldsymbol{C}^m)$ in the following way. For every $f\in L^2(S_j, \boldsymbol{C}^m, d\sigma_j)$ we put

$$(U_j f)(k) = \int_{S_j} \overline{\phi_j^{(k)}(\omega_j)} f(\omega_j) d\sigma_j\ . \tag{2.6}$$

Here the product on the right hand side is taken in the sense of the multiplication of $\overline{\phi_j^{(k)}(\omega_j)}\in \boldsymbol{C}^1$ and $f(\omega_j)\in \boldsymbol{C}^m$. Writing $f(\omega_j) = \{f_\nu(\omega_j)\}_{\nu=1,\cdots,m}$, we obtain

$$\begin{aligned}\sum_{k=1}^{\infty} |(U_j f)(k)|^2 &= \sum_{\nu=1}^{m} \sum_{k=1}^{\infty} \left|\int_{S_j} \overline{\phi_j^{(k)}(\omega_j)} f_\nu(\omega_j) d\sigma_j\right|^2 \\ &= \sum_{\nu=1}^{m} \|f_\nu\|^2_{L^2(S_j)} = \|f\|^2\ ;\end{aligned}$$

and hence it follows that U_j is a unitary operator from $L^2(S_j, \boldsymbol{C}^m, d\sigma_j)$ to $l^2(\boldsymbol{C}^m)$. Furthermore $\tilde{U}_j$ defined by

$$(\tilde{U}_j f)(\lambda, k) = (U_j f(\lambda, \cdot))(k)\ , \quad f\in L^2(\boldsymbol{R}_{\mathrm{sign}\, j}, L^2(S_j, \boldsymbol{C}^m))$$

determines a unitary operator $\tilde{U}_j: L^2(\boldsymbol{R}_{\mathrm{sign}\, j}, L^2(S_j, \boldsymbol{C}^m)) \to L^2(\boldsymbol{R}_{\mathrm{sign}\, j}, l^2(\boldsymbol{C}^m))$. We put for each $j \neq 0$,

$$\tilde{T}_j = \tilde{U}_j \Gamma_j . \tag{2.7}$$

Then it is clear that $\tilde{T}_j$ is a bounded operator from H_0 to $L^2(\boldsymbol{R}_{\mathrm{sign}\, j}, l^2(\boldsymbol{C}^m))$. Moreover the following formulas (2.8) and (2.9) can be proved easily:

$$1 - P_0 = \sum_{j=0} \tilde{T}_j^* \tilde{T}_j ; \tag{2.8}$$

$$(\tilde{T}_j L_0 f)(\lambda, k) = \lambda \tilde{T}_j f(\lambda, k) \quad \text{for all} \quad f \in D(L_0) . \tag{2.9}$$

Equation (2.8) implies

$$\|(1-P_0)f\|^2 = \sum_{j \neq 0} \|\tilde{T}_j f\|^2 \quad \text{for all} \quad f \in H_0 . \tag{2.10}$$

Now by the definition of Γ_j and $\tilde{U}_j$ we get for any $f \in H_0 \cap L^1(R^n; \boldsymbol{C}^m)$

$$\begin{aligned} \tilde{T}_j f(\lambda, k) &= \int_{S_j} \overline{\phi_j^{(k)}(\omega_j)} \left\{ (2\pi)^{-n/2} \int_{R^n} e^{-ix\cdot|\lambda|\omega_j} \hat{P}_j(\omega_j) f(x) dx \right\} d\sigma_j(\omega_j) \\ &= \int_{R^n} (2\pi)^{-n/2} \left\{ \int_{S_j} \overline{\phi_j^{(k)}(\omega_j)} e^{-ix\cdot|\lambda|\omega_j} \hat{P}_j(\omega_j) d\sigma_j(\omega_j) \right\} f(x) dx \\ &= \int_{R^n} h_j(x, \lambda, k)^* f(x) dx . \end{aligned} \tag{2.11}$$

Since $H_0 \subset L^1_{\mathrm{loc}}$, for any $f \in H_0$ and $K > 0$, we have

$$\int_{|x|<K} h_j(x, \lambda, k)^* f(x) dx \in L^2(\boldsymbol{R}_{\mathrm{sign}\, j}, l^2(\boldsymbol{C}^m), d\rho_{\mathrm{sign}\, j}) .$$

Then equations (2.10) and (2.11) imply that

$$T_j f = \underset{K\to\infty}{\mathrm{l.i.m.}} \int_{|x|<K} h_j(x, \lambda, k)^* f(x) dx$$

exists and $T_j f = \tilde{T}_j f$ for any $f \in H_0$. Hence equations (2.8), (2.9) and (2.10) imply statements (2), (3) and the first part of (1) of the theorem.

Next we prove $T' = T^*$. First of all we shall prove

$$\underset{K\to\infty}{\mathrm{l.i.m.}} \int_{\boldsymbol{R}_{\mathrm{sign}\, j} \cap \{|\lambda|<K\}} \sum_{k=1}^{K} h_j(x, \lambda, k) \hat{f}_j(\lambda, k) d\rho_{\mathrm{sign}\, j}(\lambda), \; \hat{f}_j \in L^2(\boldsymbol{R}_{\mathrm{sign}\, j}, l^2(\boldsymbol{C}^m), d\rho_{\mathrm{sign}\, j})$$

exists. For any $g \in C_0^\infty(\boldsymbol{R}^n, \boldsymbol{C}^m)$ and $\hat{f}_j(\lambda, k) \in L^2(\boldsymbol{R}_{\mathrm{sign}\, j}, l^2(\boldsymbol{C}^m)) \cap L^1(\boldsymbol{R}_{\mathrm{sign}\, j}, l^2(\boldsymbol{C}^m))$ such that $\hat{f}_j(\lambda, k) = 0$ for any $(\lambda, k) \in \boldsymbol{R}_{\mathrm{sign}\, j} \times \boldsymbol{N}$, $k > K > 0$, we get

$$\begin{aligned} &\int_{R^n} \left(g(x), \int_{\boldsymbol{R}_{\mathrm{sign}\, j}} \sum_{k=1}^{\infty} h_j(x, \lambda, k) \hat{f}_j(\lambda, k) d\rho_{\mathrm{sign}\, j}(\lambda) \right)_{C^m} dx \\ &\quad = \int_{R^n} \int_{\boldsymbol{R}_{\mathrm{sign}\, j}} \Big(\sum_{k=1}^{K} h_j(x, \lambda, k)^* g(x), \hat{f}_j(\lambda, k) \Big)_{C^m} d\rho_{\mathrm{sign}\, j}(\lambda) dx \end{aligned} \tag{2.12}$$

$$=\int_{\boldsymbol{R}_{\mathrm{sign}\,j}}\sum_{k=1}^{K}((T_jg)(\lambda, k), \hat{f}_j(\lambda, k))_{C^m}d\rho_{\mathrm{sign}\,j}(\lambda)$$
$$=(T_jg, \hat{f}_j)\ .$$

Hence by the first part of statement (1) of the theorem we have

$$\int_{\boldsymbol{R}_{\mathrm{sign}\,j}}\sum_{k=1}^{\infty}h_j(x, \lambda, k)\hat{f}_j(\lambda, k)d\rho_{\mathrm{sign}\,j}(\lambda)\in H_0$$

and

$$\left\|\int_{\boldsymbol{R}_{\mathrm{sign}\,j}}\sum_{k=1}^{\infty}h_j(x, \lambda, k)\hat{f}_j(\lambda, k)d\rho_{\mathrm{sign}\,j}(\lambda)\right\|_{H^0}^2\leqq\|\hat{f}_j\|_{L^2(\boldsymbol{R}_{\mathrm{sign}\,j}, l^2(C^m))}\ .$$

On the other hand a simple consideration shows that such functions $\hat{f}_j$ as considered above form a dense subset of $L^2(\boldsymbol{R}_{\mathrm{sign}\,j}, l^2(\boldsymbol{C}^m))\cong\bigoplus_{k=1}^{\infty}L^2(\boldsymbol{R}_{\mathrm{sign}\,j})$. Therefore for any $\hat{f}_j(\lambda, k)\in L^2(\boldsymbol{R}_{\mathrm{sign}\,j}, l^2(\boldsymbol{C}^m), d\rho_{\mathrm{sign}\,j})$

$$T_j'f=\underset{K\to\infty}{\mathrm{l.i.m.}}\int_{\boldsymbol{R}_{\mathrm{sign}\,j}\cap\{|\lambda|<K\}}\sum_{k=1}^{K}h_j(x, \lambda, k)\hat{f}_j(\lambda, k)d\rho_{\mathrm{sign}\,j}(\lambda)\in H_0$$

exists and (2.12) shows that

$$(g, T_j'f_j)_{H^0}=(T_jg, f_j)_{L^2(\boldsymbol{R}_{\mathrm{sign}\,j}, l^2(C^m))}\ , \tag{2.13}$$

for all $g\in C_0^\infty(\boldsymbol{R}^n, \boldsymbol{C}^m)$ and hence for all $g\in H^0$. Equation (2.13) implies the second part of statement (1). (Q.E.D.)

REMARK. Let $T_j(\lambda)=U_j\Gamma_j(\lambda)$, where $\Gamma_j(\lambda)$ is defined by (2.7.I). Then for any $\varepsilon>0$ we have $T_j(\lambda): H_{0,(1+\varepsilon)/2}\to l^2(\boldsymbol{C}^m)$ and it is obvious that for $f\in H_{0,(1+\varepsilon)/2}$

$$T_j(\lambda)f(k)=\int_{R^n}h_j(x, \lambda, k)^*f(x)dx\ ,$$

where the integration is absolutely convergent. Therefore by (2.4.I) the following equation holds for any $f, g\in H_{0,(1+\varepsilon)/2}$ and for any compact subset $\Delta\subset I_0=\boldsymbol{R}^1\backslash\{0\}$:

$$\begin{aligned}(E_0(\Delta)f, g)_{H_0}&=\sum_{j\neq0}(\Gamma_j^*(\chi_\Delta(\lambda)\Gamma_jf), g)_{H_0}\\&=\sum_{j\neq0}\int_{\Delta\cap\boldsymbol{R}_{\mathrm{sign}\,j}}\langle\Gamma_j^*(\lambda)\Gamma_j(\lambda)f, g\rangle_{H_{0,-(1+\varepsilon)/2}, H_{0,(1+\varepsilon)/2}}d\rho_{\mathrm{sign}\,j}\\&=\sum_{j\neq0}\int_{\Delta\cap\boldsymbol{R}_{\mathrm{sign}\,j}}\langle T_j^*(\lambda)T_j(\lambda)f, g\rangle_{H_{0,-(1+\varepsilon)/2}, H_{0,(1+\varepsilon)/2}}\frac{d\rho_{\mathrm{sign}\,j}}{d\lambda}d\lambda\ .\end{aligned}$$

On the other hand by Stone's theorem we have

$$\begin{aligned}(E_0(\Delta)f, g)_{H_0}&=\lim_{\eta\downarrow0}\frac{1}{2\pi i}\int_\Delta((R_{L_0}(\lambda+i\eta)-R_{L_0}(\lambda-i\eta))f, g)_{H_0}d\lambda\\&=\lim_{\eta\downarrow0}\frac{1}{2\pi i}\int_\Delta\langle(R_{L_0}(\lambda+i\eta)-R_{L_0}(\lambda-i\eta))f, g\rangle_{H_{0,(1+\varepsilon)/2}, H_{0,-(1+\varepsilon)/2}}d\lambda\ .\end{aligned}$$

Therefore as a $B(H_{0,(1+\varepsilon)/2}, H_{0,-(1+\varepsilon)/2})$-valued function the following equation holds for $\lambda \in I_0$:

$$(2.14) \qquad (2\pi i)^{-1}(R_{L_0}(\lambda+i0)-R_{L_0}(\lambda-i0))=\sum_{\operatorname{sign} j=\operatorname{sign}\lambda} T_j(\lambda)^* T_j(\lambda) \frac{d\rho_{\operatorname{sign} j}}{d\lambda}.$$

This fact will be used in the proof of Theorem 3.1.

§3. Eigenfunction expansions associated with L.

In this section we shall give eigenfunction expansions for L, using the perturbation method developed by Kato-Kuroda [6], Kuroda [7]. At first we record some immediate consequences of results derived in [12] which are necessary in the sequel.

PROPOSITION 3.1. *Put* $G_0(\zeta)=(L-\zeta)J^{-1}R_{L_0}(\zeta)$ *and* $G(\zeta)=(L_0-\zeta)JR_L(\zeta)$ *for* $\zeta \in C^1 \backslash R^1$. *Then the following statements hold*:

(1) *For* $\lambda \in I_0$, $G_0(\lambda\pm i0)=\lim_{\eta\downarrow 0} G_0(\lambda\pm i\eta) \in B(H_{0,\delta/2}, H_{1,\delta/2})$ *exists, and is locally Hölder continuous in* I_0, *where* lim *stands for the convergence in* $B(H_{0,\delta/2}, H_{1,\delta/2})$;

(2) *For* $\lambda \in I_1=R^1\backslash(\sigma_P(L)\cup\{0\})$, $G(\lambda\pm i0)=\lim_{\eta\downarrow 0} G(\lambda\pm i\eta)\in B(H_{1,\delta/2}, H_{0,\delta/2})$ *exists and is locally Hölder continuous in* I_1, *where* lim *stands for the convergence in* $B(H_{1,\delta/2}, H_{0,\delta/2})$;

(3) *In* $B(H_{1,\delta/2}, H_{0,\delta/2})$, $G(\lambda\pm i0)=G_0(\lambda\pm i0)^{-1}$ *for* $\lambda\in I_1$;

(4) $G_0(\lambda\pm i0)^*=J+\lambda R_{L_0}(\lambda\mp i0)(J-J^{*-1})\in B(H_{1,-\delta/2}, H_{0,-\delta/2})$ *for* $\lambda \in I_0$;

(5) $G(\lambda\pm i0)^*=J^{-1}+\lambda R_L(\lambda\mp i0)(J^{-1}-J^*)\in B(H_{0,-\delta/2}, H_{1,-\delta/2})$ *for* $\lambda\in I_1$.

PROOF. Statements (1), (2), (3) are proved in the proof of Theorem 1.3.I in [12]. (4) and (5) are obvious. (Q.E.D.)

Next two theorems are the main theorems in this paper.

THEOREM 3.1. *Let* $\alpha_j^{\pm}(x, \lambda, k)$ $(j\neq 0)$ *be an* $m\times m$-*matrix depending on* $x\in R^n$, $\lambda \in R_{\operatorname{sign} j}\backslash(\sigma_p(L)\cup\{0\})$ *and* $k\in N$ *defined by*

$$(3.1) \qquad \alpha_j^{\pm}(x, \lambda, k)=G(\lambda\pm i0)^* h_j(\cdot, \lambda, k),$$

where $G(\lambda\pm i0)^*$ *is applied to* $h_j(\cdot, \lambda, k)$ *by matrix multiplication rule. Then the following statements hold*:

(1) *Each column* $\alpha_j^{\pm}(x, \lambda, k)^{(i)}$ $(i=1, 2, \cdots, m)$ *of* $\alpha_j^{\pm}(x, \lambda, k)$ *is an* $H_{1,-\delta/2}$-*valued locally Hölder continuous function of* $\lambda\in R_{\operatorname{sign} j}\backslash(\sigma_p(L)\cup\{0\})$ *for each fixed* $k\in N$;

(2) $\alpha_j^{\pm}(x, \lambda, k)$ *can be decomposed into three parts as*

$$(3.2) \qquad \alpha_j^{\pm}(x, \lambda, k)=J^{-1}h_j(x, \lambda, k)+t_j^{\pm}(x, \lambda, k)+w_j^{\pm}(x, \lambda, k),$$

where $h_j(x, \lambda, k)$ *is the matrix defined by (2.3),* $t_j^{\mp}(x, \lambda, k)$ *is an* $H_{1,-(1+\varepsilon)/2}$*-valued continuous function of* $\lambda \in \boldsymbol{R}_{\mathrm{sign}\, j} \backslash \sigma_p(L)$ *for each fixed* $k \in N$ *and any* $\varepsilon > 0$, *and* $w_j^{\mp}(x, \lambda, k)$ *is an* $H_{1,(\delta-2)/2}$*-valued continuous function of* $\lambda \in \boldsymbol{R}_{\mathrm{sign}\, j} \backslash \sigma_p(L)$ *for each fixed* $k \in N$;

(3) $L(D)\alpha_j^{\pm}(x, \lambda, k) = \lambda \alpha_j^{\pm}(x, \lambda, k)$, *where* $L(D)$ *is applied to* $\alpha_j^{\pm}(x, \lambda, k)$ *by matrix multiplication rule and the differentiation is in the sense of distributions.*

REMARK 1. By statement (3), $\alpha_j^{\pm}(x, \lambda, k)$, as a function of x, can be considered as a (matrix-valued) eigenfunction of L corresponding to the eigenvalue λ.

REMARK 2. In the decomposition (3.2) of $\alpha_j^{\pm}(x, \lambda, k)$, $J^{-1}h_j(x, \lambda, k)$, $t_j^{\mp}(x, \lambda, k)$ and $w_j^{\mp}(x, \lambda, k)$ are considered to describe the state of incident wave, outgoing (incoming) spherical wave, and the wave damping rapidly, respectively, in the description of the stationary scattering process. Furthermore $t_j^{\mp}(x, \lambda, k)$ is the quantities which are connected with the scattering amplitudes (see (3.3) and §4, Theorem 4.2).

PROOF. By (2.3), $h_j(x, \lambda, k)^{(i)}$ is an $H_{0,-\delta/2}$-valued locally Hölder continuous function of $\lambda \in \boldsymbol{R}_{\mathrm{sign}\, j}$ for any fixed $k \in N$. Therefore statement (2) of Proposition 3.1 shows that

$$\alpha_j^{\pm}(x, \lambda, k)^{(i)} = G(\lambda \pm i0)^* h_j(x, \lambda, k)^{(i)}$$

is an $H_{1,-\delta/2}$-valued locally Hölder continuous function of $\lambda \in \boldsymbol{R}_{\mathrm{sign}\, j} \backslash \sigma_p(L)$ for each fixed $k \in N$. This proves (1). By statement (5) of Proposition 3.1,

$$\alpha_j^{\pm}(x, \lambda, k)^{(i)} = J^{-1}h_j(x, \lambda, k)^{(i)} + \lambda R_L(\lambda \mp i0)(J^{-1} - J^*)h_j(x, \lambda, k)^{(i)} .$$

Hence by Theorem 1.3.I and Proposition 2.2,

$$\begin{aligned}(L-\lambda)&\alpha_j^{\pm}(x, \lambda, k)^{(i)} \\ &= (J^*L_0J - \lambda)J^{-1}h_j(x, \lambda, k)^{(i)} + \lambda(J^{-1} - J^*)h_j(x, \lambda, k)^{(i)} \\ &= J^*L_0h_j(x, \lambda, k)^{(i)} - \lambda J^*h_j(x, \lambda, k)^{(i)} = 0 .\end{aligned}$$

This proves statement (3).

Next we prove statement (2). By statement (3) and (4) of Proposition 3.1,

$$\begin{aligned}h_j(x, \lambda, k) &= G_0(\lambda \pm i0)^* \alpha_j^{\pm}(x, \lambda, k) \\ &= J\alpha_j^{\pm}(x, \lambda, k) + \lambda R_{L_0}(\lambda \mp i0)(J - J^{*-1})\alpha_j^{\pm}(x, \lambda, k) .\end{aligned}$$

Hence

$$\alpha_j^{\pm}(x, \lambda, k) = J^{-1}h_j(x, \lambda, k) - \lambda J^{-1}R_{L_0}(\lambda \mp i0)(J - J^{*-1})\alpha_j^{\pm}(x, \lambda, k) .$$

By (A.2) and statement (1), $(J - J^{*-1})\alpha_j^{\pm}(x, \lambda, k) = (I - M(x))\alpha_j^{\pm}(x, \lambda, k) \in H_{1,\delta/2}$. Let $t_j^{\mp}(x, \lambda, k)$ be defined by

(3.3) $t_j^{\mp}(x, \lambda, k)$

$$=-\lambda J^{-1}\sum_{\mathrm{sign}\,k=\mathrm{sign}\,j}\mathscr{F}^{-1}\left[\lim_{\eta\downarrow 0}\frac{[\hat{P}_j(\xi)\mathscr{F}((1-M(y))\alpha_j^{\mp}(y, \lambda, k))(\xi)]_{\lambda_k(\xi)=\lambda}}{\lambda_k(\xi)-(\lambda\mp i\eta)}\right]$$

and let $w_j^{\mp}(x, \lambda, k)=-\lambda J^{-1}R_{L_0}(\lambda\mp i0)(J-J^{*-1})\alpha_j^{\mp}(x, \lambda, k)-t_j^{\mp}(x, \lambda, k)$, where subscript $\lambda_k(\xi)=\lambda$ denotes to take the trace on the hypersurface $\lambda_k(\xi)=\lambda$. Then using Lemma 1.1 and Lemma 1.2, we get after a simple consideration statement (2) of the theorem. (Q.E.D.)

Our expansion formulas can be stated as follows. Here we shall use the notations $\sigma_p^K(L)=\{\lambda\in R^1: |\lambda-\eta|<K^{-1}$ for some $\eta\in\sigma_p(L)\}$ and $I_{j,K}=R_{\mathrm{sign}\,j}\setminus(\{|\lambda|>K\}\cup\{|\lambda|<K^{-1}\}\cup\sigma_p^K(L))$ for $K>0$ and $j\neq 0$.

THEOREM 3.2. *Let $\alpha_j^{\pm}(x, \lambda, k)$ $(j\neq 0)$ be the matrix-valued function defined in Theorem 3.1. Then for any $f\in H$ and $K>0$, $\int_{|x|<K}\alpha_j^{\pm}(x, \lambda, k)^*M(x)f(x)dx$ belongs to $L^2(\boldsymbol{R}_{\mathrm{sign}\,j}, l^2(\boldsymbol{C}^m), d\rho_{\mathrm{sign}\,j})$ and*

$$(Z_j^{\pm}f)(\lambda, k)=\underset{K\to\infty}{\mathrm{l.i.m.}}\int_{|x|<K}\alpha_j^{\pm}(x, \lambda, k)^*M(x)f(x)dx$$

exists, where l.i.m. *stands for the convergence in $L^2(\boldsymbol{R}_{\mathrm{sign}\,j}, l^2(\boldsymbol{C}^m), d\rho_{\mathrm{sign}\,j})$. For any $\hat{f}_j(\lambda, k)\in L^2(\boldsymbol{R}_{\mathrm{sign}\,j}, l^2(\boldsymbol{C}^m), d\rho_{\mathrm{sign}\,j})$ and $K>0$, $\int_{I_{j,K}}\sum_{k=1}^{K}\alpha_j^{\pm}(x, \lambda, k)\hat{f}_j(\lambda, k)d\rho_{\mathrm{sign}\,j}(\lambda)$ belongs to H_1 and*

$$(Z_j'^{\pm}\hat{f}_j)(x)=\underset{K\to\infty}{\mathrm{l.i.m.}}\int_{I_{j,K}}\sum_{k=1}^{K}\alpha_j^{\pm}(x, \lambda, k)\hat{f}_j(\lambda, k)d\rho_{\mathrm{sign}\,j}(\lambda)$$

exists, where l.i.m. *stands for the convergence in H_1 and the integration is Bochner integral. Let*

$$Z^{\pm}: H_1\to\sum_{j\neq 0}\oplus L^2(\boldsymbol{R}_{\mathrm{sign}\,j}, l^2(\boldsymbol{C}^m), d\rho_{\mathrm{sign}\,j})$$

and

$$Z'^{\pm}: \sum_{j\neq 0}\oplus L^2(\boldsymbol{R}_{\mathrm{sign}\,j}, l^2(\boldsymbol{C}^m), d\rho_{\mathrm{sign}\,j})\to H_1$$

be operators defined by

$$Z^{\pm}f=\sum_{j\neq 0}\oplus Z_j^{\pm}f, \quad f\in H_1$$

and

$$Z'^{\pm}(\hat{f}_{-\mu}, \cdots, \hat{f}_{-1}, \hat{f}_1, \cdots, \hat{f}_\mu)$$
$$=\sum_{j\neq 0}\underset{K\to\infty}{\mathrm{l.i.m.}}\int_{I_{j,K}}\sum_{k=1}^{K}\alpha_j^{\pm}(x, \lambda, k)\hat{f}_j(\lambda, k)d\rho_{\mathrm{sign}\,j}$$

$(\hat{f}_{-\mu}, \cdots, \hat{f}_{-1}, \hat{f}_1, \cdots, \hat{f}_\mu) \in \sum_{j\neq 0} \oplus L^2(\boldsymbol{R}_{\mathrm{sign}\, j}, l^2(\boldsymbol{C}^m), d\rho_{\mathrm{sign}\, j})$, *respectively. Then the following statements hold*:

(1) *Let* P_{ac} *be the projection operator in* H_1 *onto the absolutely continuous subspace with respect to* L. *Then* $Z^\pm$ *is a partially isometric operator from* H_1 *into* $\sum_{j\neq 0} \oplus L^2(\boldsymbol{R}_{\mathrm{sign}\, j}, l^2(\boldsymbol{C}^m), d\rho_{\mathrm{sign}\, j})$ *with the initial set* $P_{ac}H_1$ *and* $Z'^\pm$ *is its adjoint*;

(2) (*Expansion formulas*) *For any* $f\in H_1$

$$\begin{aligned} P_{ac}f &= Z'^\pm Z^\pm f \\ &= \sum_{j\neq 0} \underset{K\to\infty}{\mathrm{l.i.m.}} \Big[\int_{I_{j,K}} \sum_{k=1}^{K} \alpha_j^\pm(x, \lambda, k) \\ &\quad \times \Big\{ \underset{K'\to\infty}{\mathrm{l.i.m.}} \int_{|x|<K'} \alpha_j^\pm(y, \lambda, k)^* M(y) f(y)dy \Big\} d\rho_{\mathrm{sign}\, j}(\lambda) \Big]. \end{aligned}$$

(3) (*Diagonal representation*) *For* $f\in H_1, P_{ac}f\in D(L)$ *if and only if* $\lambda Z_j^\pm f(\lambda, \cdot)\in L^2(\boldsymbol{R}_{\mathrm{sign}\, j}, l^2(\boldsymbol{C}^m), d\rho_{\mathrm{sign}\, j})$ *for each* $j\neq 0$. *Furthermore for such* f

$$(Z_j^\pm Lf)(\lambda, k) = \lambda(Z_j^\pm f)(\lambda, k)\ , \quad \lambda\in I_1\ .$$

PROOF. At first we proceed rather abstractly. Let $E(d\lambda)$ be the spectral measure associated with L. Then by Stone's theorem we get for any compact set $\Delta\subset \boldsymbol{R}^1\backslash(\sigma_p(L)\cup\{0\})\equiv I_1$ and f and $g\in H_1$,

$$\begin{aligned} (3.4)\qquad (E(\Delta)f, g)_{H_1} &= \frac{1}{2\pi i}\lim_{\eta\downarrow 0}\int_\Delta ((R_L(\lambda+i\eta)-R_L(\lambda-i\eta))f, g)_{H_1}d\lambda \\ &= \lim_{\eta\downarrow 0}\frac{\eta}{\pi}\int_\Delta (R_L(\lambda\pm i\eta)f, R_L(\lambda\pm i\eta)g)_{H_1}d\lambda \\ &= \lim_{\eta\downarrow 0}\frac{\eta}{\pi}\int_\Delta ((1-J^*J)R_L(\lambda\pm i\eta)f, R_L(\lambda\pm i\eta)g)_{H_1}d\lambda \\ &\quad + \lim_{\eta\downarrow 0}\frac{\eta}{\pi}\int_\Delta (JR_L(\lambda\pm i\eta)f, JR_L(\lambda\pm i\eta)g)_{H_0}d\lambda\ . \end{aligned}$$

If we choose f and $g\in H_{1,\delta/2}$, then by Theorem 1.3.I and (A.1) $((1-J^*J)R_L(\lambda\pm i\eta)f, R_L(\lambda\pm i\eta)g)_{H_1} = \langle(1-M(x)^{-1})R_L(\lambda\pm i\eta)f, R_L(\lambda\pm i\eta)g\rangle_{H_{1,\delta/2}, H_{1,-\delta/2}}$ converges to $\langle(1-M(x))\times R_L(\lambda\pm i0)f, R_L(\lambda\pm i0)g\rangle_{H_{1,\delta/2}, H_{1,-\delta/2}}$ uniformly on Δ as $\eta\downarrow 0$ and $\langle(1-M(x))R_L(\lambda\pm i0)f, R_L(\lambda\pm i0)g\rangle_{H_{1,\delta/2}, H_{1,-\delta/2}}$ is a continuous function of $\lambda\in\Delta$. Therefore, for such f and g the first term of the last member of (3.4) vanishes and using (2) of Proposition 3.1 we get

$$\begin{aligned} (E(\Delta)f, g)_{H_1} &= \lim_{\eta\downarrow 0}\frac{\eta}{\pi}\int_\Delta (JR_L(\lambda\pm i\eta)f, JR_L(\lambda\pm i\eta)g)_{H_0}d\lambda \\ &= \lim_{\eta\downarrow 0}\frac{\eta}{\pi}\int_\Delta (R_{L_0}(\lambda\pm i\eta)G(\lambda\pm i\eta)f, R_{L_0}(\lambda\pm i\eta)G(\lambda\pm i\eta)g)_{H_0}d\lambda \end{aligned}$$

$$=\frac{1}{2\pi i}\int_{\Delta}\langle (R_{L_0}(\lambda+i0)-R_{L_0}(\lambda-i0))G(\lambda\pm i0)f\ ,\\ G(\lambda\pm i0)g)\rangle_{H_{0,-\delta/2},H_{0,\delta/2}}d\lambda\ .$$

Hence using (2.14), we get

$$(E(\Delta)f,g)_{H_1}=\sum_{j\neq 0}\int_{\Delta\cap R_{\mathrm{sign}\,j}}(T_j(\lambda)G(\lambda\pm i0)f,\ T_j(\lambda)G(\lambda\pm i0)g)_{l^2(C^m)}d\rho_{\mathrm{sign}\,j}(\lambda)\ .$$

Put $\tilde{Z}_j^{\pm}(\lambda)=T_j(\lambda)G(\lambda\pm i0)$ $(\lambda\in I_1)$, then $\tilde{Z}_j^{\pm}(\lambda)$ is a $B(H_{1,\delta/2}, l^2(C^m))$-valued locally Hölder continuous function of $\lambda\in I_1$ and for any compact set $\Delta\subset I_1$ and f and $g\in H_{1,\delta/2}$,

$$(3.5)\qquad (E(\Delta)f,g)_{H_1}=\sum_{j\neq 0}\int_{\Delta\cap R_{\mathrm{sign}\,j}}\langle\tilde{Z}_j^{\pm}(\lambda)^*\tilde{Z}_j^{\pm}(\lambda)f,g\rangle_{H_{1,-\delta/2},H_{1,\delta/2}}d\rho_{\mathrm{sign}\,j}(\lambda)\ .$$

Hence for $f\in H_{1,\delta/2}$ and any compact set $\Delta\subset I_1$

$$(3.6)\qquad E(\Delta)f=\sum_{j\neq 0}\int_{\Delta\cap R_{\mathrm{sign}\,j}}\tilde{Z}_j^{\pm}(\lambda)^*\tilde{Z}_j^{\pm}(\lambda)f\,d\rho_{\mathrm{sign}\,j}(\lambda)\ ,$$

where the integration is Bochener integral in $H_{1,-\delta/2}$. Put $(\tilde{Z}^{\pm}f)(\cdot,k)=\sum_{\mathrm{sign}\,j=\mathrm{sign}\,\lambda}\oplus(\tilde{Z}_j^{\pm}(\lambda)f)(k)$ for $f\in H_{\delta/2}$, $\lambda\in I_1$. Then using (3.5) and (3.6) we can prove that $\tilde{Z}^{\pm}$ can be extended to $H_1\to\sum_{j\neq 0}\oplus L^2(\boldsymbol{R}_{\mathrm{sign}\,j}, l^2(C^m), d\rho_{\mathrm{sign}\,j})$ by continuity and is a partially isometric operator from H_1 into $\sum_{j\neq 0}\oplus L^2(\boldsymbol{R}_{\mathrm{sign}\,j}, l^2(C^m), d\rho_{\mathrm{sign}\,j})$. Furthermore (3.6) shows that

$$(3.7)\qquad \tilde{Z}^{\pm *}\tilde{Z}^{\pm}f=P_{ac}f\ .$$

Therefore if we can prove that

$$(\tilde{Z}^{\pm}(\lambda)f)(k)=\int_{R^n}\alpha^{\pm}(x,\lambda,k)^*M(x)f(x)dx$$

for $f\in H_{1,\delta/2}\cap L^1(\boldsymbol{R}^n, \boldsymbol{C}^m)$, statements (1), (2) and the existence of $Z_j^{\pm}f$ and $Z_j'^{\pm}\hat{f}_j$ can be proved by arguments similar to those used in the proof of Theorem 2.3. However, for $f\in H_{1,\delta/2}\cap L^1(\boldsymbol{R}^n, \boldsymbol{C}^m)$ we obtain

$$\begin{aligned}(\tilde{Z}_j^{\pm}(\lambda)f)(k)&=(T_j(\lambda)G(\lambda\pm i0)f)(k)\\ &=\langle G(\lambda\pm i0)f, h_j(y,\lambda,k)\rangle_{H_{0,\delta/2},H_{0,-\delta/2}}\\ &=\langle f, G(\lambda\pm i0)^*h_j(y,\lambda,k)\rangle_{H_{1,\delta/2},H_{1,-\delta/2}}\\ &=\langle f,\alpha_j^{\pm}(x,\lambda,k)\rangle_{H_{1,\delta/2},H_{1,-\delta/2}}\\ &=\int_{R^n}\alpha_j^{\pm}(x,\lambda,k)^*M(x)f(x)dx\ .\end{aligned}$$

Finally we shall prove (3). Let $P_{ac}f\in D(L)$, then it is obvious that there exists a sequence $f_\nu \in \mathscr{S}(\boldsymbol{R}^n, \boldsymbol{C}^m)$ $(\nu=1,2,3\cdots)$ such that $f_\nu \to f$ in H_1 and $Lf_\nu \to f$ in H_1. For $f_\nu \in \mathscr{S}$ it can be easily proved that $G(\lambda\pm i0)Lf_\nu = L_0 G(\lambda\pm i0)f_\nu$ where L_0 is applied in the sense of $\mathscr{D}'$. Therefore it is obvious that $\tilde{Z}_j^\pm(\lambda)Lf_\nu = \lambda\tilde{Z}_j^\pm(\lambda)f_\nu$, since L_0 maps $H_{0,\delta/2}$ into $H_{0,\delta/2}$ and $T_j(\lambda)L_0 g = \lambda T_j(\lambda)g$ for any $g\in H_{0,\delta/2}$. Hence the above arguments show that $\lambda(\tilde{Z}_j^\pm f)(\lambda, k)\in L^2(R^n_{\mathrm{sign}\, j}, l^2(\boldsymbol{C}^m), d\rho_{\mathrm{sign}\, j})$ for any $j\neq 0$ and $(\tilde{Z}_j^\pm Lf)(\lambda, k)=\lambda(Z_j^\pm f)(\lambda, k)$ for any $j\neq 0$. Conversely let $\lambda(Z_j^\pm f)(\lambda, k)\in L^2(\boldsymbol{R}_{\mathrm{sign}\, j}, l^2(\boldsymbol{C}^m), d\rho_{\mathrm{sign}\, j})$ for any $j\neq 0$. Put for any $K\in \boldsymbol{N}$,

$$f_K^\pm(x)=\sum_{j\neq 0}\int_{I_{j,K}}\sum_{k=1}^{K}\alpha_j^\pm(x,\lambda,k)(\tilde{Z}_j^\pm f)(\lambda,k)d\rho_{\mathrm{sign}\, j}(\lambda)\ .$$

Then it is obvious that

$$f_K^\pm(x)\to P_{\mathrm{ac}}f(x)\quad \text{in}\quad H_1\ .$$

Furthermore

$$Lf_K^\pm(x)=\sum_{j\neq 0}\int_{I_{j,K}}\sum_{k=1}^{K}\lambda\alpha_j^\pm(x,\lambda,k)(\tilde{Z}_j^\pm f)(\lambda,k)d\rho_{\mathrm{sign}\, j}(\lambda)\in H_1$$

converges to $\underset{K\to\infty}{\mathrm{l.i.m.}}\sum_{j\neq 0}\int_{I_{j,K}}\sum_{k=1}^{K}\lambda\alpha_j^\pm(x,\lambda,k)(\tilde{Z}_j^\pm f)(\lambda,k)d\rho_{\mathrm{sign}\, j}(\lambda)$ in H_1. Hence $f\in D(L)$.

(Q.E.D.)

THEOREM 3.3. (*Orthogonality of eigenfunctions.*) *The range $R(Z^\pm)$ of $Z^\pm$ is equal to the range $R(T)$ of T, where T is the operator defined in Theorem 2.3.*

PROOF. For the proof it is sufficient to show $Tf=Z^\pm Z^{\pm *}Tf$ for any $f\in H_0$ and $Z^\pm f = TT^*Z^\pm f$ for any $f\in H_1$. We shall only prove the first equation. The second one can be proved symmetrically. Put

$$\hat{g}=Tf-Z^\pm Z^{\pm *}Tf\ .$$

Since $Z^{\pm *}Z^\pm Z^{\pm *}=Z^{\pm *}$, we have $Z^{\pm *}\hat{g}=Z^{\pm *}Tf-Z^{\pm *}Tf=0$. Put $\hat{g}=\{\hat{g}_{-\mu},\cdots,\hat{g}_{-1},\hat{g}_1,\cdots,\hat{g}_\mu\}$. Then for any compact interval $\Delta\subset I_1$ and for any $p(x)\in H_{1,\delta/2}$

$$\begin{aligned}
0&=\langle E(\Delta)Z^{\pm *}\hat{g}, p\rangle=\lim_{\eta\downarrow 0}\frac{1}{2\pi i}\int_\Delta \langle (R_L(\lambda+i\eta)-R_L(\lambda-i\eta))Z^{\pm *}\hat{g}, p\rangle d\lambda\\
&=\lim_{\eta\downarrow 0}\frac{1}{2\pi i}\int_\Delta\Big\langle (R_L(\lambda+i\eta)-R_L(\lambda-i\eta))\\
&\qquad\times\Big\{\underset{K\to\infty}{\mathrm{l.i.m.}}\sum_{j\neq 0}\int_{I_{j,K}}\sum_{k=1}^{K}\alpha_j^\pm(x,\sigma,k)\hat{g}_j(\sigma,k)d\rho_{\mathrm{sign}\, j}(\sigma)\Big\}, p\Big\rangle d\lambda\\
&=\lim_{\eta\downarrow 0}\frac{1}{2\pi i}\int_\Delta d\lambda\Big\langle \underset{K\to\infty}{\mathrm{l.i.m.}}\sum_{j\neq 0}\int_{I_{j,K}}\sum_{k=1}^{K}\frac{2i\eta}{(\sigma-\lambda)^2+\eta^2}\alpha_j^\pm(x,\sigma,k)\hat{g}_j(\sigma,k)\,d\rho_{\mathrm{sign}\, j}(\sigma), p\Big\rangle,
\end{aligned}$$

where $\langle\ ,\ \rangle$ denotes the natural coupling between $H_{1,-\delta/2}$ and $H_{1,\delta/2}$. Here

$$\left\langle \underset{K\to\infty}{\mathrm{l.i.m.}} \sum_{j\neq 0} \int_{I_{j,K}} \sum_{k=1}^{K} \frac{2i\eta}{(\sigma-\lambda)^2+\eta^2} \alpha_j^{\mp}(x, \sigma, k)\hat{g}_j(\sigma, k) d\rho_{\mathrm{sign}\, j}(\sigma),\ p \right\rangle$$
$$= \lim_{K\to\infty} \sum_{j\neq 0} \int_{I_{j,K}} \sum_{k=1}^{K} \frac{2i\eta}{(\sigma-\lambda)^2+\eta^2} \langle \alpha_j^{\mp}(x, \sigma, k)\hat{g}_j(\sigma, k),\ p\rangle d\rho_{\mathrm{sign}\, j}(\sigma)$$
$$= \lim_{K\to\infty} \sum_{j\neq 0} \int_{I_{j,K}} \sum_{k=1}^{K} \frac{2i\eta}{(\sigma-\lambda)^2+\eta^2} \left(\hat{g}_j(\sigma, k),\ \int_{R^n} \alpha_j^{\mp}(x, \sigma, k)^* M(x) p(x) dx\right)_{C^m} d\rho_{\mathrm{sign}\, j}(\sigma)\,.$$

By the definition of $\tilde{Z}_j^{\mp}(\lambda)$,

$$\int_{R^n} \alpha_j^{\mp}(x, \sigma, k)^* M(x) p(x) dx = (\tilde{Z}_j^{\mp}(\sigma) p)(k) \in L^2(\boldsymbol{R}_{\mathrm{sign}\, j},\ l^2(\boldsymbol{C}^m),\ d\rho_{\mathrm{sgin}\, j}(\sigma))\,.$$

Therefore the Schwarz inequality shows

$$\sum_{k=1}^{\infty} \left(\hat{g}_j(\sigma, k),\ \int \alpha_j^{\mp}(x, \sigma, k)^* M(x) p(x) dx\right)_{C^m} \in L^1(\boldsymbol{R}_{\mathrm{sign}\, j},\ \boldsymbol{C}^1,\ d\rho_{\mathrm{sign}\, j})\,.$$

Therefore by Fubini's theorem and Lebesgue's dominated convergence theorem we get

$$0 = \langle E(\Delta) Z^{\pm *}\hat{g},\ p\rangle = \lim_{\eta\downarrow 0} \frac{1}{2\pi i} \int_{\Delta} d\lambda \left\{ \sum_{j\neq 0} \int_{R_{\mathrm{sign}\, j}} \frac{2i\eta}{(\sigma-\lambda)^2+\eta^2} \left(\hat{g}_j(\sigma, k),\right.\right.$$
$$\left.\left. \int \alpha_j^{\mp}(x, \sigma, k)^* M(x) p(x) dx\right)_{l^2(C^m)} d\rho_{\mathrm{sign}\, j}(\sigma) \right\}$$
$$= \lim_{\eta\downarrow 0} \sum_{j\neq 0} \int_{R_{\mathrm{sign}\, j}} \left\{ \int_{\Delta} \frac{\eta\, d\lambda}{\pi\{(\sigma-\lambda)^2+\eta^2\}} \right\} \left(\hat{g}_j(\sigma, k),\ \int \alpha_j^{\mp}(x, \sigma, k)^* M(x) p(x) dx\right)_{l^2(C^m)}$$
$$\times d\rho_{\mathrm{sign}\, j}(\sigma)$$
$$= \sum_{j\neq 0} \int_{R_{\mathrm{sign}\, j}\cap\Delta} \left(\hat{g}_j(\lambda, k),\ \int \alpha_j^{\mp}(x, \lambda, k)^* M(x) p(x) dx\right) d\rho_{\mathrm{sign}\, j}(\lambda)\,.$$

Since Δ can be chosen arbitrary,

$$0 = \sum_{j\neq 0} \left(\hat{g}_j(\lambda, k),\ \int \alpha_j^{\mp}(x, \lambda, k)^* M(x) p(x) dx\right)_{l^2(C^m)} = \sum_{j\neq 0} (\hat{g}_j(\lambda),\ \tilde{Z}_j^{\mp}(\lambda) p)_{l^2(C^m)}$$
$$= \sum_{j\neq 0} \langle \tilde{Z}_j^{\mp}(\lambda)^* \hat{g}_j(\lambda),\ p\rangle$$

for almost every $\lambda \in I_1$. Since $H_{1,\delta/2}$ is separable, we can conclude after a simple consideration that

$$\sum_{j\neq 0} \tilde{Z}_j^{\mp}(\lambda)^* \hat{g}_j(\lambda) = 0 \quad \text{for a.e.} \quad \lambda \in \boldsymbol{R}_{\mathrm{sign}\, j}\,.$$

Thus we get for almost every $\lambda \in \boldsymbol{R}^1$,

$$0 = \sum_{j\neq 0} \tilde{Z}_j^{\mp}(\lambda)^* \hat{g}_j(\lambda)$$
$$= \sum_{j\neq 0} [J^{-1} T_j(\lambda)^* - J^{-1}\lambda R_{L_0}(\lambda \mp i0)(J - J^{-1*}) \tilde{Z}_j^{\mp}(\lambda)^*] \hat{g}_j(\lambda)\,.$$

Hence we have for such λ

$$\sum_{j\neq 0} T_j(\lambda)^*\hat{g}_j(\lambda)=\sum_{j\neq 0}\lambda R_{L_0}(\lambda\mp i0)(J-J^{-1*})\tilde{Z}_j^{\pm}(\lambda)^*\hat{g}_j(\lambda)\ . \tag{3.8}$$

Since $(L_0-\lambda)\sum_{j\neq 0} T_j(\lambda)^*\hat{g}_j(\lambda)=0$ and $(L_0-\lambda)R_{L_0}(\lambda\mp i0)q(x)=q(x)$ for any $q(x)\in H_{0,(1+\varepsilon)/2}$, we get

$$0=\sum_{j\neq 0}\lambda(J-J^{-1*})\tilde{Z}_j^{\pm}(\lambda)^*\hat{g}_j(\lambda)\ , \tag{3.9}$$

by applying $(L_0-\lambda)$ to the both sides of (3.8). Then by (3.8) and (3.9) we obtain $\sum_{j\neq 0} T_j(\lambda)^*\hat{g}_j(\lambda)=0$, that is,

$$T^*\hat{g}=0\ . \tag{3.10}$$

Equation (3.10) implies that $\hat{g}$ is orthogonal to $R(T)$. Hence

$$\|\hat{g}\|^2+\|Tf\|^2=\|\hat{g}-Tf\|^2=\|Z^{\pm}Z^{\pm*}Tf\|^2\leqq\|Tf\|^2\ .$$

Therefore equation $\hat{g}=0$ must hold. (Q.E.D.)

REMARK. By the same line of argument used for proving (2.14), we get as an equation in $B(H_{1,\delta/2}, H_{1,-\delta/2})$

$$\frac{1}{2\pi i}(R_L(\lambda+i0)-R_L(\lambda-i0))=\sum_{\operatorname{sign} j=\operatorname{sign}\lambda} Z_j^{\pm}(\lambda)^*Z_j^{\pm}(\lambda)\frac{d\rho_{\operatorname{sign} j}}{d\lambda} \tag{3.11}$$

for any $\lambda\in I_1$.

§ 4. Representation formulas for the wave operators and the scattering operator.

In this section we shall give representation formulas for the wave operators and the scattering operator associated with the pair L and L_0 in terms of the quantities related to the eigenfunctions.

It is known (see [1], [6], [12]) that under assumptions (A.1) and (A.2) the wave operators

$$W_{\pm}(L, L_0\,; J^*)=\operatorname{s-lim}_{t\to\pm\infty} e^{itL}J^*e^{-itL_0}P_{0,ac}\ , \tag{4.1}$$

$$W_{\pm}(L_0, L\,; J)=\operatorname{s-lim}_{t\to\pm\infty} e^{itL_0}Je^{-itL}P_{1,ac} \tag{4.2}$$

exist and can be represented by the boundary values of $G(\zeta)$, $G_0(\zeta)$, $R_L(\zeta)$, and $R_{L_0}(\zeta)$ on $\boldsymbol{R}^1$ as follows:

$$\begin{aligned}&\langle W_{\pm}(L, L_0\,; J^*)E_0(\Delta)f_0, f_1\rangle_{H_{1,-\delta/2},H_{1,\delta/2}}\\&=\frac{1}{2\pi i}\int_{\Delta}\Big\langle(R_L(\lambda+i0)-R_L(\lambda-i0))G_0(\lambda\pm i0)f_0, f_1\Big\rangle_{H_{1,-\delta/2},H_{1,\delta/2}}d\lambda\ ,\end{aligned} \tag{4.3}$$

$$\begin{aligned}(4.4)\quad &\langle W_{\pm}(L_0, L\,; J)E(\Delta)g_1, g_0\rangle_{H_{0,-\delta/2}, H_{0,\delta/2}}\\ &=\frac{1}{2\pi i}\int_{\Delta}\Big\langle (R_{L_0}(\lambda+i0)-R_{L_0}(\lambda-i0))G(\lambda\pm i0)g_1, g_0\Big\rangle_{H_{0,-\delta/2}, H_{0,\delta/2}} d\lambda\,,\end{aligned}$$

where $f_0, g_0\in H_{0,\delta/2}, f_1, g_1\in H_{1,\delta/2}$ and Δ in (4.3) is any subset of $I_0\equiv \boldsymbol{R}^1\backslash\{0\}$ and Δ in (4.4) is any subset of I_1. Using equations (4.3) and (4.4), we can obtain the following theorem.

THEOREM 4.1. *Let Assumptions (A.1.I) and (A.2.I) be satisfied. Let $\Delta\subset I_0$ (or $\Delta\subset I_1$) and let $f_0, g_0\in H_{0,\delta/2}$ and $f_1, g_1\in H_{1,\delta/2}$. Then the following equations hold:*

$$\begin{aligned}(4.5)\quad \langle W_{\pm}(L, L_0\,; J^*)E_0(\Delta)f_0, f_1\rangle &= \sum_{j\neq 0}\int_{\Delta\cap R_{\mathrm{sign}\,j}}\langle \tilde{Z}_j^{\pm}(\lambda)^* T_j(\lambda)f_0, f_1\rangle d\rho_{\mathrm{sign}\,j}(\lambda)\\ &=(Z^{\pm *}TE_0(\Delta)f_0, f_1)_{H_1},\quad \Delta\subset I_0\,;\end{aligned}$$

$$\begin{aligned}(4.6)\quad \langle W_{\pm}(L_0, L\,; J)E(\Delta)g_1, g_0\rangle &= \sum_{j\neq 0}\int_{\Delta\cap R_{\mathrm{sign}\,j}}\langle T_j(\lambda)^*\tilde{Z}_j^{\pm}(\lambda)g_1, g_0\rangle d\rho_{\mathrm{sign}\,j}(\lambda)\\ &=(T^*Z^{\pm}E(\Delta)g_1, g_0)_{H_0},\quad \Delta\subset I_1\end{aligned}$$

where $\langle\,,\rangle$ in (4.5) (or in (4.6)) denotes the natural coupling between $H_{1,-\delta/2}$ and $H_{1,\delta/2}$ (or $H_{0,-\delta/2}$ and $H_{0,\delta/2}$).

PROOF. By equations (4.3) and (3.11) we get

$$\begin{aligned}&\langle W_{\pm}(L, L_0\,; J^*)E_0(\Delta)f_0, f_1\rangle\\ &=\frac{1}{2\pi i}\sum_{j\neq 0}\int_{R_{\mathrm{sign}\,j}\cap\Delta}\langle \tilde{Z}_j^{\pm}(\lambda)^*\tilde{Z}_j^{\mp}(\lambda)G_0(\lambda\pm i0)f_0, f_1\rangle\frac{d\rho_{\mathrm{sign}\,j}}{d\lambda}d\lambda\\ &=\frac{1}{2\pi i}\sum_{j\neq 0}\int_{R_{\mathrm{sign}\,j}\cap\Delta}\langle \tilde{Z}_j^{\pm}(\lambda)^* T_j(\lambda)f_0, f_1\rangle d\rho_{\mathrm{sign}\,j}(\lambda)\,.\end{aligned}$$

This relation proves equation (4.5). Equation (4.6) can be proved similarly.

(Q.E.D.)

As usual scattering operator S and scattering matrix $\hat{S}$ associated with the pair L and L_0 are defined by equations $S=W_+(L, L_0\,; J^*)^*W_-(L, L_0\,; J^*)$ and $\hat{S}=TST^*$, respectively. Then it is well known that S is a unitary operator on $P_{0,ac}H_0$ and $\hat{S}$ is a unitary operator on M, where $M=R(T)=R(Z^{\pm})$. In what follows we shall write $W_{\pm}(L, L_0 : J^*)$ as $W_{\pm}$.

The following representation formula for the scattering matrix will be proved.

THEOREM 4.2. *For $j\neq 0$, let $F_{j,i}(\lambda\,; k, k')$ be the $m\times m$ matrix depending on $\lambda\in \boldsymbol{R}_{\mathrm{sign}\,j}\cap I_1$, $k, k'\in N$ and i (sign i=sing j) defined by*

$$\begin{aligned}(4.7)\quad &F_{j,i}(\lambda\,; k, k')\\ &=\int_{S_i}\overline{\varphi_i^{(k')}(\omega_i)}\hat{P}_j(\omega_i)\Big[\mathscr{F}\Big(\lambda\frac{d\rho_{\mathrm{sign}\,j}}{d\lambda}(J^*-J^{-1})\alpha_j^{\pm}(x, \lambda, k)\Big)\Big]_{\lambda_i(\xi)=\lambda} d\sigma_i(\omega_i)\,.\end{aligned}$$

Then for any fixed λ and k, each column of $F_{j,i}(\lambda; k, k')$ belongs to $l^2(C^m)$ with *respect to $k' \in N$ and for any $\hat{f}(k') \in l^2(C^m)$ and $\lambda \in R_{\mathrm{sign}\, j} \cap I_1$,*

$$\sum_{k'=1}^{\infty} F_{j,i}(\lambda; k, k')\hat{f}(k') \in l^2(C^m)$$

with respect to $k \in N$. Let

$$\hat{t}(\lambda): \sum_{\mathrm{sign}\, j=\mathrm{sign}\,\lambda} \oplus\, l^2(C^m) \to \sum_{\mathrm{sign}\, j=\mathrm{sign}\,\lambda} \oplus\, l^2(C^m)$$

be the operator defined by

$$(4.8)\qquad (\hat{t}(\lambda)(\sum_{\mathrm{sign}\, j=\mathrm{sign}\,\lambda} \oplus \hat{f}_j))(k)$$
$$= \sum_{\mathrm{sign}\, j=\mathrm{sign}\,\lambda} \oplus\, (\sum_{\mathrm{sign}\, i=\mathrm{sign}\,\lambda} \sum_{k=1}^{\infty} F_{j,i}(\lambda; k, k')\hat{f}_i(k'))\,.$$

Then $\hat{t}(\lambda)$ is a compact operator on $\sum_{\mathrm{sign}\, j=\mathrm{sign}\,\lambda} \oplus\, l^2(C^m)$ and the scattering matrix $\hat{S}$ can be written in terms of the operator $\hat{t}(\lambda)$ as follows: For any $f_- \in M=R(Z^{\perp})=R(T)$

$$(\hat{S}\hat{f}_-)(\lambda, k)=\hat{f}_-(\lambda, k)-2\pi i[\hat{t}(\lambda)\hat{f}_-(\lambda)](k) \quad a.e. \quad \lambda \in R^1\,.$$

REMARK. $F_{j,i}(\lambda; k, k')$ is the k'-th Fourier coefficient of the numerator in (3.3) multiplied by $\dfrac{d\rho_{\mathrm{sign}\, i}}{d\lambda}$ with respect to the complete orthonormal basis $\varphi_i^{(k)}(\omega_i)$ as an element of $L^2(S_i)$.

PROOF. Let Δ be any compact interval in I_1. For any $\hat{f}_- \in R(T)$, put $T^*\hat{f}_-=f_- \in P_{0,ac}H_0$. We can choose $f \in E(\Delta)H_1$ such that $W_-^*f=E_0(\Delta)f_-$. Put $f_+=W_+^*f$. Then by the definition of S and elementary properties of the wave operators we get $SE_0(\Delta)f_-=f_+$. Therefore by definitions we get

$$SE_0(\Delta)f_- - E_0(\Delta)f_- = W_+(L_0, L; J)E(\Delta)f - W_-(L_0, L; J)E(\Delta)f\,.$$

Let $f_\nu \in H_{1,\delta/2}$ be such a sequence as $f_\nu \to f$ in H_1 as $\nu \to \infty$. Then for any $g \in H_{0,\delta/2}$

$$(SE_0(\Delta)f_- - E_0(\Delta)f_-, g)=\lim_{\nu\to\infty}(W_+(L_0, L, J)E(\Delta)f_\nu - W_-(L_0, L, J)E(\Delta)f_\nu, g)$$
$$=\lim_{\nu\to\infty}\sum_{j\neq 0}\int_{R_{\mathrm{sign}\, j}\cap\Delta} \langle T_j(\lambda)^*[\tilde{Z}_j^+(\lambda)-\tilde{Z}_j^-(\lambda)]f_\nu, g\rangle_{H_{0,-\delta/2}, H_{0,\delta/2}} d\rho_{\mathrm{sign}\, j}(\lambda)\,.$$

By statement (3) of Proposition 3.1, relation $\tilde{Z}_i^{\pm}(\lambda)=T_j(\lambda)G(\lambda\pm i0)$ and (2.14) we get as an element of $B(H_{1,\delta/2}, l^2(C^m))$,

$$\begin{aligned}\tilde{Z}_j^+(\lambda)-\tilde{Z}_j^-(\lambda) &= T_j(\lambda)[G(\lambda+i0)-G(\lambda-i0)]\\ &= T_j(\lambda)G(\lambda+i0)[G_0(\lambda-i0)-G_0(\lambda+i0)]G(\lambda-i0)\\ &= -T_j(\lambda)G(\lambda+i0)[\lambda(J^*-J^{-1})(R_{L_0}(\lambda+i0)-R_{L_0}(\lambda-i0))]G(\lambda-i0)\\ &= -2\pi i \sum_{\mathrm{sign}\, i=\mathrm{sign}\,\lambda} \tilde{Z}_j^+(\lambda)\left[\lambda(J^*-J^{-1})T_i(\lambda)^*T_i(\lambda)\frac{d\rho_{\mathrm{sign}\, i}}{d\lambda}\right]G(\lambda-i0)\end{aligned}$$

$$= -2\pi i \sum_{\operatorname{sign} i = \operatorname{sign} \lambda} \tilde{Z}_j^-(\lambda)[\lambda(J^* - J^{-1})T_i(\lambda)^*]\tilde{Z}_i^-(\lambda)\frac{d\rho_{\operatorname{sign} i}}{d\lambda}.$$

Therefore

$$\langle (S-1)E_0(\Delta)f_-, g\rangle_{H_{0,-\delta/2}, H_{0,\delta/2}}$$
$$= -\lim_{\nu\to\infty} 2\pi i \sum_{j\neq 0}\int_{R_{\operatorname{sign} j}\cap\Delta} \sum_{\operatorname{sign} i = \operatorname{sign} j}\Big\langle T_j(\lambda)^*\tilde{Z}_j^-(\lambda)[\lambda(J^*-J^{-1})T_i(\lambda)^*]\tilde{Z}_i^-(\lambda)$$
$$\times \frac{d\rho_{\operatorname{sign} i}}{d\lambda} f_\nu, g\Big\rangle d\rho_{\operatorname{sign} j}(\lambda)$$
$$= -\lim_{\nu\to\infty} 2\pi i \sum_{j\neq 0}\int_{R_{\operatorname{sign} j}\cap\Delta} \sum_{\operatorname{sign} i = \operatorname{sign} j}\langle \tilde{Z}_i^-(\lambda)f_\nu, \lambda T_i(\lambda)(J-J^{*-1})\tilde{Z}_j^+(\lambda)^* T_j(\lambda)g\rangle_{l^2(C^m)}$$
$$\times \frac{d\rho_{\operatorname{sign} j}}{d\lambda} d\rho_{\operatorname{sign} j}(\lambda).$$

Since $g\in H_{0,\delta/2}$, $\lambda T_i(\lambda)(J-J^{*-1})\tilde{Z}_j^+(\lambda)^* T_j(\lambda)g$ is an $l^2(C^m)$-valued continuous function. $\tilde{Z}_i^-(\lambda)f_\nu$ converges to $(Z_i^-f)(\lambda)$ in $L^2(\boldsymbol{R}_{\operatorname{sign} j}, l^2(C^m))$. Hence we have

(4.9) $((S-1)E_0(\Delta)f_-, g)$

$$= -2\pi i \sum_{j\neq 0}\int_{R_{\operatorname{sign} j}\cap\Delta} \sum_{\operatorname{sign} i = \operatorname{sign} j}\langle (Z_i^-f)(\lambda), \lambda T_i(\lambda)(J-J^{*-1})\tilde{Z}_j^+(\lambda)^* T_j(\lambda)g\rangle_{l^2(C^m)}$$
$$\times \frac{d\rho_{\operatorname{sign} j}}{d\lambda} d\rho_{\operatorname{sign} j}(\lambda)$$
$$= -2\pi i \sum_{j\neq 0}\int_{R_{\operatorname{sign} j}\cap\Delta} \sum_{\operatorname{sign} i = \operatorname{sign} j}\Big\langle \lambda\tilde{Z}_j^+(\lambda)(J^*-J^{-1})T_i(\lambda)^*\frac{d\rho_{\operatorname{sign} i}}{d\lambda}(Z_i^-f)(\lambda),$$
$$T_j(\lambda)g\Big\rangle d\rho_{\operatorname{sign} j}(\lambda).$$

On the other hand, by Theorem 4.1

$$E_0(\Delta)f_- = W_-^* f = \sum_{j\neq 0} T_j^* Z_j^- f.$$

Multiplying T_i on both sides, we get by Theorem 2.3 and Theorem 3.4

$$Z_i^- f = T_i E_0(\Delta)f_- = T_i E_0(\Delta)T^*\hat{f}_- = \chi_\Delta(\lambda)(\hat{f}_-)_i. \tag{4.10}$$

The left hand side of (4.9) can be rewritten as

$$\sum_{j\neq 0}\int_{R_{\operatorname{sign} j}}((T_j(S-1)E_0(\Delta)T^*\hat{f}_-)(\lambda), (T_jg)(\lambda))_{l^2(C^m)}d\rho_{\operatorname{sign} j}(\lambda) \tag{4.11}$$
$$= \sum_{j\neq 0}\int_{R_{\operatorname{sign} j}\cap\Delta}((T_j(S-1)T^*\hat{f}_-)(\lambda), T_j(\lambda)g)_{l^2(C^m)}d\rho_{\operatorname{sign} j}(\lambda).$$

By (4.9), (4.10) and (4.11) we have

$$\sum_{j\neq 0}\int_{R_{\operatorname{sign} j}\cap\Delta}\Big\langle (T_j(S-1)T^*\hat{f}_-)(\lambda) + 2\pi i \sum_{\operatorname{sign} j = \operatorname{sign} i}\lambda\tilde{Z}_j^+(\lambda)(J^*-J^{-1})T_i(\lambda)^*$$

$$\times \frac{d\rho_{\mathrm{sign}\, i}}{d\lambda}(T_i f_-)(\lambda), (T_j g)(\lambda) \Big\rangle_{l^2(C^m)} d\rho_{\mathrm{sign}\, j}(\lambda)=0 \,.$$

Since $H_{0,\delta/2}$ is a dense subset of H_0, $\{\bigoplus_{j\neq 0} T_j g\,;\, g\in H_{0,\delta/2}\}$ forms a dense subset of range T. Therefore

$$(T_j(S-1)T^*\hat{f}_-)(\lambda)+2\pi i \sum_{\mathrm{sign}\, i=\mathrm{sign}\, j} \lambda \tilde{Z}_j^+(\lambda)(J^*-J^{-1})T_i(\lambda)^* \frac{d\rho_{\mathrm{sign}\, i}}{d\lambda}(\hat{f}_-)_i=0 \,,$$

that is,

$$(\hat{S}\hat{f}_-)(\lambda)-\hat{f}_-(\lambda)=-2\pi i \bigoplus_{j\neq 0} \sum_{\mathrm{sign}\, i=\mathrm{sign}\, j} \lambda \tilde{Z}_j^+(\lambda)(J^*-J^{-1})T_i(\lambda)^* \frac{d\rho_{\mathrm{sign}\, i}}{d\lambda}(\hat{f}_-)_i$$

for a.e. $\lambda\in \boldsymbol{R}^1$.

Now, for any $\hat{f}(\lambda)\in \bigoplus_{j\neq 0} l^2(\boldsymbol{C}^m)$

$$\begin{aligned}
&\Big((\lambda\tilde{Z}_j^+(\lambda)(J^*-J^{-1})T_i(\lambda)^* \frac{d\rho_{\mathrm{sign}\, i}}{d\lambda}\hat{f}(\lambda)\Big)(k) \\
&=\Big\langle \lambda(J^*-J^{-1})T_i(\lambda)^* \frac{d\rho_{\mathrm{sign}\, i}}{d\lambda}\hat{f}(\lambda),\, \alpha_j^+(x,\lambda,k)\Big\rangle_{H_{1,\delta/2},H_{1,-\delta/2}} \\
&=\Big(\hat{f}(\lambda),\, T_i(\lambda)\Big[\lambda \frac{d\rho_{\mathrm{sign}\, i}}{d\lambda}(J-J^{-1*})\alpha_j^+(x,\lambda,k)\Big]\Big)_{l^2(C^m)} .
\end{aligned}$$

By the definition of $T_i(\lambda)$

$$\begin{aligned}
&\Big(T_i(\lambda)\Big[\lambda \frac{d\rho_{\mathrm{sign}\, i}}{d\lambda}(J-J^{-1*})\alpha_j^+(x,\lambda,k)\Big]\Big)(k') \\
&=\int \overline{\varphi_i^{(k')}(\omega_i)}P_j(\omega_i)\Big[\mathscr{F}\Big(\lambda \frac{d\rho_{\mathrm{sign}\, i}}{d\lambda}(J-J^{*-1})\alpha_j^+(x,\lambda,k)\Big)\Big]_{\lambda_i(\xi)=\lambda} d\sigma_i(\omega_i) \,.
\end{aligned}$$

Hence we finally have

$$(\hat{S}\hat{f}_-)(\lambda)=\hat{f}_-(\lambda)-2\pi i[\hat{t}(\lambda)\hat{f}_-(\lambda)] \quad \text{a.e. } \lambda\in \boldsymbol{R}^1 \,.$$

Since $T_i(\lambda)^*=\Gamma_i(\lambda)^*U_i^*$ and $\Gamma_i(\lambda)^*$ is a bounded operator from $L^2(S_i)$ to $H^s_{-(1+\varepsilon)/2}$ for any large s and any small $\varepsilon>0$, Rellich's compactness theorem shows that $\hat{t}(\lambda)$ is a compact operator. (Q.E.D.)

REMARK 1. We can prove by using elementary methods that

$$1-2\pi i\hat{t}(\lambda): \sum_{\mathrm{sign}\, j=\mathrm{sign}\,\lambda} \oplus\, l^2(\boldsymbol{C}^m) \to \sum_{\mathrm{sign}\, j=\mathrm{sign}\,\lambda} \oplus\, l^2(\boldsymbol{C}^m)$$

is a strongly continuous unitary operator valued function of $\lambda\in R^+\cap I_1$ (or $R^-\cap I_1$). Moreover we can prove the so-called phase shift formula using the method used in Ikebe [4]. We shall not discuss the subjects here.

REMARK 2. If $M(x)-I$ decreases sufficiently rapidly at infinity we can prove by using the method which will be used in the proof of Lemma 5.2 that the operator $\hat{t}(\lambda)$ is Hilbert-Schmidt type. We shall not discuss the subject here.

§ 5. Concluding remark.

If $M(x)-I$ decreases sufficiently rapidly at infinity (see Theorem 5.3) we can construct distorted plane waves in terms of $\alpha_j^{\pm}(x, \lambda, k)$. In what follows we shall show the process briefly.

Let $\phi_j^{(k)}(\omega_j)$ $(j\neq 0, k=1, 2, \cdots)$ be the normalized eigenfunctions corresponding to the eigenvalues $\gamma_j^{(k)}$ of Laplace-Beltrami operator Λ_j on C^∞-manifold S_j with metric induced from the metric in E^n. Put $\psi_j^{(k)}(\omega_j)=\sqrt[4]{|\text{grad}\,\lambda_j(\omega_j)|}\phi_j^{(k)}(\omega_j)$. Then it is easy to see that $\psi_j^{(k)}(\omega_j)$ $(k=1, 2, \cdots)$ is a complete orthonormal system in $L^2(S_j\,; d\sigma_j)$. Let $\alpha_j^{\pm}(x, \lambda, k)$ be the eigenfunction of L constructed in § 3 in terms of $\psi_j^{(k)}(\omega_j)$.

The following lemma is well known.

LEMMA 5.1. *Let p be an integer with $p>(n-1)/4$. Then for any fixed $\omega_j\in S_j$, $\sum_{k=1}^{N}\frac{\phi_j^{(k)}(\omega_j)\overline{\phi_j^{(k)}(\omega_j')}}{1+(\gamma_j^{(k)})^p}$ converges uniformly in ω_j' as $N\to\infty$.*

LEMMA 5.2. *Let p be an integer with $p>(n-1)/4$. Then for any fixed $\lambda\in \boldsymbol{R}_{\text{sign}\,j}$ and $\omega_j\in S_j$, $\sum_{k=1}^{N}\psi_j^{(k)}(\omega_j)h_j(x, \lambda, k)$ converges to $e^{ix\cdot|\lambda|\omega_j}\hat{P}_j(\omega_j)$ in $H_{0,-2p-(1+\varepsilon)/2}$ as $N\to\infty$ $(\varepsilon>0)$.*

PROOF. Let $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_n)$ be a multi-index and $|\alpha|=\sum_{j=1}^{N}\alpha_j$. Then with suitable polynomials $p_\alpha(x, \lambda)$ of x and λ of degree $|\alpha|$ and suitable matrices $Q_\alpha(\omega_j')$ depending smoothly on $\omega_j'\in S_j$, we have the following relations:

$$
\begin{aligned}
&\sum_{k=1}^{N}\psi_j^{(k)}(\omega_j)h_j(x, \lambda, k)\\
&=\sum_{k=1}^{N}\psi_j^{(k)}(\omega_j)\int_{S_j}e^{ix\cdot|\lambda|\omega_j'}\overline{\psi_j^{(k)}(\omega_j')}\hat{P}_j(\omega_j')d\sigma_j(\omega_j')\\
&=\sum_{k=1}^{N}\sqrt[4]{|\text{grad}\,\lambda_j(\omega_j)|}\int_{S_j}e^{ix|\lambda|\omega_j'}\overline{\phi_j^{(k)}(\omega_j')}\phi_j^{(k)}(\omega_j)\hat{P}_j(\omega_j')ds_j(\omega_j')\\
&=\sum_{k=1}^{N}\sqrt[4]{|\text{grad}\,\lambda_j(\omega_j)|}\int_{S_j}e^{ix|\lambda|\omega_j'}\frac{[(1+\Lambda_j^p)\overline{\phi_j^{(k)}(\omega_j')}]\phi_j^{(k)}(\omega_j)}{(\gamma_j^{(k)})^p+1}\hat{P}_j(\omega_j')ds_j(\omega_j')\\
&=\sum_{k=1}^{N}\sqrt[4]{|\text{grad}\,\lambda_j(\omega_j)|}\int_{S_j}[(\Lambda_j^p+1)e^{ix|\lambda|\omega_j'}\hat{P}_j(\omega_j')]\frac{\phi_j^{(k)}(\omega_j)\overline{\phi_j^{(k)}(\omega_j')}}{[(\gamma_j^{(k)})^p+1]}ds_j(\omega_j')\\
&=\sum_{|\alpha|=0}^{2p}\sqrt[4]{|\text{grad}\,\lambda_j(\omega_j)|}\,p_\alpha(x, \lambda)\int_{S_j}e^{ix|\lambda|\omega_j'}\sum_{k=1}^{N}\frac{\phi_j^{(k)}(\omega_j)\overline{\phi_j^{(k)}(\omega_j')}}{(\gamma_j^{(k)})^p+1}Q_\alpha(\omega_j')ds_j(\omega_j')\,.
\end{aligned}
$$

By Lemma 5.1, $\sum_{k=1}^{N}\frac{\overline{\phi_j^{(k)}(\omega_j')}\phi_j^{(k)}(\omega_j)}{(\gamma_j^{(k)})^p+1}$ converges uniformly in ω_j' for any fixed ω_j

as $N\to\infty$. Hence for any fixed λ and ω_j, $\int_{S_j} e^{ix|\lambda|\omega_j'}\sum_{k=1}^{N}\frac{\overline{\phi_j^{(k)}(\omega_j')}\phi_j^{(k)}(\omega_j)}{(\gamma_j^{(k)})^p+1}Q_a(\omega_j')ds_j(\omega_j')$ converges in $H_{0,-(1+\varepsilon)/2}$ ($\varepsilon>0$) as $N\to\infty$. Hence for any fixed λ and ω_j, $\sum_{k=1}^{N}\phi_j^{(k)}(\omega_j)\times h_j(x,\lambda,k)$ converges in $H_{0,-2p-(1+\varepsilon)/2}$ as $N\to\infty$.

It is obvious by the definition of $h_j(x,\lambda,k)$ that for any fixed λ and x, $\sum_{k=1}^{N} h_j(x,\lambda,k)\phi_j^{(k)}(\omega_j)$ converges to $e^{ix|\lambda|\omega_j}\hat{P}_j(\omega_j)$ in $L^2(S_j;\boldsymbol{C}^m:d\sigma_j)$ as $N\to\infty$. Hence the statement of the lemma holds. (Q.E.D.)

THEOREM 5.3. *Let p be an integer with $p>(n-1)/2$. Let assumptions (A.1.I) and (A.2.I) be satisfied with $\delta=4p+1+\varepsilon$ ($\varepsilon>0$). Then for any fixed $\omega_j\in S_j$ and $\lambda\in\boldsymbol{R}_{\mathrm{sign}\,j}\backslash\sigma_p(L)$, $\sum_{k=1}^{N}\phi_j(\omega_j)\alpha_j^{\pm}(x,\lambda,k)$ converges to $G(\lambda\pm i0)^*[e^{ix|\lambda|\omega_j}\hat{P}_j(\omega_j)]$ in $H_{-2p-(1+\varepsilon)/2}$ as $N\to\infty$.*

PROOF. By Proposition 3.1, $G(\lambda\pm i0)^*\in B(H_{0,-\delta/2},H_{1,\delta/2})$ for $\lambda\in\boldsymbol{R}^1\backslash\sigma_p(L)$. Hence

$$\begin{aligned}\sum_{k=1}^{N}\phi_j(\omega_j)\alpha_j^{\pm}(x,\lambda,k)&=\sum_{k=1}^{N}\phi_j(\omega_j)[G(\lambda\pm i0)^*h_j(y,\lambda,k)]\\&=G(\lambda\pm i0)^*[\sum_{k=1}^{N}\phi_j(\omega_j)h_j(y,\lambda,k)]\,.\end{aligned}$$

By Lemma 5.2, $\sum_{k=1}^{N}\phi_j(\omega_j)h_j(y,\lambda,k)$ converges to $e^{ix|\lambda|\omega_j}\hat{P}_j(\omega_j)$ in $H_{0,-\delta/2}$. Hence $\sum_{k=1}^{N}\phi_j(\omega_j)\alpha_j^{\pm}(x,\lambda,k)$ converges to $G(\lambda\pm i0)^*[e^{ix|\lambda|\omega_j}\hat{P}_j(\omega_j)]$ in $H_{1,-\delta/2}$ as $N\to\infty$.

(Q.E.D.)

Let $\xi=|\lambda|\omega_j$, then $\hat{P}_j(\omega_j)=\hat{P}_j(\xi)$ and

$$\begin{aligned}G(\lambda\pm i0)^*[e^{ix|\lambda|\omega_j}\hat{P}_j(\omega_j)]&=G(\lambda\pm i0)^*[e^{ix\xi}\hat{P}_j(\xi)]\\&=e^{ix\xi}\hat{P}_j(\xi)+[\lambda_j(\xi)R_L(\lambda_j(\xi)\mp i0)][(1-M(x)^{-1})e^{ix\xi}\hat{P}_j(\xi)]\\&\equiv\beta_j^{\pm}(x,\xi)\,.\end{aligned}$$

We call $\beta_j^{\pm}(x,\xi)$ a distorted plane wave. We can obtain the eigenfunction expansion theorem and representation formulas in terms of distorted plane waves, following the method used in the preceding sections. We shall not discuss it in detail here.

§ 6. Example.

We shall consider here Maxwell's equation in $\boldsymbol{R}^3$;

$$\frac{\partial}{\partial t}\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}E(x)&0\\0&M(x)\end{pmatrix}\begin{pmatrix}0&\mathrm{rot}\\-\mathrm{rot}&0\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}.$$

Here $u(t,x)$ and $v(t,x)$ are $\boldsymbol{C}^3$-valued function of $t\in\boldsymbol{R}^1, x\in\boldsymbol{R}^3$ whose real parts describe the strength of the electric and magnetic field, respectively, $E(x)$ and

$M(x)$ are 3×3-matrices depending on $x\in \boldsymbol{R}^3$ and are the dielectric constant and permeability of the media filling the space. We assume that the dielectric constant and permeability of the vacuum is equal to 1. Our assumptions on $M(x)$ and $E(x)$ are as follows:

ASSUMPTION (1). $M(x)$ and $E(x)$ are hermitian and positive definite and there exists a constant $C_1>0$ such that

$$C_1^{-1}|\xi|^2 \leqq {}^t\bar{\xi}\cdot M(x)\cdot \xi \leqq C_1|\xi|^2 \quad \text{for all} \quad x\in \boldsymbol{R}^3 \text{ and } \xi\in \boldsymbol{C}^3\,,$$

$$C_1^{-1}|\xi|^2 \leqq {}^t\bar{\xi}\cdot E(x)\cdot \xi \leqq C_1|\xi|^2 \quad \text{for all} \quad x\in \boldsymbol{R}^3 \text{ and } \xi\in \boldsymbol{C}^3\,.$$

ASSUMPTION (2). $M(x)$ and $E(x)$ are bounded measurable functions of $x\in \boldsymbol{R}^3$ and there exist constants $C_2>0$ and $\delta>1$ such that

$$\sup_{x\in R^3}\max_{1\leq i,j\leq 3}\,(|m_{ij}(x)-\delta_{ij}|,\ |e_{ij}(x)-\delta_{ij}|)\leqq C_2(1+|x|^2)^{\delta/2}$$

where $m_{ij}(x)$ and $e_{ij}(x)$ are (i,j)-component of $M(x)$ and $E(x)$, respectively.

We put

$$\tilde{M}(x)=\begin{pmatrix} E(x) & 0 \\ 0 & M(x)\end{pmatrix}, \qquad L_0(D)=\frac{1}{i}\begin{pmatrix} 0 & \mathrm{rot} \\ -\mathrm{rot} & 0\end{pmatrix}.$$

Then we can easily see

$$L_0(\xi)=\begin{pmatrix} 0 & L_0'(\xi) \\ {}^tL_0'(\xi) & 0\end{pmatrix},$$

where

$$L_0'(\xi)=\begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0\end{pmatrix},$$

and

$$\det(\lambda-L_0(\xi))=\lambda^2(\lambda-|\xi|)^2(\lambda+|\xi|)^2\,. \tag{6.1}$$

Therefore $L_0(D)$ is uniformly propagative system in the sense of Wilcox and assumptions (1) and (2) imply (A.1.I) and (A.2.I). Hence our results derived in the preceding sections are all valid for the Maxwell system satisfying Assumptions (1) and (2). We shall construct the eigenfunction of the unperturbed operator in what follows. Equation (6.1) shows that we should put

$$\lambda_1(\xi)=|\xi|>\lambda_0(\xi)\equiv 0>\lambda_{-1}(\xi)=-|\xi|\,;$$

$$S_1=\{\xi: |\xi|=1\}=S^2\,,\quad S_{-1}=\{\xi: -|\xi|=-1\}=S^2\,;$$

$$P_0(\xi)=\begin{pmatrix} I+L_0'(\omega)^2 & 0 \\ 0 & I+L_0'(\omega)^2\end{pmatrix}, \qquad P_\pm(\xi)=\frac{1}{2}\begin{pmatrix} -L_0'(\omega)^2 & \mp L_0'(\omega) \\ \pm L_0'(\omega) & -L_0'(\omega)^2\end{pmatrix}$$

where $\omega=\xi/|\xi|$ and I is the 3×3 unit matrix. Therefore our eigenfunction $h_{\pm 1}(x, \pm\lambda, k)$ $(\lambda>0)$ associated with the unperturbed operator are given by

$$\begin{aligned} h_{\pm 1}(x, \pm\lambda, k) &= \int_{S^2} e^{i\lambda\omega\cdot x} Y_k(\omega) P_{\pm}(\omega) d\omega \\ &= P_{\pm}\left(\frac{1}{i\lambda}\frac{\partial}{\partial x}\right) \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} e^{i\lambda|x|\cos\theta} Y_k(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\sin\theta d\phi d\theta . \end{aligned}$$

Here $Y_k(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is one of

$$\sqrt{\frac{2n+1}{4\pi}} P_n(\cos\theta), \quad \sqrt{\frac{(n-m)!(2n+1)!}{4\pi(n+m)!}} P_n^m(\cos\theta)\cos m\phi ,$$

$$\sqrt{\frac{(n-m)!(2n+1)!}{4\pi(n+m)!}} P_n^m(\cos\theta)\sin m\phi \quad n=0, 1, 2, \cdots, \quad m=1, 2, \cdots, n,$$

where $P_n(x)$ and $P_n^m(x)$ are the Legendre and the associated Legendre functions of first kind, $P_{\pm}\left(\frac{1}{i\lambda}\frac{\partial}{\partial x}\right)$ is obtained by putting $\frac{1}{i\lambda}\frac{\partial}{\partial x_j}$ instead of ω_j $(j=1, 2, 3)$ in $P_{\pm}(\omega)$.

It is well known that

$$e^{i|x|\lambda\cos\theta} = \left(\frac{2\pi}{|x|\lambda}\right)^{1/2} \sum_n \left(n+\frac{1}{2}\right) J_{n+(1/2)}(|x|\lambda) P_n(\cos\theta) ,$$

where $J_\nu(x)$ is the ν-th Bessel function. Therefore $h_{\pm 1}(x, \lambda, k)$ does not vanish for only k for which $Y_k(\omega)$ corresponds to one of $\sqrt{\frac{2n+1}{4\pi}} P_n(\cos\theta)$, and for such k

$$\begin{aligned} h_{\pm 1}(x, \lambda, k) &= P_{\pm}\left(\frac{1}{i\lambda}\frac{\partial}{\partial x}\right)\left[\left(\frac{2\pi}{|x|\lambda}\right)^{1/2}\left(k+\frac{1}{2}\right) J_{k+(1/2)}(|x|\lambda)\sqrt{\frac{4\pi}{2k+1}}\right] \\ &= \sqrt{(2k+1)\pi}\, P_{\pm}\left(\frac{1}{i\lambda}\frac{\partial}{\partial x}\right)\left[\left(\frac{2\pi}{|x|\lambda}\right)^{1/2} J_{k+(1/2)}(|x|\lambda)\right]. \end{aligned}$$

Then our eigenfunctions associated with perturbed Maxwell's equation are obtained by limiting absorption method as solutions of the equation

$$(L \mp |\xi|)u = 0 .$$

Using this eigenfunctions, we can obtain the expansion formulas and representation formulas.

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