On the number of moduli of certain algebraic surfaces of general type

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Introduction. We let S denote an irreducible hypersurface of degree $n=2r$ in the projective 3-space $\boldsymbol{P}^3$ over the field of complex numbers, defined by the equation

$$(1) \qquad g^2+Agh+Bh^2=0$$

where g, h, A, and B are homogeneous polynomials of degree $r, s, r-s$, and $2(r-s)$, respectively, with two positive integers r and s, $r>s$. Let Δ denote the curve on S defined by the equations $g=h=0$. Then Δ is contained in the singular locus of S.

We say that S is *generic* if the following conditions are satisfied:

1) S has only ordinary singularities, and is non-singular outside of Δ.

2) Δ is non-singular.

3) The normalization X of S is a non-singular algebraic surface of general type.

We note that S satisfies the conditions 1) and 2), provided that g, h, A, and B are general homogeneous polynomials. 3) is equivalent to the inequality $n>s+4$ (cf. Lemma 3).

In [4], Kodaira studied families of surfaces with ordinary singularities in $\boldsymbol{P}^3$. In particular, he proved that a generic hypersurface S defined by (1) belongs to an effectively parametrized maximal family $\mathcal{F}$ of surfaces $S_t, t\in M_1$, with ordinary singularities in $\boldsymbol{P}^3$ whose characteristic system on each S_t is complete (see [4], Theorem 8 and § 5.4). The number of effective parameters of the family $\mathcal{F}$, which we denote by $\mu(S)$, is given by

$$\mu(S)=C(r)+C(s)+C(2r-2s)-C(r-2s)-2,$$

$$\text{where} \quad C(m)=\begin{cases}(m+3)(m+2)(m+1)/6 & \text{for } m\geq 0\\ 0 & \text{for } m<0.\end{cases}$$

On the other hand, Kodaira-Spencer [5] defined the concept of *the number of moduli* $m(X)$ of a compact complex manifold X. $m(X)$ is defined only if there exists an effectively parametrized complete family $\{X_t\}_{t\in M}$ of deformations of X. In this case, we define $m(X)=\dim M$. The purpose of this paper is to prove the following

MAIN THEOREM. *Let S be a generic hypersurface in P^3 defined by the equation* (1), *X the normalization of S. Then the number of moduli $m(X)$ of X is defined, and equals*

$$\dim H^1(X, \Theta_X) = \mu(S) - 15 - 4\delta_{r,s+1}$$

where Θ_X denotes the sheaf of germs of holomorphic vector fields on X, and $\delta_{r,s+1}$ is Kronecker's delta.

The difference between $m(X)$ and $\mu(S)$ is the contribution of the number of parameters on which the natural holomorphic map $f: X \to P^3$ depends.

For $(r, s) = (3, 1)$, S is one of examples of M. Noether [6]. In this case, X is a minimal algebraic surface with geometric genus $p_g = 4$, irregularity $q = 0$, and the Chern number $c_1^2 = 6$. Hence, we have

$$m(X) = 38 = 10(p_a + 1) - 2c_1^2$$

with $p_a = p_g - q$. By the Riemann-Roch theorem, it follows that $H^2(X, \Theta_X) = 0$. In general $\dim H^2(X, \Theta_X)$ is very large.

1. Preliminaries.

Let W denote the projective 3-space P^3, and let $p: V \to W$ be the monoidal transformation with center at Δ. Then X can be identified with the proper transform of S. Let $f: X \to W$ be the restriction of p to X. The same letter f will denote the induced holomorphic map $X \to S$. We set $\tilde{\Delta} = f^{-1}(\Delta)$.

Let E be a hyperplane of W and let $\tilde{E} = f^*E$. We employ the same symbols E and $\tilde{E}$ in order to denote the restrictions of E and $\tilde{E}$ to S, Δ, and $\tilde{\Delta}$, respectively.

We cover W by a finite number of coordinate neighborhoods W_i, $i \in I$. We set

$$J = \{i \in I: \Delta \cap W_i \text{ is not empty}\},$$
$$J' = \{i \in I: \Delta \cap W_i \text{ is empty}\}.$$

We may assume that the associate line bundle $[E]$ is trivial on each W_i. Let $\{e_{ij}\}$ denote a system of transition functions for $[E]$. Then g, h, A, and B are represented, respectively, by collections $\{g_i\}$, $\{h_i\}$, $\{A_i\}$, and $\{B_i\}$ of holomorphic functions satisfying $g_i = e_{ij}^r g_j$ on $W_i \cap W_j$, etc.

For each $i \in J$, $p^{-1}(W_i)$ is covered by two open subsets

$$U_i=\{(z, u_i)\in W_i\times \boldsymbol{C}\colon\ h_i(z)u_i=g_i(z)\},$$
$$V_i=\{(z, v_i)\in W_i\times \boldsymbol{C}\colon\ g_i(z)v_i=h_i(z)\}.$$

Moreover, X is defined by the equations

$$(2)\qquad \begin{aligned} &u_i^2+A_i(z)u_i+B_i(z)=0 &&\text{on } U_i,\\ &1+A_i(z)v_i+B_i(z)v_i^2=0 &&\text{on } V_i. \end{aligned}$$

We note that $(z, u_i)\in U_i$ coincides with $(z, v_i)\in V_i$ if and only if $u_i=1/v_i$. It follows that $X\cap p^{-1}(W_i)$ is contained in U_i.

For each $i\in J'$, i.e., $W_i\cap \Delta=\varnothing$, we set $U_i=p^{-1}(W_i)$. Then X is contained in the union of U_i, $i\in I$.

LEMMA 1. *$\tilde{\Delta}$ is linearly equivalent to $s\tilde{E}$ on X.*

PROOF. On each $X\cap U_i$, $i\in J$, $\tilde{\Delta}$ is defined by the equation $h_i=0$. On the other hand, for each $i\in J'$, neither g_i nor h_i vanishes on $S\cap W_i$. Hence we can take $h_i=0$ as a local equation of $\tilde{\Delta}$ on $X\cap U_i$ for any $i\in I$. Q.E.D.

LEMMA 2. *We have* $\dim H^0(X, \mathcal{O}_X(\tilde{E}))=4+\delta_{r,s+1}$ *and* $H^1(X, \mathcal{O}_X(m\tilde{E}))=0$ *for any integer* m.

PROOF. We recall that

$$H^q(X, \mathcal{O}_X(m\tilde{E}-\tilde{\Delta}))\cong H^q(S, \mathcal{O}_S(mE-\Delta))\qquad \text{for}\quad q=0,1,2$$

where $\mathcal{O}_S(mE-\Delta)$ denotes the sheaf of germs of holomorphic sections of $m[E]$ on S which vanish on Δ (see [4]), and that the canonical bundle K of X is given by $[(n-4)\tilde{E}-\tilde{\Delta}]$.

First, we shall prove the second assertion. By the Serre duality, we have

$$\begin{aligned}\dim H^1(X, \mathcal{O}_X(m\tilde{E}))&=\dim H^1(X, \mathcal{O}_X((n-4-m)\tilde{E}-\tilde{\Delta}))\\ &=\dim H^1(S, \mathcal{O}_S((n-4-m)E-\Delta)).\end{aligned}$$

Hence it suffices to prove $H^1(S, \mathcal{O}_S(mE-\Delta))=0$ for any integer m.

In view of two exact sequences

$$0\to\mathcal{O}_W((m-n)E)\to\mathcal{O}_W(mE-\Delta)\to\mathcal{O}_S(mE-\Delta)\to 0,$$
$$0\to\mathcal{O}_W(mE-\Delta)\to\mathcal{O}_W(mE)\to\mathcal{O}_\Delta(mE)\to 0,$$

we only have to show that the restriction map

$$(3)\qquad H^0(W, \mathcal{O}_W(mE))\to H^0(\Delta, \mathcal{O}_\Delta(mE))$$

is surjective.

Let S_1 be the hypersurface in W defined by the equation $g=0$. Then we have

$H^1(S_1, \mathcal{O}_{S_1}(mE))=0$ for any integer m. From the following two exact sequences

$$0\to\mathcal{O}_{S_1}((m-s)E)\to\mathcal{O}_{S_1}(mE)\to\mathcal{O}_{\Delta}(mE)\to 0,$$
$$0\to\mathcal{O}_W((m-r)E)\to\mathcal{O}_W(mE)\to\mathcal{O}_{S_1}(mE)\to 0,$$

we infer that the restriction map (3) is surjective. This proves the second assertion.

In order to prove the first assertion, we consider the following commutative diagram

$$\begin{array}{ccccccc} 0\to H^0(S,\mathcal{O}_S(E-\Delta)) & \to & H^0(S,\mathcal{O}_S(E)) & \to & H^0(\Delta,\mathcal{O}_\Delta(E)) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0\to H^0(X,\mathcal{O}_X(\tilde{E}-\tilde{\Delta})) & \to & H^0(X,\mathcal{O}_X(\tilde{E})) & \to & H^0(\tilde{\Delta},\mathcal{O}_{\tilde{\Delta}}(\tilde{E})) & & \end{array}$$

with two exact rows. Since we have $H^1(S,\mathcal{O}_S(E))=0$, we get $0=H^1(S,\mathcal{O}_S(E-\Delta)) =H^1(X,\mathcal{O}_X(\tilde{E}-\tilde{\Delta}))$. It follows that the last horizontal map $H^0(X,\mathcal{O}_X(E))\to H^0(\tilde{\Delta}, \mathcal{O}_{\tilde{\Delta}}(E))$ is surjective. Since the first vertical map is bijective and since we have $\dim H^0(S,\mathcal{O}_S(E))=4$, it suffices to show

$$\dim H^0((\tilde{\Delta},\mathcal{O}_{\tilde{\Delta}}(\tilde{E}))=\dim H^0(\Delta,\mathcal{O}_\Delta(E))+\delta_{r,s+1}. \tag{4}$$

For this purpose, let F be the line bundle on Δ induced by $(r-s)[E]$, $F^\sharp$ its completion, $q: F^\sharp\to\Delta$ the natural projection, and let Δ_0 and Δ_∞ denote, respectively, the 0-section and the ∞-section of q. In view of (2), $\tilde{\Delta}$ is a divisor on $F^\sharp$ which is linearly equivalent to $2\Delta_0$. Moreover, $\tilde{\Delta}$ does not meet Δ_∞. Hence we have the following exact sequence:

$$0\to\mathcal{O}_{F^\sharp}(q^*E-2\Delta_0+\Delta_\infty)\to\mathcal{O}_{F^\sharp}(q^*E+\Delta_\infty)\to\mathcal{O}_{\tilde{\Delta}}(\tilde{E})\to 0.$$

On the other hand, we have $q^*F=[\Delta_0]-[\Delta_\infty]$. Hence we get

$$H^\nu(F^\sharp,\mathcal{O}_{F^\sharp}(q^*E-2\Delta_0+\Delta_\infty))=H^\nu(F^\sharp,\mathcal{O}_{F^\sharp}(q^*E-2q^*F-\Delta_\infty))=0$$

for $\nu=0,1$, and

$$H^0(F^\sharp,\mathcal{O}_{F^\sharp}(q^*E+\Delta_\infty))\cong H^0(\Delta,\mathcal{O}_\Delta(E))\oplus H^0(\Delta,\mathcal{O}_\Delta(E-F))$$

(see [8], Propositions 2.1 and 2.2). This proves the equality (4).

REMARK. In the case $r=s+1$, the complete linear system $|E|$ embeds X in $\boldsymbol{P}^4$ as a complete intersection of a quadric hypersurface and a hypersurface of degree r.

LEMMA 3. (i) *We have* $n>s+4$.

(ii) *The canonical bundle* K *of* X *is ample. In particular* X *is minimal.*

(iii) X *has the following numerical characters*:

$$p_g=\binom{2r-1}{3}-\frac{1}{2}rs(3r-s-4),$$
$$q=0,$$
$$c_1^2=2r(2r-s-4)^2.$$

PROOF. By Lemma 1, we have $K=(n-s-4)[\tilde{E}]$. Since we have assumed that X is of general type, it follows that $n>s+4$. Then, since $f:X\to S$ is finite, K is ample. The assertion (iii) follows from Lemma 2 and a classical formula for p_a (see [4], [9]).

2. Relation between deformations of S and X.

First we recall some results about characteristic systems of families of surfaces with ordinary singularities (see [4]). Let S be a surface of degree n in $W=\boldsymbol{P}^3$ with ordinary singularities. We cover W by a finite number of coordinate neighborhoods W_i, and let (x_i, y_i, z_i) be a system of local coordinates on each W_i. We may assume that S is defined, on each W_i, by an equation $\phi_i=0$, where ϕ_i is one of the following form:

$(5)_0$ $\quad\phi_i=1,$

$(5)_s$ $\quad\phi_i=z_i,$

$(5)_d$ $\quad\phi_i=y_iz_i,$

$(5)_t$ $\quad\phi_i=x_iy_iz_i,$

$(5)_c$ $\quad\phi_i=x_iy_i^2-4z$.

In the last case, we call the point $\mathfrak{c}_i$: $x_i=y_i=z_i=0$ a cuspidal point of S. Let Δ denote the double curve of S, and let $\mathcal{O}(nE-\Delta)$ denote the subsheaf of $\mathcal{O}(nE)$ consisting of germs of those holomorphic sections which vanish on Δ. Moreover let $\mathcal{O}(nE-\Delta-\sum\mathfrak{c}_i')$ denote the subsheaf of $\mathcal{O}(nE-\Delta)$ consisting of germs of those holomorphic sections ψ of $[nE]$ which vanishes on Δ and satisfy

$$\left(\frac{\partial\phi_i}{\partial y_i}\right)(\mathfrak{c}_i)=0,$$

at each cuspidal point. We let $\mathcal{O}_S(nE-\Delta-\sum\mathfrak{c}_1')$ denote the restriction of $\mathcal{O}(nE-\Delta-\sum\mathfrak{c}_1')$ to S.

Let $\{S_t\}_{t\in M}$ be a family of surfaces of degree n in W with ordinary singularities such that $S=S_0$ for $0\in M$. We let $T_0(M)$ denote the tangent space of M at 0. Then we have a *characteristic map*

$$\sigma:\ T_0(M)\to H^0(S,\mathcal{O}_S(nE-\varDelta-\sum c_i')).$$

If S_t is defined respectively by the equations $\Phi(t)=0$, then, for any $\partial/\partial t\in T_0(M)$, $\sigma(\partial/\partial t)$ is given by the restriction of $\partial\Phi(t)/\partial t|_{t=0}$ to S (see [4]).

On the other hand, the normalizations X_t of S_t describe a family $\mathcal{X}=\{X_t\}_{t\in M}$ of deformations of $X=X_0$ and f: $X\to W$ extends to a holomorphic map Ψ: $\mathcal{X}\to W\times M$ over M. Let Θ_X and Θ_W denote, respectively, the sheaves of germs of holomorphic vector fields on X and W, and let $\mathcal{T}_{X/W}$ denote the cokernel of the canonical homomorphism F: $\Theta_X\to f^*\Theta_W$. Then, we have a *characteristic map*

$$\tau:\ T_0(M)\to H^0(X,\mathcal{T}_{X/W})$$

(see [3], §1).

LEMMA 4. *There is a canonical isomorphism*

$$\mathfrak{f}:\ \mathcal{O}_S(nE-\varDelta-\sum c_i')\to f_*\mathcal{T}_{X/W}$$

which induces an isomorphism

$$\mathfrak{f}:\ H^0(S,\mathcal{O}_S(nE-\varDelta-\sum c_i'))\to H^0(X,\mathcal{T}_{X/W})$$

such that $-\tau=\mathfrak{f}\circ\sigma$.

PROOF. Let Θ_S denote the dual of the sheaf of germs of 1-differentials Ω_S^1 on S (see [2]). Then we have an exact sequence

$$0\to\Theta_S\to\Theta_{W|S}\xrightarrow{Q}\mathcal{O}_S(nE-\varDelta-\sum c_i')\to 0,$$

where Q sends $\eta\in\Gamma(U,\Theta_{W|S})$ to $\{\eta\cdot\phi_i\}$ for any open set U. Since f: $X\to S$ is finite, we get an exact sequence

$$0\to f_*\Theta_X\xrightarrow{f_*F}f_*f^*\Theta_W\xrightarrow{f_*P}f_*\mathcal{T}_{X/W}\to 0.$$

Moreover, there exists a canonical homomorphism

$$f^*:\ \Theta_{W|S}\to f_*f^*\Theta_W.$$

We shall prove that f^* induces a desired isomorphism.

We start with a lemma.

LEMMA 5. *Let U be an open set in X, and let $\tau=\tau_1\partial/\partial x_i+\tau_2\partial/\partial y_i+\tau_3\partial/\partial z_i$ be an element of $\Gamma(U\cap f^{-1}(W_i),f^*\Theta_W)$. We set*

$$\tau\cdot\phi_i=\tau_1\frac{\partial\phi_i}{\partial x_i}(f)+\tau_2\frac{\partial\phi_i}{\partial y_i}(f)+\tau_3\frac{\partial\phi_i}{\partial z_i}(f).$$

Then, we have $P\tau=0$ if and only if $\tau\cdot\phi_i=0$, where P denotes the natural projec-

tion $f^*\Theta_W \to \mathcal{T}_{X/W}$.

PROOF. We shall check the equivalence in each case in which ϕ_i is of the form $(5)_s, (5)_d, (5)_t$ or $(5)_c$.

If ϕ_i is of the form $(5)_s$, then the equivalence is clear. If ϕ_i is of the form $(5)_d$, then $f^{-1}(W_i)$ is a disjoint union of two open subsets

$$U_i = \{(x_i, y_i) \in \boldsymbol{C}^2 : (x_i, y_i, 0) \in W_i\},$$
$$V_i = \{(x_i, z_i) \in \boldsymbol{C}^2 : (x_i, 0, z_i) \in W_i\}.$$

On $U \cap U_i$, the following three conditions $P\tau = 0$, $\tau_3 = 0$, and $\tau \cdot \phi = 0$ are equivalent to each other. Similarly, $P\tau = 0$ if and only if $\tau \cdot \phi = 0$ on $U \cap V_i$.

The case $(5)_t$ is quite similar to the case $(5)_d$. If ϕ_i is of the form $(5)_c$, then

$$f^{-1}(W_i) = \left\{(u, v) \in \boldsymbol{C}^2 : \left(u^2, v, \frac{uv}{2}\right) \in W_i\right\},$$

and f is given by $x = u^2$, $y = v$, and $z = uv/2$. Let $F: \Theta_X \to f^*\Theta_W$ be the canonical homomorphism. We have

$$F\left(\frac{\partial}{\partial u}\right) = 2u\frac{\partial}{\partial x} + \frac{v}{2}\frac{\partial}{\partial z}, \qquad (6)$$
$$F\left(\frac{\partial}{\partial v}\right) = \frac{\partial}{\partial y} + \frac{u}{2}\frac{\partial}{\partial z}.$$

While we have

$$\tau \cdot \phi = \tau_1 v^2 + 2\tau_2 u^2 v - 4\tau_3 uv.$$

It is easy to check that $F(\partial/\partial u)\cdot\phi = F(\partial/\partial v)\cdot\phi = 0$. Conversely, if $\tau\cdot\phi = 0$ then τ_1/u is holomorphic and $\tau_1 v/u + 2\tau_2 u - 4\tau_3 = 0$. From (6), we get $\tau = (\tau_1/2u)F(\partial/\partial u) + \tau_2 F(\partial/\partial v)$. This proves the assertion.

PROOF OF LEMMA 4. Let $\sigma = \sigma_1 \partial/\partial x_i + \sigma_2 \partial/\partial y_i + \sigma_3 \partial/\partial z_i$ be an element of $\Gamma(U, \Theta_{W|S})$ for some open subset U of S. Then,

$$\sigma \cdot \phi_i = \sigma_1 \frac{\partial \phi_i}{\partial x_i} + \sigma_2 \frac{\partial \phi_i}{\partial y_i} + \sigma_3 \frac{\partial \phi_i}{\partial z_i}$$

is a holomorphic function on $U \cap W_i$ which vanishes on Δ. Moreover, we may assume that $\phi_i = e_{ij}^n \phi_j$ on $W_i \cap W_j$. It follows that $\sigma \cdot \phi_i = e_{ij}^n \sigma \cdot \phi_j$ on $U \cap W_i \cap W_j$. The image $Q\sigma$ is represented by the collection $\{\sigma \cdot \phi_i\}$.

Clearly, we have $(f^*\sigma)\cdot\phi_i = f^*(\sigma\cdot\phi_i)$. Hence $f^*: \Theta_W \to f_* f^* \Theta_W$ induces a homomorphism $\Theta_S \to f_*\Theta_X$. Consequently f^* induces a homomorphism

$$f: \mathcal{O}_S(nE - \Delta - \sum c_i') \to f_*\mathcal{T}_{X/W}.$$

From Lemma 5 we infer readily that f is injective. We now prove that f is surjective. Let $P\tau$ be an element of $\Gamma(f^{-1}(W_i), \mathcal{T}_{X/W})$ with $\tau=\tau_1(\partial/\partial x_i)+\tau_2(\partial/\partial y_i)+\tau_3(\partial/\partial z_i)$ in $\Gamma(f^{-1}(W_i), f^*\Theta_W)$. We shall check that $P\tau$ is in the image of f in each case $(5)_a$–$(5)_c$.

If ϕ_i is of the form $(5)_a$, there is nothing to prove. If ϕ_i is of the form $(5)_d$, we define holomorphic functions τ_2' and τ_3' on W_i by $\tau_2'=\tau_{2|V_i}$, and $\tau_3'=\tau_{3|U_i}$. Then, we have $P\tau=P(\tau_3'(\partial/\partial z_i))$ on U_i and $P\tau=P(\tau_2'(\partial/\partial z_i))$ on V_i. Setting $\sigma=\tau_2'(\partial/\partial y_i)+\tau_3'(\partial/\partial z_i)\in\Gamma(W_i, \Theta_{W|S})$, we have $P\tau=P(f^*\sigma)$.

If ϕ_i is of the form $(5)_t$, the proof is quite analogous to the above.

If ϕ_i is of the form $(5)_c$, we write $\tau_1=\sigma_1+u\sigma'$ and $\tau_3=\sigma_3+u\sigma_3'$ with $\sigma_1, \sigma_1', \sigma_3, \sigma_3'\in\Gamma(W_i, \mathcal{O}_S)$. In view of (6), we may assume that $\sigma_1'=\tau_2=0$. It follows that $P\tau=P(f^*\sigma)$ with

$$\sigma=\sigma_1\frac{\partial}{\partial x_i}+2\sigma_3'\frac{\partial}{\partial y_i}+\sigma_3\frac{\partial}{\partial z_i}.$$

In fact, we have $(\tau-f^*\sigma)\cdot\phi_i=-4\sigma_3'u^2v+4(\tau_3-\sigma_3)uv=0$. Hence the assertion follows from Lemma 5.

Next we shall prove the second assertion of Lemma 4. Let $\mathcal{F}=\{S_t\}_{t\in M}$ be a family of surfaces in W with ordinary singularities, which contains $S=S_0$, $0\in M$. Then we may assume that $\mathcal{F}\subset W\times M$ is defined, on each $W_i\times M$, by the equation

$$\Phi_i(x_i, y_i, z_i, t)=0$$

such that $\Phi_i(x_i, y_i, z_i, 0)=\phi_i(x_i, y_i, z_i)$. Moreover, we can find holomorphic functions X_i, Y_i, and Z_i of x_i, y_i, z_i, t such that

$$X_{i|t=0}=x_i,\qquad Y_{i|t=0}=y_i,\qquad Z_{i|t=0}=z_i,$$
$$\Phi_i(x_i, y_i, z_i, t)=\phi_i(X_i, Y_i, Z_i). \tag{7}$$

For any tangent vector $\partial/\partial t\in T_0(M)$, we have

$$\frac{\partial\Phi_i}{\partial t}=\frac{\partial\phi_i}{\partial x_i}\frac{\partial X_i}{\partial t}+\frac{\partial\phi_i}{\partial y_i}\frac{\partial Y_i}{\partial t}+\frac{\partial\phi_i}{\partial z_i}\frac{\partial Z_i}{\partial t},$$

by (7). This implies the equality

$$\sigma\left(\frac{\partial}{\partial t}\right)=Q\left(\frac{\partial X_i}{\partial t}\frac{\partial}{\partial x_i}+\frac{\partial Y_i}{\partial t}\frac{\partial}{\partial y_i}+\frac{\partial Z_i}{\partial t}\frac{\partial}{\partial z_i}\right). \tag{8}$$

On the other hand, let $\mathcal{X}=\{X_t\}_{t\in M}$ be the family of normalizations X_t of S_t. We may assume that $\mathcal{X}$ is covered by a finite number of coordinate neighborhoods

$\mathcal{U}_i$ such that $X_0 \cap \mathcal{U}_i = U_i = f^{-1}(W_i)$ and each $\mathcal{U}_i$ is biholomorphically equivalent to $U_i \times M = \{(\zeta_i, t): \zeta_i \in U_i, t \in M\}$. Moreover $f: X \to W$ extends to a holomorphic map $\Psi: \mathcal{X} \to W \times M$ over M. Ψ is defined, on each $\mathcal{U}_i$, by the equations

$$x_i = \Psi_i^1(\zeta_i, t), \qquad y_i = \Psi_i^2(\zeta_i, t), \qquad z_i = \Psi_i^3(\zeta_i, t)$$

where Ψ_i^λ are holomorphic functions on $\mathcal{U}_i$. By definition $\tau(\partial/\partial t)$ is given by

$$\tau\Big(\frac{\partial}{\partial t}\Big) = P\Big(\frac{\partial \Psi_i^1}{\partial t}\frac{\partial}{\partial x_i} + \frac{\partial \Psi^2}{\partial t}\frac{\partial}{\partial y_i} + \frac{\partial \Psi_i^3}{\partial t}\frac{\partial}{\partial z_i}\Big). \tag{9}$$

We note that Ψ_i^λ satisfy the equations

$$\phi_i(X_i(\Psi_i, t),\ Y_i(\Psi_i, t),\ Z_i(\Psi_i, t)) = 0. \tag{10}$$

From (10), we obtain

$$\frac{\partial \phi_i}{\partial x_i}(f)\Big(\frac{\partial \Psi_i^1}{\partial t} + \frac{\partial X_i}{\partial t}\Big) + \frac{\partial \phi_i}{\partial y_i}(f)\Big(\frac{\partial \Psi_i}{\partial t} + \frac{\partial Y_i}{\partial t}\Big) + \frac{\partial \phi_i}{\partial z_i}(f)\Big(\frac{\partial \Psi_i^3}{\partial t} + \frac{\partial Z_i}{\partial t}\Big) = 0.$$

In view of the equalities (8), (9), and Lemma 5, it follows that $-\tau(\partial/\partial t) = f\sigma(\partial/\partial t)$. This proves the second assertion.

REMARK. The first half of Lemma 4 has been obtained by J. Wahl [7].

3. Vanishing of obstructions.

Let S be a surface in W defined by the equation (1), and let X be its normalization. Our purpose of this section is to prove the following lemma.

LEMMA 6. *The coboundary map*

$$\delta:\ H^0(X, \mathcal{T}_{X/W}) \to H^1(X, \Theta_X)$$

is surjective.

PROOF. Let $\{U_i\}$ be a finite open covering of X. We may assume that $[\tilde{E}]$ is trivial on each U_i, and we let $\{e_{ij}\}$ denote a system of transition functions of $[\tilde{E}]$. We can find $f^\lambda \in H^0(X, \mathcal{O}(\tilde{E}))(\lambda = 0, 1, 2, 3)$ such that $f: X \to W$ is defined by $z \to (f^0(z), \dots, f^3(z)) \in W$. We represent each f^λ by a collection $\{f_i^\lambda\}$ of holomorphic functions on U_i which satisfy $f_i^\lambda = e_{ij} f_j^\lambda$ on $U_i \cap U_j$.

Let ρ be a cohomology class in $H^1(X, \Theta_X)$ which is represented by a 1-cocycle $\{\rho_{ij}\}$ on the nerve of the covering $\{U_i\}$. We take a system of coordinates (ζ_i^1, ζ_i^2) on each U_i, and write $\{\rho_{ij}\}$ explicitly in the form $\rho_{ij} = \sum_\alpha \rho_{ij}^\alpha \partial/\partial \zeta_i^\alpha$. For any holomorphic function h on an open subset of $U_i \cap U_j$, we set $\rho_{ij} \cdot h = \sum_\alpha \rho_{ij}^\alpha \partial h/\partial \zeta_i$, and also $\rho_{ij} \cdot \log h = (\rho_{ij} \cdot h)/h$ if h is non-vanishing.

We easily see that $\{\rho_{jk}\cdot\log e_{ij}\}$ is a 2-cocycle with coefficients in $\mathcal{O}_X$. Setting

$$\kappa_{ij}=\det\left(\frac{\partial\zeta_j^\beta}{\partial\zeta_i^\alpha}\right),\qquad \operatorname{div}\rho_{ij}=\sum_\alpha\frac{\partial\rho_{ij}^\alpha}{\partial\zeta_i^\alpha},$$

we have $\operatorname{div}\rho_{ik}-\operatorname{div}\rho_{ij}-\operatorname{div}\rho_{jk}=\rho_{jk}\cdot\log\kappa_{ij}$ on $U_i\cap U_j\cap U_k$.

By Lemma 1, we can find non-vanishing holomorphic functions γ_i on U_i such that

$$\kappa_{ij}=\gamma_i^{-1}e_{ij}^m\gamma_j$$

on $U_i\cap U_j$, with $m=n-s-4$. It follows that $\xi_{ij}=(\operatorname{div}\rho_{ij}+\rho_{ij}\cdot\log\gamma_i)/m$ satisfy

$$\xi_{ik}-\xi_{ij}-\xi_{jk}=\rho_{jk}\cdot\log e_{ij}\qquad\text{on } U_i\cap U_j\cap U_k.$$

It follows that $\{\xi_{ij}f_i^\lambda-\rho_{ij}\cdot f_i^\lambda\}$ are 1-cocycles with coefficients in $\mathcal{O}_X(\tilde{E})$. By Lemma 2, these are cohomologous to 0, so we can find holomorphic functions τ_i^λ on U_i such that

$$\tau_i^\lambda-e_{ij}\tau_j^\lambda=\xi_{ij}f_i^\lambda-\rho_{ij}\cdot f_i^\lambda$$

on $U_i\cap U_j$. We claim that this implies $F\rho=0$ in $H^1(X,f^*\Theta_W)$.

In fact, let $(w^\alpha)=(w^0,w^1,w^2,w^3)$ be a system of homogeneous coordinates on W. We set $W_\alpha=\{w\in W\colon w^\alpha\neq0\}$ and $w_\alpha^\lambda=w^\lambda/w^\alpha$ on W_α. We have

$$(11)\qquad \begin{aligned}\frac{\partial}{\partial w_\beta^\lambda}&=\frac{1}{w_\beta^\alpha}\frac{\partial}{\partial w_\alpha^\lambda}\qquad\text{for }\lambda\neq\alpha,\\ \frac{\partial}{\partial w_\beta^\alpha}&=-\frac{1}{(w_\beta^\alpha)^2}\left(\sum_{\lambda\neq\alpha}w_\beta^\lambda\frac{\partial}{\partial w_\alpha^\lambda}+\frac{\partial}{\partial w_\alpha^\beta}\right),\end{aligned}$$

on $W_\alpha\cap W_\beta$. On the other hand, f is defined by $w_\alpha^\lambda=f_i^\lambda/f_i^\alpha$ on $U_i\cap f^{-1}(W_\alpha)$. Hence we have

$$F\rho_{ij}=\frac{1}{(f_i^\alpha)^2}\sum_{\lambda\neq\alpha}(f_i^\alpha\rho_{ij}\cdot f^\lambda-f^\lambda\rho_{ij}\cdot f^\alpha)\frac{\partial}{\partial w_\alpha^\lambda}.$$

Setting

$$\eta_{i\alpha}=\frac{1}{(f_i^\alpha)^2}\sum_{\lambda\neq\alpha}(f_i^\lambda\tau_i^\alpha-f_i^\alpha\tau_i^\lambda)\frac{\partial}{\partial w_\alpha^\lambda},$$

we get $F\rho_{ij}=\eta_{i\alpha}-\eta_{j\alpha}$ on $U_i\cap U_j\cap f^{-1}(W_\alpha)$. Moreover, with the aid of (11), we infer readily that the collections $\{\eta_{i\alpha}\}_{\alpha=0,\dots,3}$ represent holomorphic sections of $f^*\Theta_W$ over U_i. This proves our claim, completing the proof of Lemma 6.

4. Proof of the Main Theorem.

By the result of Kodaira cited in Introduction, we have a family $\mathcal{F}$ of surfaces S_t, $t\in M_1$ in W with ordinary singularities containing $S=S_0$, $0\in M_1$, such that the characteristic map

$$\sigma:\ T_0(M_1)\to H^0(S, \mathcal{O}_S(nE-\Delta-\sum c_i'))$$

is bijective.

Let $\mathcal{X}_1=\{X_t\}_{t\in M_1}$ be the family of the normalizations X_t of S_t. By virtue of Lemma 4, the characteristic map

$$\tau:\ T_0(M_1)\to H^0(X, \mathcal{T}_{X/W})$$

is bijective. In view of Proposition 1.4 in [3], Lemma 6 implies that the infinitesimal deformation map

$$\rho:\ T_0(M_1)\to H^1(X, \Theta_X)$$

is surjective. From $\mathcal{X}_1$, we can derive a family $\mathcal{X}=\{X_t\}_{t\in M}$ of deformations of $X=X_0$, $0\in M$, such that $\rho:\ T_0(M)\to H^1(X,\Theta_X)$ is bijective.

By Lemma 3, K is ample. So we have $H^0(X,\Theta_X)=0$ (see [1]). It follows that the family $\mathcal{X}$ is effectively parametrized at each point t of M, provided that t is sufficiently near 0. This proves that the number of moduli $m(X)$ of X is defined and equals $\dim H^1(X,\Theta_X)$.

We now calculate $m(X)$. For this purpose, we note the following exact sequence $0\to H^0(X,f^*\Theta_W)\to H^0(X,\mathcal{T}_{X/W})\to H^1(X,\ \Theta_X)\to 0$. This follows from Lemma 6 and the vanishing of $H^0(X,\Theta_X)$. Furthermore, we have a standard exact sequence $0\to\mathcal{O}_X\to\mathcal{O}_X(\tilde{E})^4\to f^*\Theta_W\to 0$. With the aid of Lemmas 2 and 3, we obtain

$$\dim H^0(X,f^*\Theta_W)=15+4\delta_{r,s+1}.$$

Thus we conclude

$$m(X)=\dim H^1(X,\Theta_X)=\mu(S)-15-4\delta_{r,s+1},$$

where $\mu(S)$ is the number of effective parameters of the family $\mathcal{F}$.

It remains to calculate $\mu(S)$. By a result of Kodaira ([4], Theorem 4 and §5.4), we get an exact sequence

$$0\to H^0(W,\mathcal{O}_W(nE-2\Delta))\to H^0(W,\mathcal{O}_W(nE-\Delta-\sum c_i'))\to H^0(\Delta,\mathcal{N}_\Delta)\to 0$$

where $\mathcal{N}_\Delta$ denotes the sheaf of germs of sections of the normal bundle of Δ in W. We have

$$\dim H^0(\Delta,\mathcal{N}_\Delta)=C(r)+C(s)-C(r-s)-2$$

where $C(m)=(m+3)(m+2)(m+1)/6$ (see [4], § 5.4), and

$$\dim H^0(W, \mathcal{O}_W(nE-\Delta-\sum c_i'))=\mu(S)+1.$$

Letting S_1 denote the hypersurface defined by the equation $g=0$, we get two exact sequences

$$0\to\mathcal{O}_W((n-r)E-\Delta)\to\mathcal{O}_W(nE-2\Delta)\to\mathcal{O}_{S_1}(nE-2\Delta)\to 0,$$
$$0\to\mathcal{O}_W((n-2r)E)\to\mathcal{O}_W((n-r)E-\Delta)\to\mathcal{O}_{S_1}((n-r)E-\Delta)\to 0.$$

With the aid of isomorphisms $\mathcal{O}_{S_1}(nE-2\Delta)\cong\mathcal{O}_{S_1}((2r-2s)E)$ and $\mathcal{O}_{S_1}((n-r)E-\Delta)\cong\mathcal{O}_{S_1}((r-s)E)$, we readily infer that

$$\dim H^0(W, \mathcal{O}_W(nE-2\Delta))=C(2r-2s)-C(r-2s)+C(r-s)+1,$$

where we set $C(m)=0$ for $m<0$. Thus we conclude that

$$\mu(S)=C(r)+C(s)+C(2r-2s)-C(r-2s)-2.$$

This completes the proof of the Main Theorem.

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