A remark on Siegel domains of second kind

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Introduction

The purpose of this note is to give a characterization of Siegel domains of second kind. When we consider Siegel domains of second kind, we refer to them omitting the specification "of second kind". Kaup-Matsushima-Ochiai generalized the concept of Siegel domains and defined *generalized Siegel domain* to contain a *Thullen domain* $\{(z, u)\in C^2;\ \mathrm{Im}\, z-|u|^{\frac{1}{\alpha}}>0\}$ (α; positive real) [2]. Following J. Vey [1], we call them *S-domains* and define a *sweepable S-domain* $\mathscr{D}$ as follows: let Γ be a subgruop of $G(\mathscr{D})$, which is the group of all holomorphic automorphisms of $\mathscr{D}$. Then we say Γ sweeps $\mathscr{D}$ if there exists a compact subset $\mathscr{K}$ of $\mathscr{D}$ such that $\Gamma\mathscr{K}=\mathscr{D}$. We say that $\mathscr{D}$ is a *sweepable S-domain* if there exists a subgroup of $G(\mathscr{D})$ sweeping $\mathscr{D}$. J. Vey gave a characterization of Siegel domains in the category of S-domains. Namely he proved that a sweepable S-domain with non-zero exponent is a Siegel domain [1]. We shall generalize the concept of S-domains to contain a domain of the form $\{(w, u, v)\in C^3;\ \mathrm{Im}\, w-|u|^{\frac{1}{\alpha}}-|v|^{\frac{1}{\beta}}>0\}$ (α, β; non-zero reals). (We note that the domain of this type does not generally belong to the category of S-domains.) We call domains of that type *generalized S-domains* and we shall show the validity of the analogue of J. Vey's result. Our main theorem is the following

THEOREM. *A sweepable generalized S-domain is a Siegel domain.*

In §1 we will define generalized S-domains and remark that some results about vector fields on S-domains in [2], [3] can be generalized to the case of generalized S-domains. In §2 we will prove our theorem. The key to the proof is Lemma 1, from which, by applying Vey's method, Lemmas 2 and 3 follow. After then the theorem is almost automatically obtained.

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§1. **Preriminaries.** We will define generalized S-domains as follows:

DEFINITION. Let l, m, n be positive integers and a domain $\mathscr{D}$ in C^{l+m+n} is said to be a *generalized S-domain* if the following conditions are satisfied.

1) $\mathscr{D}$ is isomorphic to a bounded domain in C^{l+m+n} and $\mathscr{D}\cap(C^l\times\{0\}\times\{0\})\neq\emptyset$, where $\{0\}\times\{0\}$ denotes an origin of C^{m+n}.

2) $\mathscr{D}$ is invariant under the following affine transformations of C^{l+m+n},

$$\begin{array}{llll} \text{a)} & (z, u, v) \longrightarrow (z+a, u, v) & \text{for any} & a\in \boldsymbol{R}^l , \\ \text{b)} & (z, u, v) \longrightarrow (z, e^{it}u, v) & \text{for any} & t\in \boldsymbol{R} , \\ \text{b}') & (z, u, v) \longrightarrow (z, u, e^{it}v) & \text{for any} & t\in \boldsymbol{R} , \\ \text{c)} & (z, u, v) \longrightarrow (e^t z, e^{ct}u, e^{dt}v) & \text{for any} & t\in \boldsymbol{R} , \end{array}$$

where c, d are non-zero real numbers, called *exponents* of $\mathscr{D}$, depending only on $\mathscr{D}$.

Let $\mathscr{D}$ be a generalized S-domain with exponents c, d in C^{l+m+n}. For simplicity we denote coordinates of a point of C^{l+m+n} by $z^1, \cdots, z^l, z^{l+1}, \cdots, z^{l+m}, z^{l+m+1}, \cdots, z^{l+m+n}$. The following ranges of indices will be used throughout this paper:

$$\begin{gathered} 1\leqslant j, k, \cdots \leqslant l , \\ l+1\leqslant \alpha, \beta, \cdots \leqslant l+m , \\ l+m+1\leqslant A, B, \cdots \leqslant l+m+n , \\ 1\leqslant M, N, \cdots \leqslant l+m+n . \end{gathered}$$

∂_N will denote the holomorphic vector field $\partial/\partial z^N$.

Let $\mathscr{D}$ be a generalized S-domain in C^{l+m+n}. Then, for a fixed $a\in R^l$, $(z, u, v) \longrightarrow (z+ta, u, v)$, $t\in \boldsymbol{R}$, is a one parameter subgroup in $G(\mathscr{D})$ and determines elements of $\mathfrak{g}(\mathscr{D})$, where $\mathfrak{g}(\mathscr{D})$ is the Lie algebra of the Lie group $G(\mathscr{D})$. Let $\mathfrak{h}$ be a subspace of $\mathfrak{g}(\mathscr{D})$ and p a point of $\mathscr{D}$. We denote $\mathfrak{h}(p)$ the real subspace of $T_p^C(\mathscr{D})$ defined by

$$\mathfrak{h}(p)=\{X(p)\in T_p^C(\mathscr{D})\,;\; X\in\mathfrak{h}\} ,$$

where $T_p^C(\mathscr{D})$ is the complex tangent space at p.

Let $\mathfrak{h}^C(p)$ be the C-subspace of $T_p^C(\mathscr{D})$ generated by $\mathfrak{h}(p)$. By taking $a=(0, \cdots, \overset{k}{\check{1}}, \cdots, 0)$ in R^l, the above one parameter group defines the element ∂_k. Let ∂', ∂'' and ∂ be the elements of $\mathfrak{g}(\mathscr{D})$ determined by the one-parameter subgroups b), b′), c) respectively. Then we see

$$\begin{aligned} &\partial'=i\sum z^\alpha\partial_\alpha , \qquad \partial''=i\sum z^A\partial_A , \\ &\partial=\sum z^k\partial_k+c\sum z^\alpha\partial_\alpha+d\sum z^A\partial_A . \end{aligned} \tag{1}$$

We can prove the next theorem in the same manner as Theorem 5.1 of [3].

THEOREM A. *Any vector field* $X=\sum p^N\partial_N$ *belonging to* $\mathfrak{g}(\mathscr{D})$ *is a polynomial* *vector field. For each fixed* $(z^1, \cdots, z^l)$, p^k *(respectively* p^α, p^A*) are polynomials of* $z^{l+1}, \cdots, z^{l+m+n}$ *of total degrees* $\leqslant 1$ *(respectively* $\leqslant 2$, $\leqslant 2$*).*

Let $Z_{\mu\nu\zeta}$, $U_{\mu\nu\zeta}$ and $V_{\mu\nu\zeta}$ denote vector fields in $\mathfrak{g}(\mathscr{D})$ having the following forms,

$$Z_{\mu\nu\zeta}=\sum_k p^k_{\mu\nu\zeta}\partial_k ,$$
$$U_{\mu\nu\zeta}=\sum_\alpha p^\alpha_{\mu\nu\zeta}\partial_\alpha ,$$
$$V_{\mu\nu\zeta}=\sum_A p^A_{\mu\nu\zeta}\partial_A ,$$

where $p^N_{\mu\nu\zeta}$ is a homogeneous polynomial of degree μ in $z^1, \cdots, z^l$ and homogeneous of degree ν in $z^{l+1}, \cdots, z^{l+m}$ and homogeneous of degree ζ in $z^{l+m+1}, \cdots, z^{l+m+n}$. Then we have

$$[\partial, Z_{\mu\nu\zeta}]=(\mu-1+c\nu+d\zeta)Z_{\mu\nu\zeta} ,$$
$$[\partial, U_{\mu\nu\zeta}]=(\mu+c(\nu-1)+d\zeta)U_{\mu\nu\zeta} ,$$
$$[\partial, V_{\mu\nu\zeta}]=(\mu+c\nu+d(\zeta-1))V_{\mu\nu\zeta} ,$$
$$[\partial', Z_{\mu\nu\zeta}]=i\nu Z_{\mu\nu\zeta} ,$$
$$[\partial', U_{\mu\nu\zeta}]=i(\nu-1)U_{\mu\nu\zeta} ,$$
$$[\partial', V_{\mu\nu\zeta}]=i\nu V_{\mu\nu\zeta} ,$$
$$[\partial'', Z_{\mu\nu\zeta}]=i\zeta Z_{\mu\nu\zeta} ,$$
$$[\partial'', U_{\mu\nu\zeta}]=i\zeta U_{\mu\nu\zeta} ,$$
$$[\partial'', V_{\mu\nu\zeta}]=i(\zeta-1)V_{\mu\nu\zeta} .$$

In this place we remark that J. Vey's results of § 1, 2 in [1] are valid in our situation and we shall use freely his results.

§ 2. **Proof of the Theorem.** We have only to show that the exponents c, d are identical and then $c=\frac{1}{2}$ follows by J. Vey's results. So from now on we assume the contrary, namely assume $c\neq\frac{1}{2}$ and $c\neq d$.

LEMMA 1. *A vector field* $X\in\mathfrak{g}(\mathscr{D})$ *which is independent of* $z^1, \cdots, z^l$ *has a form*

$$X=Z_{000}+Z_{001}+U_{010}+V_{001}+V_{000} .$$

PROOF. A vector field $X\in\mathfrak{g}(\mathscr{D})$ which is independent of $z^1, \cdots, z^l$ has a form

$$X=Z_{000}+Z_{010}+Z_{001}+U_{002}+U_{020}+U_{011}+U_{010}+U_{001}+U_{000}$$
$$+V_{002}+V_{020}+V_{011}+V_{010}+V_{001}+V_{000} .$$

We have

$$\operatorname{ad}\partial(\operatorname{ad}(\partial'+\partial''))^2X=(1-c)Z_{010}+(1-d)Z_{001}+(c-2d)U_{002}$$
$$-cU_{020}-dU_{011}+cU_{000}-dV_{002}-(2c-d)V_{020}-cV_{011}+dV_{000}\ ,$$
$$(\operatorname{ad}(\partial'+\partial''))^2(\operatorname{ad}\partial')^2X=Z_{010}+U_{002}+U_{020}+U_{000}+4V_{020}+V_{011}\ ,$$
$$(\operatorname{ad}(\partial'+\partial''))^2(\operatorname{ad}\partial'')^2X=Z_{001}+4U_{002}+U_{011}+V_{002}+V_{020}+V_{000}\ .$$

Therefore we get

$$\operatorname{ad}\partial(\operatorname{ad}(\partial'+\partial''))^2X-c(\operatorname{ad}(\partial'+\partial''))^2(\operatorname{ad}\partial')^2X-d(\operatorname{ad}(\partial'+\partial''))^2(\operatorname{ad}\partial'')^2X$$
$$=(1-2c)Z_{010}+(1-2d)Z_{001}-6dU_{002}-2cU_{020}-2dU_{011}$$
$$-2dV_{002}-6cV_{020}-2cV_{011}=Y\in\mathfrak{g}(\mathscr{D})\ .$$

On the other hand $\operatorname{ad}(\partial'+\partial'')Y=iY\in\mathfrak{g}(\mathscr{D})$. By H. Cartan's principle, $Y=0$. By assumptions $cd\neq 0$, $c\neq\frac{1}{2}$, we have

$$Z_{010}=U_{002}=U_{020}=U_{011}=V_{002}=V_{020}=V_{011}=0\ .$$

Therefore we have $(\operatorname{ad}(\partial'+\partial''))^2(\operatorname{ad}\partial')^2X=U_{000}\in\mathfrak{g}(\mathscr{D})$ and since $\partial'U_{000}=-iU_{000}\in\mathfrak{g}(\mathscr{D})$, by H. Cartan's principle, we have $U_{000}=0$. Hence X has a form

$$Z_{000}+Z_{001}+U_{010}+U_{001}+V_{010}+V_{001}+V_{000}\ .$$

Then we get

$$(\operatorname{ad}\partial')^2X=-U_{001}-V_{010}\ ,\qquad \operatorname{ad}\partial(\operatorname{ad}\partial')^2X=(c-d)U_{001}-(c-d)V_{010}\ .$$

Hence we have

$$(c-d)(\operatorname{ad}\partial')^2X+\operatorname{ad}\partial(\operatorname{ad}\partial')^2X=-2(c-d)V_{010}\in\mathfrak{g}(\mathscr{D})\ .$$

By the assumption $c\neq d$, we have $V_{010}\in\mathfrak{g}(\mathscr{D})$ and $(\operatorname{ad}\partial')V_{010}=iV_{010}\in\mathfrak{g}(\mathscr{D})$. By H. Cartan's principle, $V_{010}=0$. Hence $(\operatorname{ad}\partial')^2X=-U_{001}\in\mathfrak{g}(\mathscr{D})$. As $(\operatorname{ad}\partial')U_{001}=-iU_{001}\in\mathfrak{g}(\mathscr{D})$, we have $U_{001}=0$. Namely X has a form $Z_{000}+Z_{001}+U_{010}+V_{001}+V_{000}$. q.e.d.

LEMMA 2. *∂' belongs to the center $\mathfrak{z}$ of $\mathfrak{g}(\mathscr{D})$.*

LEMMA 3. *Let $\mathscr{V}$ be a set of common zeros of the vector fields belonging to the center $\mathfrak{z}$ of $\mathfrak{g}(\mathscr{D})$. Then we have*

$$\mathscr{D}\supset\mathscr{D}\cap(C^l\times\{0\}\times C^n)\supset\mathscr{V}\supset\mathscr{D}\cap(C^l\times\{0\}\times\{0\})\ .$$

We can prove these lemmas in the same way as in the proof of J. Vey ([1] Lemmas 3.2, 3.3). So we will omit the proof.

PROOF OF THE THEOREM. We assume $c\neq\frac{1}{2}$, $c\neq d$. Since a subgroup Γ of $G(\mathscr{D})$ which sweeps $\mathscr{D}$ keeps the center $\mathfrak{z}$ of $\mathfrak{g}(\mathscr{D})$, we have $\dim\mathfrak{z}^C(p)=$constant when p moves over $\mathscr{D}$ ([1] Proposition 2.3). But this dimension is zero on $\mathscr{V}$ ($\neq\emptyset$) and non-zero outside $\mathscr{D}\cap(C^l\times\{0\}\times C^n)$. This is a contradiction. Hence we have $c=\frac{1}{2}$ or $c=d$. If $c=d$, the domain $\mathscr{D}$ is an S-domain. Therefore $\mathscr{D}$ is a Siegel domain by J. Vey's result. Next let $c=\frac{1}{2}$. If $d\neq\frac{1}{2}$, we have a contradiction by repetition of the argument which begins at §2. Hence $d=\frac{1}{2}$. Therefore $\mathscr{D}$ is a Siegel domain by J. Vey's result. q. e. d.

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