A characterization of odd order extensions of the Ree groups $^2F_4(q)$

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I. Introduction

The Ree group $^2F_4(q)$, where $q=2^{2n+1}$ $(n\geqq 0)$, is simple if $q>2$ (R. Ree, [5]), while $^2F_4(2)$ is not simple but its derived subgroup $^2F_4(2)'$ is simple (J. Tits, [10]). D. Parrott has characterized $^2F_4(2)'$ in [3] and $^2F_4(q)$ in [4] from the structure of the centralizer of a central involution. The purpose of this paper is to extend the theorem of D. Parrott in [4] to the odd order extensions of $^2F_4(q)$.

Let Γ be a finite field with $q=2^{2n+1}$ elements. Every automorphism σ of Γ canonically induces the automorphism of $^2F_4(q)$ and the semi-direct product of $\langle\sigma\rangle$ and $^2F_4(q)$ is an odd order extension of $^2F_4(q)$. Let C_q be the centralizer of a central involution in $^2F_4(q)$. Then our characterization is given by the following theorem:

THEOREM. *Let G be a finite group of even order and let τ be a central involution of G. Suppose that $C_G(\tau)$ has the following properties*:

(i) *$C_G(\tau)$ contains a normal subgroup C of odd index ρ which is isomorphic to C_q for some q.*

(ii) $O(C_G(\tau))=1$.

Then if $q>2$, one of the following cases holds:

(1) *$O(G)$ is a non-identity normal subgroup of G.*

(2) *$O(G)=1$ and there exists a non-identity subgroup E of $Z(C)$ such that $E\lhd G$.*

(3) *$G\cong\langle\sigma\rangle{}^2F_4(q)$, where $\sigma\in\mathrm{Aut}(\Gamma)$ and $|\langle\sigma\rangle|=\rho$.*

And if $q=2$, then $G=O(G)C$ or $G={}^2F_4(2)$.

COROLLARY. *If G is a finite simple group which satisfies the hypotheses of the theorem, then $q>2$ and $G\cong{}^2F_4(q)$.*

This theorem really extends the theorem of [4] (see [4, Lemma 1]).

The method used to prove the theorem is quite similar to those used by G. Thomas in [8] and M. E. Harris in [2] for the characterization of $G_2(2^n)$. We will omit the proofs if we can find nearly the same proofs of corresponding results of $G_2(2^n)$ in [8] or in [2]. But, since this work first has been done separately from the work of D. Parrott, the proof does not depend heavily on [4].[1)]

If R is a group, $R^{\#}$ stands for the set of non-identity elements in R. $L_i(R)$ and $Z_i(R)$ denote the characteristic subgroups of R defined as follows:

$$L_1(R)=R, \quad L_i(R)=[L_{i-1}(R), R] \qquad \text{for } i>1,$$
$$Z_1(R)=Z(R), \quad Z(R/Z_{i-1}(R))=Z_i(R)/Z_{i-1}(R) \qquad \text{for } i>1.$$

$\Gamma^{\times}$ denotes the multiplicative group of non-zero elements in Γ. The other notation will be standard. In particular $x^y=y^{-1}xy$ and $[x,y]=x^{-1}y^{-1}xy$ for the elements x, y of a group.

II. The group ${}^2F_4(q)$

The structures and properties of ${}^2F_4(q)$ can be found in [5]. In this section, however, we discuss these properties and explain briefly the notation we use later.

(2.1) Let G be the group ${}^2F_4(q)$, then G has a BN-pair. B is a semi-direct product of U and H, where U is a Sylow 2-subgroup of G and normal in B and $H\cong\Gamma^{\times}\times\Gamma^{\times}$. $N\cap B=H$ and $N/H=W=\langle r_1, r_2\rangle$, where $r_1^2=r_2^2=(r_1r_2)^8=1$, and hence W is a dihedral group of order 16.

$$|G|=q^{12}(q-1)(q^3+1)(q^4-1)(q^6+1),$$
$$|U|=q^{12} \quad \text{and} \quad |H|=(q-1)^2.$$

Two parabolic subgroups of G are useful; they are

$$P_1=B\cup Br_1U_1U_2 \quad \text{and} \quad P_2=B\cup Br_2U_3,$$

where U_i are subgroups of U defined in (2.2) below.

(2.2) Any element u of U can be written uniquely in the form:

$$u=\prod_{i=1}^{12}\alpha_i(\xi_i) \qquad (\xi_i\in\Gamma),$$

where the product can be taken in any order and α_i $(i=1,\dots,12)$ is an injective mapping from Γ into U; moreover for $i\neq 1, 4, 5, 6$, α_i is an additive homomorphism.

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1) The author had proved the theorem in the form of D. Parrott in [4] in May, 1973 and announced the result at the symposium held at Matsue on Aug., 1973. Soon after the symposium, he extended the result to the present form. After the publication of [4], he has changed the original paper to the present form: he has improved the proof of section V and omitted the proof in the case of $q=2$.

For this notation α_i, the reader may consult [5] p. 407. We will write the image of α_i by U_i and we define the subgroups $\mathfrak{U}_i$ $(i=1,\dots,12)$ of U as follows:

$$\mathfrak{U}_i=\prod_{j=i}^{12} U_j.$$

Another notation concerning U which is very useful in understanding the arguments in the proof can be found in [9] p. 210-05. Let $\mathfrak{X}_i, \mathfrak{Y}_i$ $(i=\pm 1,\dots,\pm 4)$ be the subgroups of G defined as follows; $\mathfrak{X}_1=U_3$, $\mathfrak{X}_2=U_9$, $\mathfrak{X}_3=U_{10}$, $\mathfrak{X}_4=U_7$, $\mathfrak{Y}_1=U_1U_2$, $\mathfrak{Y}_2=U_6U_{11}$, $\mathfrak{Y}_3=U_5U_{12}$, $\mathfrak{Y}_4=U_4U_8$, $\mathfrak{X}_{-i}=\mathfrak{X}_i^{w_0}$ and $\mathfrak{Y}_{-i}=\mathfrak{Y}_i^{w_0}$, where $w_0=(r_1r_2)^4$ and $i=1, 2, 3, 4$. Then $\langle\mathfrak{X}_i, \mathfrak{X}_{-i}\rangle\cong SL(2, q)$ and $\langle\mathfrak{Y}_i, \mathfrak{Y}_{-i}\rangle\cong {}^2B_2(q)$, where ${}^2B_2(q)$ denotes the Suzuki group. r_i $(i=1$ or $2)$ induces a permutation on $\{\mathfrak{X}_i \mid i=\pm 1,\dots,\pm 4\}$ and on $\{\mathfrak{Y}_i \mid i=\pm 1,\dots,\pm 4\}$ respectively. Roughly speaking, the action of r_1 (resp. r_2) is obtained from Fig. 1, if we consider that r_1 (resp. r_2) is a reflection defined by $r_1(\mathfrak{Y}_1)=\mathfrak{Y}_{-1}$ (resp. $r_2(\mathfrak{X}_1)=\mathfrak{X}_{-1}$). To be exact, $r_1=\alpha_1(1)\alpha_{-2}(1)\alpha_1(1)^{-1}$, $r_2=\alpha_3(1)\alpha_{-3}(1)\alpha_3(1)$ and r_i acts on α_j's according to the formula:

$$\alpha_j(\xi)^{r_i}=\alpha_{r_i(j)}(\xi),$$

where $r_i(j)$ is obtained from the following table and $r_i(-j)=-r_i(j)$.

j	1	2	4	8	5	12	6	11	3	7	9	10
$r_1(j)$	-1	-1	6	11	5	12	4	8	7	3	10	9
$r_2(j)$	4	8	1	2	6	11	5	12	-3	9	7	10

(2.3) Although the commutator relations of α_i's are calculated in [4], partly in [5] and [10], for convenience' sake, we will mention here all commutator relations of α_i's, where θ denotes the integer 2^n:

$$[\alpha_1(\xi), \alpha_3(\eta)]=\alpha_4(\xi\eta)\alpha_5(\xi^{2\theta+1}\eta^{2\theta})\alpha_7(\xi^{2\theta+2}\eta)\alpha_{11}(\xi^{4\theta+3}\eta^{2\theta+1})\alpha_{12}(\xi^{4\theta+3}\eta^{2\theta+2}),$$
$$[\alpha_1(\xi), \alpha_4(\eta)]=\alpha_5(\xi\eta^{2\theta})\alpha_6(\xi^{2\theta}\eta)\alpha_7(\xi^{2\theta+1}\eta)\alpha_9(\xi\eta^{2\theta+1})$$
$$\times\alpha_{10}(\xi^{2\theta+1}\eta^{2\theta+1})\alpha_{11}(\xi^{2\theta+2}\eta^{2\theta+1})\alpha_{12}(\xi^{2\theta+1}\eta^{2\theta+2}),$$
$$[\alpha_1(\xi), \alpha_6(\eta)]=\alpha_7(\xi\eta),$$
$$[\alpha_1(\xi), \alpha_8(\eta)]=\alpha_9(\xi\eta)\alpha_{11}(\xi^{2\theta+2}\eta)\alpha_{12}(\xi^{2\theta+1}\eta^{2\theta}),$$
$$[\alpha_1(\xi), \alpha_9(\eta)]=\alpha_{10}(\xi^{2\theta}\eta)\alpha_{11}(\xi^{2\theta+1}\eta)\alpha_{12}(\xi\eta^{2\theta}),$$
$$[\alpha_1(\xi), \alpha_{10}(\eta)]=\alpha_{11}(\xi\eta),$$
$$[\alpha_2(\xi), \alpha_3(\eta)]=\alpha_5(\xi\eta^{2\theta})\alpha_6(\xi\eta)\alpha_7(\xi^{2\theta}\eta)\alpha_8(\xi\eta^{2\theta+1})\alpha_9(\xi^{2\theta}\eta^{2\theta+1})\alpha_{11}(\xi^{2\theta+1}\eta^{2\theta+1}),$$
$$[\alpha_2(\xi), \alpha_4(\eta)]=\alpha_7(\xi\eta)\alpha_{11}(\xi^{2\theta}\eta^{2\theta+1})\alpha_{12}(\xi\eta^{2\theta+2}),$$
$$[\alpha_2(\xi), \alpha_8(\eta)]=\alpha_{10}(\xi\eta)\alpha_{11}(\xi^{2\theta}\eta)\alpha_{12}(\xi\eta^{2\theta}),$$

$$
\begin{aligned}
&[\alpha_2(\xi), \alpha_9(\eta)] = \alpha_{11}(\xi\eta),\\
&[\alpha_3(\xi), \alpha_5(\eta)] = \alpha_8(\xi\eta),\\
&[\alpha_3(\xi), \alpha_6(\eta)] = \alpha_8(\xi^{2\theta}\eta)\alpha_9(\xi\eta^{2\theta})\alpha_{12}(\xi\eta^{2\theta+1}),\\
&[\alpha_3(\xi), \alpha_7(\eta)] = \alpha_9(\xi^{2\theta}\eta)\alpha_{10}(\xi\eta^{2\theta}),\\
&[\alpha_3(\xi), \alpha_{11}(\eta)] = \alpha_{12}(\xi\eta),\\
&[\alpha_4(\xi), \alpha_5(\eta)] = \alpha_9(\xi\eta),\\
&[\alpha_4(\xi), \alpha_7(\eta)] = \alpha_{10}(\xi^{2\theta}\eta)\alpha_{11}(\xi\eta^{2\theta})\alpha_{12}(\xi^{2\theta+1}\eta),\\
&[\alpha_4(\xi), \alpha_{10}(\eta)] = \alpha_{12}(\xi\eta),\\
&[\alpha_5(\xi), \alpha_6(\eta)] = \alpha_{10}(\xi\eta),\\
&[\alpha_5(\xi), \alpha_7(\eta)] = \alpha_{11}(\xi\eta),\\
&[\alpha_6(\xi), \alpha_9(\eta)] = \alpha_{12}(\xi\eta),\\
&[\alpha_7(\xi), \alpha_8(\eta)] = \alpha_{12}(\xi\eta) \quad \text{and}\\
&[\alpha_i(\xi), \alpha_i(\eta)] = \alpha_{j(i)}(\xi^{2\theta}\eta + \xi\eta^{2\theta}),
\end{aligned}
$$

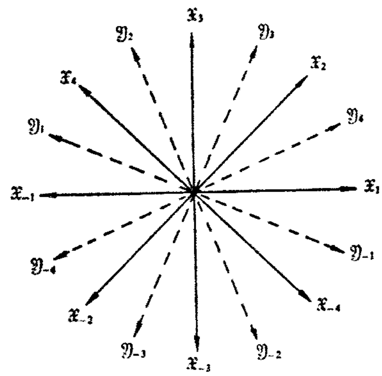

Fig. 1

where $j(i)=2, 8, 12$ or 11 corresponding to $i=1, 4, 5$ or 6 respectively. All the other commutators $[\alpha_i(\xi), \alpha_j(\eta)]$ are equal to 1.

Moreover $\alpha_i(\xi)\alpha_i(\eta) = \alpha_i(\xi+\eta)\alpha_{j(i)}(\xi\eta^{2\theta})$ for $i=1, 4, 5$ or 6.

(2.4) Every element h of H can be written with two parameters ε_1 and ε_2:

$$h = h(\varepsilon_1, \varepsilon_2), \quad \text{where} \quad \varepsilon_i \in \Gamma^{\times}.$$

The action of H on U is given by:

$$
\begin{aligned}
h\alpha_1(\xi)h^{-1} &= \alpha_1(\varepsilon_2^{2-2\theta}\xi),\\
h\alpha_2(\xi)h^{-1} &= \alpha_2(\varepsilon_2^{2\theta}\xi),\\
h\alpha_3(\xi)h^{-1} &= \alpha_3(\varepsilon_1^{2\theta-1}\varepsilon_2^{-1}\xi),\\
h\alpha_4(\xi)h^{-1} &= \alpha_4(\varepsilon_1^{2\theta-1}\varepsilon_2^{-2\theta+1}\xi),\\
h\alpha_5(\xi)h^{-1} &= \alpha_5(\varepsilon_1^{2-2\theta}\xi),\\
h\alpha_6(\xi)h^{-1} &= \alpha_6(\varepsilon_1^{2\theta-1}\varepsilon_2^{2\theta-1}\xi),\\
h\alpha_7(\xi)h^{-1} &= \alpha_7(\varepsilon_1^{2\theta-1}\varepsilon_2\xi),\\
h\alpha_8(\xi)h^{-1} &= \alpha_8(\varepsilon_1\varepsilon_2^{-1}\xi),\\
h\alpha_9(\xi)h^{-1} &= \alpha_9(\varepsilon_1\varepsilon_2^{-2\theta+1}\xi),\\
h\alpha_{10}(\xi)h^{-1} &= \alpha_{10}(\varepsilon_1\varepsilon_2^{2\theta-1}\xi),\\
h\alpha_{11}(\xi)h^{-1} &= \alpha_{11}(\varepsilon_1\varepsilon_2\xi),\\
h\alpha_{12}(\xi)h^{-1} &= \alpha_{12}(\varepsilon_1^{2\theta}\xi).
\end{aligned}
$$

And r_1 and r_2 act on H as follows:

$$h(\varepsilon_1, \varepsilon_2)^{r_1} = h(\varepsilon_1, \varepsilon_2^{-1}) \qquad \text{and}$$

$$h(\varepsilon_1, \varepsilon_2)^{r_2} = h(\varepsilon_1^{\theta}\varepsilon_2^{\theta}, \varepsilon_1^{\theta}\varepsilon_2^{-\theta}).$$

III. Some properties of C_q and preliminary lemmas

(3.1) $Z=Z(U)=U_{12}$. Every element z contained in $Z^{\sharp}$ is an involution and its centralizer in ${}^2F_4(q)$ is isomorphic to C_q and we identify C_q with the centralizer of z in ${}^2F_4(q)$. Then $Z(C_q)=Z$ and

$$C_q = UK \cup UKr_1U_1U_2,$$

where $K=\{h \in H \mid h=h(1, \varepsilon)\}$. Thus $|C_q| = q^{12}(q-1)(q^2+1)$.

(3.2) $O_2(C_q)=D=\mathfrak{U}_3$ and $C_q/D \cong {}^2B_2(q)$. In fact $U_1U_2K \cup U_1U_2Kr_1U_1U_2$ $(\cong {}^2B_2(q))$ is a complement of D in C_q. Any two distinct Sylow 2-subgroups of C_q intersect in D, since any two distinct Sylow 2-subgroups of ${}^2B_2(q)$ intersect trivially.

The following propositions (3.3)$\sim$(3.7) are easily verified using the properties in § II.

(3.3) Every involution contained in U is conjugate in U to one of the following elements:

(i)	$\alpha_2(\xi_2)\alpha_{12}(\xi_{12})$,	$\xi_2 \neq 0$,	(q^8)
(ii)	$\alpha_3(\xi_3)\alpha_{10}(\xi_{10})$,	$\xi_3 \neq 0$,	(q^6)
(iii)	$\alpha_7(\xi_7)$,		(q^8)
(iv)	$\alpha_8(\xi_8)\alpha_{11}(\xi_{11})$,	$\xi_8 \neq 0$,	(q^9)
(v)	$\alpha_9(\xi_9)$,		(q^9)
(vi)	$\alpha_{10}(\xi_{10})$,		(q^{10})
(vii)	$\alpha_{11}(\xi_{11})$,		(q^{11})
(viii)	$\alpha_{12}(\xi_{12})$,		(q^{12})

where the powers of q in the parentheses denote the order of the centralizer in U of the element on the same row.

(3.4) Every involution contained in C_q is conjugate in C_q to one of the following elements:

$$\alpha_2(1)\alpha_{12}(\eta), \quad \alpha_7(1), \quad \alpha_{10}(1), \quad \alpha_{11}(1), \quad \alpha_{12}(\xi).$$

(3.5) If A is a maximal elementary abelian subgroup in U, then A is conjugate in U to one of the following subgroups:

$$V_1 = U_2U_7U_{10}U_{11}U_{12}, \qquad V_2 = U_3U_8U_9U_{10}U_{12},$$
$$W = U_7U_9U_{10}U_{11}U_{12} \quad \text{and} \quad X = \mathfrak{U}_8.$$

Each of these has order q^5 and only W and X are normal in U.

(3.6) We define two other important subgroups of U, $M=U_1U_2\mathfrak{U}_4$ and $Y=U_{11}U_{12}$. Then $Y=Z(M)$ and M is a normal subgroup of U. Moreover, if M_1 is a subgroup of U of order q^{11} whose center has order q^2, then $M=M_1$. Therefore M char U.

(3.7) U and M have the following upper central series:

$$Z_1(U)=Z,\quad Z_2(U)=Y,\quad Z_3(U)=\mathfrak{U}_{10},\quad Z_4(U)=\mathfrak{U}_9,\quad Z_5(U)=\mathfrak{U}_7,$$
$$Z_6(U)=\mathfrak{U}_5,\quad Z_7(U)=U_2\mathfrak{U}_4,\quad \text{and} \quad Z_8(U)=U.$$
$$Z_1(M)=Y,\quad Z_2(M)=\mathfrak{U}_{10},\quad Z_3(M)=W,$$
$$Z_4(M)=U_2\mathfrak{U}_6 \quad \text{and} \quad Z_5(M)=M.$$

The following two propositions are well known, but we write them here for later use.

(3.8) *Let u and v be two involutions in a finite group G. If u is not conjugate to v in G, then there is a third involution w such that*

(i) *w commutes with both u and v, and*

(ii) *wu is conjugate in G to either u or v.*

(3.9) (M. Suzuki, [6]) *Let G be a finite not 2-closed (TI)-group in which a Sylow 2-subgroup S contains more than one involution. Then:*

(i) *G contains a subgroup J of odd order which normalizes S and acts transitively on $Z(S)^{\#}$, and moreover,*

(ii) *if S is abelian of order r, then G contains a normal subgroup of odd index, which is isomorphic to $SL(2,r)$.*

IV. The structure of $C_G(\tau)$

We assume throughout the remainder of this paper that G is a finite group which satisfies the hypotheses in the theorem stated in § I. The subgroup C of $C_G(\tau)$ is identified with C_q and letters $Z, U, K, D, V_1, V_2, W, X, Y$ and M retain the meanings given to them in § II and § III.

In this section, analyzing $C_G(\tau)$ we obtain a complement A to C in $C_G(\tau)$.

Similar arguments used in [2, Lemmas 3.1 and 3.2] yield:

(4.1) *$C \lhd C_G(Z) \lhd C_G(\tau)$, $O(C_G(Z))=1$ and $\tau \in Z$.*

(4.2) *If N is a subgroup of $C_G(\tau)$ and contains C, then $O^{2'}(N)=C$, so that we* *have especially*:

$$C=O^{2'}(C_G(Z))\triangleleft N_G(Z).$$

(4.3) $C_G(D)=Z$ *and* $C_G(X)=U_5X$.

PROOF. The proof of [2, Lemma 3.3] yields that $C_G(D)=Z$.

As $X=[D,D]$, $X\triangleleft C_G(\tau)$ and hence $C_G(X)\triangleleft C_G(\tau)$. Hence $O(C_G(X))=1$, and $C_C(X)=U_5X$ is a normal Sylow 2-subgroup of $C_G(X)$. Let R be a complement to $C_C(X)$ in $C_G(X)$. Then by definition R centralizes X and moreover R centralizes U_5X/X. In fact, if $\alpha_5(\xi)^y\equiv\alpha_5(\xi')$ (mod X) where $y\in R$, then $(\alpha_5(\xi)^2)^y=\alpha_{12}(\xi^{2\theta+1})^y=\alpha_{12}(\xi^{2\theta+1})$ and $\alpha_5(\xi')^2=\alpha_{12}(\xi'^{2\theta+1})$; however, the mapping $\xi\mapsto\xi^{2\theta+1}$ from Γ to Γ is bijective, so $[R, U_5X/X]=1$. Hence, as $|R|=$odd, R centralizes U_5X and we obtain $C_G(X)=U_5X\times R$. Then $R=O(C_G(X))=1$ so that $C_G(X)=U_5X$.

$C_G(\tau)$ operates on $C/D\cong{}^2B_2(q)$ by conjugation. This implies that there is a homomorphism φ from $C_G(\tau)$ to $\mathrm{Aut}({}^2B_2(q))$. $\mathrm{Aut}({}^2B_2(q))$ is a semi-direct product of $\mathrm{Int}({}^2B_2(q))$ and $\mathrm{Aut}(\Gamma)$, where the elements of $\mathrm{Aut}(\Gamma)$ are the automorphisms of ${}^2B_2(q)$ induced from the automorphisms of the field Γ. Let $A^*=\varphi^{-1}(\mathrm{Aut}(\Gamma))$. Then $A^*\triangleright D$, $A^*\cap C=D$ and $C_G(\tau)=A^*C$ so that $[A^*: D]=\rho=$odd. Hence Schur-Zassenhaus theorem yields that there is a complement A of D in A^*. Furthermore, the proof of [2, Lemma 3.7] is also valid in this case and after replacing A by a conjugate in A^*, we have:

(4.4) *There is a subgroup* A *of* $C_G(\tau)$ *such that* $|A|=\rho$, $C_G(\tau)=CA$, $C\cap A=1$ *and* $A\subseteq N_G(K)$. *Hence* AK *is a subgroup of* $C_G(\tau)$ *of order* $(q-1)\rho$. *Moreover there exists a homomorphism* χ: $A^*\to\mathrm{Aut}(\Gamma)$ *such that if* f *is an element in* A^*, *then*

$$\alpha_i(\xi)^f\equiv\alpha_i(\xi^{\chi(f)})\qquad(\mathit{mod}\ D)$$

for all $\xi\in\Gamma$, *where* $i=\pm1$ *or* ±2.

(4.5) *If* $q=2$, *then* $\rho=1$.

PROOF. By the construction of A^*, $A^*\subset N_G(U)$ so that $A\subset N_G(U)$. Therefore A normalizes Z, Y, $\mathfrak{U}_{10}$, $\mathfrak{U}_9$, $X=\mathfrak{U}_8$, $\mathfrak{U}_7$, $\mathfrak{U}_6$, $\mathfrak{U}_4=Z_7(U)\cap D$ and D (cf. (3.7) and (4.7) below). Since Z, Y/Z, $\mathfrak{U}_{10}/Y$, $\mathfrak{U}_9/\mathfrak{U}_{10}$, $X/\mathfrak{U}_9$, $\mathfrak{U}_7/X$, $\mathfrak{U}_4/\mathfrak{U}_6$ and $D/\mathfrak{U}_4$ are groups of order 2, A centralizes these groups. Moreover using (2.3) we can easily conclude that A also centralizes $\mathfrak{U}_6/\mathfrak{U}_7$. Therefore, since $|A|=\rho=$odd, A must centralize D and this implies that $A=1$, since $C_G(D)=Z$ (cf. (4.3)).

Therefore in the case $q=2$, the theorem in [4] can be applied and hence the conclusion of the theorem in §I is obtained. Henceforth we assume that $q>2$.

From the proofs of [8, (5.4) and (5.5)] we can also conclude that:

(4.6) *Two elements of* Y *are conjugate in* G *if and only if they are conjugate in* $N_G(M)$.

(4.7) W *and* X *are normal in* $N_G(U)$.

The proofs of [2, 3.10 and 3.11] yield:

(4.8) *If* $q>2$ *and* $\mathfrak{M}$ *is a simple subnormal subgroups of* C, *then* $\mathfrak{M}$ *is abelian.*

(4.9) *If* $N_G(M)$ *is not 2-closed, then* $N_G(M)/M$ *is a* (TI)*-group.*

(4.10) $C_C(\alpha_{10}(1))$ *is a Sylow 2-subgroup of* $C_G(\alpha_{10}(1))$ *and hence* $\alpha_{10}(1)$ *is not conjugate to any element in* Z.

PROOF. Let $T=C_C(\alpha_{10}(1))$, then $T=U_2\mathfrak{U}_4$. If T is not a Sylow 2-subgroup of $C_G(\alpha_{10}(1))$, then there is an element $g\in C_G(\alpha_{10}(1))$ such that $g\in N_G(T)-T$ and $g^2\in T$. Since $L_5(T)=Z, g\in N_G(Z)$. Then as τ is central and $C=O^{2'}(C_G(Z))\lhd N_G(Z)$ (cf. (4.2)), we obtain that $g\in C$ and hence $g\in T$. But this is a contradiction.

V. Non-simple cases

In this section we treat the case when $N_G(M)$ is 2-closed and we conclude that if $N_G(M)$ is 2-closed, then the conclusion (1) or (2) of the theorem holds.

We now assume that $N_G(M)$ is 2-closed. Then

(5.1) $N_G(M)=N_G(U)$.

From (4.6) we can conclude that:

(5.2) *No element in* $Y-Z$ *is conjugate to any element of* Z.

(5.3) *No element in* $\mathfrak{U}_9-Y$ *is conjugate to any element in* Z.

PROOF. In order to prove (5.3), it is sufficient to prove that $\alpha_{10}(1)$ is not conjugate to z for any $z\in Z$, since every element in $\mathfrak{U}_9-Y$ is conjugate to $\alpha_{10}(1)$ in C. However this fact has been proved in (4.10).

(5.4) $C_C(\alpha_7(1))$ *is a Sylow 2-subgroup of* $C_G(\alpha_7(1))$. *Therefore* $\alpha_7(1)$ *is not conjugate to* z *for any* $z\in Z^{\sharp}$.

PROOF. We let $T=C_C(\alpha_7(1))$, then $T=C_U(\alpha_7(1))=U_1U_2U_6W=U_1U_6U_9V_1$ and $Z(T)=U_7Y$. Notice that for any element $x\in U_7Y-Y$, x is conjugate to $\alpha_7(1)$ in C and for any element $y\in Y-Z$, y is conjugate to $\alpha_{11}(1)$.

Assume that $C_C(\alpha_7(1))$ is not a Sylow 2-subgroup of $C_G(\alpha_7(1))$. Then by the Sylow's theorem, we obtain an element $b\in C_G(\alpha_7(1))$ such that $b\in N_G(T)-T$ and

$b^2 \in T$. We claim that $b \notin N_G(Z)$; since, if $b \in N_G(Z)$, then from (4.2) $b \in C$ and hence $b \in T$. Therefore from (5.2) we have $Z^b \subset Z \cup (U_7Y - Y)$ and there must exist an element $z \in Z$ such that z is conjugate to $\alpha_7(1)$. Hence $(U_7Y-Y)^b \subset Z \cup (U_7Y-Y)$, which implies that $(Y-Z)^b = Y-Z$. Therefore b normalizes $\langle Y-Z \rangle = Y$ and hence Z by virtue of (5.2). But this is a contradiction.

(5.5) *$\alpha_2(1)\alpha_{12}(\xi)$ is not conjugate to z for any $z \in Z$.*

PROOF. Let $T_1 = C_C(\alpha_2(1)\alpha_{12}(\xi)) = C_U(\alpha_2(1)\alpha_{12}(\xi)) = U_1U_5U_6V_1$. Then $\Omega_1(T_1) = V_1 = U_2U_7\mathfrak{U}_{10}$. Suppose that $\alpha_2(1)\alpha_{12}(\xi)$ is conjugate to z for some $z \in Z$. Then there is an element $b \in C_G(\alpha_2(1)\alpha_{12}(\xi))$ such that $b \in N_G(T_1) - T_1$ and $b^2 \in T_1$. Therefore $b \in N_G(V_1)$. We claim that $(U_7\mathfrak{U}_{10} - \mathfrak{U}_{10})^b \neq U_7\mathfrak{U}_{10} - \mathfrak{U}_{10}$; otherwise, since $\langle U_7\mathfrak{U}_{10} - \mathfrak{U}_{10} \rangle = U_7\mathfrak{U}_{10}$, $(U_7\mathfrak{U}_{10})^b = U_7\mathfrak{U}_{10}$ and (5.2), (5.3) and (5.4) imply that $Z^b = Z$ and hence $b \in C = O^{2'}(N_G(Z))$, which contradicts $b \notin T_1$. Hence $\alpha_7(1)$ is conjugate to some involution in $\mathfrak{U}_{10} \cup (V_1 - U_7\mathfrak{U}_{10})$. But $\alpha_7(1)$ is not conjugate to $x \in \mathfrak{U}_{10}$; since $|C_U(x)| \geqq q^{10}$ while the order of T, a Sylow 2-subgroup of $C_G(\alpha_7(1))$ (cf. (5.4)), is q^8. Furthermore, if $y \in V_1 - U_7\mathfrak{U}_{10}$, then $|C_U(y)| = q^8$ but $C_U(y)$ is not isomorphic to $T = C_U(\alpha_7(1))$; in fact, $|\Omega_1(T)| > q^5$ while $|\Omega_1(C_U(y))| = q^5$. Therefore $\alpha_7(1)$ is neither conjugate to x nor y, which is a contradiction.

(5.6) *For every element $g \in G$, we have $Z^g \cap N_G(Z) \subseteq Z$.*

PROOF. This is an immediate consequence of (3.4), (5.2), (5.3) and (5.5).

(5.7) *If $N_G(M)$ is 2-closed and $q>2$, then either the conclusion* (1) *or* (2) *of the theorem holds.*

PROOF. We may assume that $O(G)=1$, since otherwise conclusion (1) holds. Let $\mathfrak{H} = \langle \tau^G \rangle$ and $\mathfrak{A} = \langle \tau^G \cap C_G(\tau) \rangle$. Then clearly $\mathfrak{A}$ is a non-identity subgroup of Z and strongly closed in $N_G(\mathfrak{A})$, since $N_G(\mathfrak{A}) \triangleright C_G(\mathfrak{A})$ and $C_G(\tau) \supseteq C_G(\mathfrak{A}) \triangleright O^{2'}(C_G(\mathfrak{A})) = C$. Hence we can use the theorem of D. Goldschmidt [1] and we can verify that the argument used in [2, p. 300] is also valid in this case, if we only notice the following fact:

(5.7.1) *Let H be a maximal subgroup of ${}^2B_2(q)$, where $q>2$, then $[{}^2B_2(q): H] \geqq q^2+1$.*

This is an immediate consequence of Suzuki [7, Theorem 9].

Therefore we can derive conclusion (2) and thus we have completed the proof in the case when $N_G(M)$ is 2-closed.

VI. Structure of $N_G(M)$

For the rest of this paper we will assume that $N_G(M)/M$ is a not-2-closed (TI)-group. The purpose of this section is to prove that $N_G(M)=Q\cdot A$, where $Q\cong P_2$ (see (2.1)), $Q\cap A=1$ and $N_G(M)\triangleright Q$ under the assumption $q>2$.

(6.1) *There is a normal subgroup L of odd index in $N_G(M)$ such that $L/M\cong SL(2,q)$* (cf. (3.9)). *Let H^* be a complement to U in $N_G(U)$ such that $H^*\supseteq AK$ and let $J=L\cap H^*$. Then $N_L(U)=JU$, $J\triangleleft H^*$ and J is a cyclic group of order $q-1$ and acts regularly on $(U/M)^{\sharp}$.*

(6.2) *J normalizes exactly one other Sylow 2-subgroup S_1 in L, and hence H^* also normalizes S_1. Since $Y=Z(M)$, clearly $Z_1=Z(S_1)\subset Y$.*

(6.3) *There is an element r in L such that $H^{*r}=H^*$, $U^r=S_1$, $J^r=J$, $r^2\in M$ and r inverts every elements of J. For r, there is a unique element $\alpha_3(\lambda)$ in $U_3^{\sharp}$ such that $(r\alpha_3(\lambda))^3\in M$* (cf. [8, (6.5)]).

(6.4) *H^* normalizes Z, Z^r and C so that $K=C\cap H^*\triangleleft H^*$.*

PROOF. Since $H^*\subset N_G(U)$, $H^*=H^{*r}$ normalizes $Z=Z(U)$ and Z^r so that $K\triangleleft H^*$ is derived from (4.2).

(6.5) *Let R be an elementary abelian subgroup of order q which is normalized by K.*

(i) *If R is contained in Y, but not in Z, then $R=U_{11}$.*

(ii) *If R is contained in $\mathfrak{U}_{10}$, but not in Y, then $R=U_{10}$.*

(iii) *If R is contained in $\mathfrak{U}_9$, but not in $\mathfrak{U}_{10}$, then $R=U_9$ if $q>8$, and $R=\{\alpha_9(\varepsilon)\alpha_{11}(\xi\varepsilon^2)\mid\varepsilon\in\Gamma\}$ for some $\xi\in\Gamma$, if $q=8$.*

(iv) *If R is contained in X, but not in $\mathfrak{U}_9$, then $R=U_8$ if $q>8$, and $R=\{\alpha_8(\varepsilon)\alpha_{10}(\xi\varepsilon^4)\mid\varepsilon\in\Gamma\}$ for some $\xi\in\Gamma$, if $q=8$.*

(v) *If R is contained in W, but not in $\mathfrak{U}_9$, then $R=\{\alpha_7(\varepsilon)\alpha_{11}(\xi\varepsilon)\mid\varepsilon\in\Gamma\}$ for some $\xi\in\Gamma$ if $q>8$, and $R=\{\alpha_7(\varepsilon)\alpha_9(\eta\varepsilon^4)\alpha_{11}(\xi\varepsilon)\mid\varepsilon\in\Gamma\}$ for some $\xi,\eta\in\Gamma$, if $q=8$.*

(vi) *If R is contained in U, but not in M, then $R=\{\alpha_3(\varepsilon)\alpha_8(\xi\varepsilon)\mid\varepsilon\in\Gamma\}$ for some $\xi\in\Gamma$ if $q>8$, and $R=\{\alpha_3(\varepsilon)\alpha_8(\xi\varepsilon)\alpha_{10}(\eta\varepsilon^4)\mid\varepsilon\in\Gamma\}$ for some $\xi,\eta\in\Gamma$ if $q=8$.*

(vii) *If R is contained in M, but not in $M\cap D=\mathfrak{U}_4$, then $R=\{\alpha_2(\varepsilon^{2\theta})\alpha_7(\xi\varepsilon)\alpha_{11}(\eta\varepsilon)\mid\varepsilon\in\Gamma\}$ for some $\xi,\eta\in\Gamma$.*

This is proved by arguments similar to those used in [8, (6.10) and (6.11)]. Cases (i)~(v) are obtained immediately and in case (vi) $R\subset V_2$ and in case (vii) $R\subset V_1$ are proved first. We must be careful about the fact that both mappings

$\varepsilon \mapsto \varepsilon^{-2\theta+1}$ and $\varepsilon \mapsto \varepsilon^{-2\theta-1}$ from Γ to Γ are not additive homomorphism unless $q=8$. We omit the proof of (6.5), but we remark here that if x is an element of H^*, then U_{11}^x satisfies the condition of (i), U_{10}^x of (ii), U_9^x of (iii), U_8^x of (iv), U_7^x of (v), U_3^x of (vi) and U_2^x of (vii) respectively.

(6.6) *H^* normalizes U_5Z and there is an element z in U_5Z such that $H^{*wz}=H^*$, where $w=r_1$.*

PROOF. Since $U_5Z=C_U(K)$, the first assertion is clear. K normalizes exactly two Sylow 2-subgroups of C, i.e. U and $U^w=(U_1U_2)^wD$, so that H^* also normalizes these two subgroups. As $K^w=K$ and $w^2=1$, we have:

$$\begin{aligned} H^{*w} &\subseteq U_G(U)\cap N_G(U^w)\cap N_G(K) \\ &= H^*U\cap H^*U^w\cap N_G(K) \\ &= H^*D\cap N_G(K)=H^*N_D(K)=H^*U_5Z. \end{aligned}$$

Therefore we obtain the desired result.

COROLLARY (6.6.1) *U_i $(8\leqq i\leqq 12)$ is normalized by H^*.*

PROOF. By (6.5), H^* normalizes U_{11} and U_{10} and hence H^* also normalizes $U_{11}^{z^{-1}w}=U_8$ and $U_{10}^{z^{-1}w}=U_9$, by virtue of (6.6.1). This result is essential only if $q=8$; since if $q>8$, this result has already been proved in [(6.5) (iii) and (iv)].

(6.7) *Let h be an element in H^*. Then there are constants $\beta_*, \gamma_*, \delta_*$ depending on h and we have:*

$$U_6^h=\begin{cases}\{\alpha_6(\varepsilon)\alpha_{10}(\beta_1\varepsilon)\alpha_{11}(\gamma_1\varepsilon^{2\theta+1})\,|\,\varepsilon\in\Gamma\} & if\quad q>8\\ \{\alpha_6(\varepsilon)\alpha_8(\delta_1\varepsilon^2)\alpha_{10}(\beta_1\varepsilon)\alpha_{11}(\gamma_1\varepsilon^{2\theta+1})\,|\,\varepsilon\in\Gamma\} & if\quad q=8.\end{cases}$$
$$U_4^h=\begin{cases}\{\alpha_4(\varepsilon)\alpha_8(\delta_2\varepsilon^{2\theta+1})\alpha_9(\beta_2\varepsilon)\,|\,\varepsilon\in\Gamma\} & if\quad q>8\\ \{\alpha_4(\varepsilon)\alpha_8(\delta_2\varepsilon^{2\theta+1})\alpha_9(\beta_2\varepsilon)\alpha_{11}(\gamma_2\varepsilon^2)\,|\,\varepsilon\in\Gamma\} & if\quad q=8.\end{cases}$$
$$U_1^h=\{\alpha_1(\varepsilon)\alpha_2(\delta_3\varepsilon^{2\theta+1})\alpha_{10}(\beta_3\varepsilon^{\theta})\alpha_{11}(\gamma_3\varepsilon^{\theta+1})\,|\,\varepsilon\in\Gamma\}.$$

PROOF. U_6^h is contained in $\mathfrak{U}_5-\mathfrak{U}_7$, since $\mathfrak{U}_5=Z_6(U)$ and $\mathfrak{U}_7=Z_5(U)$. Let x be a non-identity element in U_6^h and let $x=\prod_{i=5}^{12}\alpha_i(\xi_i)$. Then ξ_5 or $\xi_6=0$, since $(U_6^h)^2\subseteq U_{11}$. But (6.6) implies that the case $\xi_5\neq 0$ and $\xi_6=0$ cannot happen. Therefore $\xi_5=0$ and $\xi_6\neq 0$. Since K normalizes U_6^h and acts regularly on $U_6^\sharp$, we have:

$$\begin{aligned} U_6^h &= \{x^k\,|\,k\in K\}\cup\{1\} \\ &= \{\alpha_6(\varepsilon)\alpha_7(\xi_7\varepsilon^{2\theta+1})\alpha_8(\xi_8\varepsilon^{-2\theta-1})\alpha_9(\xi_9\varepsilon^{-1})\alpha_{10}(\xi_{10}\varepsilon)\alpha_{11}(\xi_{11}\varepsilon^{2\theta+1})\alpha_{12}(\xi_{12})\,|\,\varepsilon\in\Gamma\}. \end{aligned}$$

But $U_6^hU_{11}=(U_6U_{11})^h$ is a subgroup, so we have $\xi_7=\xi_1=\xi_{12}=0$ if $q\geqq 8$, and moreover $\xi_8=0$ if $q>8$, which completes the proof of U_6^h. U_4^h and U_1^h can also be treated in the same way, using U_8^h and U_2^h; hence we omit.

Recall that χ: $A^*\to\mathrm{Aut}(\Gamma)$ (cf. (4.4)).

(6.8) $\mathrm{Ker}\,\chi\cap A=1$.

PROOF. Let $f\in\mathrm{Ker}\,\chi\cap A$. Then f centralizes U_1U_2D/D and hence from (6.7) we have:

$$\alpha_1(\xi)^f=\alpha_1(\xi)\alpha_{10}(\beta\xi)\alpha_{11}(\gamma\xi).$$

Moreover (6.6.1) implies that for $\eta\in\Gamma$ there is an element $\eta_1\in\Gamma$ such that:

$$\alpha_9(\eta)^f=\alpha_9(\eta_1).$$

Then

$$[\alpha_1(\xi)^f, \alpha_9(\eta)^f]=\alpha_{10}(\xi^{2\theta}\eta_1)\alpha_{11}(\xi^{2\theta+1}\eta_1)\alpha_{12}(\xi\eta_1^{2\theta}),$$

and on the other hand, we have:

$$[\alpha_1(\xi), \alpha_9(\eta)]^f=\alpha_{10}(\xi^{2\theta}\eta)^f\alpha_{11}(\xi^{2\theta+1}\eta)^f\alpha_{12}(\xi\eta^{2\theta})^f.$$

Therefore by (6.6) we have $\alpha_{10}(\xi^{2\theta}\eta)^f=\alpha_{10}(\xi^{2\theta}\eta_1)$, $\alpha_{11}(\xi^{2\theta+1}\eta)^f=\alpha_{11}(\xi^{2\theta+1}\eta_1)$ and $\alpha_{12}(\xi\eta^{2\theta})^f=\alpha_{12}(\xi\eta_1^{2\theta})$. If $\tau=\alpha_{12}(\xi_0)$, set $\xi=\xi_0\eta^{-2\theta}$. Then the last equation implies that $\eta=\eta_1$ for every $\eta\in\Gamma$. Hence we have $\alpha_i(\xi)^f=\alpha_i(\xi)$ for all $\xi\in\Gamma$, where $i=9, 10, 11$ or 12. Moreover, from the commutator relation $[\alpha_1(\xi), \alpha_8(\eta)]$ and (6.6.1) we have also $\alpha_8(\xi)^f=\alpha_8(\xi)$ for all $\xi\in\Gamma$, so that $f\in C_G(X)=U_5X$ (cf. (4.3)). But $U_5X\cap A=1$ and hence $f=1$.

COROLLARY (6.8.1) $C_A(K)=1$.

PROOF. Let $f\in C_A(K)$. Since $\alpha_2(1)^f\equiv\alpha_2(1)$ (mod D) and K acts transitively on $U_2^\sharp$, f centralizes U_2D/D. Hence $f\in\mathrm{Ker}\,\chi\cap A=1$.

(6.9) $J\cap K=1$ *and hence* $JK=J\times K$.

PROOF. Let $h\in J\cap K$. Then $h=h^{-r}$ centralizes Z and Z^r. By (6.5) (i), we have $Z^r=U_{11}$, since $r\in N_G(M)$, $Y=Z(M)$ and $Z\neq Z^r$. However, K acts transitively on $U_{11}^\sharp$, so that $h=1$.

We define a subgroup H of H^* as follows:

$$H=J\times K.$$

Then H is an abelian subgroup of $(q-1)^2$.

The proofs of [2, Lemma 5.18 and 5.18.1] yield:

(6.10) $J\cap C_G(\tau)=J\cap AK=1$.

COROLLARY (6.10.1) *J is regular on $Z^{\sharp}$ and $H^{*}=C_{H^{*}}(\tau)J=(AK)J=A\cdot H$, where $A\cap H=1$.*

COROLLARY (6.10.2) *If Z_1 is a conjugate of Z such that $Z\cap Z_1\neq 1$, then $Z=Z_1$.*

PROOF. Set $Z_1=Z^g$ and let z_1 be a non-identity element in $Z\cap Z^g$, then $z_1=z^g$ where $z\in Z^{\sharp}$. Since z_1 is conjugate to τ and $C_G(z_1)\supseteq C$, we have $O^{2'}(C_G(z_1))=C$ and by the same reason, $O^{2'}(C_G(z))=C$, which imply that $C=C^g$ and hence $Z=Z^g$ as $Z=Z(C)$.

Thus according to (6.10.1) we may assume that:

$$\tau=\alpha_{12}(1).$$

(6.11) *Let $f\in A$. Then for all $\xi\in\Gamma$ we have:*

$$\begin{aligned}
&\alpha_i(\xi)^f=\alpha_i(\xi'), \qquad i=1,2,8,9,10,11,12,\\
&\alpha_3(\xi)^f=\alpha_3(\xi')\alpha_8(\lambda_f\xi'),\\
&\alpha_4(\xi)^f=\alpha_4(\xi')\alpha_9(\lambda_f\xi'),\\
&\alpha_5(\xi)^f=\alpha_5(\xi')\alpha_{12}(\xi'\lambda_f^{2\theta}+\xi'^{2\theta}\lambda_f),\\
&\alpha_6(\xi)^f=\alpha_6(\xi')\alpha_{10}(\lambda_f\xi'),\\
&\alpha_7(\xi)^f=\alpha_7(\xi')\alpha_{11}(\lambda_f\xi'),
\end{aligned}$$

where λ_f is a constant depending on f and $\xi'=\xi^{\chi(f)}$.

PROOF. Like the arguments used in (6.8), by using the commutator relation $[\alpha_1(\xi),\alpha_9(\eta)]=\alpha_{10}(\xi^{2\theta}\eta)\alpha_{11}(\xi^{2\theta+1}\eta)\alpha_{12}(\xi\eta^{2\theta})$ and the fact that $\alpha_{12}(1)^f=\alpha_{12}(1)$, we can prove easily that:

$$\alpha_i(\xi)^f=\alpha_i(\xi') \qquad \text{where} \qquad 8\leqq i\leqq 12.$$

As for $\alpha_i(\xi)^f$ ($i\leqq 7$), from (6.5), (6.6) and (6.7) we have at once, using suitable commutator relations:

$$\begin{aligned}
&\alpha_7(\xi)^f=\alpha_7(\xi')\alpha_9(\lambda_1\xi'^4)\alpha_{11}(\lambda_2\xi'),\\
&\alpha_6(\xi)^f=\alpha_6(\xi')\alpha_8(\lambda_3\xi'^2)\alpha_{10}(\lambda_4\xi')\alpha_{11}(\lambda_5\xi'^{2\theta+1}),\\
&\alpha_5(\xi)^f=\alpha_5(\xi')\alpha_{12}(\psi(\xi)),\\
&\alpha_4(\xi)^f=\alpha_4(\xi')\alpha_8(\lambda_6\xi'^{2\theta+1})\alpha_9(\lambda_7\xi')\alpha_{11}(\lambda_8\xi'^2),\\
&\alpha_3(\xi)^f=\alpha_3(\xi')\alpha_8(\lambda_9\xi')\alpha_{10}(\lambda_{10}\xi'^4),\\
&\alpha_2(\xi)^f=\alpha_2(\xi')\alpha_7(\lambda_{11}\xi')\alpha_{11}(\lambda_{12}\xi') \qquad \text{and}\\
&\alpha_1(\xi)^f=\alpha_1(\xi')\alpha_{10}(\lambda_{13}\xi'^{\theta})\alpha_{11}(\lambda_{14}\xi'^{\theta+1}), \qquad \text{for all } \xi\in\Gamma,
\end{aligned}$$

where λ_i's are constants depending on f and $\lambda_1=\lambda_3=\lambda_8=\lambda_{10}=0$ if $q>8$. As for $q=8$, we have $[\alpha_2(\xi)^f,\alpha_7(\eta)^f]=\alpha_{11}(\lambda_1\xi'\eta'^4)$, where $\eta'=\eta^{\chi(f)}$; however, $[\alpha_2(\xi),\alpha_7(\eta)]^f=1^f=1$ so that $\lambda_1=0$ in any case.

Thus, if we continue the calculations using commutator relations in this way, we can prove (6.11) with a little patience but without much difficulty. We omit the details.

COROLLARY (6.11.1) *$C_G(Z)=C$.*

COROLLARY (6.11.2) *$C_A(J)=1$ and $C_{H^*}(K)=C_{H^*}(J)=H$.*

COROLLARY (6.11.3) *$r\in N_G(H)$.*

(6.12) *$r^2=1$.*

PROOF. Since $r^2\in M$ and $r\in N_G(H)$, it follows that $[H,r^2]\subseteq H\cap M=1$ so that $r^2\in C_M(H)\subset C_M(K)=U_5Z$. However, J acts regularly on $Z^{\sharp}$ and every element x in U_5Z-Z has order 4 and $x^2\in Z$, hence it follows that $C_{U_5Z}(J)=1$. Therefore $r^2=1$.

(6.13) *$U_{12}^r=U_{11}$. For all $k=h(1,\varepsilon)\in K$, $[kk^r, U_3M/M]=1$, $[k^r, U_{11}]=1$ and*

(6.13.1) $k^r\alpha_{12}(\xi)k^{-r}=\alpha_{12}(\varepsilon\xi)$

and

(6.13.2) $\alpha_{11}(\xi)^r=\alpha_{12}(\lambda\xi)$ *for all* $\xi\in\Gamma$,

where λ is a constant in Γ defined in (6.3). Since J operates regulary on $U_{11}^{\sharp}=(U_{12}^r)^{\sharp}$, there exists $g\in J$ such that $\alpha_{11}(\xi)^g=\alpha_{11}(\lambda^{-1}\xi)$ for all $\xi\in\Gamma$. Then we may replace r by $t=gr$. In particular, $t^2=1$, t inverts every element in J and $(t\alpha_3(1))^3\in M$. Moreover

(6.13.3) $\alpha_{11}(\xi)^t=\alpha_{12}(\xi)$ *for all* $\xi\in\Gamma$.

PROOF. $U_{12}^r=U_{11}$ is proved in the proof of (6.9). Therefore $[K^r, U_{11}]=[K, U_{11}^r]^r=[K,Z]^r=1$. $[kk^r, U_3M/M]=1$ and (6.13.1) are proved essentially in [8, (6.15)], if we only note using the commutator relation $[\alpha_3(\xi),\alpha_{11}(1)]=\alpha_{12}(\xi)$. And (6.13.2) is proved in [8, (6.16)]. The other assertions are obvious (cf. [8, (6.18)]).

(6.14) *Let j be a generator of J, then there is a generator ω of $\Gamma^\times$ and the following equations hold for all $\xi\in\Gamma$:*

$$\alpha_1(\xi)^j=\alpha_1(\omega\xi),$$
$$\alpha_2(\xi)^j=\alpha_2(\omega^{2\theta+1}\xi),$$
$$\alpha_3(\xi)^j=\alpha_3(\omega^{-2}\xi)\alpha_8(\gamma\omega^{-2}\xi),$$
$$\alpha_4(\xi)^j=\alpha_4(\omega^{-1}\xi)\alpha_9(\gamma\omega^{-1}\xi),$$
$$\alpha_5(\xi)^j=\alpha_5(\omega^{-2\theta+1})\alpha_{12}(\gamma^{2\theta}\omega^{-2\theta+1}\xi+\gamma\omega^{2\theta-2}\xi^{2\theta}),$$
$$\alpha_6(\xi)^j=\alpha_6(\omega^{2\theta-1}\xi)\alpha_{10}(\gamma\omega^{2\theta-1}\xi),$$

$$\begin{aligned}
\alpha_7(\xi)^j &= \alpha_7(\omega^{2\theta}\xi)\alpha_{11}(\gamma\omega^{2\theta}\xi),\\
\alpha_8(\xi)^j &= \alpha_8(\omega^{-2\theta-1}\xi),\\
\alpha_9(\xi)^j &= \alpha_9(\omega^{-2\theta}\xi),\\
\alpha_{10}(\xi)^j &= \alpha_{10}(\xi),\\
\alpha_{11}(\xi)^j &= \alpha_{11}(\omega\xi),\\
\alpha_{12}(\xi)^j &= \alpha_{12}(\omega^{-1}\xi),
\end{aligned}$$

where γ is a constant depending on j.

PROOF. J operates regularly on $U_{11}^{\sharp}$, so if we let

$$\alpha_{11}(1)^j=\alpha_{11}(\omega),$$

then ω must be a generator of $\Gamma^{\times}$, and with the aid of the action of K on U, we have:

$$\alpha_{11}(\xi)^j=\alpha_{11}(\omega\xi) \qquad \text{for all} \quad \xi\in\Gamma.$$

Then

$$\begin{aligned}
\alpha_{12}(\xi)^j &= \alpha_{11}(\xi)^{tj}=\alpha_{11}(\xi)^{j^{-1}t}\\
&= \alpha_{11}(\omega^{-1}\xi)^t=\alpha_{12}(\omega^{-1}\xi).
\end{aligned}$$

Now we will prove under the assumption $q>8$. (The case when $q=8$ is a little more complicated but quite the same.) Then from (6.7) we can write

$$\alpha_6(\xi)^j=\alpha_6(\varepsilon)\alpha_{10}(\beta_1\varepsilon)\alpha_{11}(\gamma_1\varepsilon^{2\theta+1}) \qquad \text{for some} \quad \varepsilon\in\Gamma.$$

Then

$$(\alpha_6(\xi)^j)^2=\alpha_{11}(\varepsilon^{2\theta+1}).$$

On the other hand

$$(\alpha_6(\xi)^2)^j=\alpha_{11}(\xi^{2\theta+1})^j=\alpha_{11}(\omega\xi^{2\theta+1}),$$

so that $\varepsilon=\omega^{2\theta-1}\xi$. Then:

$$\begin{aligned}
(\alpha_6(\xi_1)\alpha_6(\xi_2))^j &= \alpha_6(\xi_1+\xi_2)^j\alpha_{11}(\xi_1\xi_2^{2\theta})^j\\
&= \alpha_6(\omega^{2\theta-1}(\xi_1+\xi_2))\alpha_{10}(\beta_1\omega^{2\theta-1}(\xi_1+\xi_2))\alpha_{11}(\gamma_1\omega(\xi_1+\xi_2)^{2\theta+1}+\omega\xi_1\xi_2^{2\theta})
\end{aligned}$$

and

$$\alpha_6(\xi_1)^j\alpha_6(\xi_2)^j=\alpha_6(\omega^{2\theta-1}(\xi_1+\xi_2))\alpha_{10}(\beta_1\omega^{2\theta-1}(\xi_1+\xi_2))\alpha_{11}(\gamma_1\omega(\xi_1^{2\theta+1}+\xi_2^{2\theta+1})+\omega\xi_1\xi_2^{2\theta}).$$

Therefore we have $\gamma_1(\xi_1+\xi_2)^{2\theta+1}=\gamma_1(\xi_1^{2\theta+1}+\xi_2^{2\theta+1})$ for all $\xi_1,\xi_2\in\Gamma$. Arranging the equation, we have:

$$\gamma_1(\xi_1^{2\theta}\xi_2+\xi_1\xi_2^{2\theta})=0 \qquad \text{for all} \quad \xi_1,\xi_2\in\Gamma.$$

This is impossible, unless $\gamma_1=0$; so that

$$\alpha_6(\xi)^j=\alpha_6(\omega^{2\theta-1}\xi)\alpha_{10}(\beta_1\omega^{2\theta-1}\xi).$$

In the same way, from (6.6) we have:

$$\alpha_5(\xi)^j=\alpha_5(\omega^{-2\theta+1}\xi)\alpha_{12}(\varphi(\xi)),$$

where φ is an additive homomorphism from Γ to Γ. Then

$$\begin{aligned}\alpha_{10}(\xi)^j&=[\alpha_5(1),\alpha_6(\xi)]^j\\&=[\alpha_5(\omega^{-2\theta+1})\alpha_{12}(\varphi(1)),\alpha_6(\omega^{2\theta-1}\xi)\alpha_{10}(\beta_1\omega^{2\theta-1}\xi)]\\&=\alpha_{10}(\xi).\end{aligned}$$

And from the commutator relation $[\alpha_5(1),\alpha_7(\xi)]=\alpha_{11}(\xi)$, $[\alpha_6(1),\alpha_9(\xi)]=\alpha_{12}(\xi)$ and by (6.5) (iii) and (v), we have:

$$\alpha_7(\xi)^j=\alpha_7(\omega^{2\theta}\xi)\alpha_{11}(\beta_2\omega^{2\theta}\xi)\qquad\text{and}\qquad\alpha_9(\xi)^j=\alpha_9(\omega^{-2\theta}\xi).$$

Then from $[\alpha_7(1),\alpha_8(\xi)]=\alpha_{12}(\xi)$, we have:

$$\alpha_8(\xi)^j=\alpha_8(\omega^{-2\theta-1}\xi).$$

Thus we can continue the calculation using commutator relations and we can complete the proof without difficulty. We omit the details.

(6.15) *We may assume that H^* normalizes U_i, where index i ranges from* 1 *to* 12, *and then if j is a generator of J, j acts on U_i as follows*:

$$\begin{array}{ll}\alpha_1(\xi)^j=\alpha_1(\omega\xi), & \alpha_2(\xi)^j=\alpha_2(\omega^{2\theta+1}\xi),\\ \alpha_3(\xi)^j=\alpha_3(\omega^{-2}\xi), & \alpha_4(\xi)^j=\alpha_4(\omega^{-1}\xi),\\ \alpha_5(\xi)^j=\alpha_5(\omega^{-2\theta+1}\xi), & \alpha_6(\xi)^j=\alpha_6(\omega^{2\theta-1}\xi),\\ \alpha_7(\xi)^j=\alpha_7(\omega^{2\theta}\xi), & \alpha_8(\xi)^j=\alpha_8(\omega^{-2\theta-1}\xi),\\ \alpha_9(\xi)^j=\alpha_9(\omega^{-2\theta}\xi), & \alpha_{10}(\xi)^j=\alpha_{10}(\xi),\\ \alpha_{11}(\xi)^j=\alpha_{11}(\omega\xi), & \alpha_{12}(\xi)^j=\alpha_{12}(\omega^{-1}\xi).\end{array}$$

And if $f\in A$, then:

(6.15.1) $$\alpha_i(\xi)^f=\alpha_i(\xi^{\chi(f)})$$

for all $\xi\in\Gamma$, where $i=1,\dots,12$.

PROOF. Let $H_1^*=(H^*)^u$, where $u=\alpha_5(\gamma(1+\omega^{-2\theta+1})^{-1})$, and let $J_1=H_1^*\cap L=(H^*\cap L)^u=J^u$, $A_1=A^u$, $j_1=j^u$ and $t_1=t^u$. Then it is an immediate consequence of (6.14) that J_1 normalizes all U_i $(i=1,\dots,12)$ and j_1 acts on U_i according to the formula in (6.15). Since $C_U(K)=U_5Z$, H_1^* also contains K and is a complement of U in $N_G(M)$.

In order to prove (6.15.1), we first determine the constant λ_f in (6.11). As

$J \lhd H^*$, $f^{-1}jf = j^a$ for some integer a and then $\alpha_7(\xi)^{jf} = \alpha_7(\xi)^{fj^a}$. From (6.11) and (6.14), we have:

$$\alpha_7(\xi)^{jf} = \alpha_7(\omega'^{2\theta}\xi')\alpha_{11}(\lambda_f\omega'^{2\theta}\xi' + \gamma'\omega'^{2\theta}\xi'),$$

and

$$\alpha_7(\xi)^{fj^a} = \alpha_7(\omega_1^a\xi')\alpha_{11}(\gamma\xi'(\omega_1^a + \omega_1^{a-1}\omega + \cdots + \omega_1^{a-i}\omega^i + \cdots + \omega_1\omega^{a-1}) + \lambda_f\xi'\omega^a),$$

where $\omega' = \omega^{\chi(f)}, \xi' = \xi^{\chi(f)}, \gamma' = \gamma^{\chi(f)}$ and $\omega_1 = \omega^{2\theta}$. Therefore

$$\omega' = \omega^a, \qquad \text{and} \qquad \lambda_f(\omega^{2\theta} + \omega)^a = \gamma'\omega_1^a + \gamma(\omega_1^a + \omega_1^{a-1}\omega + \cdots + \omega_1\omega^{a-1}),$$

where $\omega^{2\theta} + \omega \neq 0$, since ω is a generator of $\Gamma^\times$.

Then an easy computation shows:

$$\alpha_i(\xi)^{u^{-1}fu} = \alpha_i(\xi^{\chi(f)})$$

for all $i = 1, \ldots, 12$.

Thus $A_1 = A^u$ normalizes all U_i $(i = 1, \ldots, 12)$ and hence so does $H_1^* = A_1J_1K$.

It is readily verified that H_1^*, A_1, J_1, j_1 and t_1 satisfy all the properties of H^*, A, J, j and t which have been proved so far.

Henceforth we will write H^*, A, J, j, t instead of H_1^*, A_1, J_1, j_1 and t_1.

(6.16) *Let t be an involution defined as r in* (6.3) *and adjusted properly in* (6.13) *and* (6.15). *Then t acts on M as follows*:

$$(\alpha_1(\xi)\alpha_2(\eta))^t = \alpha_4(\xi)\alpha_8(\eta), \qquad \alpha_7(\xi)^t = \alpha_9(\xi),$$
$$(\alpha_5(\xi)\alpha_{12}(\eta))^t = \alpha_6(\xi)\alpha_{11}(\eta), \qquad \alpha_{10}(\xi)^t = \alpha_{10}(\xi)$$

for all $\xi, \eta \in \Gamma$; and $(t\alpha_3(1))^3 = 1$.

Proof. Since H^* normalizes all U_i $(i = 1, \ldots, 12)$, U_i^t is also normalized by H^*. Therefore we can apply (6.5) and, since the action of J on U is known, much stronger results are easily obtained. Thus using the characteristic subgroups of M (see (3.7)) and $\alpha_{12}(\eta)^t = \alpha_{11}(\eta)$ (see (6.13)), we can prove (6.16) by arguments similar to those used in (6.11) or (6.14). But the proof is tedious and rather straightforward, so we omit the proof of the action of t on M. For the other assertion, see the proof of [8, (6.18)].

(6.17) $N_G(M) = H^*U \cup H^*UtU_3$. *And if we set $Q = HU \cup HUtU_3$, then Q is a normal subgroup of $N_G(M)$ and $N_G(M) = QA$ where $Q \cap A = 1$. Moreover $Q \cong P_2$, where P_2 is a parabolic subgroup of ${}^2F_4(q)$ corresponding to r_2.*

Proof. This is an immediate consequence of (6.15) and (6.16) (cf. [8, (6.18)]).

Henceforth we identify $H=J\times K$ with "torus" H of ${}^2F_4(q)$ defined in (2.1).

Now we conclude this section with the following two propositions. First the proof of [2, 5.14] yields:

(6.18) $[A, \langle t, w\rangle]=1$.

(6.19) $w\in N_G(H^*)\cap N_G(H)$.

PROOF. We have already shown that $H^{*wz}=H^*$ for some $z\in U_6U_{12}$. Since $H=C_{H^*}(K)$ and wz normalizes K, it follows that $H^{wz}=H$. It is easily checked from the action of H and w on D that $C_H(wz)=C_H(U_1U_2)=C_H(U_2)=\{h(\varepsilon,1)|\varepsilon\in\Gamma^\times\}$. Moreover from the structure of ${}^2B_2(q)$, we have $(\alpha_2(1)w)^5=1$ and hence:

$$(\alpha_2(1)wz)^5=(\alpha_2(1)w)^5z^5=z^5=z.$$

Therefore $C_H(U_2)$ centralizes z, so that $z\in C_{U_6Z}(C_H(U_2))=1$, i.e. $z=1$.

VII. Identification of G

(7.1) *G has exactly two classes of involutions.*

PROOF. Since the structure of $N_G(M)$ has been determined in section VI, it is clear from (3.4) that G has at most two classes of involutions and every involution in G is conjugate to either $\alpha_{10}(1)$ or $\alpha_{12}(1)$. However $\alpha_{10}(1)\not\sim_G\alpha_{12}(1)$ has been proved in (4.10).

From now on the involution which is conjugate to τ is called central or a central involution and the one which is conjugate to $\alpha_{10}(1)$ is called non-central or a non-central involution.

(7.2) $N_G(M)=N_G(W)=N_G(Y)$.

PROOF. This follows immediately from (3.7), $C_G(Y)=C_C(Y)\cap C_G(\tau)=M$ and the fact that every central involution in W is contained in Y.

(7.3) *No involution in $C_U(\alpha_{10}(1))-Y$ is conjugate in $C_G(\alpha_{10}(1))$ to an involution of Y.*

PROOF. Set $x=\alpha_{10}(1)$ and $T=C_U(x)=U_2U_3\mathfrak{U}_5$. Every central involution in $T-Y$ is conjugate in $C_G(x)$ to $u=\alpha_8(1)\alpha_9(\xi)\alpha_{11}(\xi^{2\theta+2})$ for some $\xi\in\Gamma$. Now we will consider two cases.

Case 1. $\xi=0$. Then $u^{wt}\in Z$ and an argument similar to that used in the proof of [8, (8.3)] yield that $C_G(x)\cap C_G(u)$ is 2-closed with $T_1=U_3U_5U_6X$ as its unique Sylow 2-subgroups. On the other hand, every involution contained in Y is conjugate in $C_G(x)$ to τ and $C_G(x)\cap C_G(\tau)=C_C(x)A=TA$. T is not isomorphic to T_1, since for instance $|T|=q^{10}$ but $|T_1|=q^8$. Therefore u is not conjugate to τ in $C_G(x)$, and hence to any involution in Y.

Case 2. $\xi\neq 0$. In this case, $u^{wvwt}\in Z$ where $v=\alpha_2(\xi^{-2\theta-1})\alpha_7(\xi^{-1})$ and the same argument as in Case 1 can also be applied.

(7.4) $C_G(\alpha_{10}(1))$ *is contained in* $N_G(M)$.

PROOF. We will use the same notation x, T and T_1 as in (7.3). Let g be an element of $C_G(x)$ and let $y=\alpha_8(1)$; then y is a central involution contained in $T-Y$. Then from (7.3) it follows that y^g is not conjugate to τ in $C_G(x)$, so that there exists an involution e in $C_G(x)$ such that $[\tau, e]=[y^g, e]=1$ and $e\tau$ is conjugate to either τ or y^g in $C_G(x)$. Therefore e is contained in both T and T_1^g, since T [resp. T_1^g] is a unique Sylow 2-subgroup of $C_G(x)\cap C_G(\tau)$ [resp. $C_G(x)\cap C_G(y^g)$]. At this point we consider two cases.

Assume first that $e\tau$ is conjugate to τ in $C_G(x)$. Since $e\tau\in T$, it follows from (7.3) that $e\tau\in Y$ and hence $e\in Y$. It is known from (7.3) that no central involutions in T_1-Y can be conjugate in $C_G(x)$ to an element of Y; therefore it follows also that $e\in Y^g$ and hence $e^{g^{-1}}\in Y$. There is an element h of $N_G(M)\cap C_G(x)$ such that $e^{g^{-1}h}\in Z$. Then, as $e\in Y$, we can conclude that $Y^{g^{-1}h}\subseteq O^{2'}(C_G(z))=C$, where $z=e^{g^{-1}h}\in Z$ and hence $Y^{g^{-1}h}\subseteq C\cap C_G(x)=TA$. Since TA is 2-closed, $Y^{g^{-1}h}\subseteq T$, and from (7.3) we have $Y^{g^{-1}h}=Y$, i.e. $g^{-1}h\in N_G(Y)=N_G(M)$, so that $g\in N_G(M)$.

Next consider the case when $e\tau$ is conjugate to y^g. Then by (7.3) $e\tau\in T-Y$ and hence $e\in T-Y$ also. Since $e\tau$ is a central involution and $e\in T-Y$, it follows from the classification of involutions of $T-Y$ that either e is a central involution or a non-central involution such that $e=\alpha_2(\xi_2)\prod_{i=6}^{12}\alpha_i(\xi_i)$ $(\xi_2\neq 0)$. If e is central, then e is conjugate in $C_G(x)\cap N_G(M)$ to one of the following element:

$$u_1=\alpha_8(1)$$
$$u_2=\alpha_8(1)\alpha_9(\xi)\alpha_{11}(\xi^{2\theta+2}) \qquad \text{for some} \quad \xi\in\Gamma^\times.$$

If $e^{h'}=u_1$ for some $h'\in C_G(x)\cap N_G(M)$, then $u_1\in T_1^{gh'}$, since $e\in T_1^g$. It is easily verified that X is the unique maximal elementary abelian subgroup of T_1 which contains every central involution of T_1. Therefore $u_1\in X^{gh'}$ and moreover $X^{gh'}\subseteq C_G(x)\cap C_G(u_1)$ and hence from the proof of (7.3) it follows that $X^{gh'}\subseteq T_1$. Then the uniqueness property of X mentioned above implies $X=X^{gh'}$ and so from (7.3) we obtain $Y=Y^{gh'}$ and hence $g\in N_G(M)$. If $e^{h''}=u_2$ for some h'' in $C_G(x)\cap N_G(M)$, then by the same discussions used above we have that $D\supseteq T_2\supseteq X$ and $T_2\supset X^{gh''}$ where T_2 is the unique Sylow 2-subgroup of $C_G(x)\cap C_G(u_2)$. The uniqueness property of X in T_1 mentioned above is also true in D. Therefore $X=X^{gh''}$ and hence $g\in N_G(M)$.

Thus we have proved:

(7.4.1) *For every element g in $C_G(x)$, we can construct an involution e which is contained in both T and T_1^g; g is contained in $N_G(M)$, unless e is a non-central involution of the form*

$$e=\alpha_2(\xi_2)\prod_{i=5}^{12}\alpha_i(\xi_i) \qquad (\xi_2\neq 0)$$

with the condition that $e\tau$ is a central involution.

Now to complete the proof, assume that there is an element g in $C_G(x)-N_G(M)$. Then the involution e obtained in the above argument must be of the form $e=\alpha_2(\xi_2)\alpha_7(\xi_7)\alpha_{12}(\xi_{12})$ where $\xi_2\neq 0$ and $1=\xi_{12}+\xi_7^{2\theta+2}\xi^{-2\theta-1}$, since we can replace g by gv for any v in $T=C_G(x)\cap C$, if necessary. Then $e\in Q\cap T_1^g$ where $Q=U_2\mathfrak{U}_5$. Hence $e^t\in Q\cap T_1^{gt}$, since $Q^t=Q$. As $t\in C_G(x)\cap N_G(M)$, gt is also contained in $C_G(x)-N_G(M)$ and for gt we can construct a non-central involution e' such that $e'=\alpha_2(\eta_2)\prod_{i=5}^{12}\alpha_i(\eta_i)$ $(\eta_2\neq 0)$ and $e'\in Q\cap T_1^{gt}$. Then:

$$[e^t, e]=\alpha_{10}(\xi_2\eta_2)\alpha_{11}(\xi_7\eta_2+\xi_2\eta_2^{2\theta})\alpha_{12}(\cdots),$$

and hence $[e^t, e']$ is non-central. On the other hand, every non-central involution in T_1 is contained in either V or X and $[V\cup X, V\cup X]\subseteq Z$. This is a contradiction. Thus we have proved (7.4).

COROLLARY (7.4.2) $C_G(x)=C_Q(x)\cdot A$ *where* $C_Q(x)=JT\cup JTtU_3=O^{2'}(C_G(x))$.

COROLLARY (7.4.3) *For every element x' in $\mathfrak{U}_{10}-Y$, $C_G(x')\subset N_G(M)$.*

PROOF. Since x' is conjugate to x in $N_G(M)$, this can be immediately derived from (7.4).

(7.5) $N_G(D)=N_G(Z)=UH^*\cup UH^*wU_1U_2$,

PROOF. Since $U\subset C\lhd N_G(Z)$, it follows that

$$N_G(Z)=N_G(U)\cdot C=H^*U\cup UH^*wU_1U_2,$$

which is contained in $N_G(D)$. On the other hand, as $Z(D)=Z$, we have $N_G(Z)\supseteq N_G(D)$.

(7.6) *Let Z_1 be a conjugate of Z. Then either $Z_1\subset M$ or $Z_1\cap N_G(M)=1$; and either $Z_1\subset D$, $Z_1\subset N_G(D)-D^\sharp$ or $Z_1\cap N_G(D)=1$.*

PROOF. If $Z_1 \cap M \neq 1$, then one of the following four cases must occur.

(i) $Z_1 \cap Z \neq 1$,

(ii) $Z_1 \cap Y \neq 1$ and $Z_1 \cap Z = 1$,

(iii) $Z_1 \cap X \neq 1$ and $Z_1 \cap Y = 1$,

(iv) $Z_1 \cap M \neq 1$ and $Z_1 \cap X = 1$.

In each case we can prove that $Z_1 = Z, Z_1 \subset Y, Z_1 \subset X$ and $Z_1 \subset M$ respectively. In fact, we can use (6.10.2) to prove $Z = Z_1$ in case (i) and the other cases can be proved easily by reducing to case (i) or to the preceding case (see the proof of (7.8)).

Suppose that $Z_1 \cap M = 1$ and $Z_1 \cap N_G(M) \neq 1$. Let z' be a non-identity element in $Z_1 \cap N(M)$ and P be a Sylow 2-subgroup of $N_G(M)$ such that $P \ni z'$. Then $P^v = U$ for some v in $N_G(M)$ and $P \supset M$. Since $Z_1 \cap M = 1$, z'^v must be contained in $U - M$, which contradicts the fact that every central involution in U is contained in M. Therefore, if $Z_1 \cap M = 1$, then $Z_1 \cap N_G(M) = 1$.

By the same argument for M, we can prove that if $Z_1 \cap D \neq 1$, then $Z_1 \subset D$. Next suppose that $Z_1 \cap D = 1$ and $Z_1 \cap N_G(D) \neq 1$. Let z'' be a non-identity element in $Z_1 \cap N_G(D)$ and P_1 be a Sylow 2-subgroup of $N_G(D)$ such that $P_1 \ni z''$. If $P_1 = U$, then $Z_1 \cap M \neq 1$ and hence $Z_1 \subset M \subset U$. Thus we have $Z_1 \subset U - D^\sharp \subset N_G(D) - D^\sharp$ as $Z_1 \cap D = 1$. And if $P_1 \neq U$, then from (7.5) it follows that there exists an element v in $U_1 U_2$ such that $P_1^{vw} = U$; and hence by the preceding argument we have $Z_1^{vw} \subset U - D^\sharp$ so that $Z_1 \subset N_G(D) - D^\sharp$ as $vw \in N_G(D)$.

(7.7) *Every conjugate Z_1 of Z centralizes at least one conjugate of Z contained in M.*

PROOF. Assume first that $Z_1 \subset N_G(D)$. Then Z_1 centralizes Z, since $N_G(D) = N_G(Z), O^{2'}(N_G(Z)) = C$ and $Z(C) = Z$.

Next assume that $N_G(D) \cap Z_1 = 1$. Let z be an element in Z_1, then z is not conjugate to $x = \alpha_{10}(1)$ in G. Hence there exists an involution e such that $[e, z] = [e, x] = 1$ so that $e \in C_G(x) \subset N_G(M)$. (cf. (7.4).)

If e is central and Z_2 is a conjugate of Z such that $Z_2 \ni e$, then $[Z_1, Z_2] = 1$ and $Z_2 \subset M$ by (7.6), which proves the assertion.

If e is non-central, then one of the following seven cases must occur:

Case (i). $e \in \mathfrak{U}_{10} - Y$. Then from (7.4.3) we have $z \in C_G(e) \subset N_G(M)$ and hence from (7.6) $z \in M \subset N_G(D)$. But this contradicts the assumption that $Z_1 \cap N_G(D) = 1$ so that this case cannot happen.

Case (ii). $e \in \mathfrak{U}_9 - \mathfrak{U}_{10}$. Then $e^{uw} \in \mathfrak{U}_{10} - Y$ for some properly chosen element u in

U_1U_2 and so $z^{uw}\in C_G(e^{uw})\subset N_G(M)$. Hence $z^{uw}\in M\subset N_G(D)$ so that $z\in N_G(D)$ as $uw\in N_G(D)$. But this is again a contradiction and hence this case does not occur.

Case (iii). $e\in \mathfrak{U}_8-\mathfrak{U}_9$. Then $e^{uw}\in \mathfrak{U}_9-\mathfrak{U}_{10}$ for some u in U_1U_2. But this is clearly impossible from case (ii).

Case (iv). $e\in \mathfrak{U}_7-\mathfrak{U}_8$. Then $e\in W$ and $e^{vt}\in \mathfrak{U}_9-\mathfrak{U}_{10}$ for some $v\in U_3$. From the discussions in case (ii), it follows that $z^{vt}\in N_G(D)$ and hence by the argument of the case when $Z_1\cap N_G(D)\neq 1$ we have that Z_1^{vt} centralizes Z so that Z_1 centralizes $Z^{tv}\subset Y-Z$.

Case (v). $e\in \mathfrak{U}_3-\mathfrak{U}_7$. Then $e^{uw}\in W-\mathfrak{U}_9$ for some $u\in U_1U_2$. Hence from case (iv) we obtain that Z_1^{uw} centralizes Z_2 which is a conjugate of Z and contained in $Y-Z$. Therefore Z_1 centralizes $Z_2^{wu^{-1}}$ which is contained in $X-Y$.

Case (vi). $e\in \mathfrak{U}_2-\mathfrak{U}_3$. Then $e^{vt}\in \mathfrak{U}_8-\mathfrak{U}_9$ for some $v\in U_3$. Therefore from cases (ii) and (iii) it follows that $z^{vt}\in N_G(D)$ and hence the argument in case (iv) implies that Z_1 centralizes $Z^{tv}\subset Y-Z$.

Case (vii). $e\in N_G(M)-U$. Then $e^{vt}\in \mathfrak{U}_3-\mathfrak{U}_7$ for some $v\in U_3$. Hence we obtain from case (v) that Z_1^{vt} centralizes a conjugate Z_2 of Z contained in $X-Y$ and hence Z_1 centralizes Z_2^{tv} which is contained in M.

Thus we have completed the proof of (7.7).

(7.8) *G is the disjoint union of the following* 16 *double cosets of B where* $B=N_G(U)=H^*U$:

$$B,\quad BwU_1U_2,\quad BtU_3,\quad BwtU_3U_4U_8,\quad BtwU_1U_2U_7,\quad BwtwU_1U_2U_6U_7U_{11},$$
$$BtwtU_3U_4U_8U_9,\quad B(wt)^2U_3U_4U_5U_8U_9U_{12},\quad B(tw)^2U_1U_2U_6U_7U_{10}U_{11},$$
$$B(wt)^2wU_1U_5U_6V_1,\quad B(tw)^2tU_4U_5V_2,\quad B(wt)^3U_3U_4U_5U_6X,$$
$$B(tw)^3U_1U_2U_6U_6W,\quad B(wt)^3wM,\quad B(tw)^3tD,\quad B(wt)^4U.$$

Each of these double cosets has form $B\omega U_\omega$ where ω is in the set $\Pi=\{1,t,w,\dots,(wt)^4\}$ and U_ω is a complement to $B\cap B^\omega$ in B.

PROOF. We prove that G can be expressed as the union of these 16 subsets of G. Then nearly the same arguments as in the proofs of [8, (9.4) and (9.5)] yield that these subsets are really double cosets of the form $B\omega U_\omega$ and the sum is disjoint.

Taking an arbitrary element g in G, we consider Z^g.

Suppose first that $Z^g\subset M$.

If $Z\cap Z^g\neq 1$, then $Z=Z^g$ and hence $g\in N_G(Z)=B\cup BwU_1U_2$ (cf. (7.5)).

If $Z^g \cap Y \neq 1$ and $Z^g \cap Z = 1$, then there exists an element v in U_3 such that $Z^{gvt} \cap Z \neq 1$. Therefore $g \in BtU_3 \cup BwtU_1^t U_2^t U_3 = BtU_3 \cup BwtU_4 U_8 U_3$.

If $Z^g \cap X \neq 1$ and $Z^g \cap Y = 1$, then $Z^{guw} \cap Y \neq 1$ for some $u \in U_1 U_2$. Thus

$$g \in BtwU_7 U_1 U_2 \cup BwtwU_6 U_{11} U_7 U_1 U_2.$$

If $Z^g \cap M \neq 1$ and $Z^g \cap X = 1$, then there exists an element v in U_3 such that $Z^{gvt} \cap X \neq 1$. Hence

$$g \in BtwtU_9 U_4 U_8 U_3 \cup B(wt)^2 U_5 U_{12} U_9 U_4 U_8 U_3.$$

Next suppose that $Z^g \cap U = 1$ and $Z^g \subset N_G(D) - D^{\sharp}$. Then there exists an element u in $U_1 U_2$ such that $Z^{guw} \cap M \neq 1$ and $Z^{guw} \cap X = 1$. Hence

$$g \in B(tw)^2 U_{10} U_6 U_{11} U_7 U_1 U_2 \cup B(wt)^2 w U_5 U_{12} U_{10} U_6 U_{11} U_7 U_1 U_2.$$

Finally suppose that $Z^g \cap N_G(D) = 1$. Then from (7.7) Z^g centralizes a conjugate Z_1 of Z contained in M. Z^g does not centralize Z, since $C_G(Z) = C = O^{2'}(N_G(D))$ and $Z^g \cap N_G(D) = 1$.

If Z^g centralizes $Z_1 \subset Y - Z$, then Z^{gvt} centralizes Z where v is an element in U_3 and hence $Z^{gvt} \subset C - M^{\sharp}$, which implies that $Z^{gvt} \subset N_G(D) - D^{\sharp}$ and $Z^{gvt} \cap U = 1$. Hence

$$g \in B(tw)^2 t U_{10} U_5 U_{12} U_9 U_4 U_8 U_3 \cup B(wt)^3 U_6 U_{11} U_{10} U_5 U_{12} U_9 U_4 U_8 U_3\,.$$

If Z^g centralizes $Z_1 \subset X - Y$, then Z^{guw} centralizes $Z_1^{uw} \subset Y - Z$ for some $u \in U_1 U_2$ and $Z^{guw} \cap N_G(D) = 1$ as $uw \in N_G(D)$. Hence

$$g \in B(tw)^3 U_9 U_5 U_{12} U_{10} U_6 U_{11} U_7 U_1 U_2 \cup B(wt)^3 wM.$$

If Z^g centralizes $Z_1 \subset M - X$, then Z^{gvt} centralizes $Z_1^{vt} \subset X - Y$ for some $v \in U_3$ and we can prove that $Z^{gvt} \cap N_G(D) = 1$. In fact, suppose on the contrary that $Z^{gvt} \cap N_G(D) \neq 1$, then $Z^{gvt} \subseteq C \cap C_G(Z_1^{vt}) \subseteq D \subset N_G(M)$ and hence $Z^g \subset M$ as $vt \in N_G(M)$ and clearly this is a contradiction. Consequently applying the above argument, we have

$$g \in B(tw)^3 tD \cup B(wt)^4 U.$$

Thus all the possible conditions which are satisfied by Z^g have been considered. Therefore we have completed the proof of (7.8).

The proof of [8, (9.6) and (10.1)] yield:

(7.9) *If $\omega \in \Pi = \{1, t, w, \dots, (wt)^4\}$ then each element of $B\omega U_\omega$ can be written uniquely in the form $b\omega u$ where $b \in B$ and $u \in U_\omega$. Hence*

$$|G|=\rho q^{12}(q-1)(q^3+1)(q^4-1)(q^6+1).$$

(7.10) $(wt)^8=1$ *and hence* $\Pi=\langle w,t\rangle$ *is a dihedral group of order* 16.

Let $B_0=H\cdot U$ and $G_0=\bigcup_{\omega\in\Pi} B_0\omega U_\omega$. Then:

(7.11) (i) G_0 *is a normal subgroup of* G *and* $G_0\cong{}^2F_4(q)$.

(ii) $G=AG_0$ *and* $A\cap G_0=1$ *so that* $[G: G_0]=\rho$.

(iii) A *operates on* G_0 *faithfully and every element of* A *acts on* G_0 *as an automorphism induced from the automorphism of* Γ.

PROOF. Since the structure of G_0 has been uniquely determined and $|G_0|=|{}^2F_4(q)|$, it is easy to verify directly that G_0 is isomorphic to ${}^2F_4(q)$ by the correspondence of t to r_2. The other assertion is clear from section VI and (7.8).

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