On primitive permutation groups of degree $2p=4q+2, p$ *and* q *being prime numbers*

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1. Introduction

Transitive permutation groups of degree p, where p is a prime number such that $q=1/2(p-1)$ is also a prime number, were considered by N. Ito in his series of articles ([5], [6], [7] and others). In this note we consider primitive permutation groups of degree $2p$, where p is a prime number which satisfies the assumption of N. Ito's problem. We determine such permutation groups under the additional assumption that there exist no non-solvable transitive permutation groups of degree p except the symmetric or the alternating group of degree p.

THEOREM. *Let* G *be a primitive permutation group on* Ω, $|\Omega|=2p=4q+2$, *where* p *and* q *are prime numbers. Assume* $p>23$. *If there exist no non-solvable transitive permutation groups of degree* p *except* A_p *and* S_p, *then* $G\simeq A_{2p}$ *or* S_{2p}. *Here* A_n *and* S_n *are the alternating and the symmetric groups of degree* n.

On transitive permutation groups of degree $p=2q+1$, N. Ito has considered the case that $r=1/2(q-1)$ is also a prime in [5]. Recently this case has been settled by P. M. Neumann. In his paper [10] it is stated that if G is an insoluble transitive group of permutations of degree $p=2q+1=4r+3$, where $p>23$ and p, q, r are all primes, then $G\simeq A_p$ or S_p. From this result we obtain the following corollary.

COROLLARY. *Let* G *be a primitive permutation group of degree* $2p=4q+2=8r+6$, *where* p, q, r *are all primes, then* $G\simeq A_{2p}$, S_{2p} *or* M_{22}, $\mathrm{Aut}(M_{22})$, *where* M_{22} *and* $\mathrm{Aut}(M_{22})$ *are the Mathieu group of degree* 22 *and its automorphism group.*

The notation to be used is standard (cf. Wielandt [14]).

The author thanks Dr. E. Bannai suggesting the present formulation of Theorem. Our original one is that of Corollary.

2. Preliminaries.

Throughout this section we assume that G is a permutation group on Ω satisfying the assumption of Theorem.

LEMMA 1. *Let N be a minimal normal subgroup of G. Then N is simple and doubly transitive.*

PROOF. Since G is primitive of degree $2p$, by Wielandt [14], § 31, G is doubly transitive or $2p=s^2+1$ for some integer s. But $2p=4q+2$ yields $4q=(s-1)(s+1)$. Therefore $q=2$, this is not the case. Thus G is doubly transitive. Then N is 3/2-fold transitive since N can not be regular normal from $p\neq 2$. Hence N is a primitive or a Frobenius group by Theorem 10.4 in Wielandt [14], and the minimality of N implies that N is primitive. Therefore N is doubly transitive as well as G. Next we show that N is simple. Let P be a Sylow p-subgroup of G. If $|P|=p^2$, then G contains a cycle of degree p. So we have $G\supseteq A_{2p}$ by a theorem of Jordan (cf. Wielandt [14], 13.9). Hence we may assume that P is cyclic of order p. Then p divides the order of N to the first power, which yields N simple, since N is minimal normal.

Hence from the first we may take G simple and P of order p. Then P is self-centralizing and $N_G(P)/P$ is cyclic of order 2, q or $2q(=p-1)$. If $|N_G(P):P|=2$, then $G\simeq PSL(2, l)$, where $l=2p-1$, by the result of N. Ito [4]. Since l is a prime power such that $2p=l+1$ and $4q=l-1$, we have immediately that $l=9$. Therefore we may also assume that $|N_G(P):P|=q$ or $2q$. Let Γ_1 and Γ_2 be the orbits of P. Let Q be a Sylow q-subgroup of $N_G(P)$. Then Q is cyclic of order q and has four orbits of length q on Ω. Put them Δ_1, Δ_2, Δ_3 and Δ_4. Q leaves just two elements of Ω fixed, say 1 and 2. Then we may take $\Gamma_1=\{1\}\cup\Delta_1\cup\Delta_2$ and $\Gamma_2=\{2\}\cup\Delta_3\cup\Delta_4$. Let H be the subgroup of G which has Γ_1 and Γ_2 as its system of imprimitivity.

LEMMA 2. *Q is a Sylow q-subgroup of G.*

PROOF. Otherwise G contains an element of order q and degree $2q$ or $3q$, which implies $G\supseteq A_{2p}$ by Theorem 13.10 in Wielandt [14], since $q\geqslant 5$. This is a contradiction.

LEMMA 3. *H is solvable and has Γ_1 and Γ_2 as its orbits, and $N_H(Q)=C_H(Q)$.*

PROOF. Assume H not solvable. Then the commutator subgroup H' has Γ_1 and Γ_2 as its orbits. Hence the restriction of H' on Γ_i ($i=1$ or 2) is a non-solvable transitive permutation group of degree p. Therefore by the assumption of Theorem $H'\supseteq A_p$. Since all the subgroups of A_p of index p are conjugate, any element of H' has a same permutation structure on Γ_1 and Γ_2. Then G contains an element of degree 6. Since $p\geqslant 23$, we have $G\supseteq A_{2p}$ by a theorem of Bochert (cf. Wielandt [14], § 15). Thus H is solvable. Since H' has Γ_i as its orbit, $P \lhd H'$. Hence $N_G(P)=H$. If x is an element of H of order t exchanging Γ_1 and Γ_2, then x consists of one 2-cycle and $2p-2/t$ t-cycles since no elements of H fix more than two points on Ω. But x is also an automorphism of P, so t divides $p-1$. This implies that x is an odd permutation. The last assertion follows from $H=N_G(P)$ immediately.

LEMMA 4. *G_1 does not contain a normal subgroup whose order is prime to q. Especially $N_{12} \neq C_{12}$, where $N=N_G(Q)$ and $C=C_G(Q)$.*

PROOF. Suppose false and let L be a normal subgroup of G_1 whose order is prime to q. Let $K=LQ$. Then K is a primitive permutation group on $\Omega-\{1\}$, since $1+q$ does not divide $1+4q$, if $q \geqslant 3$. First we assume that K is not doubly transitive on $\Omega-1$. If K_2 has an orbit of length q, then L_2 acts trivially on that orbit, since $|L_2|$ is prime to q. This implies $L_2=1$ by Theorem 18.2 in Wielandt [14], that is, L is regular normal and K is a primitive Frobenius group. Hence $|L|=2p-1=4q+1$ and $|L|$ is a prime power. It follows that $|L|=9$, since p and q are prime. This is not the case. If K_2 has an orbit of length $2q$, then again by Theorem 18.2 in Wielandt [14] all the orbits of L_2 on $\Omega-\{1, 2\}$ are of length 2. Then L_2 is an elementary abelian 2-group which fixes just one point 2 of $\Omega-\{1\}$. Therefore $N_L(L_2) \subseteq L_2 = C_L(L_2)$. Since L_2 is a Sylow 2-subgroup of L, L has a normal 2-complement, which is regular normal in G_1. Therefore this case is reduced to the preceding case. Next we assume that K is doubly transitive on $\Omega-\{1\}$. Then all the orbits of L_2 on $\Omega-\{1, 2\}$ are of length 4. Hence L_2 is solvable and K_2 is solvable too. If K_2 has an abelian normal subgroup which is not semiregular on $\Omega-\{1, 2\}$, then by the theorem of O'Nan [11] we have that $PSL(n, l) \subseteq K \subseteq P\Gamma L(n, l)$ for some integer $n \geqslant 3$ and some prime power l in their usual doubly-transitive representations. But a stabilizer in $PSL(n, l)$ is not solvable, if $n \geqslant 3$ and the degree of its doubly-transitive representation is more than 13, while K_2 is solvable and $2p>13$ by the assumption of Theorem. Therefore any normal subgroup of K_2 which is abelian is semi-regular on $\Omega-\{1, 2\}$. If L_2 is divisible by 3, then a Sylow 3-subgroup of L_2 is also a Sylow 3-subgroup of K_2. Let T be a Sylow 3-subgroup of K_2 and let $F(T)$ be the fixed points of T on $\Omega-\{1\}$. Then T has just one fixed point in each orbit of L_2 of length 4. Therefore $F(T)$ consists of $q+1$ points. By Witt's lemma (cf. Wielandt [14], 9.4) $N_K(T)$ is doubly transitive on $F(T)$. Since K has a normal q-complement, so does $N_K(T)$. This implies that $N_K(T)$ is sharply doubly transitive on $F(T)$. Hence $q+1$ is a power of 2 and a Sylow 2-subgroup of $N_K(T)$ is regular on $F(T)$. Since the degree of K, $|\Omega|-1$, is odd, a Sylow 2-subgroup of $N_K(T)$ is contained in K_i and also in L_i for some point i of $\Omega-\{1\}$. Hence $q+1\leqslant 4$, which is a contradiction. Therefore $|L_2|$ is prime to 3 and L_2 is a Sylow 2-subgroup of K. If L_2 is abelian, $N_L(L_2)=C_L(L_2)$, which implies that L has a normal 2-complement. This does not occur. So $L_2'\lhd K_2$ and it is of order 4. Then $C_{L_2}(L_2')=L_2'$ by Wielandt [14] 4.4, since the non-trivial orbits of L_2 are of length 4. Therefore L_2 is a dihedral group of order 8. But any element of order 4 in L_2 consists of q 4-cycles, which is an odd permutation. Thus G_1 can not contain a normal subgroup whose order is prime to q. Since Q fixes just two points on Ω, $N_1=N_{12}$, which by Burnside's transfer argument gives the last assertion.

3. Proof of Theorem.

Let G be as in Theorem. In this section we assume that Q is a Sylow q-subgroup of G and derive a contradiction by showing that H is not solvable or that G_1 has a normal subgroup whose order is prime to q. Now we consider N as a permutation group on $\{1, 2; \Delta_1, \Delta_2, \Delta_3, \Delta_4\}$. Then the kernel of this permutation representation is Q, since otherwise there exists an element of $N-C$ which is contained in H. By Lemma 3 this does not occur. Since G is simple, we easily see that N consists of even permutations on $\{1, 2; \Delta_1, \Delta_2, \Delta_3, \Delta_4\}$. Hence $C_{12}/Q\precsim A_4$ and the following 6 cases are possible for the image of the permutation representation of C_{12} on $\{\Delta_1, \Delta_2, \Delta_3, \Delta_4\}$:

(I) A_4,

(II) $Z_2\times Z_2$,

(III) Z_3,

(IV) Z_2 and the orbits of C_{12} are $\{\Delta_1, \Delta_2\}$ and $\{\Delta_3, \Delta_4\}$,

(V) Z_2 and the orbits of C_{12} are $\{\Delta_1, \Delta_3\}$ and $\{\Delta_2, \Delta_4\}$,

(VI) 1.

Case (I). There is an element of N which exchanges 1 and 2 by Witt's lemma. Let us denote this element x. Then in this case we may assume that

$$x=(1, 2)(\Delta_1, \Delta_2),$$

since N consists of even permutations. There is an element y of C_{12} with the following cycle structure

$$y=(\Delta_1, \Delta_3)(\Delta_2, \Delta_4).$$

Then

$$xy=(1, 2)(\Delta_1, \Delta_4, \Delta_2, \Delta_3) .$$

Hence xy is contained in H and exchanges Γ_1 and Γ_2. This contradicts Lemma 3.

Case (II). From Lemma 4 $N_{12}\neq C_{12}$. Hence $N_{12}/Q\simeq A_4$. By Witt's lemma $|N:N_{12}|=2$. If N/Q is not faithful on $\{\Delta_1, \Delta_2, \Delta_3, \Delta_4\}$, then N contains an odd permutation. So N/Q is faithful on $\{\Delta_1, \Delta_2, \Delta_3, \Delta_4\}$ and $N/Q\simeq S_4$. This yields $N'/Q\simeq A_4$, but $N'\subseteq C_{12}$, which is a contradiction.

Case (III). Since $C_{12}\lhd N_{12}$, noticing that a Sylow 3-subgroup of A_4 is self-normalizing, it is obtained that $N_{12}=C_{12}$. Thus this case does not hold by Lemma 4.

Case (IV). Let $y\in C_{12}-Q$, then y has the following cycle structure.

$$y=(\Delta_1, \Delta_2)(\Delta_3, \Delta_4) .$$

Let $z\in N_{12}-C_{12}$. Since z normalizes C_{12}, we may assume that

$$z=(\Delta_1, \Delta_3)(\Delta_2, \Delta_4) .$$

Let $x\in N-N_{12}$. Since x normalizes C_{12}, we have the following two possibilities:

$$x=(1, 2)(\Delta_1, \Delta_2)$$

or

$$x=(1, 2)(\Delta_1, \Delta_3, \Delta_2, \Delta_4) .$$

But the latter contradicts Lemma 3. If the former holds, then

$$xyz=(1, 2)(\Delta_1, \Delta_3, \Delta_2, \Delta_4) .$$

Thus this case does not hold by Lemma 3 anyhow.

Case (V). Let $z\in N_{12}-C_{12}$. Since z normalizes C_{12}, we have

$$z=(\Delta_1, \Delta_2)(\Delta_3, \Delta_4)$$

or

$$z=(\Delta_1, \Delta_4)(\Delta_2, \Delta_3) .$$

Let $y\in C_{12}-Q$. Then y has the following cycle structure

$$y=(\Delta_1, \Delta_3)(\Delta_2, \Delta_4) .$$

Hence z or yz is contained in $N_H(Q)-C_H(Q)$. This contradicts Lemma 3.

Case (VI). Since N/C is cyclic, $N_{12}/Q\simeq Z_2$, Z_3 or Z_4. But the last case does not hold, since N does not contain an odd permutation. If $N_{12}/Q\simeq Z_3$, then we may assume that an element z of $N_{12}-Q$ has the following cycle structure.

$$z=(\Delta_1, \Delta_2, \Delta_3) .$$

Let x be an element of N which exchanges 1 and 2. Since x normalizes N_{12}, we may assume

$$x=(1, 2)(\Delta_1, \Delta_2) .$$

Then

$$xzx^{-1}z^{-1}=(\Delta_1, \Delta_2, \Delta_3) .$$

This can not occur, since the commutator $[x, z]$ belongs to C_{12}. Hence we have $N_{12}/Q\simeq Z_2$, also $N_1/Q\simeq Z_2$. Then by Lemma 4 and Frattini argument it is obtained that G_1 is simple and Suzuki's method ([13]) of induced characters can be used to show that G_1 has only one class of involutions. Therefore any involution of G has at most two fixed points. Then we have a contradiction by Hering [3] or by a similar argument in Nagao [9] (x), since $C_{12}=Q$. Thus we have completed the proof of Theorem.

4. Proof of Corollary.

By the result of P. M. Neumann [10] we may take $p=11$ or 23. From the proof of Theorem we may also assume that H is not solvable. Then we have $H'\simeq PSL(2, 11)$, M_{11} ($p=11$) or M_{23} ($p=23$). Referring to Hall [2] or Sims [12] it can be obtained that $G\simeq M_{22}$, if $H'\simeq PSL(2, 11)$ and that the case $H'\simeq M_{11}$ does not occur. Therefore we assume that $H'\simeq M_{23}$. Then G is triply transitive by Manning [8]. If G is not 4-ply transitive, then G_{123} must fix one more point other than 1, 2 and 3. Hence G_{12} is imprimitive and G_{12} has 22 blocks of length 2. If G_{12} acts on these blocks unfaithfully, then the kernel is elementary abelian 2-group. Hence as in Lemma 4 we have a contradiction by applying the result of O'Nan [11] or Glauberman [1]. Consequently G_{12} acts faithfully on these blocks. This implies that G_{12} acts on them as $\mathrm{Aut}(M_{22})$, since H_{12} acts on them as M_{22}. Then $(G_{12})'=H_{12}$ and by Witt's lemma $|N_G(H_{12}):H_{12}|=4$. Therefore $|C_G(H_{12})|=2$, since $|\mathrm{Aut}(M_{22}):M_{22}|=2$. But the non-identity element of $C_G(H_{12})$ consists of 23 transpositions, which can not hold, since G does not contain an odd permutation. Hence G is 4-ply transitive. Then G contains a cycle of degree 43 and a theorem of Jordan (Theorem 13.9 in Wielandt [14]) gives a contradiction. Thus the assertion of Corollary holds.

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