Topology of C^n minus a finite number of affine hyperplanes in general position

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§ 1. Statement of results

Let $L_1, \cdots, L_k$ be affine hyperplanes in the complex n-space C^n. We say they are in general position, if $\dim L_{i_1}\cap\cdots\cap L_{i_l}=n-l$ for all sequences $1\leqq i_1<\cdots<i_l\leqq k$ with $1\leqq l\leqq n+1$. When $l=n+1$, this means that $L_{i_1}\cap\cdots\cap L_{i_{n+1}}=\varnothing$. Similarly, homogeneous hyperplanes $\hat{L}_0, \hat{L}_1, \cdots, \hat{L}_k$ in C^{n+1} are said to lie in general position, if $\dim \hat{L}_{i_0}\cap\cdots\cap\hat{L}_{i_l}=n-l$ for all sequences $0\leqq i_0<\cdots<i_l\leqq k$ with $0\leqq l\leqq n$.

The purpose of the present note is to describe adequately the homotopy types of $C^n-L_p\cup\cdots\cup L_k$ and $C^{n+1}-\hat{L}_0\cup\cdots\cup\hat{L}_k$. Let $T^k=S^1\times\cdots\times S^1$ (k-times product) be the k-dimensional torus, where S^1 is the unit circle in C as usual. We denote by $\boldsymbol{k}$ the set $\{1, 2, \cdots, k\}$. If I is a subset of $\boldsymbol{k}$, we denote by $|I|$ the cardinal number of I. We define the subtorus T_I of T^k by

$$T_I=\{z \mid z=(z_1, \cdots, z_k)\in T^k\,, \quad z_j=1 \text{ for } j\notin I\}\,.$$

The dimension of T_I is equal to $|I|$.

Our main results can be formulated as follows.

THEOREM 1. *Let $L_1, \cdots, L_k$ be affine hyperplanes in C^n in general position, where $n+1\leqq k$. Then the space $X=C^n-L_1\cup\cdots\cup L_k$ has the same homotopy type as the space*

$$X_0=\bigcup_{\substack{I\in \boldsymbol{k}\\ |I|=n}} T_I\,.$$

THEOREM 2. *Let $\hat{L}_0, \cdots, \hat{L}_k$ be homogeneous hyperplanes in C^{n+1} in general position where $n+1\leqq k$. Then the space $C^{n+1}-\hat{L}_0\cup\cdots\cup\hat{L}_k$ has the same homotopy type as $X_0\times S^1$ where X_0 is as in Theorem 1.*

It is now easy to describe homotopical properties of the space $C^n-L_1\cup\cdots\cup L_k$. For instance, we have

THEOREM 3. *Let X be as in Theorem 1 and suppose that $1<n$. Then the fundamental group $\pi_1(X)$ is free abelian of rank k and the universal covering space $\tilde{X}$ of X has trivial homology in dimensions $\neq 0, n$, while the n-th homology group has free $\boldsymbol{Z}(\pi_1(X))$-resolution*

$$0\longleftarrow H_n(\tilde{X}, \boldsymbol{Z})\xleftarrow{\ \partial\ }\tilde{C}_{n+1}\xleftarrow{\ \partial\ }\tilde{C}_{n+2}\xleftarrow{\ \partial\ }\cdots\xleftarrow{\ \partial\ }\tilde{C}_k\longleftarrow 0\,,$$

where $\tilde{C}_j$ *is a free* $\boldsymbol{Z}(\pi_1(X))$*-module on* $\binom{k}{j}$ *generators. In particular if* $n+1=k$, *then* $H_n(\tilde{X}, \boldsymbol{Z})$ *is a free* $\boldsymbol{Z}(\pi_1(X))$*-module on one generator.*

THEOREM 4. *Let* X *be as in Theorem 1. If* $\mathscr{S}$ *is a non-trivial local system over* X *with stalk* $\boldsymbol{C}$, *then the homology group* $H_i(X, \mathscr{S})$ *vanishes for* $i\neq n$. *The* n*-th homology group* $H_n(X, \mathscr{S})$ *is a* $\boldsymbol{C}$*-vector space of dimension*

$$\sum_{i=1}^{k-n}(-1)^{i+1}\binom{k}{n+i}.$$

In the previous paper [2], we have treated the case $n+1=k$ and discussed applications to the Euler integral representation of hypergeometric functions. We refer to [1] for results related to Theorem 4.

Proofs of the above theorems go along the same line of idea as in [2]. The proofs of Theorem 1 and Theorem 2 are given in Section 2. Theorem 3 and Theorem 4 will be proved in Section 3.

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§2. Proof of Theorem 1

PROPOSITION (2.1). *Let* $L_1, \cdots, L_k$ *be affine hyperplanes in general position in* $\boldsymbol{C}^n$. *Then the diffeomorphism type of the space* $\boldsymbol{C}^n - L_1\cup\cdots\cup L_k$ *depends only on* n *and* k.

Proof of (2.1) will be given in Section 4.

In view of Proposition (2.1), we may take a particular set $L_1, \cdots, L_k$ to study the homotopy type of $\boldsymbol{C}^n-L_1\cup\cdots\cup L_k$. Hereafter, we shall assume that the L_i are given by

$$\begin{aligned}(2.2)\qquad & L_i:\ z_i=0 \quad\text{for}\quad 1\leqq i\leqq n\,,\\ & L_{n+j}:\ \sum a_\nu^{j-1}z_\nu=1 \quad\text{for}\quad 1\leqq j\leqq q\,,\ q=k-n\,,\end{aligned}$$

where $a, \cdots, a_n$ are real numbers such that

$$(2.3)\qquad 1>a_1>a_2>\cdots>a_n>0\,.$$

The fact that $L_1, \cdots, L_k$ given by (2.2) lie in general position follows easily from the following

LEMMA (2.4). *Let* $b_1, \cdots, b_m$ *be real numbers such that* $b_1>\cdots>b_m>0$ *and* $i_1, \cdots, i_m$ *non-negative integers such that* $i_1<\cdots<i_m$. *Then, the determinant*

$$\begin{vmatrix} b_1^{j_1} & \cdots & b_m^{j_1} \\ \vdots & & \vdots \\ b_1^{j_m} & \cdots & b_m^{j_m} \end{vmatrix}$$

is never zero and its sign equals $(-1)^{\frac{m(m-1)}{2}}$.

PROOF. We have the equality

$$\begin{vmatrix} b_1^{j_1} & \cdots & b_m^{j_1} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ b_1^{j_m} & \cdots & b_m^{j_m} \end{vmatrix} = \begin{vmatrix} 1 & \cdots\cdots & 1 \\ b_1 & \cdots\cdots & b_m \\ \vdots & & \vdots \\ b_1^{m-1} & \cdots & b_m^{m-1} \end{vmatrix} \sum K_{j_1\cdots j_m} b_1^{j_1}\cdots b_m^{j_m},$$

where the coefficients $K_{j_1\cdots j_m}$ are non-negative integers [4, Th. (7.5. B)]. But the determinant at the right-hand side is equal to $(-1)^{m(m-1)/2}\prod_{i<j}(b_i-b_j)$ by Vandermonde. Hence the lemma follows.

Next let $\varphi: C^n \to C^k$ by the affine embedding defined by $\varphi(z_1, \cdots, z_n)=(w_1, \cdots, w_k)$, where

$$\begin{aligned} &w_i=(-1)^{i-1}z_i \quad \text{for} \quad 1\leqq i\leqq n\,, \\ &w_{n+j}=(-1)^n\Big(1-\sum_{\nu=1}^{n} a_\nu^{j-1}z_\nu\Big) \quad \text{for} \quad 1\leqq j\leqq q\,. \end{aligned} \tag{2.5}$$

Thus $\varphi(L_i)$ is the intersection of $\varphi(C^n)$ with the hyperplane in C^k defined by $w_i=0$. As usual, we shall identify the complex k-space with the product of real part and imaginary part. Thus, if $w=(w_1, \cdots, w_k)\in C^k$ and $w_i=u_i+\sqrt{-1}v_i$, then w is identified with $(u, v)\in R^k\times R^k$, where $u=(u_1, \cdots, u_k)$ and $v=(v_1, \cdots, v_k)$. With this understanding, let U and V be the real and imaginary part of $\varphi(C^n)$ respectively. Since the embedding φ is defined over the reals, we have $\varphi(C^n)=U\times V$. We define X to be $\varphi(C^n)-\varphi(L_1)\cup\cdots\cup\varphi(L_k)$, which is of course homeomorphic to $C^n-L_1\cup\cdots\cup L_k$. Thus $X=(U\times V)\cap(C^*)^k$ where $C^*=C-0$. Let $X^*=(U\times R^k)\cap(C^*)^k$.

PROPOSITION (2.6). *The inclusion $X\subset X^*$ is a homotopy equivalence.*

PROOF. We consider the subspace F of $X^*\times R^k$ defined by

$$F=\{(u, v, y)\in X^*\times R^k \mid v+y\in V \text{ and } y_j=0 \text{ whenever } u_j=0\}\,.$$

Let $p: F\to X^*$ be the projection onto the first factor. We need the following

LEMMA (2.7). *There exists a cross-section*

$$s: X^*\to F$$

of the projection $p: F\to X^*$.

PROOF OF (2.7). For a subset $I\subset \boldsymbol{k}$ with $|I|\leqq n$, we put

$$X_I^* = \{(u, v) \in X^* \mid u_i = 0 \text{ for } i \in I\}$$

and

$$F_I = \{(u, v, y) \in X_I^* \times \boldsymbol{R}^k \mid v + y \in V \text{ and } y_i = 0 \text{ for } i \in I\} .$$

For the empty set $\emptyset$ we have $X_\emptyset^* = X^*$.

The proof of the following lemma is easy and is left to the reader.

LEMMA (2.8). *The projection $p: F_I \to X_I^*$ is a locally trivial fiber space which has an affine subspace of $\boldsymbol{R}^k$ of dimension $n - |I|$ as fiber.*

For $0 \leqq i \leqq n$, let $X_{(i)}^* = \bigcup_{|I| = n-i} X_I^*$ and $F_{(i)} = \bigcup_{|I| = n-i} F_I$. Then $X_{(i)}^* \subset X_{(i+1)}^*$, $F_{(i)} \subset F_{(i+1)}$, $X^* = X_{(n)}^*$ and $F \subset F_{(n)}$. By the iterated use of the obstruction theory (see e.g. [3]) applied to (2.8), we see that there exists a cross-section s of $p: F_{(n)} \to X_{(n)}^* = X^*$ such that $s(X_{(i)}^*) \subset F_{(i)}$. On the other hand, over $X_{(i)}^* - X_{(i-1)}^*$, the portion of F coincides with that of $F_{(i)}$. Thus, a cross-section s of $p: F_{(n)} \to X^*$ as above is precisely that of $p: F \to X^*$. This proves (2.7).

Using (2.7), we define a homotopy $f_t: X^* \to X^*$ by

$$f_t(u, v) = (u, v + ty(u, v)) ,$$

where $s(u, v) = (u, v, y(u, v))$. Then, f_0 is the identity and f_1 maps X^* into X. Moreover, f_t keeps X into itself since V is a linear subspace of $\boldsymbol{R}^k$, as is easily seen. It follows that the inclusion is a homotopy equivalence with homotopy inverse f_1. This completes the proof of Proposition (2.6).

Let l_i denote the hyperplane $\varphi(L_i) \cap U$ in U. These hyperplanes $l_1, \cdots, l_k$ give a cellular decomposition K of U by open convex cells. Indeed, if I is a subset of $\boldsymbol{k}$ and $\varepsilon_j = \pm 1$ for $j \notin I$, then the subset

$$\{u \in U \mid \ u_i = 0 \text{ for } i \in I,\ \varepsilon_j u_j > 0 \text{ for } j \notin I\}$$

is an open cell. Let K' be the barycentric subdivision of K and N the union of the relatively compact simplices of K' (see Fig. 1). K' should be interpreted as follows. We compactify U adding a projective $(n-1)$-space P_∞ at infinity. The resulting space is a projective n-space P^n in which each l_i is compactified into a projective $(n-1)$-space P_i. These $P_1, \cdots, P_k, P_\infty$ give us a cellular decomposition of P^n. We take its barycentric subdivision $\tilde{K}$. Then we delete P_∞ from $\tilde{K}$ and obtain K.

If I is a subset of $\boldsymbol{k}$ with $|I| = n$, let u_I denote the intersection point $\bigcap_{i \in I} l_i$. The collection $\{u_I\}_{|I|=n}$ is nothing but the 0-skeleton of K. We denote by D_I the closure of the union of those simplices τ of N whose closures contain u_I.

We shall denote by the same letter N the underlying space of the cell complex

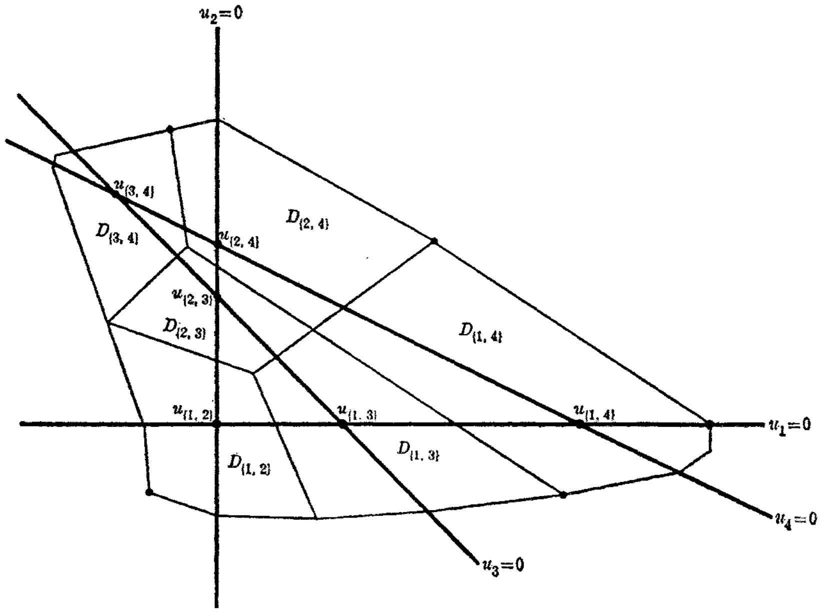

Fig. 1

N. The D_I are closed dual n-cells of N. It is easy to see that there exists a deformation retraction $\rho: U \to N$ which preserves the cells of K. In particular $\rho^{-1}(l_i)$ equals l_i itself. We set $X_2^* = \{(u, v) \in X^* | u \in N\}$. From the existence of a deformation retraction ρ with the above property, it follows easily that X_2^* is a deformation retract of X^*.

Next let X_1^* be the subspace of X_2^* consisting of the points (u, v) such that $v_j = 0$ for $j \notin I$ whenever $u \in D_I$.

PROPOSITION (2.9). *The inclusion $X_1^* \subset X^*$ is a homotopy equivalence.*

PROOF. We shall first define a map $u' = (u'_1, \cdots, u'_k): N \to [0, 1]^k$ as follows. If σ is a cell of K, let b_σ denote the barycenter of σ, J_σ the set $\{j \in \boldsymbol{k} \mid l_j \cap \bar{\sigma} \neq \emptyset\}$ and I_σ the set $\{i \in \boldsymbol{k} | \sigma \subset l_i\}$. We define u' on b_σ by

$$u'_j(b_\sigma) = \begin{cases} 1 & \text{if } j \notin J_\sigma , \\ \dfrac{|J_\sigma| - n}{|J_\sigma| - |I_\sigma|} & \text{if } j \in J_\sigma - I_\sigma , \\ 0 & \text{if } j \in I_\sigma . \end{cases}$$

The b_σ are just the 0-simplices of N. We then extend u' on each simplex of N affinely. The following is an easy consequence of the definition.

LEMMA (2.10). *If u is in D_I, then $u'_i(u) \leq u'_j(u)$ for all pairs (i, j) with*

$i\in I$ and $j\notin I$. Moreover $u_j'(u)>0$ if $u\in D_I$ and $j\notin I$.

Now let $m_q:[0,1]^k\to[0,1]$ be the function given by $m_q(u')=u_{j_q}'$ where $u_{j_1}'\geqq\cdots\geqq u_{j_q}'\geqq\cdots\geqq u_{j_k}'$ and $q=k-n$. It is a continuous function. We define a homotopy $g_t: X_2^*\to X_2^*$ by

$$(2.11)\qquad \begin{aligned} &g_t(u, v)=(u, v(t))\,,\\ &v(t)_i=\left(1-\frac{\min(u_i'(u), m_q(u'(u)))}{m_q(u'(u))}t\right)v_i\,. \end{aligned}$$

Note that $m_q(u'(u))>0$ by (2.10), so that g_t is a well-defined homotopy. It follows also from (2.10) that g_0 is the identity, that g_1 maps X_2^* into X_1^* and that g_t keeps X_1^* into itself. Therefore, the inclusion: $X_1^*\to X_2^*$ is a homotopy equivalence with homotopy inverse g_1. Since X_2^* is a deformation retract of X^*, the inclusion $X_1^*\subset X^*$ is also a homotopy equivalence. This completes the proof of Proposition (2.9).

For a given $I\subset \boldsymbol{k}$, $|I|=n$, and $j\notin I$, we define $\delta(I, j)$ by

$$\delta(I, j)=\prod_{i\in I}\operatorname{sign}(i-j)\,.$$

LEMMA (2.12). *Let $|I|=n$, $I\subset\boldsymbol{k}$, and $j\notin I$. Then, the sign of the j-th coordinate u_j of the point u_I is equal to $\delta(I, j)$.*

This follows from a bit of linear algebra together with Lemma (2.4). The details are left to the reader.

If $I\subset\boldsymbol{k}$, $|I|=n$, we define the subtorus T_I' of T^k by

$$T_I'=\{(z_1,\cdots,z_k)\in T^k\,|\,z_j=\delta(I, j)\text{ for } j\notin I\}\,,$$

and put $E_I=\{(u, v)\in X_1^*|u\in D_I\}$. Let $r:(C^*)^k\to T^k$ denote the usual projection:

$$r(w_1,\cdots,w_k)=\left(\frac{w_1}{|w_1|},\cdots,\frac{w_k}{|w_k|}\right)\,.$$

By (2.12), r maps E_I into T_I' and hence X_1^* into

$$X_1=\bigcup_{I\subset\boldsymbol{k},|I|=n}T_I'\,.$$

Let $r_1: X_1^*\to X_1$ denote the restriction of r on X_1^*.

PROPOSITION (2.13). *The projection $r_1: X_1^*\to X_1$ is a homotopy equivalence.*

PROOF. Let $I_1,\cdots,I_s$ be subsets of $\boldsymbol{k}$ such that $|I_\nu|=n$ for all ν. Notice that the intersection $\bigcap_{1\leq\nu\leq s}T_{I_\nu}'$ is non-empty if and only if the following condition (A) is satisfied.

(A) *Given $j\in\boldsymbol{k}$, $\delta(I_\nu, j)$ does not depend on the choice of ν such that $j\notin I_\nu$.*

A collection $C=\{I_1, \cdots, I_s\}$ will be called an admissible collection if it satisfies the condition (A). An admissible collection $C=\{I_1, \cdots, I_s\}$ will be called a basic collection if it satisfies the following condition (B).

(B) *If $I\subset \boldsymbol{k}$, $|I|=n$, is such that $C\cup\{I\}$ is also admissible, then the intersection $\bigcap_{I_\nu \in C} I_\nu$ is strictly greater than $(\bigcap_{I_\nu \in C} I_\nu)\cap I$.*

We shall denote by $\mathscr{B}$ the set of all basic collections $C=\{I_1, \cdots, I_s\}$. If $C\in\mathscr{B}$, then $T(C)=\bigcap_{I_\nu \in C} T_\nu'$ is a subtorus of T^k of dimension $|\bigcap_{I_\nu \in C} I_\nu|$. For $C\in\mathscr{B}$, we set

$$S_C=T(C)-\bigcup_{\substack{C'\in\mathscr{B}\\ C\subsetneqq C'}} T(C') .$$

Clearly, $\{S_C\}$ gives a stratification of X_1.

Given $C\in\mathscr{B}$, we set $N(C)=\bigcap_{I\in C} D_I$ and $S^*(C)=\{(u, v)\in X_1^*|u\in N(C)\}$. It is clear that $S^*(C)=\bigcap_{I\in C} E_I$ and that the projection $r_1: X_1^*\to X_1$ maps $S^*(C)$ into $T(C)$.

LEMMA (2.14). *The projection $r(C)=r_1|S^*(C): S^*(C)\to T(C)$ is a fiber space with a contractible fiber. Any segment in X^* joining a point of $r_1^{-1}(z)\in X_1^*$ to a point of $r(C)^{-1}(z)$, where $z\in S_C$, is contained in $r_1^{-1}(z)$.*

PROOF OF (2.14). Given $z\in X_1$, the real part of the fiber $r_1^{-1}(z)$ over z is contained in an open cell σ uniquely determined by z. Thus the real part of $r(C)^{-1}(z)$ for $z\in T(C)$ is contained in $\sigma\cap N(C)$, which is clearly contractible. From this, it follows easily that the fiber $r(C)^{-1}(z)$ is contractible.

Next let z be a point of S_C. If $w=(u, v)$ is a point in the fiber $r_1^{-1}(z)$ and $u\in D_I$, then I must be a member of C. For otherwise z should belong to a subtorus which is strictly smaller than $T(C)$. Then, any segment in U joining $u\in\sigma\cap D_I$ to a point of $\sigma\cap N(C)$ is clearly contained in $\sigma\cap D_I$. It follows that any segment in X^* joining w to a paint of $r(C)^{-1}(z)$ is contained in $r_1^{-1}(z)$. Thus Lemma (2.14) is proved.

By virtue of Lemma (2.14), there are no obstructions to extending a cross-section of $r(C)$ given on $T(C)-S_C$ over the whole torus $T(C)$. Hence we can construct a cross-section $s: X_1\to \bigcup_{C\in\mathscr{B}} S^*(C)$ of r_1 such that $s(S_C)\subset S^*(C)$, proceeding stepwise on the stratification $\{S_C\}$. Then, again by virtue of (2.14), the homotopy $tw+(1-t)sr_1(w)$ takes place in the space X_1^*. Thus, sr_1 is homotopic to the identity map of X_1^*; the projection r_1 is a homotopy equivalence and s is a homotopy inverse of r_1. This completes the proof of Proposition (2.13).

So far, we have constructed explicit homotopy equivalences $X\subset X^*$, $X^*\supset X_1^*$ and $r_1: X_1^*\to X_1$. Thus, the proof of Theorem 1 will be achieved by the following

PROPOSITION (2.15). *$X_1=\bigcup_I T_I'$ has the same homotopy type as $X_0=\bigcup_I T_I$.*

PROOF. We shall change the notations slightly and write $X(n, k)=\bigcup_{\substack{I\subset k\\ |I|=n}} T_I$ and $X'(n, k)=\bigcup_{\substack{I\subset k\\ |I|=n}} T_I'$ for $1\leqq n\leqq k$. Then, we set

$$X_a(n, k)=\bigcup_{\substack{I\subset k, |I|=n\\ k\in I}} T_I\,, \qquad X_a'(n, k)=\bigcup_{\substack{I\subset k, |I|=n\\ k\in I}} T_I'\,,$$

and, when $n<k$,

$$X_b(n, k)=\bigcup_{\substack{I\subset k, |I|=n\\ k\notin I}} T_I\,, \qquad X_b'(n, k)=\bigcup_{\substack{I\subset k, |I|=n\\ k\notin I}} T_I'\,.$$

Clearly we have

$$\begin{aligned}
&X(n, k)=X_a(n, k)\cup X_b(n, k)\,,\\
&X'(n, k)=X_a'(n, k)\cup X_b'(n, k)\,,\\
&X_a(n, k)=X(n-1, k-1)\times S^1\,,\\
&X_b(n, k)=X(n, k-1)\times 1\,,\\
&X_a'(n, k)=X'(n-1, k-1)\times S^1\,, \quad \text{where} \quad X'(0, k-1)=1\,,\\
&X_b'(n, k)=X'(n, k-1)\times(-1)^n\,.
\end{aligned}$$

Moreover $X(n-1, k)\subset X(n, k)$ and $X'(n-1, k)\subset X'(n, k)$.

The condition $n+1\leqq k$ is indispensable in Theorem 1, but not in Proposition (2.15). Indeed we shall prove

PROPOSITION (2.16). *For each pair (n, k) of integers with $1\leqq n\leqq k$, we can associate a homotopy equivalence $f(n, k): X(n, k)\rightarrow X'(n, k)$ such that*
1) $f(n, k)|X(n-1, k)=f(n-1, k): X(n-1, k)\rightarrow X'(n-1, k)$ and
2) if we regard $f(n, k)$ as a map $X(n, k)\rightarrow T^k$, then $f(n, k)$ is homotopic to the inclusion $i: X(n, k)\rightarrow T^k$.

PROOF. We shall proceed by double induction on (n, k).
Case: $n<k$. Assume that we have $f(m, k-1)$ satisfying the conditions 1) and 2) for $1\leqq m\leqq k-1$. We define $f(n, k)$ to be equal to $f(n-1, k-1)\times(-1)^n$ on $X_a(n, k)=X(n-1, k-1)\times S^1$ and to $f(n, k-1)\times(-1)^n$ on $X_b(n, k)=X(n, k-1)\times 1$. Using the condition 1) for $f(n, k-1)$, we see easily that $f(n, k)$ is a well-defined map and that $f(n, k)$ itself satisfies the condition 1).

To prove the condition 2) for $f(n, k)$, we note that $f(n, k)$ is homotopic to i if and only if $f(n, k)_*=i_*: H_*(X(n, k), \boldsymbol{Z})\rightarrow H_*(T^k, \boldsymbol{Z})$, since T^k is an Eilenberg-MacLane space. By definition, $X(n, k)$ is the n-skeleton of the usual economical cell decomposition of T^k so that i_* identifies $H_q(X(n, k))$ with $H_q(T^k)$ for $q\leqq n$. By the Künneth formula, $f(n, k)_*$ restricted on $H_*(X_a(n, k))$ is identified with

$f(n-1, k-1)_* \otimes (-1)^n_* : H_*(X(n-1, k)) \otimes H_*(S^1) \to H_*(T^{k-1})_* \otimes H_*(S^1) = H_*(T^k)$. But the multiplication by $(-1)^n$ on S^1 has degree 1, so that $(-1)^n_*$ is the identity. Using the condition 2) for $f(n-1, k)$, we see that $f(n-1, k-1)_* \otimes (-1)^n_*$ coincides with i_*. In a similar way, we see that $f(n, k)_*$ and i_* restricted on $H_*(X_b(n, k))$ coincide. By an argument using Mayer-Vietoris sequence, it then follows that $f(n, k)_* = i_*$ on $H_*(X(n, k))$. Thus, $f(n, k)$ satisfies the condition 2).

Case: $n=k$. Assume we already have $f(k-1, k)$ satisfying 1) and 2). T^k has a cell decomposition $T^k = X(k-1, k) \cup e^k$. Since $i: X(k-1, k) \to T^k$ has the obvious extension $i: X(k, k) = T^k \to T^k$ and $f(k-1, k)$ is homotopic to i by 2), it follows from the homotopy extension property that $f(k-1, k)$ has an extension $f(k, k)$: $X(k, k) \to T^k$ such that $f(k, k)$ is homotopic to i. Such an $f(k, k)$ obviously satisfies the conditions 1) and 2). This completes the proof of Proposition (2.16), and hence of Proposition (2.15) and Theorem 1.

Finally we shall prove Theorem 2. Let $\hat{L}_0, \hat{L}_1, \cdots, \hat{L}_k$ be homogeneous hyperplanes in C^{n+1} in general position and $P_0, P_1, \cdots, P_k$ the corresponding projective hyperplanes in CP^n. As usual, we identify $CP^n - P_0$ with C^n. Then $L_1 = P_1 - P_0, \cdots, L_k = P_k - P_0$ are affine hyperplanes in general position in $C^n = CP^n - P_0$. If $\pi: C^{n+1} - 0 \to CP^n$ denotes the Hopf fibering then we have

$$\pi^{-1}(C^n - L_1 \cup \cdots \cup L_k) = C^{n+1} - \hat{L}_0 \cup \hat{L}_1 \cup \cdots \cup \hat{L}_k .$$

Since the Hopf fibering restricted on $C^n = CP^n - P_0$ is trivial it follows that $C^{n+1} - \hat{L}_0 \cup \hat{L}_1 \cup \cdots \cup \hat{L}_k$ is diffeomorphic to $(C^n - L_1 \cup \cdots \cup L_k) \times C^*$. Hence, by Theorem 1, $C^{n+1} - \hat{L}_0 \cup \hat{L}_1 \cup \cdots \cup \hat{L}_k$ has the same homotopy type as $X_0 \times S^1$.

§ 3. Homology with local system coefficients

First, we shall investigate the universal covering of $X = C^n - L_1 \cup \cdots \cup L_n$. Let X_0 be the space as in Section 1. Since X_0 is the n-skeleton of the usual economical cell decomposition of T^k, the homomorphism $\pi_1(X_0) \to \pi_1(T^k)$, induced by the inclusion, is surjective for $n \geqq 1$ and bijective for $n > 1$. Therefore, if p_0: $R^k \to T^k$, $p_0(t_1, \cdots, t_k) = (e^{2\pi i t_1}, \cdots, e^{2\pi i t_k})$, is the usual projection, then $\tilde{X}_0 = p_0^{-1}(X_0)$ is the universal covering space for $n > 1$. We have obtained

PROPOSITION (3.1). *Let $\tilde{X}_0$ be the subspace of R^k consisting of the elements $x = (x_1, \cdots, x_k)$ such that $x_j \in Z$ for $j \notin I$ for some $I \subset k$, $|I| = n$. If $n > 1$, then $\tilde{X}_0$ is the universal covering space of X_0 with projection p_0.*

In passing, we note the following

PROPOSITION (3.2). *If $n > 1$, then the universal covering space of C^n—*

$L_1\cup\cdots\cup L_k$, where $L_1,\cdots,L_k$ are affine hyperplanes in general position, can be embedded in C^k as a complex analytic submanifold.

PROOF. We consider the commutative diagram

$$\begin{array}{ccc} C^k & \xrightarrow{\tilde r} & R^k \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p_0} \\ (C^*)^k & \xrightarrow{r} & T^k \end{array}$$

where $p(w_1,\cdots,w_k)=(e^{2\pi i w_1},\cdots,e^{2\pi i w_k})$, $\tilde r(u,v)=u$ and r is as in Section 2. The map p is also a universal covering map. Now, as was shown in Proposition (2.16), X_0 is deformed into X_1 in T^k. Therefore from (3.1) it follows that $p_0^{-1}(X_1)$ is the universal covering of X_1. Since $r:((C^*)^k, X_1^*)\to(T^k, X_1)$ is a homotopy equivalence by (2.13), $p^{-1}(X_1^*)$ is the universal covering of X_1^*. By (2.6) and (2.11), the map $g_1: X\to X_1^*$ is a homotopy equivalence and the composition $X\xrightarrow{g_1}X_1^*\subset(C^*)^k$ is homotopic to the inclusion $X\subset(C^*)^k$. Hence it follows finally that $p^{-1}(X)\subset C^k$ is the universal covering of X which is identified with $C^n-L_1\cup\cdots\cup L_k$ through φ as in Section 2. This proves (3.2).

We turn to homological properties of X and proceed to prove Theorems 3 and 4. Changing the notations slightly, we denote by $X(n)$ the subspace $\bigcup_{|I|=n} T_I$ of T^k and by $\tilde X(n)$ the subspace $p_0^{-1}(X(n))$ of R^k where $1\leqq n\leqq k$. $p_0:\tilde X(n)\to X(n)$ is the projection of a regular covering with group $H_1(T^k, Z)$ which is canonically identified with $H_1(X(n), Z)$. If $n>1$, then $\pi_1(X(n))=H_1(X(n), Z)$ and the covering $p_0:\tilde X(n)\to X(n)$ is universal as in (3.1). For $I\subset k=\{1, 2,\cdots,k\}$ we set

$$\sigma_I=\{x\mid x\in R^k\,,\quad 0<x_i<1 \text{ for } i\in I,\ x_j=0 \text{ for } j\notin I\}\,.$$

σ_I is an open cell of dimension $|I|$. The totality of the transforms $\sigma_I\cdot\gamma$, $|I|\leqq n$, $\gamma\in H_1(T^k, Z)$, forms a cell decomposition of $\tilde X(n)$. Let $\tilde C_p$ denote the free $Z(H_1(T^k, Z))$-module generated by the σ_I with $|I|=p$ and $\partial:\tilde C_p\to\tilde C_{p-1}$ the usual boundary operator ($Z(H_1(T^k, Z))$-module map) defined by

$$\partial\sigma_I=\sum_{i\in I}\varepsilon(I, i)(\sigma_{I-\{i\}}-\sigma_{I-\{i\}}\cdot\gamma_i)$$

where $\varepsilon(I, i)=\prod_{i'\in I, i'\neq i}\operatorname{sign}(i'-i)$ and γ_i is the generator of $H_1(T^k, Z)$ determined by $1\times\cdots\times1\times S^1\times1\times\cdots\times1$, S^1 being placed at the i-th factor. From the above cell decomposition we obtain the chain complex $C_*(\tilde X(n))$ given by

$$C_p(\tilde X(n))=\begin{cases}\tilde C_p\,, & 0\leqq p\leqq n\,,\\ 0\,, & \text{otherwise}\,.\end{cases}$$

PROPOSITION (3.3). *The p-th homology group $H_p(\tilde{X}(n), \boldsymbol{Z})$ vanishes for $p\neq 0$, n. The n-th homology group has a free $\boldsymbol{Z}(H_1(T^k, \boldsymbol{Z}))$-resolution*

$$0 \longleftarrow H_n(\tilde{X}(n), \boldsymbol{Z}) \xleftarrow{\partial} \tilde{C}_{n+1} \xleftarrow{\partial} \cdots \xleftarrow{\partial} \tilde{C}_k \longleftarrow 0 .$$

PROOF. Since $C_p(\tilde{X}(n))=C_p(\tilde{X}(k))$ for $p\leqq n$ and $C_p(\tilde{X}(n))=0$ for $n<p$, we see that $H_p(\tilde{X}(n), \boldsymbol{Z})\cong H_p(\tilde{X}(k), \boldsymbol{Z})$ for $p<n$ and $H_n(\tilde{X}(n), \boldsymbol{Z})=Z(\tilde{C}_n)=$the cycles of $\tilde{C}_n$. But $\tilde{X}(k)=\boldsymbol{R}^k$. Hence we have $H_p(\tilde{X}(k), \boldsymbol{Z})=0$ for $0<p$ and the sequence

$$0 \longleftarrow H_n(\tilde{X}(n), \boldsymbol{Z})=Z(\tilde{C}_n) \xleftarrow{\partial} \tilde{C}_{n+1} \longleftarrow Z(\tilde{C}_{n+1}) \longleftarrow 0$$

is exact. Iterating the process we obtain the desired resolution.

Theorem 3 is an immediate consequence of Theorem 1 and Proposition (3.3).

Next, let $\mathscr{S}$ be a local system with stalk $\boldsymbol{C}$ on $X(n)$ with characteristic homomorphism $\theta:\pi_1(X(n))\to \boldsymbol{C}^*$. Since θ factors through $\pi_1(X(n))\to H_1(X(n), \boldsymbol{Z})$ and $H_1(X(n), \boldsymbol{Z})=H_1(T^k, \boldsymbol{Z})$, the local system $\mathscr{S}$ is the restriction of a unique local system on T^k which we shall also denote by $\mathscr{S}$.

As is well-known, the homology $H_*(X(n), \mathscr{S})$ of $X(n)$ with coefficients in $\mathscr{S}$ can be computed from the chain complex $C_*(X(n), \mathscr{S})$, where $C_p(X(n), \mathscr{S})$ is the $\boldsymbol{C}$-vector space with basis $\{\sigma_I \mid |I|=p\}$ for $0\leqq p\leqq n$, $C_p(X(n), \mathscr{S})=0$ otherwise, and the boundary operator ∂ is given by

$$\partial\sigma_I=\sum_{i\in I}\varepsilon(I, i)(1-\theta(\gamma_i))\sigma_{I-\{i\}} ,$$

cf. [2].

PROPOSITION (3.4). *If $\mathscr{S}$ is a non-trivial local system with stalk $\boldsymbol{C}$ over $X(n)$ then the homology group $H_p(X(n), \mathscr{S})$ vanishes for $p\neq n$. The n-th homology group $H_n(X(n), \mathscr{S})$ is a $\boldsymbol{C}$-vector space of dimension $\sum_{i=1}^{k-n}(-1)^{i+1}\binom{k}{n+i}$.*

REMARK (3.5). *Suppose that the affine hyperplanes $L_1, \cdots, L_k$ in C^n are defined over the reals and lie in general position. Let l_i be the real part of L_i, i. e. $l_i=\boldsymbol{R}^n\cap L_i$. It can be shown without difficulty that the number of the relatively compact domain in $\boldsymbol{R}^n$ bounded by some of the l_i is precisely equal to $\sum_{i=1}^{k-n}(-1)^{i+1}\binom{k}{n+i}$.*

PROOF OF PROPOSITION (3.4). Since $C_p(X(n), \mathscr{S})=C_p(X(k), \mathscr{S})$ for $p\leqq n$ and $C_p(X(n), \mathscr{S})=0$ for $n<p$, we have $H_p(X(n), \mathscr{S})=H_p(X(k), \mathscr{S})$ for $p<n$ and $H_n(X(n), \mathscr{S})=Z(C_n(X(k), \mathscr{S}))=$the cycles of $C_n(X(k), \mathscr{S})$. By assumption $\mathscr{S}$ is non-trivial. Hence there is a γ_i for which $\theta(\gamma_i)\neq 1$. We may assume without loss of generality that $\theta(\gamma_k)\neq 1$. We have a homotopy operator $s: C_p(X(k), \mathscr{S})\to$

$C_{p+1}(X(k), \mathscr{S})$ defined by

$$s(\sigma_I)=\begin{cases}\dfrac{(-1)^p}{1-\theta(\gamma_k)}\sigma_{I\cup\{k\}}\,, & \text{if } k\notin I\,,\\ 0\,, & \text{if } k\in I\,,\end{cases}$$

where $|I|=p$, $0\leq p\leq k$. It can be seen easily that the equality $\partial s+s\partial=1$ holds for all p. Therefore $H_p(X(k), \mathscr{S})=0$ and we have a short exact sequence

$$0 \longleftarrow Z(C_p(X(k), \mathscr{S})) \xleftarrow{\partial} C_{p+1}(X(k), \mathscr{S}) \longleftarrow Z(C_{p+1}(X(k), \mathscr{S}) \longleftarrow 0$$

for all p. It follows that $H_p(X(n), \mathscr{S})=H_p(X(k), \mathscr{S})=0$ for $p<n$ and

$$\dim H_n(X(n), \mathscr{S})=\dim Z(C_n(X(k), \mathscr{S}))=\sum_{i=1}^{k-n}(-1)^{i+1}\dim C_{n+i}(X(k), \mathscr{S})$$
$$=\sum_{i=1}^{k-n}(-1)^{i+1}\binom{k}{n+i}\,.$$

This proves (3.4).

Theorem 4 is a direct consequence of Theorem 1 and Proposition (3.4).

§4. Proof of (2.1)

Proposition (2.1) seems to be known by experts. But the author was not able to find a proof in the literature. In the following, a proof will be presented.

If I is a subset of $\boldsymbol{k}$, then we put

$$\boldsymbol{C}_I=\{w \mid w\in \boldsymbol{C}^k,\ w_i=0 \text{ for } i\in I\}\,.$$

If I consists of a single element i, then we write simply $\boldsymbol{C}_i$ for $\boldsymbol{C}_I$. We shall denote by $E^0(k, n)$ the set of all n-dimensional affine subspaces M in $\boldsymbol{C}^k$ which satisfy the condition:

(P) $\dim M\cap \boldsymbol{C}_I=n-|I|$ for all $I\subset \boldsymbol{k}$ with $1\leq|I|\leq n+1$.

An n-dimensional affine subspace M in $\boldsymbol{C}^k$ given by a system of linear equations

$$b_{i0}+\sum_{j=1}^{k} b_{ij}w_j=0\,,\quad i=1,\cdots,q\,, \tag{4.1}$$

where $n+q=k$, is in $E^0(k, n)$ if and only if the matrix $B=(b_{ij})_{\substack{1\leq i\leq q\\ 0\leq j\leq k}}$ satisfies the condition:

(Q) $\det B_{j_1\cdots j_q}\neq 0$ for all sequences $0\leq j_1<\cdots<j_q\leq k$,

where

$$B_{j_1\cdots j_q}=\begin{bmatrix} b_{1j_1} & \cdots & b_{1j_q}\\ \vdots & & \vdots\\ b_{qj_1} & \cdots & b_{qj_q}\end{bmatrix}.$$

We denote by $M^0(q, k+1)$ the set of $q\times(k+1)$ matrices $B=(b_{ij})$ which satisfy the condition (Q). Let $p: M^0(q, k+1)\to E^0(k, n)$ be the map which assigns to B the affine subspace determined by the equation (4.1). Note that the set $E(k, n)$ of all n-dimensional affine subspaces of C^k can be identified with the total space of the universal q-vector bundle over the complex Grassmann manifold $G(k, n)$. We regard $E^0(k, n)$ as an open submanifold of $E(k, n)$. Moreover $M^0(q, k+1)$ is also an open submanifold of the space of all $q\times(k+1)$ matrices and it is easy to see that $p: M^0(q, k+1)\to E^0(k, n)$ is a smooth fiber bundle.

We put

$$A(k, n)=\{(M, w)\,|\,(M, w)\in E^0(k, n)\times C^k,\ w\in M\}\ .$$

Let $\pi: A(k, n)\to E^0(k, n)$ be the restriction of the projection of $E^0(k, n)\times C^k$ into the first factor. The fiber $\pi^{-1}(M)$ over $M\in E^0(k, n)$ is canonically identified with the affine subspace M itself.

Now let $L_1, \cdots, L_k$ be affine hyperplanes in C^n in general position. If L_i is given by a linear equation

$$a_{i0}+\sum_{j=1}^{n} a_{ij}z_j=0\ ,$$

then we define an affine embedding $\varphi: C^n\to C^k$ by $\varphi(z_1, \cdots, z_n)=(w_1, \cdots, w_k)$, where

$$w_i=a_{i0}+\sum_{j=1}^{n} a_{ij}z_j\ .$$

φ maps L_i onto C_i and it is obvious that the image $\varphi(C^n)$ satisfy the condition (P). Thus, if we put $A^0(k, n)=A(k, n)\cap(E^0(k, n)\times(C^*)^k)$, where $C^*=C-0$, then $\varphi(C^n-L_1\cup\cdots\cup L_k)$ is isomorphic to a fiber of $\pi^0=\pi|A^0(k, n): A^0(k, n)\to E^0(k, n)$. Since the base space $E^0(k, n)$ is connected, the proof of (2.1) is reduced to the following

LEMMA (4.2). *$\pi^0: A^0(k, n)\to E^0(k, n)$ is a smooth (locally trivial) fiber bundle.*

PROOF OF (4.2). Put $A_i=A(k, n)\cap(E^0(k, n)\times C_i)$. Take a point $y_0\in E^0(k, n)$ and let O be a small open neighborhood of y_0 such that there is a smooth cross-section s of $p: M^0(q, k+1)\to E^0(k, n)$ over O. We shall construct a local triviality $\psi: \pi^{-1}(O)\to O\times\pi^{-1}(y_0)$ such that $\psi(\pi^{-1}(O)\cap A_i)=O\times(\pi^{-1}(y_0)\cap A_i)$ for all i, which clearly suffices to prove (4.2). Let $\mathscr{X}(O)$ and $\mathscr{X}(\pi^{-1}(O))$ denote the set of smooth vector fields on O and $\pi^{-1}(O)$ respectively. Constructing a map ψ as above is easily seen to be equivalent to constructing a map $\mathscr{X}(O)\to\mathscr{X}(\pi^{-1}(O))$, $v\mapsto\tilde{v}$, satisfying the following four conditions.

1) $\tilde{v}$ and v are π-related, i.e.

$$v_{\pi(x)}=\pi_*(\tilde{v}_x) .$$

2) $\tilde{v}$ depends smoothly on v.

3) If $c(t)$ is an integral curve of v and $\tilde{c}(t)$ an integral curve of $\tilde{v}$ such that $\pi\tilde{c}(0)=c(0)$ then $\tilde{c}(t)$ is defined for $t\in[0, a]$ whenever $c(t)$ is defined for $t\in[0, a]$.

4) Regarding $\tilde{v}$ as a vector field in $E^0(k, n)\times C^k$, let us write $\tilde{v}$ as $\tilde{v}=(v, \tilde{v}^{(1)}, \cdots, \tilde{v}^{(k)})$, where $\tilde{v}^{(i)}$ is the projection of $\tilde{v}$ into the i-th coordinate of C^k. If $x=(M, w_1, \cdots, w_k)\in\pi^{-1}(O)\subset(k, n)\times C^k$, then $\tilde{v}_x^{(i)}=0$ whenever $w_i=0$.

Now let $s(y)$ be given by the matrix $s(y)=(b_{ij}(y))$, $1\leqq i\leqq q$, $0\leqq j\leqq k$. Then, $(y, w)\in O\times C^k$ belongs to $\pi^{-1}(y)\subset A(k, n)$ by (4.1) if and only if the relations

$$b_{i0}(y)+\sum_{j=1}^{k} b_{ij}(y)w_j=0\,,\quad 1\leqq i\leqq q$$

hold. Let $y(t)$ be an integral curve of v. A curve in $A(k, n)$ covering $y(t)$ is of the form $(y(t), w(t))$ where $w(t)$ satisfies the relations

$$b_{i0}(y(t))+\sum_{j=1}^{k} b_{ij}(y(t))w_j(t)=0 .$$

Differentiating this we get

$$\frac{db_{i0}(y(t))}{dt}+\sum\frac{db_{ij}(y(t))}{dt}w_j(t)+\sum b_{ij}(y(t))\frac{dw_j(t)}{dt}=0 . \tag{4.3}$$

The cross-section s induces a vector field $s^*(v)$ on $s(O)$. Let it be given by $s^*(v)_{s(y)}=(v_{ij}(y))$, where we identified the tangent spaces of $M(q, k+1)$ with $M(q, k+1)$ itself as usual. We consider the equations in the unknowns $\tilde{v}^{(1)}, \cdots, \tilde{v}^{(k)}$

$$v_{i0}(y)+\sum_{j=1}^{k} v_{ij}(y)w_j+\sum_{j=1}^{k} b_{ij}(y)\tilde{v}^{(j)}=0 \tag{4.4}$$

with the additional conditions that $\tilde{v}^{(j)}=0$ whenever $w_j=0$.

For a subset $I\subset\boldsymbol{k}=\{1, 2, \cdots, k\}$ we put $A_I=\bigcap_{i\in I}A_i$. Note that A_I is empty if $|I|>n$. We put $U=\pi^{-1}(O)$ and $B_I=U\cup A_I$. Let U_I be a small open neighborhood of $B_I-\bigcup_{I\subset J}B_J$ in $U-\bigcup_{I\subset J}B_J$. We may take U_I small enough so that we have $U_I\cap U_{I'}=\emptyset$ if $|I|=|I'|$, $I\neq I'$. Then $\{U_I\}_{0\leqq|I|\leqq n}$ is an open covering of U. Let $\{\lambda_I\}$ be a smooth partition of unity subordinate to the covering $\{U_I\}$. We choose a map α of $\{I\}_{0\leqq|I|\leqq n}$ into $\{I\}_{|I|=n}$ such that $I\subset\alpha(I)$. Given I with $0\leqq|I|\leqq n$, the equations (4.4) has a unique solution $(\tilde{v}_I^{(1)}, \cdots, \tilde{v}_I^{(k)})$ such that $\tilde{v}_I^{(j)}=0$ if $j\in\alpha(I)$, since $(b_{ij}(y))\in M^0(q, k+1)$. Then the vector field $\tilde{v}_I=(v, \tilde{v}_I^{(1)}, \cdots, \tilde{v}_I^{(k)})$ on U_I clearly satisfies the desired conditions 1), 2) and 4).

We now put $\tilde{v}=\sum\lambda_I\tilde{v}_I$. It is easy to see that $\tilde{v}$ satisfies 1) and 2). The condition 4) is also preserved because of the particular form of the covering $\{U_I\}$.

Next we observe that $(\tilde{v}^{(1)}, \cdots, \tilde{v}^{(k)})$ is also a solution of (4.4) where $\tilde{v}=(v, \tilde{v}^{(1)}, \cdots, \tilde{v}^{(k)})$. Thus, if $(y(t), w(t))$ is an integral curve of $\tilde{v}$, then $w(t)$ is a solution of linear differential equations (4.3). It follows that $w(t)$ is defined whenever $y(t)\in O$. Thus $(y(t), w(t))$ is defined whenever $y(t)$ is defined. This completes the proof of (4.2), and hence of (2.1).

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