A p-adic analogue of the Γ-function*

By Yasuo MORITA

In this paper, we shall construct a p-adic analogue of the classical Γ-function.

In §1, we shall construct a p-adic analogue of $(-1)^z\Gamma(z)$ as a continuous function on $\boldsymbol{Z}_p$ with values in $\boldsymbol{Q}_p$. In §2, we shall obtain an interesting theorem, using the results of Kubota-Leopoldt [3]. In §3, we shall prove that the p-adic analogue of $(-1)^z\Gamma(z)$ is analytic.

The author would like to express his deep gratitude to Professor M. Sato, whose philosophy had a very strong influence on the author when he was studying this topic, and to Professor K. Iwasawa, who pointed out to the author the importance of this problem.

Notation. Let $\boldsymbol{Z}, \boldsymbol{Q}, \boldsymbol{Z}_p, \boldsymbol{Q}_p$ be the ring of rational integers, the field of rational numbers, the ring of p-adic integers, and the field of p-adic numbers, respectively. For any ring R, we denote by $R^\times$ the multiplicative group of all units in R. For any two integers a and b, we denote by (a, b) the greatest common divisor of a and b.

§1. $(-1)^z\Gamma_p(z)$ as a continuous function

Let p be a prime number, n a natural number. We define a function $\Gamma_p(n)$ on the set $\boldsymbol{N}$ of all natural numbers by

$$\Gamma_p(n) = \prod_{\substack{1\leq t\leq n-1 \\ (t,p)=1}} t\,.$$

It is obvious that values of $\Gamma_p(n)$ belong to the group $\boldsymbol{Z}_p^\times$ of p-adic units.

Let α, v be natural numbers. We assume $\alpha\neq 2$ if $p=2$. Put

$$I = \Gamma_p(n+p^\alpha v)/\Gamma_p(n)\,.$$

Then we have

$$\begin{aligned} I &= \prod_{\substack{n\leq t\leq n+p^\alpha\cdot v-1 \\ (t,p)=1}} t \\ &\equiv \Big(\prod_{\substack{0\leq t\leq p^\alpha-1 \\ (t,p)=1}} t\Big)^v \pmod{p^\alpha} \\ &\equiv \Big(\prod_{t\in(\boldsymbol{Z}/p^\alpha\boldsymbol{Z})^\times} t\Big)^v \pmod{p^\alpha}\,. \end{aligned}$$

* This work was supported in part by National Science Foundation grant GP-36418X1.

Now we consider the following map:

$$\iota : (\boldsymbol{Z}/p^{\alpha}\boldsymbol{Z})^{\times} \ni g \mapsto g^{-1} \in (\boldsymbol{Z}/p^{\alpha}\boldsymbol{Z})^{\times} .$$

Then we have

$$I \equiv \Big(\prod_{\substack{t \in (\boldsymbol{Z}/p^{\alpha}\boldsymbol{Z})^{\times} \\ \iota(t) = t}} t \Big)^{v} \pmod{p^{\alpha}}$$

$$\equiv \Big(\prod_{\substack{t \in (\boldsymbol{Z}/p^{\alpha}\boldsymbol{Z})^{\times} \\ t^{2} = 1}} t \Big)^{v} \pmod{p^{\alpha}} .$$

If p is an odd prime number, $(\boldsymbol{Z}/p^{\alpha}\boldsymbol{Z})^{\times}$ is a cyclic group. Hence 1 and -1 are the only elements t of $(\boldsymbol{Z}/p^{\alpha}\boldsymbol{Z})^{\times}$ such that $t^{2}=1$. On the other hand, if $p=2$,

$$(\boldsymbol{Z}/p^{\alpha}\boldsymbol{Z})^{\times} \cong (\boldsymbol{Z}/2\boldsymbol{Z}) \times (\boldsymbol{Z}/2^{\alpha-2}\boldsymbol{Z}) .$$

Hence the product of all elements t of $(\boldsymbol{Z}/2^{\alpha}\boldsymbol{Z})^{\times}$ satisfying $t^{2}=1$ is 1. Therefore we have proved the following

LEMMA 1. *Let n, α, v be as above. Then we have*

$$\Gamma_{p}(n+p^{\alpha}\cdot v)/\Gamma_{p}(n) \equiv \begin{cases} (-1)^{v} & \text{if } p \neq 2 , \\ 1 & \text{if } p=2 . \end{cases} \pmod{p^{\alpha}}$$

Since the right hand side of the above equality is obviously equal to $(-1)^{p^{\alpha}\cdot v}$, this lemma implies the following

THEOREM 1. *Let $p, \Gamma_{p}(n), \alpha$ be as before. Let n_{1}, n_{2} be two natural numbers satisfying $n_{1} \equiv n_{2} \pmod{p^{\alpha}}$. Then we have*

$$(-1)^{n_{1}}\Gamma_{p}(n_{1}) \equiv (-1)^{n_{2}}\Gamma_{p}(n_{2}) \pmod{p^{\alpha}} .$$

In particular, $(-1)^{n}\Gamma_{p}(n)$ can be extended to a continuous function from $\boldsymbol{Z}_{p}$ to $\boldsymbol{Z}_{p}$.

REMARK. Since $\Gamma_{p}(n)$ satisfies

$$(-1)^{n+1}\Gamma_{p}(n+1) = \begin{cases} -n(-1)^{n}\Gamma_{p}(n) & \text{if } n \notin p\boldsymbol{Z}_{p} \\ -(-1)^{n}\Gamma_{p}(n) & \text{if } n \in p\boldsymbol{Z}_{p} \end{cases}$$

for any natural number n, this functional equation holds for any $n \in \boldsymbol{Z}_{p}$. In particular,

$$\Gamma_{p}(-n) = \prod_{\substack{-n \leqq t \leqq 0 \\ (t, p)=1}} t^{-1}$$

for any negative integer $-n$.

§2. An application of the results of Kubota-Leopoldt

Let p be a prime number, $\boldsymbol{C}_{p}$ the completion of the algebraic closure of $\boldsymbol{Q}_{p}$.

Put $q=4$, if $p=2$, and $q=p$ otherwise. Let $|\ \ |$ be the valuation of $\boldsymbol{C}_p$ such that $|p|=p^{-1}$. We denote by $\mathscr{F}$ the ring consisting of all power series $A(u)=\sum_{n\geqq 0} a_n(u-1)^n$ $(a_n\in \boldsymbol{Q}_p)$ that converge on the set $U=\{u\in \boldsymbol{C}_p \mid |u-1|\leqq |q|\}$.

For any $A(u)$ of $\mathscr{F}$, we define $\|A\|$ by

$$\|A\|=\max_n |a_n q^n| .$$

Then it is known that $\|\ \ \|$ is a norm of $\mathscr{F}$ and $\|\ \ \|$ satisfies

$$\text{(i)} \qquad \left|\frac{1}{k!} A^{(k)}(u)q^k\right| \leqq \|A\|$$

and

$$\text{(ii)} \qquad \left|\frac{q^k}{k!}\right| \|A^{(k)}\| \leqq \|A\| ,$$

where k is a non-negative integer, u is an element of U, and $A^{(k)}$ is the k-th derivative of A (cf. Kubota-Leopoldt [3]).

If $p\neq 2$, we define $\omega(x)$ $(x\in \boldsymbol{Z}_p)$ by

$$\omega(x)=\lim_{n\to\infty} x^{p^n} .$$

Then $\omega(x)$ induces a Dirichlet character modulo p. If $p=2$, let $\omega(x)$ be the quadratic character with conductor 4 such that $\omega(x)\equiv x \pmod 4$. For any p-adic unit x (p is any prime number), we define $\langle x\rangle$ by $x=\omega(x)\cdot\langle x\rangle$. Then $\langle\ \rangle$ induces a map from $\boldsymbol{Z}_p^\times$ to U.

Let χ be a Dirichlet character, f the conductor of χ. Let $\tilde{f}$ be the smallest common multiple of f and q. Further, for any subset I of $\boldsymbol{Z}$, $\sum_{x\in I}^*$ denotes the sum over all x such that x is an element of I prime to p. For any natural number n, we define a linear operator $\mathfrak{M}_\chi^n$ on $\mathscr{F}$ with values in $\boldsymbol{C}_p$ by

$$\mathfrak{M}_\chi^n(A)=\frac{1}{\tilde{f}q^n}\sum_{x=1}^{\tilde{f}q^n}{}^* \chi(x)A(\langle x\rangle) \quad (A\in\mathscr{F}) .$$

Then Kubota-Leopoldt proved that (i) for any $A\in\mathscr{F}$, the sequence $\{\mathfrak{M}_\chi^n(A)\}_{n=1}^\infty$ has a limit $\mathfrak{M}_\chi(A)$ in $\boldsymbol{C}_p$; (ii) there exists a constant C_χ depending only on χ satisfying

$$|\mathfrak{M}_\chi^n(A)|\leqq C_\chi\|A\|$$

and

$$|\mathfrak{M}_\chi^{n+1}(A)-\mathfrak{M}_\chi^n(A)|\leqq C_\chi\|A\|\,|\tilde{f}q^{n-1}| .$$

In particular, $|\mathfrak{M}_\chi(A)|\leqq C_\chi\|A\|$ and hence $\mathfrak{M}_\chi$ is a bounded linear operator on $\mathscr{F}$ with values in $\boldsymbol{C}_p$.

Let z be an element of qZ_p. For any natural number n and for any element A of $\mathscr{F}$, put

$$\mathfrak{M}^n_{\chi,z}(A)=\frac{1}{\bar{f}q^n}\sum_{x=1}^{\bar{f}q^n}{}^{*}\,\chi(x)A(\langle x+z\rangle)\,.$$

By the Taylor expansion, we have

$$\begin{aligned}A(\langle x+z\rangle)&=A(\langle x\rangle+\omega^{-1}(x)z)\\&=\sum_{m\geq 0}\frac{1}{m!}A^{(m)}(\langle x\rangle)\omega^{-m}(x)z^m\,.\end{aligned}$$

Hence we have

$$\begin{aligned}\mathfrak{M}^n_{\chi,z}(A)&=\sum_{m\geq 0}\left\{\frac{1}{\bar{f}q^n}\sum_{x=1}^{\bar{f}q^n}{}^{*}\,\chi\omega^{-m}(x)A^{(m)}(\langle x\rangle)\right\}\frac{z^m}{m!}\\&=\sum_{m\geq 0}\mathfrak{M}^n_{\chi\omega^{-m}}(A^{(m)})\frac{z^m}{m!}\,.\end{aligned}$$

Since $\omega(x)$ is a Dirichlet character, some power of ω is the trivial character. Hence the quantity

$$C^*_\chi=\max_{0\leqq m<\infty}C_{\chi\omega^{-m}}<\infty$$

is defined. Hereafter we assume that $\sum_{m\geq 0}\left\|\frac{q^m}{m!}A^{(m)}(u)\right\|$ is finite. Since

$$\begin{aligned}\sum_{m\geq 0}\left\|\frac{q^m}{m!}A^{(m)}(u)\right\|&=\sum_{m\geq 0}\left\|\sum_{k\geq 0}\binom{k+m}{k}a_{k+m}(u-1)^k q^m\right\|\\&=\sum_{m\geq 0}\max_k\left|\binom{k+m}{k}a_{k+m}q^{k+m}\right|\\&\leqq\sum_{m\geq 0}\max_k|a_{k+m}q^{k+m}|\,,\end{aligned}$$

this condition is satisfied if $A(u)$ is convergent for $|u-1|\leqq|q|^{1-\varepsilon}$ with a positive number ε. Then we have

$$\begin{aligned}|\mathfrak{M}^{n+1}_{\chi,z}(A)-\mathfrak{M}^n_{\chi,z}(A)|&\leqq\max_{m\geq 0}|\mathfrak{M}^{n+1}_{\chi\omega^{-m}}(A^{(m)})-\mathfrak{M}^n_{\chi\omega^{-m}}(A^{(m)})|\left|\frac{z^m}{m!}\right|\\&\leqq\max_{m\geq 0}C^*_\chi\|A^{(m)}\||\bar{f}q^{n-1}|\left|\frac{q^m}{m!}\right|\\&\leqq C^*_\chi\|A\||\bar{f}q^{n-1}|\,.\end{aligned}$$

Hence the limit

$$\mathfrak{M}_{\chi,z}(A)=\lim_{n\to\infty}\mathfrak{M}^n_{\chi,z}(A)$$

exists. Furthermore, since

$$\sum_{m\geq 0}\left|\mathfrak{M}^{n}_{\chi\omega^{-m}}(A^{(m)})\frac{z^m}{m!}\right|\leqq\sum_{m\geq 0}C^*_\chi\|A^{(m)}\|\left|\frac{q^m}{m!}\right|$$
$$=C^*_\chi\sum_{m\geq 0}\left\|\frac{q^m}{m!}A^{(m)}\right\|<\infty$$

for any natural number n, we have

$$\mathfrak{M}_{\chi,z}(A)=\sum_{m\geq 0}\lim_{n\to\infty}\mathfrak{M}^{n}_{\chi\omega^{-m}}(A^{(m)})\frac{z^m}{m!}$$
$$=\sum_{m\geq 0}\mathfrak{M}_{\chi\omega^{-m}}(A^{(m)})\frac{z^m}{m!}\ .$$

Therefore we can define an analytic function $F_{A,\chi}$ from $q\boldsymbol{Z}_p$ to $\boldsymbol{C}_p$ by

$$F_{A,\chi}(z)=\mathfrak{M}_{\chi,z}(A)-\mathfrak{M}_\chi(A)\ .$$

Now we assume that z is an integer satisfying $z\equiv 0 \pmod{\bar{f}}$. Then we have

$$\chi(x+z)=\chi(z)\ .$$

Therefore we have

$$F_{A,\chi}(z)=\lim_{n\to\infty}\frac{1}{\bar{f}q^n}\sum_{x=1}^{\bar{f}q^n}{}^*\chi(x)A(\langle x+z\rangle)-\lim_{n\to\infty}\frac{1}{\bar{f}q^n}\sum_{x=1}^{\bar{f}q^n}{}^*\chi(x)A(\langle x\rangle)$$
$$=\lim_{n\to\infty}\frac{1}{\bar{f}q^n}\sum_{x=1}^{\bar{f}q^n}{}^*\{\chi(x+z)A(\langle x+z\rangle)-\chi(x)A(\langle x\rangle)\}$$
$$=\lim_{n\to\infty}\frac{1}{\bar{f}q^n}\sum_{x=0}^{z}{}^*\{\chi(x+\bar{f}q^n)A(\langle x+\bar{f}q^n\rangle)-\chi(x)A(\langle x\rangle)\}$$
$$=\sum_{x=0}^{z}{}^*\chi(x)\lim_{n\to\infty}\frac{1}{\bar{f}q^n}\{A(\langle x+\bar{f}q^n\rangle)-A(\langle x\rangle)\}$$
$$=\sum_{x=0}^{z}{}^*\chi(x)\lim_{n\to\infty}\frac{1}{\bar{f}q^n}\{A(\langle x\rangle+\omega^{-1}(x)\bar{f}q^n)-A(\langle x\rangle)\}$$
$$=\sum_{x=0}^{z}{}^*\chi\omega^{-1}(x)\lim_{h\to 0}\frac{1}{h}\{A(\langle x\rangle+h)-A(\langle x\rangle)\}$$
$$=\sum_{x=0}^{z}{}^*\chi\omega^{-1}(x)A^{(1)}(\langle x\rangle)\,;$$

here $\sum_{x=0}^{z}{}^*$ should be replaced by $-\sum_{x=z}^{0}{}^*$ if z is a negative integer. Therefore we have obtained the following

THEOREM 2. *Let* $p, \boldsymbol{C}_p, q, \omega, \chi, \bar{f}, \sum^*$ *be as before. Let* $A(u)$ *be an analytic function* $\sum_{n\geq 0}a_n(u-1)^n$ $(a_n\in\boldsymbol{Q}_p)$ *that is convergent on* $U_\epsilon=\{u\in\boldsymbol{C}_p\,|\,|u-1|\leqq|q|^{1-\epsilon}\}$

for a positive number ε. *Then*

$$F_{A,\chi}(z)=\lim_{n\to\infty}\frac{1}{\bar{f}q^n}\sum_{x=1}^{\bar{f}q^n}{}^{*}\,\chi(x)A(\langle x+z\rangle)-\lim_{n\to\infty}\frac{1}{\bar{f}q^n}\sum_{x=1}^{\bar{f}q^n}\chi(x)A(\langle x\rangle)\,,$$

determines a well-defined analytic function from $q\boldsymbol{Z}_p$ *to* $\boldsymbol{C}_p$. *Furthermore, if* z *is an integer satisfying* $z\equiv 0\ (\mathrm{mod}\ \bar{f})$, *we have*

$$F_{A,\chi}(z)=\begin{cases}\displaystyle\sum_{x=0}^{z}{}^{*}\,\chi\omega^{-1}(x)A^{(1)}(\langle x\rangle) & \text{if}\quad z\geqq 0\,,\\ \displaystyle-\sum_{x=z}^{0}{}^{*}\,\chi\omega^{-1}(x)A^{(1)}(\langle x\rangle) & \text{if}\quad z<0\,.\end{cases}$$

REMARK. It should be noted that, if $A(u)=\sum_{n\geqq 0}a_n(u-1)^n\in\mathscr{F}$ is convergent for $|u-1|<\rho$, then

$$F_{A,\chi}(z)=\sum_{m\geqq 1}\mathfrak{M}_{\chi\omega^{-m}}(A^{(m)})\frac{z^m}{m!}$$

is convergent for $|z|<\rho$.

REMARK. If $|z|<|q|$, the linear functional $\mathfrak{M}_{\chi,z}: A(u)\mapsto\mathfrak{M}_{\chi,z}(A)$ is well-defined and continous on $\mathscr{F}$.

§ 3. $(-1)^z\Gamma_p(z)$ as an analytic function

Let p be a prime number. Put

$$A(u)=u(\log u-1)\,,$$

$$\log u=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(u-1)^n\,.$$

Then $A(u)$ is an element of $\mathscr{F}$, and

$$A^{(1)}(u)=\log u\,.$$

Let ω be the Dirichlet character modulo q that was defined in §2. If follows from Theorem 2 that $F_{A,\omega}(z)$ is an analytic function from $q\boldsymbol{Z}_p$ to $\boldsymbol{Q}_p$. Moreover, if z is a non-negative integer (resp. a negative integer), we have

$$F_{A,\omega}(z)=\sum_{x=0}^{z}{}^{*}\log\langle x\rangle$$

$$(\text{resp. } F_{A,\omega}(z)=-\sum_{x=z}^{0}{}^{*}\log\langle x\rangle)\,.$$

Let $(-1)^z\Gamma_p(z)$ be the continuous function from $\boldsymbol{Z}_p$ to $\boldsymbol{Q}_p$ that was defined in §1. Then the preceding facts imply that $\log\langle(-1)^{z+1}\Gamma_p(z+1)\rangle$ is an analytic function from $q\boldsymbol{Z}_p$ to $\boldsymbol{C}_p$. Since $(-1)^{z+1}\Gamma_p(z+1)$ $(z\in 2q\boldsymbol{Z}_p)$ belongs to $U'=\{u\in\boldsymbol{Z}_p \mid |u-1|\leqq |2q|\}$ (cf. Theorem 1), and since $\exp(u)=\sum_{m=0}^{\infty}\frac{u^m}{m!}$ satisfies

$$\exp(\log u)=u$$

for any $u\in U'$, we see that

$$(-1)^{z+1}\Gamma_p(z+1)=\exp[\log\{(-1)^{z+1}\Gamma_p(z+1)\}]$$

is an analytic function on $2q\boldsymbol{Z}_p$ with values in $\boldsymbol{Q}_p$. Therefore we have proved the following

THEOREM 3. *Let p be a prime number, $(-1)^z\Gamma_p(z)$ the function defined in §1. Put $q=p$, if $p\neq 2$, and $q=4$ otherwise. Then $(-1)^{z+1}\Gamma_p(z+1)$ is an analytic function from $2q\boldsymbol{Z}_p$ to $\boldsymbol{Q}_p$. Namely, $(-1)^{z+1}\Gamma_p(z+1)$ can be expanded as a convergent power series $\sum_{n\geqq 0}\gamma_n(z-z_0)^n$ $(\gamma_n\in\boldsymbol{Q}_p)$ for every $z_0\in 2q\boldsymbol{Z}_p$.*

REMARK. It follows from the calculations in §2 that

$$\log\langle\Gamma_p(z+1)\rangle=\mathfrak{M}_1(\log u)z+\sum_{m\geqq 2}\mathfrak{M}_{\omega^{1-m}}((-1)^m(m-2)!\,u^{-m+1})\frac{z^m}{m!}\ .$$

We note that

(i) $$\log\langle\Gamma_p(1)\rangle=0\,;$$

(ii) $$-\left[\frac{d}{dz}\log\langle\Gamma_p(z+1)\rangle\right]_{z=0}=-\mathfrak{M}_1(\log u)\ ,$$

which is a p-adic analogue of the Euler number;

(iii) $$\left(\frac{d}{dz}\right)^2\log\langle\Gamma_p(z+1)\rangle=\mathfrak{M}_{\omega^{-1},z}(u^{-1})\ ,$$

which is a p-adic analogue of $\left(\frac{d}{dz}\right)^2\log\Gamma(z+1)=\sum_{n=1}^{\infty}\frac{1}{(n+z)^2}$ (cf. Appendix);

(iv) the above Taylor expansion of $\log\langle\Gamma_p(z+1)\rangle$ is convergent for any $z\in\boldsymbol{C}_p$ such that $|z|<1$.

REMARK. We can prove that $(-1)^{z+1}\Gamma_p(z+1)$ gives a uniform analytic function on $\{z\in\boldsymbol{C}_p \mid |z|\leqq|2q|\}$ (cf. Krasner [2]).

§4. Further discussions of analytic properties

Let p, q, A etc. be as in Theorem 2. Let $\bar{f}$ be a natural number divisible by q, ν an element of $\boldsymbol{Z}/\bar{f}\boldsymbol{Z}$ such that $\nu \pmod p\in(\boldsymbol{Z}/p\boldsymbol{Z})^\times$. For any natural number n, put

$$\mathfrak{N}_\nu^n(A)=\frac{1}{\bar{f}q^n}\sum_{\substack{1\leq x\leq \bar{f}q^n\\ x\equiv\nu(\mathrm{mod}\,\bar{f})}} A(\langle x\rangle)\,.$$

Then, following step-by-step the proof of Satz 1 of Kubota-Leopoldt [3], we can prove that (i) the sequence $\{\mathfrak{N}_\nu^n(A)\}_{n=1}^\infty$ has a limit $\mathfrak{N}_\nu(A)$ in $\boldsymbol{Q}_p$; (ii) there exists a constant C_ν depending only on (f and) ν satisfying

$$|\mathfrak{N}_\nu^n(A)|\leqq C_\nu\|A\|$$

and

$$|\mathfrak{N}_\nu^{n+1}(A)-\mathfrak{N}_\nu^n(A)|\leqq C_\nu\|A\|\,|\bar{f}q^{n-1}|\,.$$

Therefore, as in §2, we can prove that (i)

$$\mathfrak{N}_{z,\nu}(A)=\lim_{n\to\infty}\frac{1}{\bar{f}q^n}\sum_{\substack{1\leq x\leq \bar{f}q^n\\ x\equiv\nu(\mathrm{mod}\,\bar{f})}} A(\langle x+z\rangle)$$

is a well-defined analytic function on $q\boldsymbol{Z}_p$ with values in $\boldsymbol{C}_p$; (ii)

$$\mathfrak{N}_{z,\nu}(A)=\sum_{m\geqq 0}\omega^{-m}(\nu)\mathfrak{N}_\nu(A^{(m)})\frac{z^m}{m!}\ ;$$

(iii) if z is an integer satisfying $z\equiv 0\ (\mathrm{mod}\,\bar{f})$, then

$$\mathfrak{N}_{z,\nu}(A)-\mathfrak{N}_\nu(A)=\begin{cases}\displaystyle\sum_{\substack{0\leqq x\leqq z\\ x\equiv\nu(\mathrm{mod}\,\bar{f})}}\omega^{-1}(\nu)A^{(1)}(\langle x\rangle) & \text{if}\quad z\geqq 0\\ \displaystyle -\sum_{\substack{z\leqq x\leqq 0\\ x\equiv\nu(\mathrm{mod}\,\bar{f})}}\omega^{-1}(\nu)A^{(1)}(\langle x\rangle) & \text{if}\quad z<0\,;\end{cases}$$

(iv) if χ is a Dirichlet character modulo $\bar{f}$, then

$$\mathfrak{M}_{\chi,z}(A)=\sum_{\substack{\nu\in Z/\bar{f}Z\\ \nu(\mathrm{mod}\,p)\in(Z/pZ)^\times}}\chi(\nu)\mathfrak{N}_{z,\nu}(A)\,.$$

Let ν, ν' be elements of $\boldsymbol{Z}/\bar{f}\boldsymbol{Z}$ such that $\nu\,(\mathrm{mod}\,p)$, $\nu'\,(\mathrm{mod}\,p)\in(\boldsymbol{Z}/p\boldsymbol{Z})^\times$. Let μ,μ' be natural numbers satisfying $1\leqq\mu$, $\mu'\leqq\bar{f}-1$, $\mu\,(\mathrm{mod}\,\bar{f})=\nu$ and $\mu'\,(\mathrm{mod}\,\bar{f})=\nu'$. Suppose $\mu\leqq\mu'$. Then we have $\mathfrak{N}_{z,\nu}(A)=\mathfrak{N}_{z+\mu-\mu',\nu'}(A)$. Therefore $\mathfrak{N}_{z,\nu}(A)$ is, as a function in z, well-defined on $\{z\in\boldsymbol{Z}_p|z\not\equiv-\nu\,(\mathrm{mod}\,p)\}$.

Let $A(u)=u(\log u-1)$ be as in §3. Put

$$f_\nu(z)=\omega(\nu+z)\mathfrak{N}_{z,\nu}(A)-\omega(\nu)\mathfrak{N}_\nu(A)$$

for each $\nu\in(\boldsymbol{Z}/q\boldsymbol{Z})^\times$. Then the above remarks show that (i) $f_\nu(z)$ is well-defined on $\{z\in\boldsymbol{Z}_p|z\not\equiv-\nu\,(\mathrm{mod}\,p)\}$; (ii) if z is a non-negative integer (resp. a negative integer) satisfying $z\not\equiv-\nu\ (\mathrm{mod}\,p)$, then

$$f_\nu(z)=\omega(\nu+z)\mathfrak{N}_{z+\nu}(A)-\omega(\nu)\mathfrak{N}_\nu(A)+\log\langle\prod_{\substack{1\leq x\leq z\\ x\equiv\nu+z \pmod q}} x\rangle$$

$$\text{(resp. } f_\nu(z)=\omega(\nu+z)\mathfrak{N}_{z+\nu}(A)-\omega(\nu)\mathfrak{N}_\nu(A)+\log\langle\prod_{\substack{z\leq x\leq 0\\ x\equiv\nu+z \pmod q}} x^{-1}\rangle);$$

(iii) $\log(-1)^{z+1}\Gamma_p(z+1)=\sum_{\nu\in(Z/qZ)^\times} f_\nu(z)$ for any $z\in 2q\boldsymbol{Z}_p$.

Let z be an element of $q\boldsymbol{Z}_p$. Then we have

$$f_\nu(z)=\sum_{m\geq 1}\omega^{1-m}(\nu)\mathfrak{N}_\nu(A^{(m)})\frac{z^m}{m!}.$$

Hence

$$\left(\frac{d}{dz}\right)^2 f_\nu(z)=\sum_{m\geq 0}\omega^{-1-m}(\nu)\mathfrak{N}_\nu(A^{(m+2)})\frac{z^m}{m!}$$
$$=\omega^{-1}(\nu)\mathfrak{N}_{z,\nu}(u^{-1})$$
$$=\lim_{n\to\infty}\frac{1}{q^{n+1}}\sum_{\substack{1\leq x\leq q^{n+1}\\ x\equiv\nu \pmod q}}\frac{1}{x+z}.$$

Put

$$g_n(z)=\frac{1}{q^{n+1}}\sum_{\substack{1\leq x\leq q^{n+1}\\ x\equiv\nu \pmod q}}\frac{1}{x+z}.$$

Then $g_n(z)$ is a rational function in z such that its poles are in the set $\{z\in\boldsymbol{Z}_p | z+\nu\equiv 0 \bmod q\}$ and such that the sequence $\{g_n(z)\}_{n\geq 1}$ converges to $\left(\frac{d}{dz}\right)^2 f_\nu(z)$ for each $z\in q\boldsymbol{Z}_p$.

Now let z be an element of the complete algebraic closure $\boldsymbol{C}_p$ of $\boldsymbol{Q}_p$ such that $|z+\nu_0|>|q|$, where ν_0 is the smallest non-negative integer satisfying $\nu_0\equiv\nu \pmod q$. Then we have

$$g_n(z)=\frac{1}{q^{n+1}}\sum_{\substack{1\leq x\leq q^{n+1}\\ x\equiv\nu \pmod q}}\frac{1}{(x-\nu_0)+(z+\nu_0)}$$
$$=\frac{1}{q^{n+1}}\sum_{\substack{1\leq x\leq q^{n+1}\\ x\equiv\nu \pmod q}}\sum_{m\geq 0}\frac{(-1)^m(x-\nu_0)^m}{(z+\nu_0)^{m+1}}$$
$$=\sum_{m\geq 0}\left\{\frac{1}{q^{n+1}}\sum_{\substack{1\leq x\leq q^{n+1}\\ x\equiv\nu \pmod q}}(x-\nu_0)^m\right\}\frac{(-1)^m}{(z+\nu_0)^{m+1}}$$
$$=\sum_{m\geq 0}\left\{\frac{1}{q^n}\sum_{1\leq x\leq q^n}x^m\right\}\frac{(-1)^m q^{m-1}}{(z+\nu_0)^{m+1}}.$$

Therefore we obtain

$$\lim_{n\to\infty} g_n(z)=\lim_{n\to\infty}\sum_{m\geq 0}\left\{\frac{1}{q^n}\sum_{1\leq x\leq q^n}x^m\right\}\frac{(-1)^m q^{m-1}}{(z+\nu_0)^{m+1}}$$

$$= \sum_{m\geq 0}\left\{\lim_{n\to\infty}\frac{1}{q^n}\sum_{1\leq x\leq q^n} x^m\right\}\frac{(-1)^m q^{m-1}}{(z+\nu_0)^{m+1}}$$
$$=\frac{q^{-1}}{z+\nu_0}+\frac{1}{2}\,\frac{-1}{(z+\nu_0)^2}+\sum_{m\geq 2}\mathfrak{M}_{\omega^m}(u^m)\frac{(-1)^m q^{m-1}}{1-p^{m-1}}\,\frac{1}{(z+\nu_0)^{m+1}}\,.$$

Furthermore, for any positive number ε, this convergence is uniform in z for any $z\in \boldsymbol{C}_p$ such that $|z+\nu_0|\geq |q|^{1-\varepsilon}$. Therefore $\left(\frac{d}{dz}\right)^2 f_\nu(z)$ is a uniform analytic function on $\{z\in \boldsymbol{C}_p\cup\{\infty\}\,|\,|z+\nu_0|>|q|\}$ in the sense of Krasner [2].

REMARK. If z is an element of $\{z\in \boldsymbol{C}_p\cup\{\infty\}\,|\,|z|>1\}$, we have

$$\left(\frac{d}{dz}\right)^2\log\langle\Gamma_p(z+1)\rangle=\sum_{m\geq 0}(-1)^m\mathfrak{M}_{\omega^m}(u^m)\frac{1}{z^{m+1}}\,.$$

We note that this formula is a p-adic analogue of the well-known asymptotic formula

$$\left(\frac{d}{dz}\right)^2\log\Gamma(z+1)\sim\sum_{m\geq 0}(-1)^m(-m)\zeta(1-m)\,\frac{1}{z^{m+1}}\,.$$

Appendix

Let $\zeta(s,z)$ be the Hurwitz zeta function. Hence

$$\zeta(s,z)=\sum_{n=0}^{\infty}\frac{1}{(z+n)^s}$$

for any complex number s with $\mathrm{Re}(s)>1$. We define polynomials $\psi_n(z)$ in z by

$$\frac{t^2e^{-zt}}{e^{-t}-1}=\sum_{m=1}^{\infty}\frac{\psi_m(z)(-t)^m}{m!}\,.$$

Then it is known (cf. e.g., Whittaker-Watson [4]) that if s is zero or a negative integer $(=-m)$, we have

$$\zeta(-m,z)=-\frac{\psi_{m+2}(z)}{(m+1)(m+2)}\,.$$

Let p be a prime number and put

$$\zeta'(s,z+1)=\sum_{\substack{1\leq n<\infty\\(n,p)=1}}\frac{1}{(z+n)^s}\,.$$

Then we have

$$\zeta'(s,z+1)=\zeta(s,z+1)-p^{-s}\zeta\left(s,\frac{z}{p}+1\right).$$

Hence

$$\zeta'(-m, z+1)=\zeta(-m, z+1)-p^m\zeta\left(-m, \frac{z}{p}+1\right).$$

Now, as a formal power series in z and t, we have

$$\begin{aligned}\frac{e^{-zt}}{e^{-t}-1}-\frac{e^{-nt}e^{-zt}}{e^{-t}-1}&=\frac{e^{-zt}(1-e^{-nt})}{e^{-t}-1}\\&=-e^{-zt}\sum_{k=0}^{n-1}e^{-kt}\\&=-\sum_{k=0}^{n-1}e^{-(z+k)t}\\&=-\sum_{m=0}^{\infty}\sum_{k=0}^{n-1}(z+k)^m\frac{(-t)^m}{m}\end{aligned}$$

for any non-negative integer n. Therefore we have

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{k=0}^{n-1}(z+k)^m\frac{(-t)^m}{m}\\&\qquad=-\frac{te^{-zt}}{e^{-t}-1}n+(\text{a function in } n \text{ of degree } \geqq 2).\end{aligned}$$

Now we assume that z is an element of $\boldsymbol{Q}_p$ such that $|z|\leqq|q|$ and n is a power p^α of p. Then, taking the limit as $\alpha\to+\infty$, we obtain

$$-\frac{te^{-zt}}{e^{-t}-1}=\sum_{m=0}^{\infty}\left\{\lim\frac{1}{p^\alpha}\sum_{k=0}^{p^\alpha-1}(z+k)^m\right\}\frac{(-t)^m}{m!}.$$

Hence we have

$$\zeta(-m, z)=-\frac{1}{m+1}\lim_{n\to\infty}\frac{1}{p^n}\sum_{k=0}^{p^n-1}(z+k)^{m+1}.$$

Therefore

$$\begin{aligned}\zeta'(-m, z+1)&=\zeta(-m, z+1)-p^m\zeta\left(-m, \frac{z}{p}+1\right)\\&=-\frac{1}{m+1}\lim_{n\to\infty}\frac{1}{p^n}\sum_{x=1}^{p^n}{}^{*}(z+x)^{m+1}\\&=-\frac{1}{m+1}\mathfrak{M}_{\omega^{m+1},z}(u^{m+1}).\end{aligned}$$

This proves the following

THEOREM. *Let $p, \chi, \omega, z, \mathfrak{M}_{\chi,z}$ be as in §2. Let $\zeta'(s, z+1)$ be as above. Let m be zero or a non-negative integer. Then we have*

$$\zeta'(-m, z+1)=-\frac{1}{m+1}\mathfrak{M}_{\omega^{m+1},z}(u^{m+1}).$$

References

[1] Iwasawa, K., Lectures on p-adic L-functions, Ann. of Math. Studies, No. 74, Princeton Univ. Press, Princeton, 1972.

[2] Krasner, M., Prolongement analytique uniforme et multiforme dans les corps valués complets, Colloque Int. C.N.R.S., 143, Paris, 1966.

[3] Kubota, T. and H. W. Leopoldt, Eine p-adische Theorie der Zetawerte, I, J. Reine und angew. Math., **214/215** (1964), 328-339.

[4] Whittaker, E. T. and G. N. Watson, A course of modern analysis, Cambridge University Press, London, 1902.

(Received March 25, 1975)

Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Tokyo
113 Japan