The conjugacy classes of the finite Ree groups of type (F_4)

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In the paper [3], the author has determined the conjugacy classes of the finite Chevalley groups of type $F_4(2^n)$. From the results of that paper many informations about the finite Ree groups of type F_4 —${}^2F_4(q)$— can be obtained. In this paper, using those informations we will determine the conjugacy classes of ${}^2F_4(q)$.

After some preliminaries in § 1, we shall settle the case of the unipotent classes in § 2, in which some examples of $H^1(\sigma, Z/Z^0)$ will also be given. In § 3, the semisimple classes and the general classes, i.e. the classes of type $x=x_s x_u$ $(x_s\neq 1, x_u\neq 1)$ where x_s (resp. x_u) is the semisimple (resp. unipotent) part of x, will be treated. Moreover the calculation of $H^1(\sigma, W)$ will lead us to the classification of maximal torus; with respect to this problem, see [4], p. 191 and p. 213.

§ 1. Preliminaries

(1.1) Throughout this paper, k denotes the finite field with $l=2^{2n+1}$ elements, $\bar{k}$ the algebraic closure of k, $q=\sqrt{l}$ and $\theta=2^n$. G (resp. $\bar{G}$) always denotes the Chevalley group of type (F_4) over k (resp. $\bar{k}$). Under this situation we shall follow the notations in [3], § 1; for example, $\bar{B}, \bar{H}, \bar{U}, \bar{N}$ are the subgroups of $\bar{G}$, which are a Borel subgroup, a maximal torus splitting over the prime field, the maximal unipotent subgroup of $\bar{B}$ and the normalizer of $\bar{H}$ in $\bar{G}$ respectively, and B, H, U, N are rational points over k of B, H, U, N, respectively. Φ is a root system of type (F_4) and Δ is a set of simple roots whose elements are

$$e_2-e_3, \quad e_3-e_4, \quad e_4, \quad \frac{1}{2}(e_1-e_2-e_3-e_4).$$

We set $P_r=\sum_{\alpha\in\Phi} \boldsymbol{Z}\alpha$ and $V=\boldsymbol{R}\otimes_{\boldsymbol{Z}} P_r$. Then V is a Euclidean space of four dimension and $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis of V.

(1.2) There is an abstract outer automorphism σ of $\bar{G}$, also of G, expressed explicitly in Ree [2], p. 405, Theorem 2.7. For a subgroup X of $\bar{G}$, X_σ denotes the stabilizer of σ in X, i.e.

$$X_\sigma=\{x\in X \mid \sigma(x)=x\}.$$

Then $G_\sigma=\tilde{G}_\sigma$ holds and G_σ is a Ree group whose conjugacy classes we are going to study.

(1.3) We shall use the following notations used in [2], p. 407;

$$\begin{aligned}
\alpha_1(t)&=x_4(t^\theta)x_{3-4}(t)x_3(t^{\theta+1}),\\
\alpha_2(t)&=x_3(t^\theta)x_{3+4}(t),\\
\alpha_3(t)&=x_{1-2-3-4}(t^\theta)x_{2-3}(t),\\
\alpha_4(t)&=x_{1-2-3+4}(t^\theta)x_{2-4}(t)x_{1+2-3-4}(t^{\theta+1}),\\
\alpha_5(t)&=x_2(t^\theta)x_{1-2}(t)x_1(t^{\theta+1}),\\
\alpha_6(t)&=x_{1-2+3-4}(t^\theta)x_{2+4}(t)x_{1+2+3+4}(t^{\theta+1}),\\
\alpha_7(t)&=x_{1-2+3+4}(t^\theta)x_{2+3}(t),\\
\alpha_8(t)&=x_{1+2-3-4}(t^\theta)x_{1-3}(t),\\
\alpha_9(t)&=x_{1+2-3+4}(t^\theta)x_{1-4}(t),\\
\alpha_{10}(t)&=x_{1+2+3-4}(t^\theta)x_{1+4}(t),\\
\alpha_{11}(t)&=x_{1+2+3+4}(t^\theta)x_{1+3}(t),\\
\alpha_{12}(t)&=x_1(t^\theta)x_{1+2}(t).
\end{aligned}$$

$\alpha_i(t)$ is an element of G_σ if $t\in k$.

(1.4) Any element of G_σ can be written uniquely in the form $g=uhn_wv$, where $u\in U_\sigma, h\in H_\sigma, v\in(U_w)_\sigma$ and $\{n_w\}_{w\in W_\sigma}$ is a representative system of $W_\sigma\cong N_\sigma/H_\sigma$ in N_σ. Hence

$$G_\sigma=\bigcup_{w\in W_\sigma} U_\sigma H_\sigma n_w(U_w)_\sigma.$$

If we express an element $h(\chi)$ of H as

$$h(\chi)=(z_1,z_2,z_3,z_4),$$

where $\chi\in\mathrm{Hom}(P_r,k^*)$ and $\chi(e_i)=z_i$ $(i=1,2,3,4)$, then $h(\chi)$ is an element of H_σ if and only if it takes the following form:

$$h(\chi)=(\varepsilon_1,\varepsilon_1^{2\theta-1},\varepsilon_2,\varepsilon_2^{2\theta-1}),\quad \text{where}\quad \varepsilon_i\in k^*.$$

Any element u of U_σ can be written uniquely as follows ([2], p. 407, (3.4)):

$$u=\prod_{i=1}^{12}\alpha_i(t_i),\quad \text{where}\quad t_i\in k.$$

W_σ is isomorphic to the dihedral group of order 16 and is generated by $r_a=r_{2-3}r_{1-2-3-4}$ and $r_b=(r_4r_{3-4})^2=r_3r_4$, which satisfy the following relations:

$$r_a^2=1,\quad r_b^2=1 \quad \text{and} \quad (r_a r_b)^8=1.$$

(1.5) For the determination of the conjugacy classes, the following propositions are often used and well known.

PROPOSITION 1.1 ([4], p. 176, 3.4). *Let C be a conjugacy class of $\bar{G}$ stable under σ. Then*

(a) *C contains an element stabilized by σ, and*

(b) *if x is such an element, then the conjugacy classes of G_σ contained in $G_\sigma \cap C$ correspond bijectively to $H^1(\sigma, Z_{\bar{G}}(x)/Z_{\bar{G}}(x)^0)$.*

PROPOSITION 1.2 ([4], p. 197, 3.9). *If t is a semisimple element of $\bar{G}$, then $Z_{\bar{G}}(t)$ is a connected reductive group.*

§2. Unipotent classes

(2.1) From Proposition 1.1, it follows that in order to determine the unipotent classes the following calculations are needed and sufficient:

(1) To list up the unipotent classes $\{C_i\}$ of $\bar{G}$ such that $\sigma(C_i)=C_i$.

(2) For each C_i, to find an element u_i contained in $C_i \cap G_\sigma$.

(3) For each u_i, to calculate $H^1(\sigma, Z_i/Z_i^0)$, where $Z_i=Z_{\bar{G}}(u_i)$.

(4) To find a representative system $\{u_{i,j}\}$ $(u_{i,j} \in C_i)$ which correspond to the elements of $H^1(\sigma, Z_i/Z_i^0)$.

(5) To calculate $Z_{G_\sigma}(u_{i,j})$.

(1) and (3) are almost settled in [3], Theorem 2.1 and (2) and (4) are theoretically solved (cf. (2.4) below).

(2.2) For convenience' sake, we summarize and rewrite the results about the unipotent classes of G in Table I (cf. [3], §2).

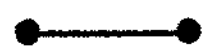
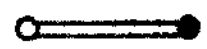
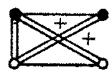
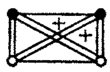
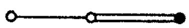
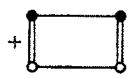
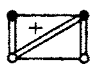
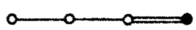
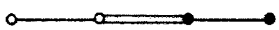
(2.3) Table I shows that there are 10 unipotent classes in $\bar{G}$ which are stable under σ. If $Z/Z^0 \cong 1$ or Z_2, then σ acts trivially on Z/Z^0. Hence we have only to calculate $H^1(\sigma, Z/Z^0)$ for C_{14}, C_{17} and C_{19}.

For the representative elements contained in $C_i \cap G_\sigma$ $(i=14, 17, 19)$, we can choose

$$u_1=\alpha_5(1)\alpha_6(1) \in C_{14} \cap G_\sigma,$$
$$u_2=\alpha_2(1)\alpha_4(1) \in C_{17} \cap G_\sigma,$$
$$u_3=\alpha_1(1)\alpha_3(1) \in C_{19} \cap G_\sigma.$$

We set $Z_i=Z_{\bar{G}}(u_i)$ $(i=1,2,3)$, and for an element $x \in Z_i$, $\bar{x}$ denotes the canonical

Table I.

C_i	$\sigma(C_i)$	rep. element	diagram	Z/Z^0
C_0	C_0	$x_0=1$	∅	1
C_1	C_2	x_1	○	1
C_2	C_1	x_2	●	1
C_3	C_3	x_3		1
C_4	C_4	x_4	○ ●	1
C_5	C_6	x_5 or x_6		Z_2
C_6	C_5	x_7 or x_8		Z_2
C_7	C_7	x_9 or x_{10}		Z_2
C_8	C_9	x_{11}		1
C_9	C_8	x_{12}		1
C_{10}	C_{11}	x_{13}		1
C_{11}	C_{10}	x_{14}		1
C_{12}	C_{12}	x_{15}		1
C_{13}	C_{13}	x_{16}		1
C_{14}	C_{14}	x_{17}, x_{18}, or x_{19}		$\mathfrak{S}_3$
C_{15}	C_{16}	x_{20} or x_{21}		Z_2
C_{16}	C_{15}	x_{22} or x_{23}		Z_2
C_{17}	C_{17}	x_{24}, x_{25}, x_{26}, x_{27} or x_{28}		D_8
C_{18}	C_{18}	x_{29} or x_{30}	 or 	Z_2
C_{19}	C_{19}	x_{31}, x_{32}, x_{33} or x_{34}		Z_4

Remarks on Table I: (a) x_i in the column of "representative element" denotes the element defined in Theorem 2.1 of [3]. (b) The column of "Z/Z^0" denotes the group $Z_{\bar{G}}(x)/Z_{\bar{G}}(x)^0$ for $x\in C_i$. $\mathfrak{S}_3$ is the symmetric group of degree 3 and D_8 is the dihedral group of order 8. (c) From each class C_i, we can take an element x such that $x=\prod_{i=1}^{j} x_{\alpha_i}(1)$ $(1\leq j\leq 4)$ where $\alpha_i\in\Phi$ and $\{\alpha_1,\dots,\alpha_j\}$ are linearly independent. This element is not always equal to "rep-

(Continued on next page)

image of x in Z_i/Z_i^0. These notations, u_i, Z_i $(i=1,2,3)$ and $\bar{x}$, are used only in (2.3).

First we calculate Z_1/Z_1^0 and Z_3/Z_3^0. We set $v=\alpha_3(1)\alpha_4(1)$, then

$$Z_1/Z_1^0=\langle\bar{v}, \bar{r}_a\rangle=\mathfrak{S}_3,$$

since $\bar{v}^2=\bar{r}_a^2=1$ and $(\bar{v}\bar{r}_a)^3=1$; and

$$Z_3/Z_3^0=\langle\bar{u}_3\rangle=\boldsymbol{Z}_4.$$

Hence σ acts trivially on Z_1/Z_1^0 and Z_3/Z_3^0, and so the elements of $H^1(\sigma, Z_1/Z_1^0)$ or $H^1(\sigma, Z_3/Z_3^0)$ correspond bijectively to the conjugacy classes of $\mathfrak{S}_3$ or $\boldsymbol{Z}_4$, respectively.

Next we consider Z_2/Z_2^0. We define v_i $(i=1,2,3)$ as follows.

$$\begin{aligned}v_1&=x_{2-3}(1)x_4(1),\\ v_2&=x_{1-2-3-4}(1)x_{3-4}(1),\\ v_3&=x_{2-4}(1)x_{1-2-3+4}(1).\end{aligned}$$

Then we have

$$Z_2/Z_2^0=\langle\bar{v}_1, \bar{v}_2\rangle\cong D_8,$$

since $\bar{v}_i^2=1$ $(i=1,2,3)$, $(\bar{v}_1\bar{v}_2)^4=1$ and $(\bar{v}_1\bar{v}_2)^2=\bar{v}_3$ hold. σ interchanges v_1 and v_2, hence it follows that $H^1(\sigma, Z_2/Z_2^0)$ consists of three elements:

$$\{\bar{1}, \bar{v}_1\bar{v}_2, \bar{v}_2\bar{v}_1, \bar{v}_3\},\quad \{\bar{v}_1, \bar{v}_2\}\quad\text{and}\quad \{\bar{v}_1\bar{v}_3, \bar{v}_2\bar{v}_3\}.$$

From the results above and Table I, it is easily deduced that G_σ has 19 unipotent classes.

(2.4) Step (2) and step (4) in (2.1) are not so easy. Step (2) is solved as follows: if $C=\sigma(C)$ and x is an element of C, then there is an element g in $\bar{G}$ such that $x=g\sigma(x)g^{-1}$. From the theorem of Lang-Steinberg, we have $g=z\sigma(z)^{-1}$ for some $z\in\bar{G}$ (cf. [4], I, 2.2). Then $z^{-1}xz$ is a desired element. Step (4) is solved as follows: let g be an element of $Z=Z_{\bar{G}}(u)$ $(u\in G_\sigma)$, then the corresponding element of G_σ to the element of $H^1(\sigma, Z/Z^0)$ represented by g is obtained by $z^{-1}uz$ where z is deter-

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(Continued)
resentative element". For such an element, we can associate a "diagram". This diagram consists of nodes and bonds; the nodes correspond to the roots $\{\alpha_1, \ldots, \alpha_j\}$ and the node ○ (resp. ●) stands for a long (resp. short) root; the number of bonds which bind the nodes corresponding to the roots α and β is equal to $n_{\alpha,\beta}n_{\beta,\alpha}$, where $n_{\alpha,\beta}=\dfrac{2(\alpha,\beta)}{(\alpha,\alpha)}$; the sign + above the bonds (for example C_3: $\circ\overset{\alpha+\beta}{=\!=}\bullet$) shows that $(\alpha,\beta)>0$. Roughly speaking, the automorphism σ interchanges long roots and short roots, hence $\sigma(C_i)$ can be read off from the diagram.

mined by the relation $z\sigma(z)^{-1}=g$ (cf. [4], I, 2.7).

Difficulties lie on the way of finding the element g such that $x=g\sigma(x)g^{-1}$ in step (2), and finding z such that $z\sigma(z)^{-1}=g$ in steps (2) and (4).

But we can find representative elements in some way: in some cases by using above method and in some cases by the examination of all the possible elements. We omit the process how to get the representative elements, but as a result we get the following theorem.

THEOREM 2.1. *G_σ has 19 unipotent classes. Their representative elements, the orders of their centralizers and the class of G which contains the class of G_σ are given in Table II.*

The calculations of the centralizers in G_σ are straightforward and quite similar to those in G and much simpler. Hence we omit the proof, but we explain some properties of $Z_{G_\sigma}(u_i)$.

$$u_1 \in Z(U_\sigma)$$
$$Z_{G_\sigma}(u_1)=U_\sigma H_2 \cup U_\sigma H_2 r_b(U_{r_b})_\sigma,$$

where $H_2=\{(1,1,\varepsilon,\varepsilon^{2\theta-1})\mid \varepsilon\in k^*\}$, and $(U_{r_b})_\sigma=\{\alpha_1(t_1)\alpha_2(t_2)\mid t_i\in k\}$.

$$Z_{G_\sigma}(u_2)=U'H'\cup U'H'r_a(U_{r_a})_\sigma,$$

where $U'=\{\prod_{i\neq 1,4}\alpha_i(t_i)\mid t_i\in k\}$, $H'=\{(\varepsilon,\varepsilon^{2\theta-1},\varepsilon^{-2\theta-1},\varepsilon^{-1})\mid \varepsilon\in k^*\}$, and $(U_{r_a})_\sigma=\{\alpha_3(t)\mid t\in k\}$.

$$Z_{G_\sigma}(u_3)=U''H_2\cup U''H_2 r_b(U_{r_b})_\sigma,$$

where $U''=\{\alpha_5(\eta)\prod_{i\neq 3,4,6,7}\alpha_i(t_i)\mid \eta=0 \text{ or } 1, t_i\in k\}$.

$$Z_{G_\sigma}(u_4)=Z_{G_\sigma}(u_3).$$

Now we define the "parabolic" subgroups P_a, P_b of G_σ as follows:

$$P_a=B_\sigma\cup B_\sigma r_a(U_{r_a})_\sigma,$$
$$P_b=B_\sigma\cup B_\sigma r_b(U_{r_b})_\sigma.$$

Then $Z_{G_\sigma}(u_2)\subset P_a$ and $Z_{G_\sigma}(u_1), Z_{G_\sigma}(u_3)\subset P_b$.

$$Z_{G_\sigma}(u_i)\cap U_\sigma \subsetneqq Z_{G_\sigma}(u_i)\subset P_a \quad (i=7,9).$$

For the other u_i, i.e. $i=5,6,8,10,11,\dots,18$,

$$Z_{G_\sigma}(u_i)\subset U_\sigma.$$

REMARK. It is known that the number of the unipotent elements in G_σ is q^{48}. Hence it is possible to prove Theorem 2.1 directly without the calculations of $H^1(\sigma, Z/Z^0)$.

Table II.

rep. element u_i	$\lvert Z_{G_\sigma}(u_i)\rvert$	$x_j \underset{G}{\sim} u_j$
$u_0=1$	$q^{24}(q^2-1)(q^6+1)(q^8-1)(q^{12}+1)$	1
$u_1=\alpha_{12}(1)$	$q^{24}(q^2-1)(q^4+1)$	x_3
$u_2=\alpha_{10}(1)$	$q^{20}(q^4-1)$	x_4
$u_3=\alpha_5(1)\alpha_{12}(1)$	$2q^{14}(q^2-1)(q^4+1)$	x_{10}
$u_4=\alpha_5(1)$	$2q^{14}(q^2-1)(q^4+1)$	x_{10}
$u_5=\alpha_7(1)\alpha_8(1)$	q^{16}	x_{15}
$u_6=\alpha_5(1)\alpha_7(1)$	q^{14}	x_{16}
$u_7=\alpha_5(1)\alpha_6(1)$	$6q^{12}$	x_{17}
$u_8=\alpha_5(1)\alpha_6(1)\alpha_8(1)$	$2q^{12}$	x_{17}
$u_9=\alpha_2(1)\alpha_6(1)\alpha_8(1)$	$3q^{12}$	*) x_{19} or x_{17}
$u_{10}=\alpha_2(1)\alpha_4(1)$	$2q^8$	x_{26}
$u_{11}=\alpha_2(1)\alpha_4(1)\alpha_5(1)$	$4q^8$	x_{28}
$u_{12}=\alpha_2(1)\alpha_4(1)\alpha_5(1)\alpha_8(1)$	$4q^8$	x_{28}
$u_{13}=\alpha_2(1)\alpha_3(1)$	$2q^6$	x_{29}
$u_{14}=\alpha_2(1)\alpha_3(1)\alpha_5(1)$	$2q^6$	x_{29}
$u_{15}=\alpha_1(1)\alpha_3(1)$	$4q^4$	x_{34}
$u_{16}=\alpha_1(1)\alpha_2(1)\alpha_3(1)$	$4q^4$	x_{32}
$u_{17}=\alpha_1(1)\alpha_3(1)\alpha_5(1)$	$4q^4$	x_{34}
$u_{18}=\alpha_1(1)\alpha_2(1)\alpha_3(1)\alpha_5(1)$	$4q^4$	x_{32}

*) If X^3+X+1 is an irreducible polynomial in $k[X]$, then $u_9 \sim x_{19}$, while if X^3+X+1 is reducible, then $u_9 \sim x_{17}$.

§3. Semisimple classes

For the determination of the semisimple classes, the program showed in [§2, (2.1)] is also valid in this case, but much simpler way exists: the calculation of $H^1(\sigma, W)$.

(3.1) Each element of W acts canonically on $P_r=\sum_{\alpha\in\Phi} Z\alpha$ and on $V=R\otimes_Z P_r$. W acts on V faithfully as the orthogonal transformations. Let φ be this representation of W into $O(V)$. σ acts also on V and we write the corresponding element in $GL(V)$ as σ^*. It is well known that $\sigma^*=q\tau$ where $\tau\in O(V)$. τ is expressed relative to a basis $\{e_1, e_2, e_3, e_4\}$ as follows:

$$\tau=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

It is easily verified that $\varphi(\sigma(w))=\tau\varphi(w)\tau^{-1}$ for all $w\in W$.

Now we recall the definition of $H^1(\sigma, W)$. $H^1(\sigma, W)$ is the equivalence classes of W under the equivalence relation $\approx$; for $a,b\in W$, $a\approx b$ holds if and only if for

Table III.

rep. element w_i	$f_{w_i\tau}(q)=\det(q\cdot I-w_i\tau)$	$\lvert Z_W(w_i\tau)\rvert$
$w_1=\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix}$	$(q^2-1)^2$	16 (W_σ)
$w_2=\begin{pmatrix}1&&&\\&&1&\\&1&&\\&&&1\end{pmatrix}$	q^4-1	4 $(Z_2\times Z_2)$
$w_3=\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&-1\end{pmatrix}$	$(q^2-1)(q^2-\sqrt{2}q+1)$	8 $(Z_2\times Z_4)$
$w_4=\begin{pmatrix}1&&&\\&1&&\\&&-1&\\&&&1\end{pmatrix}$	$(q^2-1)(q^2+\sqrt{2}q+1)$	8 $(Z_2\times Z_4)$
$w_5=\begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&1\end{pmatrix}$	q^4+1	16 $(Z_4\times Z_4)$
$w_6=\begin{pmatrix}1&&&\\&-1&&\\&&1&\\&&&-1\end{pmatrix}$	$(q^2-\sqrt{2}q+1)^2$	96
$w_7=\begin{pmatrix}-1&&&\\&1&&\\&&-1&\\&&&1\end{pmatrix}$	$(q^2+\sqrt{2}q+1)^2$	96
$w_8=\begin{pmatrix}&&1&\\&&&1\\-1&&&\\&-1&&\end{pmatrix}$	$(q^2+1)^2$	48

Table III. (Continued)

rep. element w_i	$f_{w_i\tau}(q)=\det(q\cdot I-w_i\tau)$	$\lvert Z_W(w_i\tau)\rvert$
$w_9=\begin{pmatrix} 1 & & & \\ & & -1 & \\ & 1 & & \\ & & & 1 \end{pmatrix}$	q^4-q^2+1	6 (Z_6)
$w_{10}=\begin{pmatrix} 1 & & & \\ & & 1 & \\ & & & 1 \\ & 1 & & \end{pmatrix}$	$q^4-\sqrt{2}\,q^3+q^2-\sqrt{2}\,q+1$	12 (Z_{12})
$w_{11}=\begin{pmatrix} & & 1 & \\ & 1 & & \\ -1 & & & \\ & & & 1 \end{pmatrix}$	$q^4+\sqrt{2}\,q^3+q^2+\sqrt{2}\,q+1$	12 (Z_{12})

some $c\in W$, $a=c^{-1}b\sigma(c)$ holds. $a\approx b$ is clearly equivalent to

$$\varphi(a)\tau=\varphi(c)^{-1}\varphi(b)\tau\varphi(c).$$

In the sequence we identify the element of W with an element of $O(V)$ through the isomorphism φ. Then if we set $\tilde{W}=\langle W,\tau\rangle$, we have $\tilde{W}=W\cup W\tau$ since $\tau^2=1$ and τ normalizes W. The argument above shows that the mapping from W to $W\tau$: $w\longmapsto w\tau$ induces the bijection between $H^1(\sigma, W)$ and the classes of $W\tau$ under conjugations of W.

THEOREM 3.1. $\lvert H^1(\sigma, W)\rvert=11$ *and the representative elements are given in Table III.*

PPOOF. We shall calculate the classes of $W\tau$ under conjugations of W. The characteristic polynomial $f_{w_i\tau}(q)=\det(q\cdot I-w_i\tau)$ are all different for $i=1,\dots,11$. Hence any of the two in $\{w_i\tau \mid i=1,\dots,11\}$ are not conjugate under W. On the other hand, the calculations of $\lvert Z_W(w_i\tau)\rvert$ shows that:

$$\sum_{i=1}^{11}\lvert W\rvert/\lvert Z_W(w_i\tau)\rvert=1152=\lvert W\tau\rvert,$$

and this proves the theorem.

Theorem 3.1 completes the classification of the maximal tori of G_σ: that is, let T be a maximal torus of $\bar{G}$ stabilized by σ, then T_σ is conjugate in $\bar{G}$ to one of the subgroups $T(i)$ $(i=1,\dots,11)$ of $\bar{H}$, where $T(i)=\{h\in\bar{H} \mid h={}^{w_i\sigma}h\}$ ([4], II, 1.3.). We will write explicitly the elements of $T(i)$:

$$
\begin{aligned}
T(1)&=\{(\varepsilon_1, \varepsilon_1^{2\theta-1}, \varepsilon_2, \varepsilon_2^{2\theta-1})\}\cong(\boldsymbol{Z}_{q^2-1})^2,\\
T(2)&=\{(\varepsilon^{\theta+1}\kappa^{\theta}, \varepsilon^{\theta}\kappa^{1-\theta}, \varepsilon^{\theta}\kappa^{-1-\theta}, \varepsilon^{1-\theta}\kappa^{-\theta})\}\cong\boldsymbol{Z}_{q^4-1},\\
T(3)&=\{(\varepsilon, \varepsilon^{2\theta-1}, \gamma, \gamma^{-q^2})\}\cong\boldsymbol{Z}_{q^2-1}\times\boldsymbol{Z}_{(q^2-\sqrt{2}q+1)},\\
T(4)&=\{(\varepsilon, \varepsilon^{2\theta-1}, \delta, \delta^{-q^2})\}\cong\boldsymbol{Z}_{q^2-1}\times\boldsymbol{Z}_{(q^2+\sqrt{2}q+1)},\\
T(5)&=\{(\gamma, \gamma^{-q^2}, \delta, \delta^{-q^2})\}\cong\boldsymbol{Z}_{q^4+1},\\
T(6)&=\{(\gamma_1, \gamma_1^{-q^2}, \gamma_2, \gamma_2^{-q^2})\}\cong(\boldsymbol{Z}_{(q^2-\sqrt{2}q+1)})^2,\\
T(7)&=\{(\delta_1, \delta_1^{-q^2}, \delta_2, \delta_2^{-q^2})\}\cong(\boldsymbol{Z}_{(q^2+\sqrt{2}q-1)})^2,\\
T(8)&=\{(\kappa_1, \kappa_2, (\kappa_1\kappa_2)^{\theta}, (\kappa_1\kappa_2^{-1})^{\theta}\}\cong(\boldsymbol{Z}_{q^2+1})^2,\\
T(9)&=\{(\rho, \rho^{-4\theta^3+2\theta-1}, \rho^{1-2\theta}, \rho^{4\theta^3-4\theta^2+1})\}\cong\boldsymbol{Z}_{(q^4-q^2+1)},\\
T(10)&=\{(\lambda, \lambda^{q^2-q^4}, \lambda^{2\theta-1}, \lambda^{4\theta^3-1})\}\cong\boldsymbol{Z}_{(q^4-\sqrt{2}q^3+q^2-\sqrt{2}q+1)},\\
T(11)&=\{(\mu^{q^4-q^2}, \mu, \mu^{2\theta+1}, \mu^{-4\theta^3-1})\}\cong\boldsymbol{Z}_{(q^4+\sqrt{2}q^3+q^2+\sqrt{2}q+1)},
\end{aligned}
$$

where $\varepsilon^{q^2-1}=\varepsilon_1^{q^2-1}=\varepsilon_2^{q^2-1}=1$, $\kappa^{q^2+1}=\kappa_1^{q^2+1}=\kappa_2^{q^2+1}=1$, $\gamma^{q^2-\sqrt{2}q+1}=\gamma_1^{q^2-\sqrt{2}q+1}=\gamma_2^{q^2-\sqrt{2}q+1}=1$, $\delta^{q^2+\sqrt{2}q+1}=\delta_1^{q^2+\sqrt{2}q+1}=\delta_2^{q^2+\sqrt{2}q+1}=1$, $\rho^{q^4-q^2+1}=1$, $\lambda^{q^4-\sqrt{2}q^3+q^2-\sqrt{2}q+1}=1$ and $\mu^{q^4+\sqrt{2}q^3+q^2+\sqrt{2}q+1}=1$, and $\boldsymbol{Z}_n=\boldsymbol{Z}/n\boldsymbol{Z}$.

(3.2) Every semisimple element of G_σ is conjugate to an element of $\bigcup_{i=1}^{11} T(i)$ in $\bar{G}$. If $s_1, s_2\in G_\sigma$, $t_1, t_2\in\bigcup_{i=1}^{11} T(i)$ and $s_i \underset{\bar{G}}{\sim} t_i$ $(i=1,2)$, then $s_1 \underset{G_\sigma}{\sim} s_2$ if and only if $t_1 \underset{W}{\sim} t_2$ (This is an immediate consequence of Prop. 1.1, Prop. 1.2 and the uniqueness of Bruhat decomposition.): so that, in order to classify the semisimple elements in G_σ, we have only to classify the elements of $\bigcup_{i=1}^{11} T(i)$ under conjugation by W. The representative elements of semisimple classes in G_σ can be taken from the elements in $\bigcup_{i=1}^{11} T(i)$ in the sense written above.

THEOREM 3.2. *The representative elements of the semisimple classes of G_σ in $\bigcup_{i=1}^{11} T(i)$, the number of classes and the orders of their centralizers are given in Table IV.*

PROOF. The representative elements and the number of classes are obtained immediately, so that the problem left is to calculate the orders of centralizers. Since later we shall determine the unipotent classes in the centralizers, now we examine the structure of the centralizers in detail.

Let s be a semisimple element contained in G_σ and $s \underset{\bar{G}}{\sim} t$, $t\in T(i)$. Then

$$Z_{G_\sigma}(s)\cong Z_{\bar{G}}(t)\cap G(w_i\sigma)$$

where $G(w_i\sigma)=\{g\in\bar{G}\mid g={}^{w_i\sigma}g\}$. $Z_{\bar{G}}(t)$ are calculated by means of [1], §2.

Case 1. $t=t_j$ $(j=3, 6, 8, 10, 11, 12, 13, 14, 15, 16$ or $17)$.

In this case t is a regular element, hence contained in a unique maximal torus

Table IV.

rep. element t_i	the number of classes f_i	order of centralizer	$(*_1)$ h_j	$(*_2)$ m_j
$t_0=(1, 1, 1, 1)$	1	$q^{24}(q^2-1)(q^6+1)(q^8-1)(q^{12}+1)$	h_0	19
$t_1=(1, 1, \varepsilon, , \varepsilon^{2\theta-1})$	$\frac{1}{2}(q^2-2)$	$q^4(q^2-1)^2(q^4+1)$	h_6	4
$t_2=(\varepsilon, \varepsilon^{2\theta-1}, \varepsilon^{2\theta+1}, \varepsilon)$	$\frac{1}{2}(q^2-2)$	$q^2(q^2-1)(q^4-1)$	h_7	2
$t_3=(\varepsilon_1, \varepsilon_1^{2\theta-1}, \varepsilon_2, \varepsilon^{2\theta-1})$ $\varepsilon_1\neq\varepsilon^{\pm1}, \varepsilon_2^{\pm(2\theta\pm1)}$	$\frac{1}{16}(q^2-2)(q^2-8)$	$(q^2-1)^2$	h_{12}	1
$t_4=(\omega, \omega^{1-\theta}, \omega^{-1-\theta}, \omega^{-\theta})$ $\omega^3=1,\ \omega\neq1$	1	$q^6(q^4-1)(q^6+1)$	h_{15}	5
$t_5=(\kappa, \kappa^{1-\theta}, \kappa^{-1-\theta}, \kappa^{-\theta})$ $\kappa^3\neq1$	$\frac{1}{2}(q^2-2)$	$q^2(q^2+1)(q^4-1)$	h_{19}	2
$t_6=(\varepsilon^{\theta+1}\kappa^{\theta}, \varepsilon^{\theta}\kappa^{1-\theta}, \varepsilon^{\theta}\kappa^{-1-\theta}, \varepsilon^{1-\theta}\kappa^{-\theta})$	$\frac{1}{4}q^2(q^2-2)$	q^4-1	h_{44}	1
$t_7=(1, 1, \gamma, \gamma^{-q^2})$	$\frac{1}{4}(q^2-\sqrt{2}\,q)$	$q^4(q^2-\sqrt{2}\,q+1)(q^2-1)(q^4+1)$	h_{53}	4
$t_8=(\varepsilon, \varepsilon^{2\theta-1}, \gamma, \gamma^{-q^2})$	$\frac{1}{8}(q^2-2)(q^2-\sqrt{2}\,q)$	$(q^2-1)(q^2-\sqrt{2}\,q+1)$	h_{66}	1
$t_9=(1, 1, \delta, \delta^{-q^2})$	$\frac{1}{4}(q^2+\sqrt{2}\,q)$	$q^4(q^2+\sqrt{2}\,q+1)(q^2-1)(q^4+1)$	h_{53}	4
$t_{10}=(\varepsilon, \varepsilon^{2\theta-1}, \delta, \delta^{-q^2})$	$\frac{1}{8}(q^2-2)(q^2+\sqrt{2}\,q)$	$(q^2-1)(q^2+\sqrt{2}\,q+1)$	h_{56}	1
$t_{11}=(\gamma, \gamma^{-q^2}, \delta, \delta^{-q^2})$	$\frac{1}{16}q^2(q^2-2)$	q^4+1	h_{63}	1
$t_{12}=(\gamma_1, \gamma^{-q^2}, \gamma_2, \gamma_2^{-q^2})$ $\gamma_1\neq\gamma_2^{\pm1}, \gamma^{\pm q^2}$	$\frac{1}{96}(q^2-\sqrt{2}\,q-4)(q^2-\sqrt{2}\,q)$	$(q^2-\sqrt{2}\,q+1)^2$	h_{63}	1

Table IV. (Continued)

rep. element t_i	the number of classes f_i	order of centralizer	(*1) h_j	(*2) m_j
$t_{13}=(\delta_1, \delta_1^{-q^2}, \delta_2, \delta_2^{-q^2})$ $\delta_1\neq\delta_2^{\pm 1}, \delta_2^{\pm q^2}$	$\frac{1}{96}(q^2+\sqrt{2}q-4)(q^2+\sqrt{2}q)$	$(q^2+\sqrt{2}q+1)^2$	h_{63}	1
$t_{14}=(\kappa_1, \kappa_2, (\kappa_1\kappa_2)^\theta, (\kappa_1\kappa_2^{-1})^\theta)$	$\frac{1}{48}(q^2-2)(q^2-8)$	$(q^2+1)^2$	h_{24}	1
$t_{15}=(\rho, \rho^{-4\theta^3+2\theta-1}, \rho^{1-2\theta}, \rho^{4\theta^3-4\theta^2+1})$	$\frac{1}{6}(q^2-2)(q^2+1)$	q^4-q^2+1	h_{75}	1
$t_{16}=(\lambda, \lambda^{q^2-q^4}, \lambda^{2\theta-1}, \lambda^{4\theta^3-1})$	$\frac{1}{12}(q^4-\sqrt{2}q^3+q^2-\sqrt{2}q)$	$q^4-\sqrt{2}q^3+q^2-\sqrt{2}q+1$	h_{76}	1
$t_{17}=(\mu^{q^4-q^2}, \mu, \mu^{2\theta+1}, \mu^{-4\theta^3-1})$	$\frac{1}{12}(q^4+\sqrt{2}q^3+q^2+\sqrt{2}q)$	$q^4+\sqrt{2}q^3+q^2+\sqrt{2}q+1$	h_{76}	1

(*1) $t_i \underset{G}{\sim} h$, where h is one of the element of type h_j determined in ([3], §3).

(*2) m_j is the number of unipotent classes in $Z_{w_j\sigma}(t_i)$, $t_i\in T(j)$.

$\bar{H}$ of $\bar{G}$ and $Z_{\bar{G}}(t)=\bar{H}$. Clearly $Z_{\bar{G}}(t)\cap G(w_i\sigma)=T(i)$.

Case 2. $t=t_j$ ($j=1,7$ or 9).

$$Z_{\bar{G}}(t)\cong S_p(4,\bar{k})\times S,$$

where S is a torus contained in $\bar{H}$ and $\dim S=2$.

$$Z_{\bar{G}}(t)\cap G(w_i\sigma)\cong {}^2B_2(q)\times S(i),$$

where ${}^2B_2(q)$ denotes a Suzuki group of order $q^4(q^2-1)(q^4+1)$ and $S(i)$ is a cyclic group of odd order $q^2-1, q^2-\sqrt{2}\,q+1$ or $q^2+\sqrt{2}\,q+1$ according to $i=1,7$ or 9.

Case 3. $t=t_j$ ($j=2$ or 5).

We examine the case where $t=t_2$ (for the notations used here and in Case 4 see [1], §2):

$$\begin{aligned}
&t_2=(\varepsilon,\varepsilon^{2\theta-1},\varepsilon^{2\theta+1},\varepsilon) \quad\text{and}\quad G(w_i\sigma)=G_\sigma,\\
&\Phi_\chi^+=\{1+2-3+4,\ 1-4\},\\
&W_\chi=\langle r_{1+2-3+4}\rangle\times\langle r_{1-4}\rangle\cong(Z_2)^2,\\
&Z_{\bar{G}}(t_2)=\bigcup_{w\in W_\chi}\bar{U}^\chi\bar{H}n_w\bar{U}_w^\chi,\\
&Z_{\bar{G}}(t_2)\cap G_\sigma=(\bar{U}^\chi)_\sigma H_\sigma\cup(\bar{U}^\chi)_\sigma H_\sigma(r_{1+2-3+4}r_{1-4})(\bar{U}^\chi)_\sigma,
\end{aligned}$$

where $(\bar{U}^\chi)_\sigma=\{\alpha_9(t)\mid t\in k\}$.

The case where $t=t_5$ can be dealt with in the same way.

Case 4. $t=t_4\in T(2)$.

First we assume that $\theta\equiv 1 \pmod 3$, then $t=(\omega,1,\omega,\omega^{-1})$; hence

$$\begin{aligned}
&\Phi_\chi^+=\{2,\ 1\pm2+3-4,\ 1-3,\ 3+4,\ 1+4\},\\
&W_\chi\cong\mathfrak{S}_3\times\mathfrak{S}_3,\\
&(W_\chi)_{w_2\sigma}=\{w\in W_\chi\mid w={}^{w_2\sigma}w\}=\{1,r_{1+2+3-4}r_{1+4}\}\cong \boldsymbol{Z}_2,\\
&(\bar{U}_\chi)_{w_2\sigma}=\{x_2(s_1)x_{1-2+3-4}(s_1^l)x_{1+2+3-4}(s_2)x_{1-3}(s_1^{2\theta})x_{3+4}(s_1^{2\theta l})x_{1+4}(s_2^{2\theta})\mid s_1\in \boldsymbol{F}_{l^2},\ s_2^l+s_2+s_1^{l+1}=0\},\\
&(\bar{H})_{w\sigma}=T(2).
\end{aligned}$$

Hence $|(\bar{U}_\chi)_{w_2\sigma}|=l^3=q^6$ and $|(\bar{H})_{w\sigma}|=l^2-1=q^4-1$.

$$Z_{\bar{G}}(t)\cap G(w_2\sigma)=(\bar{U}_\chi)_{w_2\sigma}T(2)\cup(\bar{U}_\chi)_{w_2\sigma}T(2)(r_{1+2+3-4}r_{1+4})(\bar{U}_\chi)_{w_2\sigma},$$

and we have

$$|Z_{\bar{G}}(t)\cap G(w_2\sigma)|=l^3(l^2-1)(1+l^3)=q^6(q^4-1)(q^6+1).$$

The case where $\theta\equiv -1 \pmod 3$ are settled in the same way and in this case $t=(\omega^{-1},\omega^{-1},1,\omega)$: hence $\Phi_\chi^+=\{3,\ 1+2\pm3-4,\ 1-2,\ 2+4,\ 1+4\}$ and

$$(W^\chi)_{w_2\sigma}=\{1,r_{1+2+3-4}r_{1+4}\}\cong \boldsymbol{Z}_2 \text{ etc.}$$

(3.3) In order to determine the remaining classes, it is sufficient to determine the unipotent classes in the group $Z_G(t)\cap G(w_i\sigma)$, where $t\in T(i)$. In Case 1 or Case 3 in the proof of Theorem 3.2, the unipotent classes are obtained immediately: in former case only one class exists, i.e. unit element and in latter case two classes. In Case 2 we have only to determine the unipotent classes of ${}^2B_2(q)$, but this is easily determined and well known.

Now let us determine the unipotent classes of $Z=Z_{G(w_2\sigma)}(t)$, $t=(\omega,1,\omega,\omega^{-1})$ and $\theta\equiv 1 \pmod 3$.

Every unipotent element is conjugate in Z to an element of $(\bar{U}_\chi)_{w_2\sigma}$, so that it is sufficient to classify the elements of $(\bar{U}_\chi)_{w_2\sigma}$ under conjugation by Z. From the proof of Case 4 in Theorem 3.2, every element u in $(\bar{U}_\chi)_{w_2\sigma}$ can be written uniquely in the form:

$$u=x_2(s_1)x_{1-2+3-4}(s_1^l)x_{1+2+3-4}(s_2)x_{1-3}(s_1^{2\theta})x_{3+4}(s_1^{2\theta l})x_{1+4}(s_2^{2\theta})$$

where $s_1\in F_{l^2}$, $s_2^l+s_2+s_1^{l+1}=0$ (hence $s_2\in F_{l^2}$).

First let us consider the case where $s_1=0$. Then if $s_2=0$ $u\sim 1$ and if $s_2\neq 0$, $u\sim x_{1+2+3-4}(1)x_{1+4}(1)$ by the conjugation of an element in $T(2)$.

Next we consider the case where $s_1\neq 0$. It is known that by the conjugation of the elements in $T(2)$ s_1 can take all values in $\{s_1\varepsilon^\theta\kappa^{1-\theta}\}$. Let η be a generator of $F_{l^2}^*$, then we can write

$$\varepsilon=\eta^{(l+1)a},\quad \kappa=\eta^{(l-1)b}$$

where a, b are integers. Then $\varepsilon^\theta\kappa^{1-\theta}=\eta^{(l+1)\theta a+(l-1)(1-\theta)b}$. The greatest common divisor of $(l+1)\theta$ and $(l-1)(1-\theta)$ is 3, so that s_1 can take one and only one value in $\{1, \eta, \eta^2\}$. As for s_2, by conjugation under an element of $(\bar{U}^\chi)_{w_2\sigma}$, s_2 can take one and only one value in $\{\tau_0,\tau_1,\tau_2\}$, where τ_i is a fixed element in F_{l^2}, which satisfy $\tau_i^l+\tau_i+\eta^i=0$. Hence if $s_1\neq 0$, then u is conjugate to

$$x_2(\eta^i)x_{1-2+3-4}(\eta^{il})x_{1+2+3-4}(\tau_i)x_{1-3}(\eta^{2\theta})x_{3+4}(\eta^{2\theta l})x_{1+4}(\tau_i^{2\theta})$$

for some i ($i=1,2$ or 3).

The case where $t=(\omega^{-1},\omega^{-1},1,\omega)$, $\theta\equiv -1 \pmod 3$, can be dealt with in the same way, so we omit.

The orders of the centralizers of these classes can be easily calculated.

The whole number of conjugacy classes in G_σ is obtained by $\sum_{j=0}^{17} f_j\times m_j$, where m_j is the number of unipotent classes in $Z_{G(w_i\sigma)}(t_j)$, ($t_j\in T(i)$) and f_j is the number of classes of t_j. Thus we have proved the following theorem.

Table V.

rep. element	order of centralizer
$t_1x_1(1)x_{1+2}(1)$	$q^4(q^2-1)$
$t_1x_{1-2}(1)x_2(1)x_1(1)$	$2q^2(q^2-1)$
$t_1x_{1-2}(1)x_2(1)x_{1+2}(1)$	$2q^2(q^2-1)$
$t_2x_{1+2-3+4}(1)x_{1-4}(1)$	$q^2(q^2-1)$
$t_4x_{1+2+3-4}(1)x_{1+4}(1)$	$q^6(q^2+1)$
$t_4x_{\alpha_1}(1)x_{\alpha_2}(1)x_{1+2+3-4}(\tau_0)x_{\beta_1}(1)x_{\beta_2}(1)x_{1+4}(\tau_0^{2\theta})$	$3q^4$
$t_4x_{\alpha_1}(\eta)x_{\alpha_2}(\eta^l)x_{1+2+3-4}(\tau_1)x_{\beta_1}(\eta^{2\theta})x_{\beta_2}(\eta^{2\theta l})x_{1+4}(\tau_1^{2\theta})$	$3q^4$
$t_4x_{\alpha_1}(\eta^2)x_{\alpha_2}(\eta^{2l})x_{1+2+3-4}(\tau_2)x_{\beta_1}(\eta^{4\theta})x_{\beta_2}(\eta^{4\theta l})x_{1+4}(\tau_2^{2\theta})$	$3q^4$
If $\theta\equiv1$ (mod. 3), $\alpha_1=2$, $\alpha_2=1-2+3-4$, $\beta_1=1-3$, $\beta_2=3+4$.	
If $\theta\equiv-1$ (mod. 3), $\alpha_1=3$, $\alpha_2=1+2-3-4$, $\beta_1=2+4$, $\beta_2=1-2$.	
$t_5x_{1+2+3-4}(1)x_{1+4}(1)$	$q^2(q^2+1)$
$t_7x_1(1)x_{1+2}(1)$	$q^4(q^2-\sqrt{2}q+1)$
$t_7x_{1-2}(1)x_2(1)x_1(1)$	$2q^2(q^2-\sqrt{2}q+1)$
$t_7x_{1-2}(1)x_2(1)x_{1+2}(1)$	$2q^2(q^2-\sqrt{2}q+1)$
$t_9x_1(1)x_{1+2}(1)$	$q^4(q^2+\sqrt{2}q+1)$
$t_9x_{1-2}(1)x_2(1)x_1(1)$	$2q^2(q^2+\sqrt{2}q+1)$
$t_9x_{1-2}(1)x_2(1)x_{1+2}(1)$	$2q^2(q^2+\sqrt{2}q+1)$

(*) The number of classes represented by t_jx is the same number of t_j written in Table IV.

THEOREM 3.3. *The remaining class representatives and the orders of their centralizers are given in Table V. And the total number of conjugacy classes in G_σ is q^4+4q^2+17.*

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