# *On the structure of polarized varieties with $\Delta$-genera zero*

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**Introduction**

In this paper we determine the structure of every polarized variety $(V, L)$ such that $\dim H^0(V, \mathcal{O}_V(L)) = \dim V + d(V, L)$. Here by a polarized variety is meant a pair $(V, L)$ consisting of an irreducible reduced compact complex space $V$ and an ample (not necessarily very ample) line bundle $L$ on $V$, and $d(V, L)$ is the Chern number $(c_1(L))^n\{V\}$, where $n = \dim V$ (we remark that if $L$ is the bundle attached to a hyperplane section of a subvariety $V$ in a projective space, then $d(V, L)$ is the degree of $V$). Our results seem to be natural generalizations of those of Nagata [14] and Swinnerton-Dyer [16].

In Section 1 we shall introduce the concept of $\Delta$-genus of $(V, L)$, which is defined to be $\Delta(V, L) = n + d(V, L) - \dim H^0(V, L)$, and shall state the key lemma: $\dim Bs|L| < \Delta(V, L)$, where $Bs|L|$ denotes the set of base points of $|L|$. Section 6 is devoted to the proof of this inequality. In Section 2 we shall establish theorems characterizing projective spaces and hyperquadrics, which are quite analogous to those of Kobayashi-Ochiai [10]. In Section 3 we shall study the structure of polarized manifolds of $\Delta$-genus zero. In Section 4 we shall give a structure theorem for singular polarized varieties of $\Delta$-genus zero. Section 5 is devoted to the study of (global) deformations of such polarized varieties.

The theory of $\Delta$-genus itself has a much wider range of applications (see [1], [2], [3] and forthcoming papers of the present author). For example, it furnishes us with a somewhat systematic method for characterizing several types of polarized varieties such as

a) complete intersections in projective spaces with $\Delta$-genera $\leq 3$,

b) projective fiber bundles over $\boldsymbol{P}^1$,

c) two-sheeted cyclic branched coverings of $\boldsymbol{P}^n$ or hyperquadrics, etc.

Moreover, using this method, we can determine the structure of some kinds of polarized manifolds such as

d) polarized $K3$-surfaces $(M, L)$ with $d(M, L) \leq 8$,

e) polarized $K3$-surfaces $(M, L)$ with not very ample $L$,

f) polarized surfaces $(M, L)$ such that $L=K_M$ and $d(M, L)=c_1^2\leqq 2p_g-3$, etc.

In this article, however, we shall deal only with the simplest cases. Nevertheless the results are very fundamental since they will play an essential role at various stages of the applications mentioned above.

## Notation, Convention and Terminology

Basically we employ a notation analogous to that of EGA [4], and [5]. A variety is an irreducible reduced complex analytic space, and is assumed to be compact unless otherwise explicitly stated. A non-singular variety is called a manifold. We don't discriminate a vector bundle from the locally free sheaf associated with it.

Let $\mathcal{F}$ be an analytic coherent sheaf on an analytic space $S$.

$\mathrm{Sing}(S)$: The set of singular points of $S$, which turns out to be an analytic subset of $S$.

$h^p(S, \mathcal{F}):=\dim H^p(S, \mathcal{F})$, which is finite if $S$ is compact, then

$\chi(S, \mathcal{F}):=\sum_{p=0}^{n}(-1)^p h^p(S, \mathcal{F})$, where $n=\dim S$.

$h^p(S):=h^p(S, I_S)$, where $I_S$ is the trivial line bundle of $S$.

$\boldsymbol{P}_S(\mathcal{F})$: The projective fiber space $\mathrm{Proj}(\bigoplus_{k=0}^{\infty} S^k\mathcal{F})$ over $S$ (see EGA II).

$H(\mathcal{F})$: The tautological (ample) line bundle on $\boldsymbol{P}_S(\mathcal{F})$.

We remark that if $E$ is a vector space with $(\alpha_0, \cdots, \alpha_r)$ being a coordinate system of its dual space, then $\boldsymbol{P}(E)$ is in a canonical one-one correspondence with the set of ratios $\{(\alpha_0: \cdots :\alpha_r)\}$. Then $H_\alpha=H(E)$ turns out to be the line bundle on $\boldsymbol{P}_\alpha\simeq\boldsymbol{P}^r$ defined by a hyperplane.

$Q_C(S)$: The monoidal transform of $S$ with center $C$.

$\{Z\}$: The homology class associated with an analytic cycle $Z$.

$[\Lambda]$: The line bundle associated with a linear system $\Lambda$ of Cartier divisors on $S$.

$Bs\Lambda$: The set of base points of $\Lambda$.

$\rho_\Lambda$: The rational mapping associated with $\Lambda$, which turns out to be a morphism if $Bs\Lambda=\emptyset$.

$|L|$: The complete linear system of Cartier divisors associated with a line bundle $L$ on $S$.

$L_T$, $\Lambda_T$: The pull back of $L$, $\Lambda$ to a space $T$ by a given morphism $T\to S$.

We use additive notation for tensor products of line bundles, and use multiplicative notation for cup products of their Chern classes.

$K_M$: The canonical line bundle of a manifold $M$.

$b_i(M)$: The $i$-th Betti number of $M$.
$c_i(M)$: The $i$-th Chern class of $M$.
We often use an abbreviated form of this notation, e.g., $L$ instead of $L_T$, $H^p(\mathcal{F})$ instead of $H^p(S, \mathcal{F})$, when there is no danger of confusion.

## § 1. $\Delta$-genera of polarized varieties

DEFINITION 1.1. We call a pair $(V, L)$ consisting of a variety $V$ and a line bundle $L$ on $V$ a *prepolarized variety*. $(V, L)$ is said to be isomorphic to $(V', L')$ if there is a biholomorphic morphism $f: V\to V'$ such that $L\cong f^*L'$. A prepolarized variety $(V, L)$ is called a polarized variety if $L$ is ample.

DEFINITION 1.2. Expanding the Hilbert polynomial $\chi(V, tL)$[1] into the following form $\sum_{j=0}^{n}\chi_j(V, L)t^{[j]}/j!$ with $n=\dim V$ and $t^{[j]}=\prod_{\alpha=0}^{j-1}(t+\alpha)$, we define the following two invariants of a prepolarized variety $(V, L)$: $d(V, L)=\chi_n(V, L)$ and $g(V, L)=1-\chi_{n-1}(V, L)$.

REMARK. The Riemann-Roch Theorem states that $d(V, L)=L^n=(c_1(L))^n\{V\}$. Moreover, if $V$ is non-singular, $2g(V, L)-2=(K_V+(n-1)L)L^{n-1}$.

PROPOSITION 1.3. *Let $(V, L)$ be a prepolarized variety and let $D$ be an irreducible reduced member of $|L|$. Then $\chi_r(D, L_D)=\chi_{r+1}(V, L)$ for $r\geqq 0$. In particular, $d(D, L_D)=d(V, L)$ and $g(D, L_D)=g(V, L)$.*

PROOF. The exact sequence $0\to\mathcal{O}_V((t-1)L)\to\mathcal{O}_V(tL)\to\mathcal{O}_D(tL_D)\to 0$ yields the equality $\chi(D, tL)=\chi(V, tL)-\chi(V, (t-1)L)$. Our conclusion follows from this after some elementary calculations.

DEFINITION 1.4. The *$\Delta$-genus* of a prepolarized variety $(V, L)$ is defined to be $\dim V+d(V, L)-h^0(V, L)$, and is denoted by $\Delta(V, L)$.

PROPOSITION 1.5. *Let $(V, L)$ and $D$ be as in Proposition 1.3. Then $0\leqq\Delta(V, L)-\Delta(D, L_D)\leqq h^1(V)\leqq h^1(D)+h^1(V, -L)$. Moreover, the following conditions are equivalent to each other: a) $\Delta(D, L_D)=\Delta(V, L)$, b) $H^0(V, L)\to H^0(D, L_D)$ is surjective, c) $|L|_D=|L_D|$.*

PROOF. The exact sequence $0\to\mathcal{O}_V\to\mathcal{O}_V(L)\to\mathcal{O}_D(L)\to 0$ yields the long exact sequence $0\to H^0(V)\to H^0(V, L)\to H^0(D, L)\to H^1(V)$. The equivalence of a), b) and c) follows from this since $\dim\operatorname{Coker}(H^0(V, L)\to H^0(D, L))=\Delta(V, L)-\Delta(D, L)$. Moreover we have $0\leqq\Delta(V, L)-\Delta(D, L)\leqq h^1(V)$. The remaining inequality is obtained

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[1] Using the desingulization theory of Hironaka, we can show that $\chi(V, tL)$ is a polynomial in $t$ for any (possibly non-algebraic) prepolarized variety $(V, L)$.

similarly from the exact sequence $0 \to \mathcal{O}_V(-L) \to \mathcal{O}_V \to \mathcal{O}_D \to 0$.

DEFINITION 1.6. We say that a line bundle $L$ on a variety $V$ is *fully generating* if the canonical homomorphism $m_t: H^0(V, tL) \otimes H^0(V, L) \to H^0(V, (t+1)L)$ is surjective for every $t \geq 1$.

PROPOSITION 1.7. *Let $(V, L)$ and $D$ be as in Proposition 1.3. Suppose in addition that $H^0(V, L) \to H^0(D, L_D)$ is surjective and that $L_D$ is fully generating. Then a) $r_t: H^0(V, tL) \to H^0(D, tL_D)$ is surjective for any $t \geq 1$, and b) $L$ itself is fully generating.*

PROOF. We use the following commutative diagram:

$$\begin{array}{ccccc} & & H^0(V, tL)\otimes H^0(V, L) & \xrightarrow{r_t\otimes r_1} & H^0(D, tL)\otimes H^0(D, L) \\ & \nearrow & \downarrow{\scriptstyle m_t} & & \downarrow{\scriptstyle m'_t} \\ H^0(V, tL) & \longrightarrow & H^0(V, (t+1)L) & \xrightarrow{r_{t+1}} & H^0(D, (t+1)L) \end{array}$$

The dotted arrow is defined by $a \mapsto a \otimes \delta$, where $\delta \in H^0(V, L)$ is the defining section of $D$. We show a) by induction on $t$. Assuming a) for $t=k$, we infer that $r_k \otimes r_1$ is surjective. Hence the composition $m'_k \circ (r_k \otimes r_1) = r_{k+1} \circ m_k$ is also surjective, and so is $r_{k+1}$. Moreover, combined with the exactness of the lower row, this yields the surjectivity of $m_k$, too.

COROLLARY 1.8. *Let $V$, $L$ and $D$ be as above. Then $L$ is very ample if $L$ is ample. If in addition $h^p(D, tL)=0$ for $p>0$, $t \geq t_0 \in \boldsymbol{Z}$, then $h^p(V, tL)=0$ for $p>0$, $t \geq t_0-1$.*

PROOF. The very ampleness is obvious. The long exact sequence associated with the exact sequence $0 \to \mathcal{O}_V((t-1)L) \to \mathcal{O}_V(tL) \to \mathcal{O}_D(tL) \to 0$ proves $h^p(V, (t-1)L) = h^p(V, tL)$ for $p>0$, $t \geq t_0$ since $r_t$ is surjective for $t \in \boldsymbol{Z}$. Repeating such arguments we obtain $h^p(V, tL) = h^p(V, lL)$ for $p>0$, $t, l \geq t_0-1$, and the latter vanishes for $l \gg 0$.

Now we state a key result of our theory. A proof of it will be given in the last section.

THEOREM 1.9. *Let $\Lambda$ be a linear system of effective Cartier divisors on a variety $V$. If $[\Lambda]$ is ample, the following inequality holds:* $\dim Bs\Lambda < \Delta(V, [\Lambda]) + (\dim|\Lambda| - \dim \Lambda)$, *where* $\dim \varnothing$ *is defined to be* $-1$.

COROLLARY 1.10. *$\Delta(V, L) \geq 0$ for any polarized variety $(V, L)$. Moreover, if $\Delta(V, L)=0$, then $Bs|L| = \varnothing$.*

## § 2. Characterization of projective spaces and hyperquadrics

THEOREM 2.1. *For an $n$-dimensional polarized variety $(V, L)$ the following conditions are equivalent to each other*:

a) $(V, L) \simeq (\boldsymbol{P}^n, H)$, *where $H$ is the hyperplane section,*

b) $d(V, L)=1$ *and* $\Delta(V, L)=0$,

b′) $d(V, L)=1$ *and* $Bs|L|=\varnothing$,

c) *$V$ is non-singular and* $K_V+(n+1)L=0$,

c′) *$V$ is non-singular and $-c_1(K_V+nL)$ is representable by a closed $(1, 1)$-form which is positive almost everywhere on $V$,*

c″) *$V$ is non-singular,* $d(V, L)=1$ *and* $g(V, L)\leqq 0$,

d) *$V$ is normal and there exists a member $D\in|L|$ such that* $(D, L)\simeq(\boldsymbol{P}^{n-1}, H)$.

PROOF. It is clear that a) implies all the other conditions. The implication c)$\to$c′) is also clear. From c′) we infer that $h^p(V, tL)=0$ except for $p=0$, $t\geqq 0$ or $p=n$, $t\leqq-(n+1)$ using the vanishing theorem of Kodaira [11]. So the polynomial $\chi(t)=\chi(V, tL)$, which is of degree $n$, is equal to zero for $-n\leqq t\leqq -1$ and $\chi(0)=1$. Therefore $\chi(t)=(1/n!)\prod_{j=1}^{n}(t+j)$, from which the condition c″) follows immediately. Moreover, b) is also valid since $h^0(V, L)=\chi(1)=n+1$. From c″) we obtain similarly $\chi(t)=0$ for $-n\leqq t\leqq -1$. Hence $h^0(V, K_V+(n+1)L)=h^n(V, -(n+1)L)\neq 0$ or, equivalently, $\chi(-(n+1))\neq 0$, because otherwise $\chi(t)$ would vanish identically. This implies c) since $(K_V+(n+1)L)L^{n-1}\leqq 0$ and $L$ is ample. From b) we obtain b′) using Corollary 1.10. b′) gives rise to a finite morphism $\rho=\rho_{|L|}: V\to \boldsymbol{P}^N$. Letting $W=\rho(V)$ we have $\deg W=1$ and hence $W=\boldsymbol{P}^n$. Applying Zariski's Main Theorem we obtain $V\simeq W$, the condition a). Finally we show that d) implies b). In fact, we can apply Proposition 1.5 since $H^1(V, -L)=0$ (see Mumford [13], Theorem 2).

THEOREM 2.2. *For an $n$-dimensional polarized variety $(V, L)$ the following conditions are equivalent to each other*:

a) *$V$ is isomorphic to a hyperquadric in $\boldsymbol{P}^{n+1}$ and $L$ is the hyperplane section,*

b) $d(V, L)=2$ *and* $\Delta(V, L)=0$.

*If $V$ is non-singular, each of the following is also equivalent to a) and b)*:

c) $K_V+nL=0$,

c′) $d(V, L)=2$ *and* $g(V, L)\leqq 0$.

We prove only c′)$\to$c) since the implications c)$\to$b)$\leftrightarrow$a)$\to$c′) are obtained by a similar method to that in Theorem 2.1. If c) is false, $h^0(V, K_V+nL)=h^n(V, -nL)=0$ since $(K_V+nL)L^{n-1}\leqq 0$. Then $\chi(t)=0$ for $-n\leqq t\leqq -1$ and so $\chi(t)$ has the form

$(\alpha/n!)\prod_{j=1}^{n}(t+j)$, $\alpha\in\boldsymbol{Z}$. By definition this means that $\alpha=d(V, L)=2$ and $g(V, L)=-1$, hence $(K_V+(n+1)L)L^{n-1}=0$. Since $L$ is positive, $K_V+(n+1)L=0$ if $h^0(V, K_V+(n+1)L)>0$. This contradicts

$$h^0(V, K_V+(n+1)L)=h^n(V, -(n+1)L)=(-1)^n\chi(-(n+1))=2 .$$

## § 3. The non-singular case

LEMMA 3.1. *Let* $(C, L)$ *be a polarized curve with* $\Delta(C, L)=0$. *Then* $C\simeq\boldsymbol{P}^1$.

PROOF. Letting $\pi:\tilde{C}\to C$ be the normalization of $C$, we have $\Delta(\tilde{C}, L)\leqq 0$. Therefore $\tilde{C}\simeq\boldsymbol{P}^1$ and $|L|_C=|L_C|$. Since $L_C$ is very ample, this implies $\tilde{C}\simeq C$.

Throughout this section let $(M, L)$ be a polarized manifold with $\Delta(M, L)=0$, $\dim M=n$ and $d(M, L)=d$.

THEOREM 3.2. $g(M, L)=0$, $h^p(M, tL)=0$ *for* $p>0$, $t\geqq 0$ *and* $L$ *is very ample and is fully generating.*

We prove this theorem by induction on $n$. When $n=1$, the assertion follows from the above lemma. When $n\geqq 2$, a general member $D$ of $|L|$ is non-singular since $Bs|L|=\varnothing$. Applying Proposition 1.5 we have $0\leqq\Delta(D, L)\leqq\Delta(M, L)=0$. Therefore $g(D, L)=0$, $h^p(D, tL)=0$ for $p>0$, $t\geqq 0$ and $L_D$ is fully generating. So $g(M, L)=g(D, L)=0$ and the other conclusions follow from Proposition 1.7 and Corollary 1.8.

To establish a structure theorem for such polarized manifolds, we need the following lemmata.

LEMMA 3.3. $h^0(M, K_M+nL)=d-1$.

PROOF. We use induction on $n$ since the result is obvious for $n=1$. For $n\geqq 2$, with $D$ being a non-singular member of $|L|$, we have the exact sequence

$$H^0(M, K_M+(n-1)L)\to H^0(M, K_M+nL)\to H^0(D, K_M+nL)\to H^1(M, K_M+(n-1)L) .$$

Since $(K_M+(n-1)L)L^{n-1}=2g-2<0$ we infer $H^0(M, K_M+(n-1)L)=0$, while $H^1(M, K_M+(n-1)L)=0$ (see Kodaira [11]). Hence

$$h^0(M, K_M+nL)=h^0(D, K_M+nL)=h^0(D, K_D+(n-1)L)=d-1 ,$$

where $K_D$ denotes the canonical bundle of $D$.

LEMMA 3.4. $\dim Bs|K_M+nL|<n-1$ *if* $d\geqq 3$.

PROOF. For $n=1$ everything is clear. For $n\geqq 2$ letting $D$ be as above we have $|K_M+nL|_D=|K_D+(n-1)L_D|$ and

$$\dim Bs|K_M+nL|\leqq 1+\dim(Bs|K_M+nL|\cap D)=1+\dim Bs|K_D+(n-1)L_D|\ .$$

Thus we can prove the lemma by induction on $n$.

LEMMA 3.5. *If $n=2$ and $d\geqq 3$, then $(K_M+2L)^2=0$ except for the case in which $(M, L)\simeq(\boldsymbol{P}^2, 2H)$.*

PROOF. The above lemma yields $(K_M+2L)^2\geqq 0$, which implies that $K_M^2=c_1^2\geqq 8$. On the other hand $c_1^2+c_2=12\chi(M)=12$ and $c_2=2+b_2\geqq 3$ (see Kodaira [12]). Combining these we have $c_1^2=8$ or 9. $(K_M+2L)^2=0$ if $c_1^2=8$. If $c_1^2=9$, then $b_2=1$ and the quotient of $H^2(M;\boldsymbol{Z})\simeq H^1(M, \mathcal{O}_M^*)$ by its torsion subgroup is a cyclic group. Letting $F\in H^1(M, \mathcal{O}_M^*)$ be a representative element of its integral base such that $FL>0$, we infer readily that $L=2F$ and $K_M=-3F$. Hence Theorem 2.1 proves $(M, F)\simeq(\boldsymbol{P}^2, H)$.

LEMMA 3.6. *If $n\geqq 3$ and $d\geqq 3$, then $(K_M+nL)^2L^{n-2}=0$.*

PROOF. We may assume that $n=3$, since the induction on $n$ works for $n\geqq 4$. By $D$ denoting a non-singular member of $|L|$, it suffices to show that $(D, L)\not\simeq(\boldsymbol{P}^2, 2H)$. Assume $(D, L)\simeq(\boldsymbol{P}^2, 2H)$. Then Lefschetz's hyperplane section theorem asserts that $H_2(D;\boldsymbol{Z})\to H_2(M;\boldsymbol{Z})$ is surjective, while $H_2(D;\boldsymbol{Z})\simeq\boldsymbol{Z}$. Therefore $H_2(D;\boldsymbol{Z})\simeq H_2(M;\boldsymbol{Z})$ and

$$H^1(M, \mathcal{O}_M^*)\simeq H^2(M;\boldsymbol{Z})\simeq H^2(D;\boldsymbol{Z})\simeq H^1(D, \mathcal{O}_D^*)\ .$$

So $H$ is the restriction of a line bundle on $M$, which is also denoted by $H$, by abuse of notation. Then $L=2H$, $K_M+L=K_D=-3H$ and consequently $K_M=-5H$. This contradicts Theorem 2.1, c′).

LEMMA 3.7. *If $d\geqq 3$ and $(K_M+nL)^2L^{n-2}=0$, then $Bs|K_M+nL|=\varnothing$, and $W=\rho(M)$ is a curve, where $\rho=\rho_{|K_M+nL|}$.*

PROOF. With the help of Lemma 3.4 we can choose two members $D_1$, $D_2$ of $|K_M+nL|$ such that $\dim(D_1\cap D_2)<n-1$. If $D_1\cap D_2\neq\varnothing$, then $(K_M+nL)^2L^{n-2}=L^{n-2}\{D_1\cap D_2\}>0$ since $L$ is ample. Hence $D_1\cap D_2=\varnothing$ and consequently $\dim W=1$.

THEOREM 3.8. *If $d\geqq 3$, then there is a vector bundle $E$ on $\boldsymbol{P}^1$ such that $(M, L)\simeq(\boldsymbol{P}(E), H(E))$ except the case $(M, L)\simeq(\boldsymbol{P}^2, 2H)$. Moreover, $E$ is a direct sum of line bundles of positive degrees.*

PROOF. Assume $(M, L)\not\simeq(\boldsymbol{P}^2, 2H)$. Then from preceding lemmata we infer that $\rho=\rho_{|K_M+nL|}$ is a morphism onto a curve $W\subset\boldsymbol{P}_\alpha^{d-2}$. Letting $w=\deg W$ and $X$ be a general fiber of $\rho$, we have $d-2=(K_M+nL)L^{n-1}=wL^{n-1}X\geqq w$. On the other hand, $0\leqq\Delta(W, H_\alpha)\leqq 1+w-(d-1)=w-(d-2)$. Combining these inequalities we get

$w=d-2$, $L^{n-1}X=1$ and $\Delta(W, H_\alpha)=0$, which implies $W\simeq \boldsymbol{P}^1$ (see Lemma 3.1). Any fiber $Y$ of $\rho$ is irreducible and reduced since $L^{n-1}Y=L^{n-1}X=1$. Therefore Theorem 2.1, $b'$) proves that $(Y, L)\simeq(\boldsymbol{P}_\beta^{n-1}, H_\beta)$. Combined with $H^2(\boldsymbol{P}^1, \mathcal{O}^*)=0$, this implies the existence of a vector bundle $E$ on $W\simeq\boldsymbol{P}^1$ such that $(M, L)\simeq(\boldsymbol{P}(E), H(E))$. Moreover, $E$ is a direct sum of line bundles (see Grothendieck [6]), each one of which is ample since so is $E$ (see Hartshorne [7], Propositions (2.2) and (3.2)).

REMARK. This theorem, combined with Theorems 2.1 and 2.2, gives a complete enumeration of polarized manifolds with $\Delta$-genera zero.

COROLLARY 3.9. *If $d\geqq 3$, then $d\geqq n$ and the equality holds if and only if $(M, L)\simeq(\boldsymbol{P}_\zeta^{n-1}\times\boldsymbol{P}_\xi^1, H_\zeta+H_\xi)$.*

PROOF. Clearly we may assume $(M, L)\not\simeq(\boldsymbol{P}^2, 2H)$. Letting $E=\bigoplus_{j=1}^{n}L_j$ with $d_j=\deg L_j\geqq 1$ we have $d(M, L)=\deg(\det E)=\sum d_j\geqq n$. The equality holds if and only if $d_j=1$ for every $j$, which is equivalent to $(M, L)\simeq(\boldsymbol{P}_\zeta^{n-1}\times\boldsymbol{P}_\xi^1, H_\zeta+H_\xi)$.

## § 4. The singular case

REMARK 4.1. Let $E$ be a vector space with $E^*$ being its dual space and let $P=\boldsymbol{P}(E^*)$. Then there is a canonical one-one correspondence between subsets of $P$ and '*conic*' subsets of $E$, i.e., subsets which are closed under scalar multiplication. If both $X, Y\subset E$ are conic, then $X+Y=\{x+y|x\in X, y\in Y\}$ is also conic. With the help of the above correspondence, this gives rise to an operation $*$ upon subsets of $P$. We summarize a few properties of it.

a) $x*x=x$ *for* $x\in P$.

b) $x*y$ $(x\neq y\in P)$ *is the line which passes through* $x$ *and* $y$.

c) $S*\emptyset=S$ *for any* $S\subset P$.

d) $S*T=\bigcup_{x\in S, y\in T} x*y$ *if neither* $S$ *nor* $T\subset P$ *is empty.*

e) $S*T=T*S$ *for* $S, T\subset P$.

f) $(S*T)*U=S*(T*U)$ *for* $S, T, U\subset P$.

The proofs are easy.

DEFINITION 4.2. For a subset $S$ of $P$ we denote the set $\{x\in S|x*S=S\}$ by *Ridge* $(S)$. Note that $x\in$ *Ridge* $(S)$ if and only if $x*y\subset S$ for every $y\in S$.

PROPOSITION 4.3. i) *$S$ is linear if and only if Ridge $(S)=S$.*

ii) *Ridge $(S)$ is a linear subset.*

PROOF. i) is clear. As for ii), letting $x, y\in$ *Ridge* $(S)$, for every $z\in x*y$, we have $z*S\subset(x*y)*S=x*(y*S)=x*S=S$. This proves ii).

LEMMA 4.4. *Let $S\subset P$, $R=Ridge(S)$ and let $H$ be a hyperplane such that $H\not\supset R$. Let $S_H=S\cap H$, $R_H=R\cap H$ and $v$ be a point on $R$ such that $R=v*R_H$. Then $S=v*S_H$ and $Ridge(S_H)=R_H$.*

PROOF. For any $x\in S/(S_H\cup\{v\})$ let $y$ be the point $(x*v)\cap H$. Then $x\in v*y\subset v*((v*S)\cap H)\subset v*S_H$. Therefore $S=v*S_H$, since $v*S_H\subset v*S=S$. For $x\in Ridge(S_H)$, we have $x*S=x*(S_H*v)=(x*S_H)*v=S_H*v=S$. Hence $x\in R$. For $y\in R_H$, we have $y*S_H\subset(y*S)\cap(y*H)=S\cap H=S_H$. Combining these we infer that $Ridge(S_H)=R_H$.

PROPOSITION 4.5. *Let $S$, $R$ be as above and let $T$ be a linear subspace of $P$ such that* $\dim T+\dim R=\dim P-1$ *and $T\cap R=\varnothing$. Then $S=S_T*R$ and $Ridge(S_T)=\varnothing$, where $S_T=S\cap T$.*

PROOF. We use induction on $r=\dim R$, since the assertion is obvious for $r\leqq 0$. When $r>0$, take a hyperplane $H$ such that $T\subset H\not\supset R$. Then the induction hypothesis proves that $Ridge(S_T)=\varnothing$ and $S_H=S_T*R_H$. Letting $v$ be as in Lemma 4.4, we have $S=v*S_H=v*(R_H*S_T)=(v*R_H)*S_T=R*S_T$.

PROPOSITION 4.6. *Let $W$ be a subvariety of $P$ and let $x\in W/Ridge(W)$. Then* $\deg(x*W)\leqq\deg W-1$, *and the equality holds if and only if $W$ is non-singular at $x$.*

COROLLARY 4.7. *Suppose that $\Delta(W, H)=0$. Then $Sing(W)\subset Ridge(W)$.*

The proofs are easy. For details, see Swinnerton-Dyer [16], p. 406.

THEOREM 4.8. *Let $(V, L)$ be a polarized variety with $\Delta(V, L)=0$. Then $V$ is normal, locally Macaulay, and $L$ is very ample.*

PROOF. Since $Bs|L|=\varnothing$, we have a morphism $\rho=\rho_{|L|}: V\to P=\mathbf{P}(H^0(V, L))$. Letting $W=\rho(V)$ and $H=H(H^0(V, L))$ we infer that $\Delta(W, H)=0$ and $\deg\rho=1$. Recalling Proposition 4.5 we write $W=M*Ridge(W)$ where $Ridge(M)=\varnothing$. Since $\Delta(M, H)=0$, this implies that $M$ is non-singular. Now, with the help of the theory of local cohomology (see Hartshorne [8] and EGA, III, §2.1), we infer from Theorem 3.2 that $W$ is locally Macaulay. Since codim $Sing(W)\geqq 2$, this implies that $W$ is normal (see EGA, IV, §5.8). Now we can apply Zariski's Main Theorem to obtain that $V\simeq W$.

COROLLARY 4.9. *Let $(V, L)$ be as above. Then $L$ is fully generating and $H^p(V, tL)=0$ for $p>0$, $t\geqq 0$.*

The proof is quite similar to that of Theorem 3.2.

COROLLARY 4.10. *Let $(V, L)$ be as above and suppose in addition that $R=Ridge(V)\neq\varnothing$. Then every line bundle on $V$ is an integral multiple of $L$.*

PROOF. We regard $V$ as a subset of $P=\boldsymbol{P}(H^0(V, L))$. Let $M$ be a submanifold of $V$ described in the proof of Theorem 4.8. Then the proper transform $V'\subset Q_R(P)$ of $V$ is isomorphic to $\boldsymbol{P}_M(L_M\oplus I_M^{r+1})$, where $I_M^{r+1}$ is the trivial vector bundle on $M$ of rank $r+1$, $r=\dim R$. Moreover, $L_{V'}=H(L_M\oplus I_M^{r+1})$, and the submanifold $\boldsymbol{P}(I_M^{r+1})$ of $\boldsymbol{P}(L_M\oplus I_M^{r+1})=V'$ is the exceptional divisor $E=V'\cap E_R$ on $V'$, where $E_R=Q_R(R)\subset Q_R(P)$. Now it is easy to see that $F_E$ is an integral multiple of $L_E$ for any line bundle $F$ on $V$. Recall that

$$H^1(V', \mathcal{O}_{V'}^*)\simeq H^1(M, \mathcal{O}_M^*)\oplus \boldsymbol{Z}[L_{V'}]\simeq H^1(E, \mathcal{O}_E^*)$$

and that $H^1(V, \mathcal{O}_V^*)\to H^1(V', \mathcal{O}_{V'}^*)$ is injective since $V$ is normal. Combining these observation we obtain the result.

## §5. The relative case

DEFINITION 5.1. By a *family of polarized varieties* we mean a quadruple $(\mathcal{V}, \mathcal{L}, \pi, S)$ of two varieties $\mathcal{V}$, $S$, which may not be compact, a proper flat morphism $\pi:\mathcal{V}\to S$, and a line bundle $\mathcal{L}$ on $\mathcal{V}$ which is ample relative to $\pi$, such that every (ideal theoretical) fiber $V_s=\pi^{-1}(s)$ over $s\in S$ is irreducible and reduced. For each $s\in S$, $(V_s, L_s)$ turns out to be a polarized variety where $L_s$ is the restriction of $\mathcal{L}$ to $V_s$.

THEOREM 5.2. *Let $(\mathcal{V}, \mathcal{L}, \pi, S)$ be a family of polarized varieties. Then $\Delta(V_s, L_s)$ is a lower-semi-continuous function of $s\in S$.*

This is a corollary to the results of Schneider [15].

PROPOSITION 5.3. *Let $(\mathcal{V}, \mathcal{L}, \pi, S)$ be as above and suppose that $\Delta(V_0, L_0)=0$ for a point $0\in S$. Then $\Delta(V_s, L_s)=0$ for any $s\in S$.*

PROOF. Using Corollary 4.9 we infer that $h^p(V_s, L_s)=0$ for $p>0$ in a neighbourhood of 0. Hence $h^0(V_s, L_s)$ is locally constant and $\Delta(V_s, L_s)=0$ in a neighbourhood of 0. Since $\Delta(V_s, L_s)\geqq 0$ for any $s\in S$, Theorem 5.2 yields the assertion.

COROLLARY 5.4. *If in addition $(V_0, L_0)\simeq(\boldsymbol{P}^n, H)$, then there is a vector bundle $E$ on $S$ such that $(\mathcal{V}, \mathcal{L})\simeq(\boldsymbol{P}(E), H(E))$.*

PROOF. In view of the results of Schneider [15], we infer that $\pi_*\mathcal{O}_{\mathcal{V}}(t\mathcal{L})$ $(t\in \boldsymbol{Z})$ is a locally free sheaf on $S$. Let $E$ be a vector bundle on $S$ such that $\mathcal{O}_S(E)\simeq\pi_*\mathcal{O}_{\mathcal{V}}(\mathcal{L})$. Then $\pi_*\mathcal{O}_{\mathcal{V}}(t\mathcal{L})$ is canonically isomorphic to $\mathcal{O}_S(\mathrm{S}^tE)$ for every $t\geqq 0$. Hence $\boldsymbol{P}(E)\simeq\mathrm{Proj}\,(\bigoplus_{k=0}^{\infty}\mathrm{S}^k\mathcal{O}_S(E))\simeq\mathrm{Proj}\,(\bigoplus_{k=0}^{\infty}\pi_*\mathcal{O}_{\mathcal{V}}(t\mathcal{L}))$ and $\mathcal{V}$ turns to be a subspace of this by definition of the relative ampleness. Since the inclusion morphism $f_s$ restricted to each fiber $V_s$ is nothing but the morphism $\rho_{|L_s|}$, $f:\mathcal{V}\to \boldsymbol{P}(E)$ is an isomorphism.

Analogously we can prove the following

COROLLARY 5.5. *Let* $(\mathcal{V}, \mathcal{L}, \pi, S)$ *be as in Proposition 5.3 and suppose in addition that* $d(V_0, L_0)=2$. *Then there exists a vector bundle* $E$ *on* $S$ *and a divisor* $W$ *on* $\mathcal{P}=\boldsymbol{P}(E)$ *such that* $(\mathcal{V}, \mathcal{L})\simeq(W, H(E))$. *Moreover, restricted to each fiber of* $\mathcal{P}$, $W$ *is a hyperquadric.*

PROPOSITION 5.6. *Let* $(\mathcal{V}, \mathcal{L}, \pi, S)$ *be as in Proposition 5.3 and suppose in addition that* $d(V_0, L_0)\geqq 3$ *and that* $V_0$ *is a smooth variety whose canonical bundle* $K_{V_0}$ *is the restriction of a line bundle* $\mathcal{K}$ *on* $\mathcal{V}$. *Then every fiber* $V_s$ *is smooth.*

PROOF. Putting $K_s=\mathcal{K}_{V_s}$ we have

$$(K_s+(n-1)L_s)L_s^{n-1}\{V_s\}=(K_0+(n-1)L_0)L_0^{n-1}\{V_0\}=2g(V_0, L_0)-2=-2\,.$$

If $V_s$ is singular, we use Corollary 4.10 to infer that $-2$ is an integral multiple of $d(V_s, L_s)$, which contradicts the assumption $d(V_0, L_0)\geqq 3$. So $V_s$ is smooth.

COROLLARY 5.7. *Suppose in addition that* $(V_0, L_0)\not\simeq(\boldsymbol{P}^2, 2H)$ *and that* $S$ *is smooth. Then* $\mathcal{V}$ *is a* $\boldsymbol{P}^{n-1}$*-bundle over a* $\boldsymbol{P}^1$*-bundle over* $S$.

## §6. Proof of the fundamental inequality

The purpose of this section is to prove the estimate

$$\dim Bs\Lambda<\Delta(V, [\Lambda])+(\dim|\Lambda|-\dim\Lambda)$$

stated in Theorem 1.9. From now on this inequality is denoted by $(F)$.

LEMMA 6.1. *Let* $D$ *be an effective Cartier divisor on a variety* $W$ *such that* $[D]$ *is ample. Suppose that the support of* $D$ *is a union of two analytic subsets* $D_1$, $D_2$ *neither of which contains the other. Then* $\dim(D_1\cap D_2)\geqq\dim W-2$.

We prove this lemma by induction on $w=\dim W$. For $w\leqq 1$, everything is trivial. For $w=2$, take a non-singular model $W'$ of $W$ (see Hironaka [9]). Using the vanishing theorem of Kodaira-Mumford (see [13], Theorem 2), we infer that $H^1(W', -[D_{W'}])=0$. This implies $h^0(D_{W'})=0$ and hence the support of $D_{W'}$ is connected. Therefore $D$ is also connected and this implies $D_1\cap D_2\neq\varnothing$. For $w\geqq 3$, take a generic hyperplane section $H$ such that $\dim(D_1\cap D_2\cap H)=\dim(D_1\cap D_2)-1$ and that neither of $(D_1\cap H)$ nor $(D_2\cap H)$ contains the other. Applying the induction hypothesis we obtain $\dim H-2\leqq\dim(D_1\cap D_2\cap H)$. The estimate follows from this.

PROOF OF THEOREM 1.9. We shall establish the inequality $(F)$ by induction on $n=\dim V$. If $n=1$, $(F)$ follows easily from Lemma 3.1. We consider the case $n\geqq 2$. Since $L=[\Lambda]$ is ample, $\dim Bs\Lambda-\dim Bs|\Lambda|\leqq\dim|\Lambda|-\dim\Lambda$. Therefore we may assume $\Lambda=|L|$. Now, we take a non-singular model $\pi: V'\rightarrow V$ of $V$ such that $\pi^*\Lambda=E+\Lambda'$, where $E$ is the fixed part of $\pi^*\Lambda$ and $\Lambda'$ is a linear system of effective Cartier divisors free of base points. Let $W$ be the image of the morphism $\rho=\rho_{\Lambda'}: V'\rightarrow P^N$, $N=\dim\Lambda$. Note that $[\Lambda']=\rho^*H$ and $Bs\Lambda=\pi(E)$. Then one of the following conditions is valid: a) $\dim Bs\Lambda=n$, b) $\dim Bs\Lambda=n-1$ and $\dim W=1$, c) $\dim Bs\Lambda\leqq n-2$ and $\dim W=1$, d) $\dim Bs\Lambda=n-1$ and $\dim W\geqq 2$, e) $\dim Bs\Lambda\leqq n-2$ and $\dim W\geqq 2$. We shall show that the inequality $(F)$ holds in each of these cases.

Case a). This condition implies $V=Bs\Lambda$ and $h^0(V, L)=0$. Hence $\Delta(V, L)=n+d(V, L)>n$.

Case b). Putting $w=\deg W$ we have $0\leqq\Delta(W, H)\leqq 1+w-(N+1)=1+w-h^0(V, L)$. Let $X$ be a general fiber of $\rho$. Then $L^{n-1}H=wL^{n-1}X\geqq w$ since $L^{n-1}X>0$. Moreover, $L^{n-1}E>0$ because $\dim\pi(E)=n-1$. Combining these inequalities we obtain that $d(V, L)=L^n>L^{n-1}H\geqq w\geqq h^0(V, L)-1$. This implies $(F)$.

Case c). Similarly we have $0\leqq\Delta(W, H)\leqq 1+w-h^0(V, L)$ and $d(V, L)=L^n\geqq L^{n-1}H$. $(F)$ follows from these.

Case d). Let $S$ be a general member of $\Lambda'$ and let $D$ be the corresponding member of $\Lambda$. Note that $D=\pi(E)\cup\pi(S)$. $S$ is connected since $H^1(V', -[\Lambda'])=0$ (see Mumford [13], Theorem 2). So $G=\pi(S)$ is irreducible. Moreover, Lemma 6.1 proves that $\dim(\pi(E)\cap G)=\dim Bs\Lambda_G\geqq n-2$. On the other hand, $\dim\Lambda_G=\dim\Lambda-1=h^0(V, L)-2$. Now we apply the induction hypothesis to the pair $(G, \Lambda_G)$ to obtain that $n-2\leqq\dim Bs\Lambda_G<n-1+d(G, L)-(\dim\Lambda_G+1)$. The estimate $(F)$ follows from this since $d(V, L)\geqq d(G, L)+d(\pi(E), L)>d(G, L)$.

Case e). Let $S$, $D$ be as in case d). Then $D$ is irreducible and $D_{red}=G=\pi(S)$. Hence $Bs\Lambda=Bs\Lambda_G$ and $\dim\Lambda_G=\dim\Lambda-1=h^0(V, L)-2$. Applying the induction hypothesis to $(G, \Lambda_G)$ we obtain $\dim Bs\Lambda_G<n-1+d(G, L)-(\dim\Lambda_G+1)$. This proves $(F)$.

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