On power series with integer coefficients

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1. Let T be the set of power series with integer coefficients whose radii of convergence are equal to 1. Then the following Theorem of G. Pólya and F. Carlson is well known (cf. [1]): *A power series $f\in T$ is a rational function, or it has no analytic continuation across the unit circle. Moreover, if $f(X)$ is a rational function, it can be expressed in the form $P(X)/(1-X^m)^n$, where m and n are natural numbers, and $P(X)$ is a polynomial with integer coefficients.*

The analogous assertion for a power series with integer coefficients whose radius of convergence is strictly smaller than 1 is false, unless some additional condition is imposed on the coefficients.

We propose here two kinds of such conditions: Let $f(X)$ be a convergent power series $\sum_{n=0}^{\infty} A_n X^n$ $(A_n \in \boldsymbol{Z})$ such that its radius of convergence is strictly smaller than 1. Then one condition is

(D): $A_n \in \boldsymbol{Z}$ and A_n divides A_{n+1} $(n=0,1,2,\cdots)$, and the other is

(K): A_n is the n-th denominator of the continued fraction of some irrational real number θ.

First, we shall give a necessary and sufficient condition for $f(X)$ to be a rational function in each of the two cases (Proposition 1, 2). Then we shall show that these rational functions have similar properties to those of Pólya-Carlson, and that the possible poles lie at the division points (i.e. Farey points) of the circle of convergence of $f(X)$.

These phenomena seem to suggest the validity of an analogy of the theorem of Pólya-Carlson under condition (D) or (K), and a possibility of a generalization of the circle method for some special functions of that type. But it seems fairly difficult to prove this conjecture. We shall treat only case (D) with some further conditions (Proposition 3, 4), and give an example in case (K).

2. First we investigate the condition for $f(X)$ to be a rational function.

PROPOSITION 1. *Let $f(X)=\sum_{n=0}^{\infty} A_n X^n$ such that $A_n=\prod_{k=0}^{n} a_k$, and $a_k \in \boldsymbol{Z}-\{0\}$. Then $f(X)$ is a rational function if and only if there exists a number k_0 such that a_k is periodic for $k\geq k_0$.*

PROOF. Let $f(X)$ be a rational function. Then $f(X)=h(X)/g(X)$ with $g(X)$, $h(X)\in \boldsymbol{Z}[X]$. Since $g(X)f(X)-h(X)=0$, there exist a natural number n_0 and integers $N_0, N_1, \cdots, N_q$ ($N_0\neq 0, N_q\neq 0$) such that for any $n\geq n_0$

$$N_0A_{n+q}+\cdots+N_qA_n=0 . \tag{1}$$

Since $A_n=\prod_{k=0}^{n} a_k$, we have

$$N_0a_{n+1}a_{n+2}\cdots a_{n+q}+\cdots+N_{q-1}a_{n+1}+N_q=0 . \tag{1'}$$

Hence a_{n+1} must divide N_q for all $n\geq n_0$. It follows that the q-ple $\mathfrak{A}_s=(a_s, a_{s+1}, \cdots, a_{s+q-1})$ has only a finite number of possibilities, and consequently there exist two numbers s_1, s_2, such that $\mathfrak{A}_{s_1}=\mathfrak{A}_{s_2}$. Now a_n's satisfy recursive relation (1'), so the sequence $\{a_n\}$ is periodic from a sufficiently large number n on. The converse is obvious. Q.E.D.

In preparation to treat case (K), we introduce some notations. Take and fix a sequence of integers $u_1, u_2, u_3, \cdots$. We define $\langle u_1, u_2, \cdots, u_v\rangle=U_v$ inductively as follows:

$$U_1=u_1 , \quad U_2=u_2u_1+1 , \quad U_v=u_vU_{v-1}+U_{v-2} , \qquad (v\geq 3) .$$

We extend the definition formally by $U_0=1$, $U_{-1}=0$.

Let θ be a real irrational number whose continued fractional expansion is $[b_0, b_1, b_2, \cdots]$. Put $A_v=\langle b_0, \cdots, b_v\rangle$, $B_v=\langle b_1, \cdots, b_v\rangle$, and more generally $A_{s,t}=\langle b_t, \cdots, b_{t+s}\rangle$, $B_{s,t}=\langle b_{t+1}, \cdots, b_{t+s}\rangle$. Hence B_n is the n-th denominator of θ.

Now our proposition is stated as follows:

PROPOSITION 2. *Let B_n be the n-th denominator of the continued fraction of a real irrational number θ. Then $f(X)=\sum_{n=1}^{\infty} B_nX^n$ is a rational function if and only if the number θ is a real quadratic irrational number.*

PROOF. Let θ be a real quadratic irrational number such that its continued fractional expansion $[b_0, b_1, b_2, \cdots]$ has a pre-period of length k_0 and a periodic cycle of length k. We need the following lemma which is essentially contained in [3], §12, (30). But we give a proof for the convenience of the reader.

LEMMA 1. *Let $h>k_0$. Then*

$$B_{h+2k}=(A_{k-1,h}+B_{k-2,h})B_{h+k}+(-1)^{k+1}B_h ,$$

and $A_{k-1,h}+B_{k-2,h}$ is constant for $h>k_0$.

PROOF OF LEMMA 1. The following relations are known (cf. [2] §5); $B_{v+t-1}=B_{t-1}A_{v-1,t}+B_{t-2}B_{v-1,t}(v, t\geq 1)$. Putting $v=k$, $t=h+k+1$, we have

$$B_{h+2k}=B_{h+k}A_{k-1,h+k+1}+B_{h+k-1}B_{k-1,h+k+1}\ .$$

Put $I=B_{h+2k}-(A_{k-1,h}+B_{k-2,h})B_{h+k}$. Then

$$I=B_{h+k}(A_{k-1,h+k+1}-A_{k-1,h}-B_{k-2,h})+B_{h+k-1}B_{k-1,h+k+1}\ .$$

Here we note the facts that $\langle u_1, \cdots, u_v\rangle=u_1\langle u_2, \cdots, u_v\rangle+\langle u_3, \cdots, u_v\rangle$ (cf. [2] § 5, (28)), and by definitions

$$\langle u_1, \cdots, u_v\rangle=u_v\langle u_1, \cdots, u_{v-1}\rangle+\langle u_1, \cdots, u_{v-2}\rangle\ .$$

Since $b_{i+k}=b_i$ for $i>k_0$,

$$\begin{aligned}A_{k-1,h+k+1}-A_{k-1,h}-B_{k-2,h}&=\langle b_{h+k+1}, \cdots, b_{h+2k}\rangle-\langle b_h, \cdots, b_{h+k-1}\rangle-\langle b_{h+1}, \cdots, b_{h+k-2}\rangle\\&=b_{h+k}\langle b_{h+1}, \cdots, b_{h+k-1}\rangle-\langle b_h, \cdots, b_{h+k-1}\rangle\\&=-\langle b_{h+2}, \cdots, b_{h+k-1}\rangle\ .\end{aligned}$$

So we have

$$\begin{aligned}I&=(-1)(\langle b_{h+2}, \cdots, b_{h+k-1}\rangle B_{h+k}-\langle b_{h+2}, \cdots, b_{h+k}\rangle B_{h+k-1})\\&=B_{h+k-1}\langle b_{h+2}, \cdots, b_{h+k}\rangle-\langle b_{h+2}, \cdots, b_{h+k-1}\rangle(b_{h+k}B_{h+k-1}+B_{h+k-2})\\&=B_{h+k-1}\langle b_{h+2}, \cdots, b_{h+k-2}\rangle-B_{h+k-2}\langle b_{h+2}, \cdots, b_{h+k-1}\rangle\ .\end{aligned}$$

For general $i\leq k$, we obtain inductively

$$\begin{aligned}I&=(-1)^i(\langle b_{h+2}, \cdots, b_{h+k-i}\rangle B_{h+k-i+1}-\langle b_{h+2}, \cdots, b_{h+k-i+1}\rangle B_{h+k-i})\\&=(-1)^{k+1}B_h\ .\end{aligned}$$

Last assertion follows from easy calculations. Q.E.D.

By virtue of this lemma, put $A_{k-1,h}+B_{k-2,h}=a$ ($h>k_0$). Then $f(X)$ can be expressed in the form $P(X)/(X^{2k}-aX^k+(-1)^k)$ with $P(X)\in \boldsymbol{Z}[X]$.

Conversely, take any continued fraction $[b_0, b_1, b_2, \cdots]$, and consider the power series $f(X)=\sum_{n=1}^{\infty} B_nX^n$ with $B_n=\langle b_1, \cdots, b_n\rangle$. If $f(X)$ is a rational function, then, as in the proof of Proposition 1, we have

$$N_0B_{n+q}+N_1B_{n+q-1}+ \cdots +N_qB_n=0\ , \qquad \text{for } n\geq n_0\ , \tag{1}$$

with $N_1, \cdots, N_q\in \boldsymbol{Z}$ ($N_0\neq 0$, $N_q\neq 0$), and B_n's satisfy

$$B_{n+2}=b_{n+2}B_{n+1}+B_n\ . \tag{2}$$

Since B_n's are natural numbers which increase monotonously as n increases, we obtain

$$b_{n+q}B_{n+q-1}<B_{n+q}<((|N_1|+|N_2|+ \cdots +|N_q|)\ /\ |N_0|)B_{n+q-1}\ .$$

Hence we have $b_{n+q}<(|N_1|+ \cdots +|N_q|)\ /\ |N_0|$. Thus b_n's are bounded for $n\geq n_0+q$.

Since they are natural numbers, b_n's have only a finite number of possibilities for $n \geq n_0+q$. Now, by substituting the relation (2) into (1), we obtain

$$Q_n B_{n+1} + R_n B_n = 0\ , \tag{3}$$

where $Q_n = N_0\langle b_{n+2}, \cdots, b_{n+q}\rangle + N_1\langle b_{n+2}, \cdots, b_{n+q-1}\rangle + \cdots + N_{q-2}b_{n+2} + N_{q-1}$, and $R_n = N_0\langle b_{n+3}, \cdots, b_{n+q}\rangle + N_1\langle b_{n+3}, \cdots, b_{n+q-1}\rangle + \cdots + N_{q-3}b_{n+3} + N_{q-2} + N_q$. Now b_n's, and also Q_n's and R_n's have only a finite number of possibilities. But $B_n \to \infty$ as $n \to \infty$, and G.C.M. of B_n and B_{n+1} is equal to that of B_2 and B_1. So relation (3) implies $Q_s = R_s = 0$ for all sufficiently large s. Thus b_{s+q} is recursively determined by $b_{s+2}, \cdots, b_{s+q-1}$ from the relation $Q_s = 0$. Now we can conclude the periodicity of the sequence $\{b_s\}$ as in the proof of Proposition 1. Q.E.D.

REMARK 1. The quadratic equation $X^2 - aX + (-1)^k = 0$ is the characteristic equation of θ (cf. [3], § 10).

REMARK 2. In case that $f(X)$ is a rational function, possible poles of $f(X)$ lie at Farey points of two concentric circles.

3. Now we study the nature of non-rational function $f(X)$ under condition (D) or (K). In particular, we wish to show the validity of the analogy of Pólya-Carlson's Theorem. But we can solve this problem only in case (D) with some more additional conditions.

Let $f(X)$ be a power series with $1/\rho$ (<1) as its radius of convergence such that its coefficients satisfy condition (D). Let $\boldsymbol{P}$ be the set of all the rational primes, and we arrange primes $P \in \boldsymbol{P}$ in the ascending order i.e. $P_1 = 2$, $P_2 = 3$, $P_3 = 5, \cdots$. We denote by $v_P(n)$ the exponent of the P-part of the coefficient A_n. Then by virtue of condition (D), $v_P(n)$ increases monotonously (in a wider sense) as n increases. We denote $v_P(n)/n$ by $\tilde{v}_P(n)$.

PROPOSITION 3. *If* $\lim_{n\to\infty} \tilde{v}_P(n) = 0$ *for all prime* $P \in \boldsymbol{P}$, *then the circle of convergence of* $f(X)$ *is the natural boundary.*

PROOF. Let N be the cardinality of $\{i; i \leq n, A_{i-1} \neq A_i\}$. Hence $N \leq \sum_{P \in \boldsymbol{P}} v_P(n)$. So if we ascertain the relation $\lim_{n\to\infty} (\sum_{P \in \boldsymbol{P}} v_P(n)/n) = 0$, we can apply Fabry's Lückensatz for the function $(1-X)f(X)$, and we obtain the expected conclusion. Since the radius of convergence of $f(X)$ is $1/\rho$, the numbers $\tilde{v}_P(n)$ satisfy $\sum_{i=1}^{\infty} (\log P_i)\tilde{v}_{P_i}(n) < c$ with some constant c. So for any given $\varepsilon > 0$, there exists a number t such that $\sum_{i \geq t} \tilde{v}_{P_i}(n) < c/\log P_t < \varepsilon$. Therefore, by the assumption that $\tilde{v}_P(n) \to 0$ as $n \to \infty$, we have $\sum_{P \in \boldsymbol{P}} \tilde{v}_P(n) < 2\varepsilon$ for any sufficiently large n. Q.E.D.

We treat below another type, which, in a sense, make a contrast to the preceeding case.

PROPOSITION 4. *Let $f(X)=\sum_{n=0}^{\infty} A_nX^n$ ($A_n \in \boldsymbol{Z}$) be a power series such that A_n divides A_{n+1}. Let $1/\rho$ be the radius of convergence of $f(X)$ and assume that $f(X)$ satisfies the following properties:*

There exists a set $\boldsymbol{S}$ of finite number of primes such that the coefficients A_n's satisfy

(i) $\lim_{n\to\infty} |A_n^{(S)}|^{1/n}=\rho$, *where* $A_n^{(S)}=\prod_{P\in S} P^{v_P(n)}$,

(ii) *for all $P\in \boldsymbol{S}$, $\bar{v}_P(n)$ oscillates slowly (i.e. $\bar{v}_P(n)/\bar{v}_P(m)$ tends to 1 when the two natural numbers m and n, satisfying the inequality $m<n<cm$ with some constant c, tend to infinity).*

Then $f(X)$ is a rational function, or it has no analytic continuation across its circle of convergence.

PROOF. First we quote some results of G. Pólya. Let $\Delta_n^{(N)}$ be the determinant

$$\begin{vmatrix} A_n & A_{n+1} & \cdots & A_{n+N} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ A_{n+N} & \cdots & \cdots & A_{n+2N} \end{vmatrix}.$$

Then the following two lemmas are known.

LEMMA 2. (cf. [4]). *Let $f(X)$ be a power series whose radius of convergence is $1/\rho$. If $f(X)$ has an analytic continuation across its circle of convergence, then $\overline{\lim}_{n\to\infty} |\Delta_n^{(N)}|^{1/(n+N)(N+1)}<\rho$ holds for $N=n$, and $n-1$.*

LEMMA 3. (cf. [1]). *For a power series $f(X)$ to be a rational function, it is necessary and sufficient that there exists a number n_0 such that $\Delta_n^{(n)}=\Delta_n^{(n-1)}=0$ for all $n\geq n_0$.*

Now let $f(X)=\sum_{n=0}^{\infty} A_nX^n$ be a power series whose coefficients satisfy the assumption of Proposition 4, and assume that $f(X)$ is not a rational function. Then by virtue of Lemma 3, there are infinitely many number n such that $\Delta_n^{(n)}\neq 0$ or $\Delta_n^{(n-1)}\neq 0$.

We assume first $\Delta_n^{(n)}\neq 0$ for infinitely many n. Since the determinant $\Delta_n^{(n)}$ is a sum of terms of the form

$$(\pm)A_{i_1}\cdots A_{i_{n+1}} \quad (i_1+i_2+\cdots+i_{n+1}=2n(n+1),\ n\leq i_j\leq 3n),$$

we have the exponent of the P-part of

$$\Delta_n^{(n)} \geq \min_{\substack{(i_1, \cdots, i_{n+1}) \\ i_1 + \cdots + i_{n+1} = 2n(n+1), \\ n \leq i_j \leq 3n.}} (v_P(i_1) + \cdots + v_P(i_{n+1})) .$$

We denote this minimum by $V_P(n, n)$.

In case $P \in S$, by $n \leq i_j \leq 3n$ and condition (ii) of the proposition, we have

$$(v_P(i_1) + \cdots + v_P(i_{n+1}))/2n(n+1)$$
$$= (\tilde{v}_P(i_1)i_1 + \tilde{v}_P(i_2)i_2 + \cdots + \tilde{v}_P(i_{n+1})i_{n+1})/2n(n+1) \to \tilde{v}_P(n) , \quad \text{as } n \to \infty . \quad (4)$$

Since $\Delta_n^{(n)}$ is an integer, and $P^{V_P(n,n)}$ divides $\Delta_n^{(n)}$, we have $\prod_{P \in S} P^{V_P(n,n)} \leq |\Delta_n^{(n)}|$, if $\Delta_n^{(n)} \neq 0$. By condition (i) and relation (4), we have $|\prod_{P \in S} P^{V_P(n,n)}|^{1/2n(n+1)} \to \rho$ as $n \to \infty$, so $\overline{\lim}_{n\to\infty} |\Delta_n^{(n)}|^{1/2n(n+1)} \geq \rho$. By Lemma 2, $f(X)$ has no analytic continuation across its circle of convergence. It is obvious that the same reasoning works if we assume that $\Delta_n^{(n-1)} \neq 0$ for infinitely many n. Q.E.D.

4. It seems plausible that Proposition 4 remains true in a wider situation. But even in the case such that the set S contains only two prime numbers, the problem turns out to be a difficult one if these two primes satisfy only condition (i). In this case, the problem seems to be related with diophantine properties of prime powers, such as Pillai's Theorem with respect to the growth order of $p^x - q^y$, where p and q are prime numbers and two integers x and y tend to infinity.

In case (K), the situation is even worse, and we can treat only some cases which can be reduced to the Lückensatz. For example, take a continued fraction $[b_0, b_1, b_2, \cdots]$ with the following properties:

(I) $b_i = a$ or b, where a and b are fixed two natural numbers;

(II) Let $A(n) = \#\{i; i \leq n, b_i = a\}$ and $B(n) = \#\{i; i \leq n, b_i = b\}$, then $A(n)/B(n) \to 0$ as $n \to \infty$.

Then, it is easy to see that the power series $f(X) = \sum_{n=1}^{\infty} B_n X^n$ with $B_n = \langle b_1, \cdots, b_n \rangle$ satisfies the following assertions.

(1) The radius of convergence of $f(X)$ is $((b^2+4)^{1/2} - b)/2$.

(2) The power series $(1 - bX - X^2)f(X)$ satisfies the Lücken condition of Fabry, and its radius of convergence is also $((b^2+4)^{1/2} - b)/2$.

So $f(X)$ has no analytic continuation across its circle of convergence.

References

[1] Carlson, F., Über Potenzreihen mit ganzzahligen Koeffizienten, Math. Z. **9** (1921), 1-13.

[2] Perron, O., Die Lehre von der Kettenbruchen, Teubner 1929.

[3] Perron, O., Grundlagen für eine Theorie der Jacobischen Kettenbruchen algorithums, Math. Ann. **64** (1907), 1-76.
[4] Pólya, G., Über gewisse notwendigen Diterminantenkriterien für die Fortsetzbarkeit einen Potenzreihe, Math. Ann. **99** (1928), 608-706.
[5] Bieberbach, L., Analytisch Fortsetzung, Springer 1955.

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