On involutive systems of partial differential equations in two independent variables

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§0. Introduction.

The general theory of involutive systems of partial differential equations has been developed by many mathematicians (E. Cartan, E. Kähler, M. Kuranishi, M. Matsuda et al). On the other hand, in the classical theory of solving Cauchy's problem for a single differential equation in two independent variables by integrating a system of ordinary differential equations, we find Darboux's method as the most powerful one (cf. E. Goursat [8], A. R. Forsyth [7]). The principal purpose of this memoir is firstly the investigation on the structure of involutive systems of partial differential equations with one unknown function of two independent variables and secondly the extension of Darboux's method to those involutive systems. In this way, we obtain a modern point of view for understanding Darboux's method of integration, from which it is possible to clarify the relation between his method and the method by integrable systems given recently by M. Matsuda of solving Cauchy's problem for Monge-Ampère's equation (cf. Remark in §7).

All notions occurring in this memoir are assumed to be in the category of real or complex analyticity though all arguments except when the Cauchy-Kowalewskaja theorem is applied can be done in the category of infinite differentiability.

A subsheaf of ideals (which is locally finitely generated) in $\mathcal{O}(J^m)$, the sheaf of germs of analytic functions defined on the space of m-jets $J^m(M, N, \rho)$ on a fibered manifold (M, N, ρ), is called a system of differential equations of order m in (M, N, ρ). Let Φ be a system of differential equations of order m. Suppose that it has only one unknown function of two independent variables, that is, Φ is a system of order m in (M, N, ρ) in which $\dim M=3$, $\dim N=2$. In this case, we obtain a simple criterion in order that Φ is involutive (Theorem I). We define the characteristic polynomial of Φ at an integral point as the highest common

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factor of the characteristic polynomials of all equations in Φ, and then characteristic directions of Φ are defined, corresponding to the roots of the characteristic polynomial (§3). Suppose that Φ is involutive. We show that the degree of the characteristic polynomial, in other words, the number of characteristic directions at an integral point is equal to $m+1-r$, where r is, roughly speaking, the number of independent equations exactly of order m in Φ (Theorem II). Next we propose the problem of finding a method of constructing a new involutive system which contains Φ. To consider this problem, the characteristic systems of Φ (in the sense of Monge) are introduced; these are Pfaffian systems defining singular elements of a differential system canonically associated with Φ. A function u defined on an open set in $J^m(M, N, \rho)$ is called a (relative) invariant of a characteristic system if du vanishes in consequence of that system (and $u=0$). u is said to be independent of Φ if the equation $u=0$ cannot be reduced to an equation of order lower than m in consequence of the equations of Φ. Let $u_1, \cdots, u_s$ be s functions defined on an open set $\mathscr{U}$ in $J^m(M, N, \rho)$. We define $\Phi(u_1, \cdots, u_s)$ to be the subsheaf of ideals in $\mathscr{O}(\mathscr{U})$ generated by Φ and $u_1, \cdots, u_s$. Let u be a function of m-jets which is independent of Φ. Then $\Phi(u)$ is involutive if and only if u is a relative invariant of a characteristic system of Φ, and in this case the degree of the characteristic polynomial of $\Phi(u)$ is less than that of Φ by one (Proposition 8). Furthermore, if functions of m-jets $u_1, \cdots, u_s$ are relative invariants of s different characteristic systems of Φ respectively and if each u_α is independent of Φ, then $\Phi(u_1, \cdots, u_s)$ is an involutive system and the degree of the characteristic polynomial of $\Phi(u_1, \cdots, u_s)$ is less than that of Φ by s (Proposition 9). These results are extended to the general case when functions u's are invariants of higher order (Theorems III and IV). Thus we obtain an answer to the problem above proposed.

Finally we consider Cauchy's problem for an involutive system Φ. By the first existence theorem of Cartan-Kähler, Cauchy's problem has a unique solution. The problem to engage our attention is that of finding a method of solving Cauchy's problem by integrating a system of ordinary differential equations. Let ν denote the degree of the characteristic polynomial of Φ at an integral point.

1°) *When* $\nu=0$, Φ *is a completely integrable system.*

2°) *When* $\nu=1$, *the solution of Cauchy's problem can be reduced to the integration of a system of ordinary differential equations.*

This is a conclusion of the fact that a differential system canonically associated with Φ (see §1) has one-dimensional Cauchy's characteristics (E. Cartan [5]).

3°) *When* $\nu>1$, *the solution of Cauchy's problem cannot be reduced to the* *integration of a system of ordinary differential equations in general. However, in this case, we can extend Darboux's method as follows.*

"If $\nu-1$ different characteristic systems of Φ have respectively two independent invariants which are independent of Φ, then the solution of Cauchy's problem can be reduced to the integration of a system of ordinary differential equations."

In the case where the system considered is derived from a single equation, this is the classical theorem due to Darboux: It is Goursat who completed it in this style for a single equation.

The precise statements of these results are found in the text.

The author obtained, at first, the results of this memoir by applying the theory of differential systems due to E. Cartan ([1]-[5]) as were announced in his previous paper [10]. In formulating the results in the style of this memoir where fruitful results on involutive systems after E. Cartan are applied to our problem, many important suggestions were given by M. Matsuda to whom the author wishes to express his sincere gratitude.

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§1. A criterion of involution.

Let (M, N, ρ) be an analytic fibered manifold where ρ is the projection from M onto N and let $J^m(M, N, \rho)$ be the space of m-jets of cross-sections of (M, N, ρ). Let $\mathscr{U}$ be an open set in $J^m(M, N, \rho)$. The sheaf of germs of analytic functions on $\mathscr{U}$ will be denoted by $\mathscr{O}(\mathscr{U})$. A subsheaf of ideals of $\mathscr{O}(\mathscr{U})$ (which is locally finitely generated) is called a system of differential equations of order m in (M, N, ρ). $\mathscr{U}$ is called the domain of that system. Let Φ be a system of differential equations of order m in (M, N, ρ). A point X in $J^m(M, N, \rho)$ is called an integral point of Φ if every φ in Φ_X, the ring of germs of sections of Φ at X, vanishes at X. Let us denote by $I\Phi$ the set of all integral point of Φ. An integral point X is called an *ordinary* integral point if $\Phi=0$ gives a regular local equation of $I\Phi$ around X. A cross-section f of (M, N, ρ) is said to be a solution of Φ if and only if every m-jet $j_a^m(f)$ of f at a, where a varies in the domain of f, is an integral point of Φ; that is, $\varphi(j_a^m(f))=0$ for any section φ of Φ defined around $j_a^m(f)$.

Let ρ_k^m be the projection from $J^m(M, N, \rho)$ onto $J^k(M, N, \rho)$ defined by

$$\rho_k^m(j_a^m(f))=j_a^k(f) \qquad (m \geqq k \geqq 0)$$

and ρ^m_{-1} denote $\rho\circ\rho^m_0$. For each point X and $\bar{X}=\rho^m_{m-1}X$, we have the injection i from $\mathcal{O}_X(J^{m-1})$ into $\mathcal{O}_X(J^m)$ defined by $i\varphi=\varphi\circ\rho^m_{m-1}$. We identify $\mathcal{O}_X(J^{m-1})$ with its image $i\mathcal{O}_X(J^{m-1})$ in $\mathcal{O}_X(J^m)$.

The kernel of the mapping

$$d\rho^m_{m-1} : T_X(J^m) \longrightarrow T_X(J^{m-1})$$

is denoted by $Q_X(J^m)$ $(m\geqq 0)$. $Q_X(J^m)$ can be canonically identified with $Q_b(M)\otimes S^m(T^*_a(N))$, where $a=\rho^m_{-1}X$ and $Q_b(M)$ is the kernel of $d\rho$ at b, $b=\rho^m_0 X$. The subspace of $C_X(\Phi)$ of $Q_X(J^m)$ is defined, for each integral point X of Φ, by

$$C_X(\Phi)=\{\mathscr{X}\in Q_X(J^m);\ \mathscr{X}(\varphi)=0 \quad \text{for every} \quad \varphi\in\Phi_X\}\,.$$

$C_X(\Phi)$ is considered to be a subspace of $Q_X(J^{m-1})\otimes T^*_a(N)$. Now we recall the notion of involutive subspaces of $Q_X(J^{m-1})\otimes T^*_a(N)$ (cf. M. Kuranishi [12], §6). Let A be a subspace of $Q_X(J^{m-1})\otimes T^*_a(N)$. The prolongation of pA of A is defined by

$$pA=A\otimes T^*_a(N)\cap Q_X(J^{m-1})\otimes S^2(T^*_a(N))\,.$$

$Q_X(J^{m-1})\otimes T^*_a(N)$ can be canonically identified with $\mathrm{Hom}(T_a(N), Q_X(J^{m-1}))$. For a system of vectors $v_1,\cdots,v_q$ in $T_a(N)$, we denote by $A(v_1,\cdots,v_q)$ the subspace of A composed of all vectors which annihilate $v_1,\cdots,v_q$. Let g_q denote the minimal dimension of $A(v_1,\cdots,v_q)$ when $(v_1,\cdots,v_q)$ varies arbitrary $(q\geqq 1)$ and g_0 denote $\dim A$. A is called an involutive subspace of $Q_X(J^{m-1})\otimes T^*_a(N)$ if and only if $\dim pA$ is equal to $\sum_{q=0}^{n} g_q$, here $n=\dim N$.

The (total) prolongation $p\Phi$ of Φ is defined as follows. Let φ be an analytic function defined on an open set $\mathscr{U}$ in $J^m(M,N,\rho)$ and ξ be a vector field on $\rho^m_{-1}\mathscr{U}\subset N$. Let φ_ξ denote an analytic function on $(\rho^{m+1}_m)^{-1}\mathscr{U}$ defined by

$$\varphi_\xi(j^{m+1}_a(f))=\xi(\varphi(j^m_a(f)))\,.$$

For each open set $\tilde{\mathscr{U}}$ in $J^{m+1}(M,N,\rho)$, we denote by $\mathscr{G}(\tilde{\mathscr{U}})$ the ideal of $\mathcal{O}(\tilde{\mathscr{U}})$ generated by all sections of the sheaf Φ over $\rho^{m+1}_m\tilde{\mathscr{U}}$ and all φ_ξ constructed from every section φ of Φ over $\rho^{m+1}_m\tilde{\mathscr{U}}$ and every vector field ξ on $\rho^{m+1}_{-1}\tilde{\mathscr{U}}$. Let $\rho_{\tilde{\mathscr{U}}\tilde{\mathscr{V}}}$ denote the restriction mapping from $\mathscr{G}(\tilde{\mathscr{V}})$ to $\mathscr{G}(\tilde{\mathscr{U}})$, where $\tilde{\mathscr{U}}\subset\tilde{\mathscr{V}}$. Then $\{\mathscr{G}(\tilde{\mathscr{U}}),\rho_{\tilde{\mathscr{U}}\tilde{\mathscr{V}}}\}$ forms a presheaf over $J^{m+1}(M,N,\rho)$. The *prolongation* $p\Phi$ is defined to be the sheaf over $J^{m+1}(M,N,\rho)$ associated with the presheaf $\{\mathscr{G}(\tilde{\mathscr{U}}),\rho_{\tilde{\mathscr{U}}\tilde{\mathscr{V}}}\}$. $p\Phi$ is a system of differential equations of order $m+1$ which has the same solutions as Φ. From the definitions we see that for each point $\tilde{X}$ in $(\rho^{m+1}_m)^{-1}X\cap I(p\Phi)$

$$C_{\tilde{X}}(p\Phi)=pC_X(\Phi)\,.$$

Let $\tilde{X}$ be a point of $(\rho_m^{m+1})^{-1}X$. $(p\Phi)_{\tilde{X}}\cap\mathcal{O}_X(J^m)$ is independent of the choice of such $\tilde{X}$. Φ is said to be *p-closed* at X if $(p\Phi)_{\tilde{X}}\cap\mathcal{O}_X(J^m)$ is contained in Φ_X (M. Matsuda [13]). Let X be an ordinary integral point. Then, in order that Φ is p-closed at X, it is necessary and sufficient that there exists a neighbourhood $\mathcal{U}$ of X such that the mapping ρ_m^{m+1} from $I(p\Phi)\cap(\rho_m^{m+1})^{-1}\mathcal{U}$ to $I\Phi\cap\mathcal{U}$ is surjective (M. Matsuda [13]).

Involutive systems of partial differential equations were characterized by M. Kuranishi [12]. Here we state a criterion of involution given by M. Matsuda [13].

THEOREM A. *Φ is involutive at X_0 if and only if the following four conditions are satisfied:*

(i) *X_0 is an ordinary integral point of Φ.*

(ii) *$\dim pC_X(\Phi)$ remains constant on a neighbourhood of X_0 in $I\Phi$.*

(iii) *$C_{X_0}(\Phi)$ is an involutive subspace of $Q_{\bar{X}_0}(J^{m-1})\otimes T^*_{a_0}(N)$* $(\bar{X}_0=\rho_{m-1}^m X_0,\ a_0=\rho_{-1}^m X_0)$.

(iv) *Φ is p-closed at X_0.*

REMARK. If Φ is involutive at X_0, then Φ is involutive at each point in a neighbourhood of X_0 in $I\Phi$.

Finally we recall the link between involutive systems of differential equations and involutive differential ideals. Let $(x_1,\cdots,x_n,y_1,\cdots,y_l)$ be a coordinate system of M on U ($n=\dim N$, $n+l=\dim M$) such that there exists a coordinate system $(x'_1,\cdots,x'_n)$ of N on ρU satisfying $x_i=x'_i\circ\rho$ $(1\leq i\leq n)$, and let

$$(x_i,y_\lambda,p_\lambda^i,p_\lambda^{i_1i_2},\cdots,p_\lambda^{i_1\cdots i_m};\quad 1\leq i,i_\alpha\leq n,1\leq\lambda\leq l)$$

be a coordinate system on $(\rho_0^m)^{-1}U$, where

$$p_\lambda^{i_1\cdots i_\alpha}(j_a^m(f))=\frac{\partial^\alpha f_\lambda}{\partial x_{i_1}\cdots\partial x_{i_\alpha}}(a),\qquad f_\lambda\text{ being }y_\lambda\text{-coordinate of }f.$$

We shall call the Pfaffian forms on $J^m(M,N,\rho)$

$$\begin{cases}dy_\lambda-\sum_{i=1}^n p_\lambda^i dx_i & (1\leq\lambda\leq l),\\ dp_\lambda^{i_1\cdots i_\alpha}-\sum_{i=1}^n p_\lambda^{i_1\cdots i_\alpha i}dx_i & (1\leq\lambda\leq l,1\leq i_1,\cdots,i_\alpha\leq n,1\leq\alpha<m)\end{cases}$$

the contact forms of orders up to and including m. Let $\mathcal{M}$ be an n-dimensional manifold in $J^m(M,N,\rho)$ satisfying $\dim(\rho_{-1}^m)_*T(\mathcal{M})=n$. The contact forms of orders up to and including m are characterized by the fact that the restrictions of those

Pfaffian forms to $\mathscr{M}$ vanish if and only if $\mathscr{M}$ is a manifold derived from a cross-section f of (M, N, ρ) defined on an open set V in N; $\mathscr{M}=\{j_a^m(f); a\in V\}$.

One can associate with Φ the differential ideal $\Sigma(\Phi)$ defined on the domain of Φ in $J^m(M, N, \rho)$ generated by Φ and the contact forms of orders up to and including m. Every solution of Φ defines an n-dimensional manifold $\mathscr{M}$ in $J^m(M, N, \rho)$ satisfying $\dim(\rho_{-1}^m)_*T(\mathscr{M})=n$ which is an integral manifold of $\Sigma(\Phi)$. Conversely, such $\mathscr{M}$ defines a solution of Φ. We shall also call such an integral manifold of $\Sigma(\Phi)$ in $J^m(M, N, \rho)$ a solution of Φ.

$\Sigma(\Phi)$ is said to be *involutive at X with respect to N* if there exists at least one n-dimensional integral element E_n of origin X satisfying $\dim(\rho_{-1}^m)_*E_n=n$ and if every such integral element E_n is an ordinary integral element in the sense that there exists a chain of integral elements of $\Sigma(\Phi)$ $E_0\subset E_1\subset\cdots\subset E_n$ which ends with E_n, where $E_0, E_1, \cdots, E_{n-1}$ are regular integral elements of dimension $0, 1, \cdots, n-1$ respectively (such a chain is called a regular chain) (cf. E. Cartan [5], Chap. V; M. Kuranishi [11], Definition I.17).

Comparing the criterion for a differential ideal of higher order to be involutive given by M. Matsuda [14] with the conditions in Theorem A, we can show the following (cf. M. Kuranishi [12], Appendix).

THEOREM B. *Φ is involutive at X_0 if and only if $\Sigma(\Phi)$ is involutive at X_0 with respect to N.*

§2. Involutive systems in two independent variables.

Hereafter, in this memoir, we shall consider only systems of partial differential equations with a single unknown function of two independent variables; We shall always assume that $\dim M=3$ and $\dim N=2$. Let Φ be a system of order m. Let $(x, y, z, p_{i,k};\ 1\leq i+k\leq m)$ be such a coordinate system of $J^m(M, N, \rho)$ on $(\rho_0^m)^{-1}U$ as is stated in §1, where (x, y, z) is a coordinate system of M on an open set U and $p_{i,k}(X)=\partial^{i+k}z(x, y)/\partial x^i\partial y^k$ when $X=j_a^m(f)$ $(a=\rho_{-1}^m X)$, $z(x, y)$ being z-coordinate of f. Using this coordinate system, the contact forms of orders up to and including m are given by

$$dz-p_{1,0}dx-p_{0,1}dy\,,\quad dp_{i,k}-p_{i+1,k}dx-p_{i,k+1}dy\qquad(1\leq i+k<m)\,.$$

$\Sigma(\Phi)$ is generated by these Pfaffian forms and Φ as a differential ideal.

Let us deduce a necessary and sufficient condition for Φ to be involutive. Let X be an integral point contained in the above coordinate neighbourhood. For brevity, we shall write $p_\beta = p_{m-\beta\cdot\beta}$ $(0 \leqq \beta \leqq m)$. The following is immediately obtained from the definition

$$(1)\quad \begin{cases} C_X(\Phi) = \left\{ \sum_{\beta=0}^{m} \zeta_\beta \frac{\partial}{\partial p_\beta} ; \quad \sum_{\beta=0}^{m} \frac{\partial F}{\partial p_\beta} \zeta_\beta = 0 \quad \text{for every} \quad F \in \Phi_X \right\} , \\ pC_X(\Phi) = \left\{ \sum_{\beta=0}^{m} \varphi_\beta \frac{\partial}{\partial p_\beta} \otimes dx + \sum_{\beta=0}^{m} \psi_\beta \frac{\partial}{\partial p_\beta} \otimes dy ; \right. \\ \quad (*) \quad \left. \begin{bmatrix} \varphi_\beta = \psi_{\beta-1} \quad (0 < \beta \leqq m) , \\ \sum_{\beta=0}^{m} \frac{\partial F}{\partial p_\beta} \varphi_\beta = \sum_{\beta=0}^{m} \frac{\partial F}{\partial p_\beta} \psi_\beta = 0 \quad \text{for every} \quad F \in \Phi_X \end{bmatrix} \right\} . \end{cases}$$

Let π_m^* denote the projection from $T_X^*(J^m)$ onto $Q_X^*(J^m)$, the dual space of $Q_X(J^m)$. We denote by $\langle v_\lambda ;\ \lambda \in \Lambda \rangle$ the vector space spanned by the vectors $\{v_\lambda ;\ \lambda \in \Lambda\}$. To calculate the dimensions of the above spaces, we introduce the following notations.

$$r_m(\Phi;\ X) = \dim \langle \pi_m^* dF ;\ F \in \Phi_X \rangle ,$$
$$r_{m+1}(\Phi;\ X) = \dim \langle \pi_{m+1}^* dF ;\ F \in (p\Phi)_{\tilde{X}} \rangle ,$$

where $\tilde{X}$ is a point of $(\rho_m^{m+1})^{-1} X$. We shall call $r_k(\Phi;\ X)$ $(k = m, m+1)$ the *rank of order* k of Φ at X. It is noticed that $r_{m+1}(\Phi;\ X)$ is well-defined since the right side of the above definition is independent of the choice of $\tilde{X}$ over X. From (1), we immediately obtain the following equalities.

$$(2)\quad \begin{cases} \dim C_X(\Phi) = m + 1 - r_m(\Phi;\ X) , \\ \dim pC_X(\Phi) = m + 2 - r_{m+1}(\Phi;\ X) . \end{cases}$$

In the case when $\dim M = 3$ and $\dim N = 2$, we have a simple criterion for a subspace of $Q_X(J^m)$ to be an involutive subspace of $Q_{\bar{X}}(J^{m-1}) \otimes T_a^*(N)$ $(\bar{X} = \rho_{m-1}^m X$, $a = \rho_{-1}^m X)$.

LEMMA 1. *A proper subspace A of $Q_X(J^m)$ is an involutive subspace of $Q_{\bar{X}}(J^{m-1}) \otimes T_a^*(N)$ if and only if* $\dim pA$ *is equal to* $\dim A$.

PROOF. By definition, $g_0 = \dim A$. $g_1 = \min \{\dim A(v);\ v \in T_a(N)\}$ is equal to zero. Since $g_1 \geqq g_2 = \min \{\dim A(v_1, v_2);\ v_1, v_2 \in T_a(N)\}$, g_2 is equal to zero. Therefore, $\dim pA$ is equal to $g_0 + g_1 + g_2$ if and only if $\dim pA$ is equal to $\dim A$. Q.E.D.

From (2) and Lemma 1, we obtain the following

PROPOSITION 1. *$C_X(\Phi)$ is an involutive subspace of $Q_{\bar{X}}(J^{m-1}) \otimes T_a^*(N)$ if and only if*

$$(3)\quad r_{m+1}(\Phi;\ X) = r_m(\Phi;\ X) + 1 .$$

Let X_0 be an ordinary integral point of Φ. Assume that Φ is p-closed at X_0

and that $r_m(\Phi; X)$ is constant on a neighbourhood of X_0 in $I\Phi$. If (3) is valid on a neighbourhood of X_0 in $I\Phi$, then $C_{X_0}(\Phi)$ is an involutive subspace of $Q_{X_0}(J^{m-1})\otimes T^*_{a_0}(N)$ and $\dim pC_X(\Phi)$ remains constant on that neighbourhood. Hence, by Theorem A, Φ is involutive at X_0. Conversely, suppose that Φ is involutive at X_0. Then $C_X(\Phi)$ is an involutive subspace of $Q_X(J^{m-1})\otimes T^*_a(N)$ for each point X on a neighbourhood of X_0 in $I\Phi$. Hence, by Proposition 1, (3) is valid for each point X on that neighbourhood in $I\Phi$. Since it follows from condition (ii) in Theorem A that $r_{m+1}(\Phi; X)$ is constant on a neighbourhood of X_0 in $I\Phi$, $r_m(\Phi; X)$ remains constant on a neighbourhood of X_0 in $I\Phi$. Thus we have proved the following theorem.

THEOREM I. *Φ is involutive at X_0 if and only if the following four conditions are satisfied:*

(i) *X_0 is an ordinary integral point of Φ.*

(ii) *$r_m(\Phi; X)$ remains constant on a neighbourhood of X_0 in $I\Phi$.*

(iii) *$r_{m+1}(\Phi; X)=r_m(\Phi; X)+1$ on a neighbourhood of X_0 in $I\Phi$.*

(iv) *Φ is p-closed at X_0.*

REMARK 1. Condition (iii) in Theorem I is especially important in the forthcoming investigations. Roughly speaking, condition (iii) is nothing else than the condition that the number of independent equations exactly of order $m+1$ obtained by differentiating once the equations in Φ is greater than the number of independent equations exactly of order m in Φ by one. The systems considered by G. Cerf ([6], Chap. I) are involutive systems.

REMARK 2. Our condition (iii) is very useful. Starting with Theorem I, one can obtain a brief proof of the prolongation theorem which is different from the standard one (cf. M. Kuranishi [12], M. Matsuda [13]).

Let us parametrize the general solution of an involutive system. Suppose that Φ is involutive at X_0. Then $\Sigma(\Phi)$ is involutive at X_0 with respect to N (Theorem B). Hence, for any two-dimensional integral manifold of $\Sigma(\Phi)$ passing in a sufficiently small neighbourhood of X_0, we can apply the Cartan-Kähler theorem (E. Cartan [5], Chap. IV, part II; E. Kähler [9]). By means of this parametrization of integral manifolds of $\Sigma(\Phi)$, we can parametrize the solution of Φ: "*The general solution of Φ passing through a neighbourhood of X_0 depends upon*

$$s_0=\delta+r-m-3 \quad \textit{arbitrary constants},$$
$$s_1=m+1-r \quad \textit{arbitrary functions of one argument},$$

where $\delta=\dim I\Phi$ *at* X_0, $r=r_m(\Phi; X_0)$."

The numbers s_α are called the characters (of $\Sigma(\Phi)$) in Cartan's terminology. It is remarked that we can parametrize the general solution of Φ by means of the parametrization of integral manifolds of $p\Sigma(\Phi)$. In this case, the number s_1 of arbitrary functions of one argument occurring in the general solution is unchanged, but the number s_0 of arbitrary constants occurring in it augments in general.

§ 3. Characteristic directions.

Let us first define the characteristic polynomial of Φ at an integral point X. Let F be an analytic function defined on a neighbourhood of X in $J^l(M, N, \rho)$ satisfying $F(X)=0$ which cannot be considered as a function of $(l-1)$-jets. The *characteristic polynomial* of F at X is by definition (cf. E. Goursat [8], § 209) the element of $S^l(T^*_a(N))$ $(a=\rho^m_{-1}X)$ defined, using the local coordinate system around X, by

$$\frac{\partial F}{\partial p_{l,0}}(X)dy^l-\frac{\partial F}{\partial p_{l-1,1}}(X)dy^{l-1}dx+\cdots+(-1)^l\frac{\partial F}{\partial p_{0,l}}(X)dx^l .$$

We define the *characteristic polynomial of Φ at $X\in I\Phi$* to be the highest common factor of the characteristic polynomials at X of all local sections of Φ around X. When X is an ordinary integral point and Φ is p-closed at X, the characteristic polynomial of Φ at X is nothing else than the highest common factor of the polynomials

$$F^{(0)}dy^m-F^{(1)}dy^{m-1}dx+\cdots+(-1)^mF^{(m)}dx^m , \qquad F\in\Phi_X ,$$

where $F^{(\beta)}=\partial F(X)/\partial p_{m-\beta,\beta}$. When the characteristic polynomial at X is of degree q, it is an element of $S^q(T^*_a(N))$. We shall call a vector $v\,(\neq 0)\in T_a(N)$ a *characteristic vector of Φ at X* if the characteristic polynomial vanishes at $v\otimes v\otimes\cdots\otimes v$ (q product). When $v=\lambda\partial/\partial x+\mu\partial/\partial y$, v is a characteristic vector if and only if the characteristic polynomial represented by the same coordinate system vanishes when $(dy, dx)=(\mu, \lambda)$ is substituted into the characteristic polynomial. The direction of a characteristic vector of Φ at X is called a *characteristic direction of Φ at X*. We shall sometimes identify a characteristic direction c defined by v with the one-dimensional subspace c of $T_a(N)$ spanned by v. The characteristic directions play an important role in the theory of characteristic systems (in the sense of Monge) which will be discussed later on.

THEOREM II. *Suppose that Φ is involutive at X. Then the degree of the* *characteristic polynomial of Φ at X is equal to* $\dim C_X(\Phi)=m+1-r_m(\Phi; X)$.

To prove this theorem, we shall prepare an algebraic lemma concerning polynomials which is fundamental in the subsequent considerations. Let K be a field. Let us consider r homogeneous polynomials in two indeterminates ξ, η:

$$P_\alpha(\xi,\eta)=a_0^{(\alpha)}\xi^m+a_1^{(\alpha)}\xi^{m-1}\eta+\cdots+a_m^{(\alpha)}\eta^m \qquad (\alpha=1,2,\cdots,r),$$

where $a_i^{(\alpha)}\in K$. We shall denote by $\langle P_1,P_2,\cdots,P_r\rangle$ the vector space spanned by the polynomials $P_1,P_2,\cdots,P_r$ over K in the ring $K[\xi,\eta]$ of polynomials and denote by $(P_1,P_2,\cdots,P_r)$ the highest common factor of r polynomials $P_1,P_2,\cdots,P_r$; $(P_1,P_2,\cdots,P_r)$ is a generator of the ideal of $K[\xi,\eta]$ generated by $P_1,P_2,\cdots,P_r$.

LEMMA 2. *Assume that*

$$\dim\langle\xi P_1,\eta P_1,\cdots,\xi P_r,\eta P_r\rangle=\dim\langle P_1,\cdots,P_r\rangle+1.$$

Then

$$\deg(P_1,\cdots,P_r)=m+1-\dim\langle P_1,\cdots,P_r\rangle.$$

Original proof of the author heavily depends upon the general theory of resultants. We shall present here an elementary proof given by Matsuda.

PROOF. Without loss of generality, we may assume that $\dim\langle P_1,\cdots,P_r\rangle$ is equal to r. Let us prove this by induction on the degree m. When $m=1$, the assertion is obviously valid. Assume that the assertion is valid in the case where the degree is less than m. Clearly $r\leqq m+1$. When $r=m+1$, $\deg(P_1,\cdots,P_r)$ is equal to zero and this is equal to $m+1-r$. Hence we may assume that $r\leqq m$. The assumption implies that the following formula is valid (in which the order of the polynomials is interchanged if necessary):

$$\eta P_\alpha=b_1^{(\alpha)}\xi P_1+\cdots+b_r^{(\alpha)}\xi P_r+b^{(\alpha)}\eta P_1 \qquad (\alpha=2,3,\cdots,r).$$

Hence

$$(b_1^{(\alpha)}\xi+b^{(\alpha)}\eta)P_1+b_2^{(\alpha)}\xi P_2+\cdots+(b_\alpha^{(\alpha)}\xi-\eta)P_\alpha+\cdots+b_r^{(\alpha)}\xi P_r=0 \qquad (\alpha=2,3,\cdots,r).$$

We denote by $\Delta(\xi,\eta)$ the determinant of the matrix $(b_i^{(\alpha)}\xi-\eta\delta_i^\alpha;\ 1<\alpha,i\leqq r)$. Then the above equations imply that

$$\Delta(\xi,\eta)P_\alpha(\xi,\eta)=H_\alpha(\xi,\eta)P_1(\xi,\eta) \qquad (\alpha=2,3,\cdots,r),$$

where $H_\alpha\in K[\xi,\eta]$. By assumption $\deg\Delta(=r-1)<m$. Therefore there exists a factor of P_1 which is not a factor of Δ. Since that factor is a common factor of $P_1,\cdots,P_r$, $P=(P_1,\cdots,P_r)$ is of degree greater than zero. We define polynomials

Q_α by $PQ_\alpha=P_\alpha$ $(1\leqq\alpha\leqq r)$. Then $n=\deg Q_\alpha<m$ and $\dim\langle Q_1,\cdots,Q_r\rangle=r$. Furthermore, $\xi Q_1,\xi Q_2,\cdots,\xi Q_r,\eta Q_1$ are linearly independent of one another and

$$\eta Q_\alpha=b_1^{(\alpha)}\xi Q_1+\cdots+b_r^{(\alpha)}\xi Q_r+b^{(\alpha)}\eta Q_1 \qquad (\alpha=2,3,\cdots,r).$$

Hence, by the induction assumption, $\deg(Q_1,\cdots,Q_r)$ is equal to $n+1-r$. On the other hand, $\deg(Q_1,\cdots,Q_r)$ is equal to zero. Therefore $\deg f$ is equal to $m-n=m+1-r$. Q.E.D.

PROOF OF THEOREM II. We write $r=r_m(\Phi;X)$. We can choose a system of local generators $\{F_1,F_2,\cdots,F_r,f_1,\cdots,f_t\}$ of Φ at X in such a manner that $\mathrm{rank}\,(\partial(F_1,\cdots,F_r)/\partial(p_{m,0},\cdots,p_{0,m}))(X)$ is equal to r and $\pi_m^*df_a=0$ on a neighbourhood of X in $I\Phi$. Then the characteristic polynomial of Φ at X is the highest common factor of the polynomials

$$(4)\quad G_\alpha(dy,dx)=F_\alpha^{(0)}dy^m-F_\alpha^{(1)}dy^{m-1}dx+\cdots+(-1)^mF_\alpha^{(m)}dx^m \qquad (\alpha=1,2,\cdots,r),$$

where $F_\alpha^{(\beta)}=\partial F_\alpha(X)/\partial p_{m-\beta,\beta}$. $\dim\langle G_1,\cdots,G_r\rangle$ is equal to r. It is easy to see that $r_{m+1}(\Phi;X)$ is equal to $\dim\langle dyG_1,dxG_1,\cdots,dyG_r,dxG_r\rangle$. Since, by Theorem I, $r_{m+1}(\Phi;X)$ is equal to $r+1$, we can apply Lemma 2 to the polynomials $G_1,\cdots G_r$. Hence the highest common factor of $G_1,\cdots,G_r$ is of degree $m+1-r$. Thus we have shown the desired result. Q.E.D.

Suppose that Φ is involutive at X. Theorem II implies that the number of characteristic directions of Φ at X is equal to $m+1-r$. Here, the number of characteristic directions is counted with their multiplicities equal to the multiplicities of the corresponding linear factor of the characteristic polynomial and when every notions are assumed to be in the category of real analyticity, the "imaginary" directions are also counted. Remarking the parametrization of the general solution of Φ stated at the end of §2, we obtain from Theorem II the following fact which is intuitively conjectured.

"*The number of arbitrary functions of one argument occurring in the general solution of Φ passing through a neighbourhood of X is equal to the number of characteristic directions of Φ at X.*"

§4. Algebraic considerations concerning characteristics.

For a vector $v\in T_a(N)$ $(a=\rho_{-1}^m X)$, the linear mapping $\sigma(v)$ from $Q_X(J^m)\otimes T_a^*(N)$ to $Q_X(J^m)$ is defined by extending linearly the mapping

$$\sigma(v)(w\otimes v^*)=v^*(v)w\,,\quad \text{where}\quad w\in Q_X(J^m),\ v^*\in T_a^*(N)\,.$$

Using the coordinate system around $X, \sigma(v)$ can be defined as follows: When $v=\lambda\partial/\partial x+\mu\partial/\partial y$,

$$\sigma(v)\Big(\sum_{\beta=0}^{m}\varphi_\beta\frac{\partial}{\partial p_\beta}\otimes dx+\sum_{\beta=0}^{m}\psi_\beta\frac{\partial}{\partial p_\beta}\otimes dy\Big)=\sum_{\beta=0}^{m}(\lambda\varphi_\beta+\mu\psi_\beta)\frac{\partial}{\partial p_\beta}.$$

For any vector $v\in T_a(N)$, the mapping $\sigma(v)$ induces a mapping from $pC_X(\Phi)$ to $C_X(\Phi)$. This is a direct consequence of (1).

PROPOSITION 2. *Suppose that Φ is involutive at X. Let v be a non-zero vector of $T_a(N)$. The mapping $\sigma(v)$ from $pC_X(\Phi)$ to $C_X(\Phi)$ is surjective if and only if v is not a characteristic vector of Φ at X. When v is a characteristic vector, the image of $\sigma(v)$ is a subspace of $C_X(\Phi)$ of codimension* 1.

PROOF. Since Φ is involutive at X, $\dim pC_X(\Phi)$ is equal to $\dim C_X(\Phi)$ (Lemma 1). The kernel of $\sigma(v)$ is the space

$$\Big\{\sum_{\beta=0}^{m}\varphi_\beta\frac{\partial}{\partial p_\beta}\otimes dx+\sum_{\beta=0}^{m}\psi_\beta\frac{\partial}{\partial p_\beta}\otimes dy\in pC_X(\Phi)\ ;\quad \lambda\varphi_\beta+\mu\psi_\beta=0\quad(0\leqq\beta\leqq m)\Big\}.$$

This space is of dimension at most 1 and we readily see that it is of dimension 1 if and only if $v=\lambda\partial/\partial x+\mu\partial/\partial y$ is a characteristic vector of Φ at X (cf. (*) in (1)). Hence the assertion is proved. Q.E.D.

From now on, in this section, we shall always assume that Φ is involutive at X when nothing is expressly stated. Let $\{F_1,\cdots,F_r,f_1,\cdots,f_t\}$ be such a system of local generators of Φ at X as in the proof of Theorem II. Then we have the following.

$$C_X(\Phi)=\Big\{\sum_{\beta=0}^{m}\zeta_\beta\frac{\partial}{\partial p_\beta}\ ;\quad \sum_{\beta=0}^{m}F_\alpha^{(\beta)}\zeta_\beta=0\qquad(1\leqq\alpha\leqq r)\Big\},$$

$$pC_X(\Phi)=\left\{\sum_{\beta=0}^{m}\varphi_\beta\frac{\partial}{\partial p_\beta}\otimes dx+\sum_{\beta=0}^{m}\psi_\beta\frac{\partial}{\partial p_\beta}\otimes dy\ ;\quad (*)\ \begin{bmatrix}\varphi_\beta=\psi_{\beta-1}\qquad(0<\beta\leqq m),\\ \sum_{\beta=0}^{m}F_\alpha^{(\beta)}\varphi_\beta=\sum_{\beta=0}^{m}F_\alpha^{(\beta)}\psi_\beta=0\qquad(1\leqq\alpha\leqq r)\end{bmatrix}\right\}.$$

Let v be a characteristic vector of Φ at X. Let us denote by $C(v)$ the image of $\sigma(v)$ in Proposition 2. We can choose a system of scalars $(e^0(v),e^1(v),\cdots,e^m(v))$ in such a manner that

$$\begin{cases}C(v)=\Big\{\sum_{\beta=0}^{m}\zeta_\beta\frac{\partial}{\partial p_\beta}\ ;\quad \sum_{\beta=0}^{m}F_\alpha^{(\beta)}\zeta_\beta=0\quad(1\leqq\alpha\leqq r),\quad \sum_{\beta=0}^{m}e^\beta(v)\zeta_\beta=0\Big\},\\ \operatorname{rank}(F_\alpha^{(\beta)},e^\beta(v)\ ;\ 0\leqq\beta\leqq m,\ 1\leqq\alpha\leqq r)=r+1.\end{cases}$$

When $v=\lambda\partial/\partial x+\mu\partial/\partial y$, such $(e^0(v),\cdots,e^m(v))$ is characterized by the following condition:

$$(5)\qquad \begin{cases}\lambda\sum_{\beta=0}^{m}e^{\beta}(v)\varphi_{\beta}+\mu\sum_{\beta=0}^{m}e^{\beta}(v)\psi^{\beta}\equiv 0 \qquad (\mathrm{mod}\ (*))\,, \\ \mathrm{rank}\,(F_{\alpha}^{(\beta)},e^{\beta}(v)\ ;\ 0\leqq\beta\leqq m\,,\ 1\leqq\alpha\leqq r)=r+1\,.\end{cases}$$

LEMMA 3. *Suppose that Φ is involutive at X. The subspace C' of $C_X(\Phi)$ of codimension 1 defined by*

$$(6)\qquad C'=\left\{\sum_{\beta=0}^{m}\zeta_{\beta}\frac{\partial}{\partial p_{\beta}}\ ;\quad \sum_{\beta=0}^{m}F_{\alpha}^{(\beta)}\zeta_{\beta}=0\quad(1\leqq\alpha\leqq r)\,,\quad \sum_{\beta=0}^{m}e^{\beta}\zeta_{\beta}=0\right\}$$

is an involutive subspace of $Q_X(J^{m-1})\otimes T_a^*(N)$ if and only if there exists $(\lambda,\mu)\neq 0$ such that

$$(7)\qquad \lambda\sum_{\beta=0}^{m}e^{\beta}\varphi_{\beta}+\mu\sum_{\beta=0}^{m}e^{\beta}\psi_{\beta}\equiv 0 \qquad (\mathrm{mod}\ (*)).$$

PROOF. Clearly, pC' is given by

$$pC'=\left\{\sum_{\beta=0}^{m}\varphi_{\beta}\frac{\partial}{\partial p_{\beta}}\otimes dx+\sum_{\beta=0}^{m}\psi_{\beta}\frac{\partial}{\partial p_{\beta}}\otimes dy\ ;\ (*),\ \ \sum_{\beta=0}^{m}e^{\beta}\varphi_{\beta}=\sum_{\beta=0}^{m}e^{\beta}\psi_{\beta}=0\right\}.$$

Hence $\dim pC'$ is equal to $m-r$ if and only if there exists $(\lambda,\mu)\neq 0$ such that (7) is valid. Since $\dim C'$ is equal to $m-r$, we obtain the required result by Lemma 1. Q.E.D.

By Lemma 3, it follows from (5) that $C(v)$ is an involutive subspace of $Q_X(J^{m-1})\otimes T_a^*(N)$, where v is a characteristic vector. Furthermore, the following proposition can be shown. Let c be a characteristic direction of Φ at X. We define the space $C(c)$ to be $C(v)$, where v $(\neq 0)$ is a vector belonging to c. Obviously $C(c)$ is determined only by c.

PROPOSITION 3. *Suppose that Φ is involutive at X. In order that a subspace C' of $C_X(\Phi)$ of codimension 1 is an involutive subspace of $Q_X(J^{m-1})\otimes T_a^*(N)$, it is necessary and sufficient that there exists a characteristic direction c of Φ at X such that C' coincides with $C(c)$.*

PROOF. It remains to prove the necessity. Let C' be defined by (6). By Lemma 3, C' is an involutive subspace if and only if (7) is valid for certain $(\lambda,\mu)\neq 0$. Condition (7) is equivalent to

$$(8)\qquad \lambda\xi R-\mu\eta R\equiv 0 \qquad (\mathrm{mod}\ \xi G_1,\eta G_1,\cdots,\xi G_r,\eta G_r)\,,$$

where G_α are the polynomials (4) in which dy, dx are replaced by ξ, η respectively and $R(\xi,\eta)=\sum_{\beta=0}^{m}(-1)^\beta e^\beta \xi^{m-\beta}\eta^\beta$. If we denote by G the highest common factor of $G_1,\cdots,G_r$, that is, the characteristic polynomial of Φ at X, then (8) can be written in the form

$$(8')\qquad (\lambda\xi-\mu\eta)R\equiv 0 \qquad (\mathrm{mod}\, G)\,.$$

From (8), we also have

$$(9)\qquad \dim\langle \xi G_1,\eta G_1,\cdots,\xi G_r,\eta G_r,\xi R,\eta R\rangle=\dim\langle G_1,\cdots,G_r,R\rangle+1\,.$$

Hence, by Lemma 2, $\deg(G_1,\cdots,G_r,R)$ is equal to $m-r$; $\deg(G,R)$ is less than $\deg G$ by one. This fact and (8′) imply that $(\lambda\xi-\mu\eta)$ is a factor of G; the vector $v=\lambda\partial/\partial x+\mu\partial/\partial y$ is a characteristic vector of Φ at X. By the fact already stated (see (5)), (7) shows that C' coincides with $C(v)$. Q.E.D.

NOTE 1. $(G_1,\cdots,G_r,R)$ is obtained from $G=(G_1,\cdots,G_r)$ by excluding the linear factor $(\lambda\xi-\mu\eta)$.

NOTE 2. C' defined by (6) is an involutive subspace if and only if (9) is valid.

The above proposition can be generalized in some sense to the following

PROPOSITION 4. *Suppose that Φ is involutive at X. Let $c_1,c_2,\cdots,c_s$ be s different characteristic directions of Φ at X. Then $C(c_1)\cap C(c_2)\cap\cdots\cap C(c_s)$ is an involutive subspace of $Q_X(J^{m-1})\otimes T_a^*(N)$ of dimension* $\dim C_X(\Phi)-s$.

PROOF. Let $C(c_i)$ be given by

$$C(c_i)=\left\{\sum_{\beta=0}^{m}\zeta_\beta\frac{\partial}{\partial p_\beta}\,;\ \sum_{\beta=0}^{m}F_\alpha^{(\beta)}\zeta_\beta=0\ (1\leqq\alpha\leqq r)\,,\ \sum_{\beta=0}^{m}e_i^\beta\zeta_\beta=0\right\}.$$

Then $\tilde C=C(c_1)\cap\cdots\cap C(c_s)$ is the subspace

$$\left\{\sum_{\beta=0}^{m}\zeta_\beta\frac{\partial}{\partial p_\beta}\,;\ \sum_{\beta=0}^{m}F_\alpha^{(\beta)}\zeta_\beta=0\ (1\leqq\alpha\leqq r)\,,\ \sum_{\beta=0}^{m}e_i^\beta\zeta_\beta=0\ \ (1\leqq i\leqq s)\right\}$$

and $p\tilde C$ is given by

$$p\tilde C=\left\{\sum_{\beta=0}^{m}\varphi_\beta\frac{\partial}{\partial p_\beta}\otimes dx+\sum_{\beta=0}^{m}\phi_\beta\frac{\partial}{\partial p_\beta}\otimes dy\,;\ (*),\ \sum_{\beta=0}^{m}e_i^\beta\varphi_\beta=\sum_{\beta=0}^{m}e_i^\beta\phi_\beta=0\quad(1\leqq i\leqq s)\right\}.$$

We write $R_i(\xi,\eta)=\sum_{\beta=0}^{m}(-1)^\beta e_i^\beta\xi^{m-\beta}\eta^\beta$ $(i=1,2,\cdots,s)$. Then we obtain

$$\begin{cases}\dim\tilde C=\mathrm{m}+1-\dim\langle G_1,\cdots,G_r,R_1,\cdots,R_s\rangle\,,\\ \dim p\tilde C=m+2-\dim\langle\xi G_1,\eta G_1,\cdots,\xi G_r,\eta G_r,\xi R_1,\eta R_1,\cdots,\xi R_s,\eta R_s\rangle.\end{cases}$$

Hence, by Lemma 1, $\tilde C$ is an involutive subspace of $Q_X(J^{m-1})\otimes T_a^*(N)$ if and only if

$$\text{(10)}\qquad \begin{aligned}&\dim\langle \xi G_1, \eta G_1, \cdots, \xi G_r, \eta G_r, \xi R_1, \eta R_1, \cdots, \xi R_s, \eta R_s\rangle\\ &\quad=\dim\langle G_1, \cdots, G_r, R_1, \cdots, R_s\rangle+1\,.\end{aligned}$$

Since $C(c_i)$ is an involutive subspace,

$$\dim\langle \xi G_1, \eta G_1, \cdots, \xi G_r, \eta G_r, \xi R_i, \eta R_i\rangle=\dim\langle G_1, \cdots, G_r, R_i\rangle+1$$

and $(G_1, \cdots, G_r, R_i)$ is obtained from $(G_1, \cdots, G_r)$ by excluding the linear factor $(\lambda_i\xi-\mu_i\eta)$, c_i being supposed to be defined by $\lambda_i\partial/\partial x+\mu_i\partial/\partial y$ (the proof of Proposition 3 and Notes 1, 2). Equality (10) follows from this fact, and hence $\tilde{C}$ is an involutive subspace. Moreover, since $c_1, \cdots, c_s$ are distinct from one another,

$$\deg(G_1, \cdots, G_r, R_1, \cdots, R_s)=m+1-r-s\,.$$

Therefore, using Lemma 2, it is shown that

$$\dim\langle G_1, \cdots, G_r, R_1, \cdots, R_s\rangle=r+s\,.$$

This implies that $\dim\tilde{C}$ is equal to $\dim C_X(\Phi)-s$. Thus the proof is complete.

Q.E.D.

Let E_2^0 be a two-dimensional subspace of $T_X(J^m)$ satisfying $\dim(\rho^m_{-1})_*E_2^0=2$ on which the contact forms of orders up to and including m vanish. Using the coordinate system of $J^m(M, N, \rho)$ around X, such E_2^0 is spanned by two vectors of the form

$$\frac{d}{dx}+\sum_{\beta=0}^{m}\varphi_\beta^0\frac{\partial}{\partial p_\beta}\,,\quad \frac{d}{dy}+\sum_{\beta=0}^{m}\psi_\beta^0\frac{\partial}{\partial p_\beta}\,,$$

where

$$\frac{d}{dx}=\frac{\partial}{\partial x}+p_{1,0}\frac{\partial}{\partial z}+\sum_{i+k=1}^{m-1}p_{i+1,k}\frac{\partial}{\partial p_{i,k}}\,,\quad \frac{d}{dy}=\frac{\partial}{\partial y}+p_{0,1}\frac{\partial}{\partial z}+\sum_{i+k=1}^{m-1}p_{i,k+1}\frac{\partial}{\partial p_{i,k}}\,.$$

E_2^0 is an integral element of $\Sigma(\Phi)$ if and only if

$$\text{(11)}\qquad \begin{cases}\varphi_\beta^0=\psi_{\beta-1}^0 \qquad (0<\beta\leqq m)\,,\\ \dfrac{dF}{dx}+\displaystyle\sum_{\beta=0}^{m}\frac{\partial F}{\partial p_\beta}\varphi_\beta^0=\frac{dF}{dy}+\sum_{\beta=0}^{m}\frac{\partial F}{\partial p_\beta}\psi_\beta^0=0 \quad \text{for every} \quad F\in\Phi_X\,.\end{cases}$$

Since $\Sigma(\Phi)$ is involutive at X with respect to N (Theorem B), there exists at least one two-dimensional integral element E_2 of $\Sigma(\Phi)$ of origin X satisfying $\dim(\rho^m_{-1})_*E_2=2$.

DEFINITION. Let c be a characteristic direction of Φ at X. We define the subspace $B(c)$ of $T_X(J^m)$ to be the set of all vectors each of which is contained in a two-dimensional integral element E_2 of $\Sigma(\Phi)$ satisfying $\dim(\rho^m_{-1})_* E_2=2$ and is also contained in $(\rho^m_{-1})^{-1}_* c$. (See the following proposition.)

PROPOSITION 5. *Suppose that Φ is involutive at X. Let c be a characteristic direction of Φ at X. Then* $\dim B(c)$ *is equal to* $m+1-r_m(\Phi; X)$ *and*

$$B(c)=\left\{\lambda\left(\frac{d}{dx}+\sum_{\beta=0}^{m}\varphi_\beta\frac{\partial}{\partial p_\beta}\right)+\mu\left(\frac{d}{dy}+\sum_{\beta=0}^{m}\psi_\beta\frac{\partial}{\partial p_\beta}\right);\quad \lambda\frac{\partial}{\partial x}+\mu\frac{\partial}{\partial y}\in c,\right.$$

$$(**)\qquad \left.\begin{cases}\varphi_\beta=\psi_{\beta-1} & (0<\beta\leqq m),\\ \dfrac{dF}{dx}+\displaystyle\sum_{\beta=0}^{m}\frac{\partial F}{\partial p_\beta}\varphi_\beta=\frac{dF}{dy}+\sum_{\beta=0}^{m}\frac{\partial F}{\partial p_\beta}\psi_\beta=0 & \text{for every}\quad F\in\Phi_X\end{cases}\right\}.$$

PROOF. The first part is an immediate consequence of the last and Proposition 1. If a vector Δ is contained in such a two-dimensional integral element as stated in the definition, Δ can be written in the form

$$\lambda\left(\frac{d}{dx}+\sum_{\beta=0}^{m}\varphi_\beta\frac{\partial}{\partial p_\beta}\right)+\mu\left(\frac{d}{dy}+\sum_{\beta=0}^{m}\psi_\beta\frac{\partial}{\partial p_\beta}\right),$$

where $\varphi_\beta, \psi_\beta$ satisfy (**) (cf. (11)). Since $(\rho^m_{-1})_*\Delta=\lambda\partial/\partial x+\mu\partial/\partial y$, Δ belongs to $B(c)$ if and only if $\lambda\partial/\partial x+\mu\partial/\partial y$ belongs to c. Q.E.D.

DEFINITION. Let c be a characteristic direction of Φ at X. We define the subspace $D(c)$ of $T^*_X(J^m)$ as the annihilator of $B(c)$.

We shall denote by $\{\Sigma_1(\Phi)\}_X$ the subspace of $T^*_X(J^m)$ spanned by the germs of linear forms of $\Sigma(\Phi)$ at X.

PROPOSITION 6. *Suppose that Φ is involutive at X. Let c be a characteristic direction of Φ at X. Then $D(c)$ is spanned by*

$$(12)\qquad \begin{cases}\{\Sigma_1(\Phi)\}_X,\quad \lambda dy-\mu dx,\\ \omega(c)=\displaystyle\sum_{\beta=0}^{m}e^\beta dp_\beta-\Big(\sum_{\beta=0}^{m}e^\beta\varphi^0_\beta\Big)dx-\Big(\sum_{\beta=0}^{m}e^\beta\psi^0_\beta\Big)dy,\end{cases}$$

where $\lambda\partial/\partial x+\mu\partial/\partial y$ is a (non-zero) vector belonging to c and $(e^0,\cdots,e^m)$ is a system of scalars satisfying (5), *constructed from that vector, and $(\varphi^0_\beta,\psi^0_\beta)$ is a system of scalars satisfying* (11).

PROOF. If $(\varphi_\beta,\psi_\beta)$ is one more system of scalars satisfying (11), then from (5) we obtain

$$\sum_{\beta=0}^{m}e^\beta(\varphi_\beta-\varphi^0_\beta)dx+\sum_{\beta=0}^{m}e^\beta(\psi_\beta-\psi^0_\beta)dy\equiv 0\qquad(\mathrm{mod}\ \lambda dy-\mu dx).$$

It is easy to show that all elements in (12) annihilate $B(c)$. On the other hand, the codimension (in $T_X^*(J^m)$) of the space spanned by (12) is obviously equal to $m+1-r$ and $\dim B(c)$ is equal to $m+1-r$. Hence $D(c)$ is spanned by (12). Q.E.D.

Let E_1 be a linear integral element of $\Sigma(\Phi)$ of origin X spanned by a vector

$$\lambda\frac{d}{dx}+\mu\frac{d}{dy}+\sum_{\beta=0}^{m}\zeta_\beta\frac{\partial}{\partial p_\beta}, \qquad (\lambda,\mu)\neq 0 .$$

The polar element $H(E_1)$ of E_1 is defined to be a linear subspace of $T_X(J^m)$ spanned by every linear integral element E_1' such that the space spanned by E_1 and E_1' is an integral element of $\Sigma(\Phi)$. Let E_1' be spanned by a vector

$$\lambda'\frac{d}{dx}+\mu'\frac{d}{dy}+\sum_{\beta=0}^{m}\zeta_\beta'\frac{\partial}{\partial p_\beta} .$$

In order that E_1' is contained in $H(E_1)$, it is necessary and sufficient that

$$(13)\qquad \begin{cases} -\lambda'\zeta_\alpha-\mu'\zeta_{\alpha+1}+\lambda\zeta_\alpha'+\mu\zeta_{\alpha+1}'=0 & (0\leqq\alpha<m)\,,\\ \dfrac{dF}{dx}\lambda'+\dfrac{dF}{dy}\mu'+\displaystyle\sum_{\beta=0}^{m}\frac{\partial F}{\partial p_\beta}\zeta_\beta'=0 & (F\in\Phi_X)\,. \end{cases}$$

When E_1 is a linear integral element satisfying $\dim(\rho_{-1}^m)_*E_1=1$, $\dim H(E_1)$ is equal to or less than 3.

LEMMA 4. *Suppose that Φ is involutive at X. Let E_1 be a linear integral element of $\Sigma(\Phi)$ of origin X satisfying* $\dim(\rho_{-1}^m)_*E_1=1$. *Then,* $\dim H(E_1)$ *is equal to* 3 *if and only if E_1 is contained in $B(c)$ for a certain characteristic direction c of Φ at X.*

PROOF. When E_1 is contained in a two-dimensional integral element E_2 of $\Sigma(\Phi)$ satisfying $\dim(\rho_{-1}^m)_*E_2=2$, the system (13) with unknown $(\lambda',\mu',\zeta_0',\cdots,\zeta_m')$ induces no linear relation between λ' and μ'. Hence, calculating the rank of (13) and remembering the definition of $B(c)$, we get the desired result. Q.E.D.

Suppose now that $r_m(\Phi;X)$ is equal to m, the order of Φ. Then, by Theorem II, there is only one characteristic direction c of Φ at X. The space $B(c)$ is of dimension 1 (Proposition 5). The subspace $H(X)$ of $T_X(J^m)$, spanned by all linear integral element of $\Sigma(\Phi)$ of origin X, is of dimension 3. If E_1 is contained in $B(c)$, that is, $E_1=B(c)$, then $\dim H(E_1)$ is equal to 3 (Lemma 4). Since $H(X)$ contains $H(E_1)$, $H(X)$ coincides with $H(E_1)$. From this fact it follows the following result.

PROPOSITION 7. *Suppose that Φ is involutive at X and that $r_m(\Phi;X)$ is equal to m. Then $B(c)$ is an integral element E_1 of dimension* 1 *and for each linear element E_1' contained in $H(X)$, the element spanned by E_1 and E_1' is an integral element of $\Sigma(\Phi)$; $E_1=B(c)$ is characteristic in the sense of E. Cartan (E. Cartan*

[1], *part VIII*).

This proposition indicates that a system Φ of order m satisfying $r_m(\Phi; X)=m$ has a special property: This fact is fundamental in Darboux's method and its extension which will be discussed in §7.

§5. Characteristic systems and their invariants.

We shall now carry out analytical considerations. Let $\mathscr{V}$ be a (connected) open set in $I\Phi$. Suppose that Φ is involutive at each point on $\mathscr{V}$. In this case $\mathscr{V}$ is a manifold in $J^m(M, N, \rho)$. Let $c(X)$ (briefly c) be a characteristic direction field (briefly a characteristic direction) of Φ defined on $\mathscr{V}$; $c(X)$ is a law which assigns to each point $X \in \mathscr{V}$ a characteristic direction of Φ at X. We shall assume that $c(X)$ is *analytic* on $\mathscr{V}$ in the sense that there exists an analytic vector field $v(X)$ on $\mathscr{V}$ which defines $c(X)$. For such $c(X)$, we have the law D(c) which assigns to each point $X \in \mathscr{V}$ the subspace $D(c(X))$ of codimension $m+1-r$ ($r = r_m(\Phi; X)$) defined in the preceding section 4. D(c) is analytic on $\mathscr{V}$ in the sense that we can find a Pfaffian system, defined and analytic on $\mathscr{V}$, such that for each point $X \in \mathscr{V}$ it defines $D(c(X))$ (cf. Proposition 6). Thus D(c) can be considered as a Pfaffian system defined and analytic on $\mathscr{V}$.

DEFINITION. D(c) considered as a Pfaffian system is called the *characteristic system (of order m) of Φ on $\mathscr{V}$ corresponding to a characteristic direction c.* We shall denote it by D$^m(c)$.

Let $\{F_1, \cdots, F_r, f_1, \cdots, f_t\}$ be such a system of local generators of Φ around $X_0 \in \mathscr{V}$ as in the proof of Theorem II. When $v(X)=\lambda(X)\partial/\partial x+\mu(X)\partial/\partial y$, the characteristic system D$^m(c)$ defined around X_0 in $I\Phi$ is nothing else than the Pfaffian system

$$\begin{cases} \lambda dy-\mu dx,\ \omega(c),\ dF_\alpha \qquad (\alpha=1, 2, \cdots, r)\,, \\ dz-p_{1,0}dx-p_{0,1}dy,\ dp_{i,k}-p_{i+1,k}dx-p_{i,k+1}dy \qquad (1 \leqq i+k < m)\,, \end{cases}$$

where $\omega(c)$ is a Pfaffian form defined on a neighbourhood of X_0 in $I\Phi$ which is defined at each point as stated in Proposition 6; it can be defined in such a manner that $\omega(c)$ is analytic on that neighbourhood.

Next we define characteristic systems of higher order. By Theorem B, $\Sigma(\Phi)$ is involutive at each point on $\mathscr{V}$ with respect to N. Hence by the theorem due to E. Cartan and Y. Matsushima [17], the $(n-m)$-th prolonged system $p^{n-m}\Sigma(\Phi)$ is involutive at each point on $\mathscr{V}^n$ with respect to N, where $\mathscr{V}^n = I(p^{n-m}\Phi) \cap (\rho^n_m)^{-1}\mathscr{V}$ $(n \geqq m)$. $(\mathscr{V}^n, \mathscr{V}, \rho^n_m)$ forms a fibered manifold. $p^{n-m}\Sigma(\Phi)$ coincides with $\Sigma(p^{n-m}\Phi)$

and $p^{n-m}\Phi$ is involutive at each point on $\mathscr{Y}^n$ (Theorem B). It is clear that the characteristic polynomial of $p^{n-m}\Phi$ at X^n coincides with that of Φ at X, where X^n is a point on $\mathscr{Y}^n$ satisfying $\rho^n_m X^n = X$; In other words, the prolongation $p^{n-m}\Phi$ has the same characteristic directions at X^n as those of Φ at X.

DEFINITION. We define the *characteristic system of order n of Φ over $\mathscr{Y}$ corresponding to a characteristic direction c* to be the characteristic system (of order n) of $p^{n-m}\Phi$ defined on $\mathscr{Y}^n$ corresponding to c. We shall denote it by $D^n(c)$.

Let u be an analytic function of n-jets defined on a neighbourhood of $X^n \in \mathscr{Y}^n$.

DEFINITION. u is called an *invariant of* $D^n(c)$ *at* X^n if

$$du \equiv 0 \quad (\mathrm{mod}\, D^n(c)) \text{ on a neighbourhood of } X^n \text{ in } \mathscr{Y}^n.$$

A function u satisfying $du \neq 0$ at X^n is called a *relative invariant of* $D^n(c)$ *at* X^n if and only if u vanishes at X^n and

$$du \equiv 0 \quad (\mathrm{mod}\, D^n(c)) \text{ on a neighbourhood of } X^n \text{ in } \mathscr{Y}^n(u),$$

where $\mathscr{Y}^n(u)$ denotes the set of all points in $\mathscr{Y}^n$ at which u vanishes.

This denomination is justified by the fact that the value of an invariant remains constant on any integral manifold (in $I\Phi$) of the characteristic system. As may be expected, it can be proved the following fact.

"*A function u of n-jets is an invariant of* $D^n(c)$ *at* $X^n_0 \in \mathscr{Y}^n$ *if and only if u is an invariant of* $D^{n+1}(c)$ *at each point* $X^{n+1}_0 \in (\rho^{n+1}_n)^{-1} X^n_0 \cap \mathscr{Y}^{n+1}$."

In fact, let c be defined by $\lambda\partial/\partial x + \mu\partial/\partial y$. By Proposition 5, u is an invariant of $D^n(c)$ at X^n_0 if and only if for each point X^n in a neighbourhood of X^n_0 in $\mathscr{Y}^n$

$$\lambda\left(\left\{\frac{du}{dx}\right\}_n + \sum_{\beta=0}^{n} \frac{\partial u}{\partial p_{n-\beta,\beta}} \varphi_\beta\right) + \mu\left(\left\{\frac{du}{dy}\right\}_n + \sum_{\beta=0}^{n} \frac{\partial u}{\partial p_{n-\beta,\beta}} \psi_\beta\right) \equiv 0$$

$$\left(\mathrm{mod}\left[\begin{array}{l} \varphi_\beta = \psi_{\beta-1} \quad (0<\beta\leqq n)\,, \\ \left\{\dfrac{dF}{dx}\right\}_n + \displaystyle\sum_{\beta=0}^{n} \frac{\partial F}{\partial p_{n-\beta,\beta}} \varphi_\beta = \left\{\dfrac{dF}{dy}\right\}_n + \displaystyle\sum_{\beta=0}^{n} \frac{\partial F}{\partial p_{n-\beta,\beta}} \psi_\beta = 0\,, \quad F \in (p^{n-m}\Phi)_{X^n} \end{array}\right.\right). \tag{14}$$

Here

$$\left\{\frac{d}{dx}\right\}_n = \frac{\partial}{\partial x} + p_{1,0}\frac{\partial}{\partial z} + \sum_{i+k=1}^{n-1} p_{i+1,k}\frac{\partial}{\partial p_{i,k}}\,,$$

$$\left\{\frac{d}{dy}\right\}_n = \frac{\partial}{\partial y} + p_{0,1}\frac{\partial}{\partial z} + \sum_{i+k=1}^{n-1} p_{i,k+1}\frac{\partial}{\partial p_{i,k}}\,.$$

On the other hand, by replacing n in (14) by $n+1$, we can readily show that u is an invariant of $D^{n+1}(c)$ at X^{n+1}_0 that contains no $p_{n+1-\beta,\beta}$ $(0\leqq\beta\leqq n+1)$ if and only if

$$\lambda\left\{\frac{du}{dx}\right\}_{n+1}+\mu\left\{\frac{du}{dy}\right\}_{n+1}=0 \text{ on a neighbourhood of } X_0^{n+1} \text{ in } I(p^{n+1-m}\Phi)\,.$$

Comparing this and (14), we immediately obtain the desired result.

By the above fact, we may forget the order of a characteristic system. We shall denote by $\mathbf{D}(c)$ the characteristic system corresponding to a characteristic direction c without specifying the order. If a function u of n-jets which cannot be considered as a function of $(n-1)$-jets is an invariant of $\mathbf{D}(c)$, we shall say that u is *of order n*. We shall also say that such a function u is *a relative invariant of order n of* $\mathbf{D}(c)$ *at* X^n if it is a relative invariant of $D^n(c)$ at X^n.

REMARK. We can define characteristic systems of Φ (in the sense of Monge) in line with E. Cartan's ([5], Chap. IV, part III). That is, if we explicitly write down the condition that a linear integral element of $\Sigma(\Phi)$ is singular, then we obtain a set of Pfaffian systems, corresponding to characteristic directions of Φ. The minimal dimension of $H(E_1)$, when E_1 is an arbitrary linear integral element contained in a two-dimensional integral element E_2 of $\Sigma(\Phi)$ satisfying $\dim(\rho^m_{-1})_*E_2=2$, is equal to 2 (cf. (13)). Therefore such E_1 is singular if and only if $\dim H(E_1)$ is equal to 3 (cf. the argument around (13) in §4). Lemma 4 shows that both definitions are equivalent to each other. In the case of a single equation, characteristic systems defined above coincide with those given by E. Goursat [8]. It is also noticed that characteristic systems thus defined are always well-defined only for systems of partial differential equations in two independent variables.

§6. Fundamental theorems regarding the invariants.

We shall now consider the problem of constructing a new involutive system of partial differential equations which contains a given involutive system. In the investigation of this problem, the invariants of characteristic systems play an important role.

Let $u_1, \cdots, u_s$ be analytic functions defined respectively on an open set $\mathscr{U}_1$ of $J^{n_1}(M, N, \rho), \cdots$, on an open set $\mathscr{U}_s$ of $J^{n_s}(M, N, \rho)$. We write $n=\max\{n_\alpha\,;\ 1\leqq\alpha\leqq s\}$, $\tilde{\mathscr{U}}=\bigcap_{\alpha=1}^{s}(\rho^n_{n_\alpha})^{-1}\mathscr{U}_\alpha$. We define $\Phi^n(u_1, \cdots, u_s)$ to be the subsheaf of ideals of $\mathscr{O}(\tilde{\mathscr{U}})$ generated by $p^{n-m}\Phi$, $p^{n-n_1}(u_1), \cdots, p^{n-n_s}(u_s)$, where we denote by (u_α) the subsheaf of ideals of $\mathscr{O}(\mathscr{U}_\alpha)$ generated by u_α $(1\leqq\alpha\leqq s)$.

We shall say that a function u defined on a neighbourhood of a point X^l in $J^l(M, N, \rho)$ is *independent of* Φ *at* X^l $(l\geqq m)$ if and only if

$$\pi_l^* du \not\equiv 0 \qquad (\mathrm{mod}\ \pi_l^* dF\,; \quad F\in (p^{l-m}\Phi)_{X^l})\,,$$

where π_l^* denotes the projection from $T^*_{X^l}(J^l)$ to $Q^*_{X^l}(J^l)$ as before.

From now on, we shall always assume that characteristic directions of Φ occurring in our considerations are analytic in the sense stated at the beginning of § 5. We shall first try to construct a new involutive system of the same order which contains Φ.

PROPOSITION 8. *Suppose that Φ is involutive at X_0. Let u be a function of m-jets defined on a neighbourhood of X_0 which is independent of Φ at X_0. Then $\Phi^m(u)$ is involutive at X_0 if and only if u is a relative invariant of order m of a characteristic system of Φ at X_0. In this case, the degree of the characteristic polynomial of $\Phi^m(u)$ at X_0 is less than that of Φ at X_0 by one; If u is a relative invariant of $\mathrm{D}^m(c)$, then the characteristic directions of $\Phi^m(u)$ at X_0 are obtained from those of Φ by omitting c (more precisely, by decreasing the multiplicity of c by one).*

PROOF. Sufficiency: Since u is independent of Φ at X_0, X_0 is an ordinary integral point of $\Phi^m(u)$ and $r_m(\Phi^m(u); X)$ is equal to $r_m(\Phi; X)+1$ on a neighbourhood of X_0 in $I\Phi^m(u)$, which remains constant on that neighbourhood. By Proposition 5, u is a relative invariant at X_0 of a characteristic system corresponding to a characteristic direction defined by $\lambda\partial/\partial x+\mu\partial/\partial y$ if and only if there exists a neighbourhood $\mathscr{W}$ of X_0 in $I\Phi^m(u)$ such that

$$(15)\qquad \lambda\Big(\frac{du}{dx}+\sum_{\beta=0}^{m}\frac{\partial u}{\partial p_\beta}\varphi_\beta\Big)+\mu\Big(\frac{du}{dy}+\sum_{\beta=0}^{m}\frac{\partial u}{\partial p_\beta}\psi_\beta\Big)\equiv 0 \quad (\mathrm{mod}\ (**))\ \text{on}\ \mathscr{W}\,.$$

This implies that

$$(16)\qquad \lambda\Big(\sum_{\beta=0}^{m}\frac{\partial u}{\partial p_\beta}\varphi_\beta\Big)+\mu\Big(\sum_{\beta=0}^{m}\frac{\partial u}{\partial p_\beta}\psi_\beta\Big)\equiv 0 \quad (\mathrm{mod}\ (*))\ \text{on}\ \mathscr{W}\,.$$

By Lemma 3, this shows that $C_X(\Phi^m(u))$ is an involutive subspace of $Q_X(J^{m-1})\otimes T^*_a(N)$ at each point on $\mathscr{W}$. Hence, by Proposition 1,

$$r_{m+1}(\Phi^m(u); X)=r_m(\Phi^m(u); X)+1 \quad \text{on}\ \mathscr{W}\,.$$

From (15) we also see that the mapping ρ_m^{m+1} from $I\{p\Phi^m(u)\}\cap(\rho_m^{m+1})^{-1}\mathscr{W}$ to $\mathscr{W}$ is surjective. Hence $\Phi^m(u)$ is p-closed at X_0 (cf. § 1). Thus the four conditions in Theorem I are satisfied for $\Phi^m(u)$. Hence $\Phi^m(u)$ is involutive at X_0.

Necessity: Since $C_X(\Phi^m(u))$ is an involutive subspace of $Q_X(J^{m-1})\otimes T^*_a(N)$ at each point on a neighbourhood of X_0 in $I\Phi^m(u)$, we can find a neighbourhood $\mathscr{W}$

of X_0 in $I\Phi^m(u)$ and a pair of functions $(\lambda, \mu)\neq 0$ defined on $\mathscr{W}$ such that u is independent of Φ at each point on $\mathscr{W}$ and that (16) is valid. We may also assume that $\mathscr{W}$ is such that Φ is p-closed at each point on $\mathscr{W}$. Then, by the same argument as in the proof of Proposition 3, we can show that $\lambda\partial/\partial x+\mu\partial/\partial y$ is a characteristic vector of Φ at each point on $\mathscr{W}$ as follows. Writing

$$H(\xi, \eta)=\sum_{\beta=0}^{m}(-1)^{\beta}\partial u/\partial p_{\beta}\xi^{m-\beta}\eta^{\beta},$$

we have from (16) that

$$(\lambda\xi-\mu\eta)H(\xi, \eta)\equiv 0 \quad (\operatorname{mod} G_1, \cdots, G_r) \text{ on } \mathscr{W}.$$

Accordingly, since u is independent of Φ at each point on $\mathscr{W}$, $(\lambda\xi-\mu\eta)$ is a factor of the characteristic polynomial $(G_1, \cdots, G_r)$ of Φ at each point on $\mathscr{W}$ and the characteristic polynomial of $\Phi^m(u)$ at each point on $\mathscr{W}$ is nothing else than $(G_1, \cdots, G_r, H)$, which is obtained by dividing $(G_1, \cdots, G_r)$ by $(\lambda\xi-\mu\eta)$ within a non-zero constant factor. In order to complete the proof of necessity, we must prove further that (15) is valid for such $\mathscr{W}$. Since $\Phi^m(u)$ is p-closed at each point on $\mathscr{W}$, the following system of linear equations in $p_{m+1-\beta,\beta}$ $(0\leqq\beta\leqq m+1)$ has a solution for each point X on $\mathscr{W}$:

$$\begin{cases}\displaystyle\sum_{\beta=0}^{m}\frac{\partial u}{\partial p_{m-\beta,\beta}}p_{m+1-\beta,\beta}+\frac{du}{dx}=0, & \displaystyle\sum_{\beta=0}^{m}\frac{\partial u}{\partial p_{m-\beta,\beta}}p_{m-\beta,\beta+1}+\frac{du}{dy}=0, \\ \displaystyle\sum_{\beta=0}^{m}\frac{\partial F}{\partial p_{m-\beta,\beta}}p_{m+1-\beta,\beta}+\frac{dF}{dx}=0, & \displaystyle\sum_{\beta=0}^{m}\frac{\partial F}{\partial p_{m-\beta,\beta}}p_{m-\beta,\beta+1}+\frac{dF}{dy}=0 \qquad (F\in\Phi_X).\end{cases}$$

This fact and (16) imply that (15) is valid. Hence u is a relative invariant of the characteristic system corresponding to the characteristic direction defined by $\lambda\partial/\partial x+\mu\partial/\partial y$ at X_0.

The last statement follows from the fact already shown that the characteristic polynomial $(G_1, \cdots, G_r, H)$ of $\Phi^m(u)$ at X_0 is obtained from the characteristic polynomial $(G_1, \cdots, G_r)$ of Φ at X_0 by dividing it by $(\lambda\xi-\mu\eta)$. Q.E.D.

The following is in a certain sense a generalization of Proposition 8.

PROPOSITION 9. *Suppose that Φ is involutive at X_0. Let* $\mathrm{D}^m(c_1), \cdots, \mathrm{D}^m(c_s)$ *be s different characteristic systems of Φ defined on a neighbourhood of X_0 in $I\Phi$. Let $u_1, \cdots, u_s$ be a relative invariant of* $\mathrm{D}^m(c_1)$ *at $X_0, \cdots,$ a relative invariant of* $\mathrm{D}^m(c_s)$ *at X_0 respectively. If each u_α is independent of Φ at X_0 $(1\leqq\alpha\leqq s)$, then the system $\Phi^m(u_1, \cdots, u_s)$ is involutive at X_0. In this case, the degree of the characteristic polynomial of $\Phi^m(u_1, \cdots, u_s)$ at X_0 is less than that of Φ at X_0 by* *s; The characteristic directions of $\Phi^m(u_1, \cdots, u_s)$ at X_0 are obtained from those of Φ at X_0 by decreasing the multiplicities of $c_1, \cdots, c_s$ by one respectively.*

PROOF. By Proposition 8, $\Phi^m(u_1)$ is involutive at X_0 and the characteristic directions of $\Phi^m(u_1)$ are obtained from those of Φ by decreasing the multiplicity of c_1 by one. Next let us prove that $\Phi^m(u_1, u_2)$ is involutive at X_0. Clearly every relative invariant of $D^m(c_2)$ at X_0 is also a relative invariant of the characteristic system of $\Phi^m(u_1)$ at X_0 corresponding to the characteristic direction c_2. Further, u_2 is independent of $\Phi^m(u_1)$ at X_0. In fact, the characteristic polynomial of $\Phi^m(u_1, u_2)$ at X_0 is of degree $m+1-r_m(\Phi^m(u_1, u_2); X_0)$ (Theorem II). Since c_1 and c_2 are distinct from each other, that degree must be equal to $m-1-r_m(\Phi; X_0)$. Hence $r_m(\Phi^m(u_1, u_2); X_0)$ is equal to $r_m(\Phi; X_0)+2$. This implies that

$$\pi_m^* du_2 \not\equiv 0 \qquad (\mathrm{mod}\ \pi_m^* du_1, \pi_m^* dF;\quad F \in \Phi_{X_0}).$$

That is, u_2 is independent of $\Phi^m(u_1)$ at X_0. Therefore we can apply Proposition 8; We obtain that $\Phi^m(u_1, u_2)$ is involutive at X_0 and the characteristic directions of $\Phi^m(u_1, u_2)$ are obtained from those of Φ by decreasing the multiplicities of c_1 and c_2 by one respectively. Proceeding step by step, we can complete the proof.

Q.E.D.

Let us extend the above two propositions to the general case where invariants of higher order occur.

THEOREM III. *Suppose that Φ is involutive at X_0. Let X_0^n be a point of $I(p^{n-m}\Phi) \cap (\rho_m^n)^{-1} X_0$ $(n \geqq m)$ and u be a function of n-jets defined on a neighbourhood of X_0^n which is independent of Φ at X_0^n. Then $\Phi^n(u)$ is involutive at X_0^n if and only if u is a relative invariant of order n of a characteristic system of Φ at X_0^n. In this case, the degree of the characteristic polynomial of $\Phi^n(u)$ at X_0^n is less than that of Φ at X_0 by one; If u is a relative invariant of $D^n(c)$, then the characteristic directions of $\Phi^n(u)$ at X_0^n are obtained from those of Φ at X_0 by decreasing the multiplicity of c by one.*

PROOF. $p^{n-m}\Phi$ is involutive at X_0^n and the characteristic directions of $p^{n-m}\Phi$ at X_0^n coincide with those of Φ at X_0. Since the characteristic systems of order n of Φ are by definition those (of order n) of $p^{n-m}\Phi$, this theorem is an immediate consequence of Proposition 8. Q.E.D.

THEOREM IV. *Suppose that Φ is involutive at X_0. Let $\mathbf{D}(c_1), \cdots, \mathbf{D}(c_s)$ be s different characteristic systems of Φ over a neighbourhood of X_0 in $I\Phi$ and let $u_1, \cdots, u_s$ be a relative invariant of order n_1 of $\mathbf{D}(c_1)$ at $X_1^{n_1}, \cdots$, a relative* *invariant of order* n_s *of* $\mathbf{D}(c_s)$ *at* $X_s^{n_s}$ *respectively, where* $X_1^{n_1}, \cdots, X_s^{n_s}$ *are a point of* $I(p^{n_1-m}\Phi), \cdots$, *a point of* $I(p^{n_s-m}\Phi)$ *respectively such that* $\rho_m^{n_\alpha} X_\alpha^{n_\alpha} = X_0$ $(1 \leq \alpha \leq s)$ *and* $\rho_{n_\beta}^{n_\alpha} X_\alpha^{n_\alpha} = X_\beta^{n_\beta}$ *for any pair* $\{\alpha, \beta\}$ *satisfying* $n_\alpha \geqq n_\beta$ $(n_1, \cdots, n_s \geqq m)$. *If each* u_α *is independent of* Φ *at* $X_\alpha^{n_\alpha}$ $(1 \leq \alpha \leq s)$, *then* $\Phi^n(u_1, \cdots, u_s)$ *is involutive at* $\tilde{X}_0$, *where* n *denotes* $\max\{n_\alpha;\ 1 \leq \alpha \leq s\}$ *and* $\tilde{X}_0$ *is the point of* $J^n(M, N, \rho)$ *satisfying* $\rho_{n_\alpha}^n \tilde{X}_0 = X_\alpha^{n_\alpha}$ $(1 \leq \alpha \leq s)$. *In this case, the degree of the characteristic polynomial of* $\Phi^n(u_1, \cdots, u_s)$ *at* $\tilde{X}_0$ *is less than that of* Φ *at* X_0 *by* s; *The characteristic directions of* $\Phi^n(u_1, \cdots, u_s)$ *at* $\tilde{X}_0$ *are obtained from those of* Φ *at* X_0 *by decreasing the multiplicities of* $c_1, \cdots, c_s$ *by one respectively.*

PROOF. We may assume that $n_1 \leqq n_2 \leqq \cdots \leqq n_s$ without loss of generality. By Theorem III, $\Phi^{n_1}(u_1)$ is involutive at $X_1^{n_1}$ and its characteristic directions at $X_1^{n_1}$ are obtained from those of Φ at X_0 by decreasing the multiplicity of c_1 by one. Proceed step by step, as in the proof of Proposition 9, using Theorem III successively, then we obtain the desired result. Q.E.D.

An answer of the problem proposed at the beginning of this section is given by Theorem IV; When there exist such invariants $u_1, \cdots, u_s$ as in Theorem IV, we can construct a new involutive system $\Phi^n(u_1, \cdots, u_s)$ whose local solutions are also those of the original system Φ. It is also noticed that Theorems III and IV indicate the structure of an involutive system to some extent.

REMARK. The coverse of Theorem IV is not valid in general. An involutive system cannot be obtained, in general, from a single equation by the method stated above. For example, let us consider the involutive system of order 4

$$p_{4,0} = p_{3,1} = p_{2,2} = p_{1,3} = 0 .$$

This system cannot be obtained from a single equation, say, $p_{4,0}=0$; $p_{3,1}$ is a relative invariant of a characteristic system of $p_{4,0}=0$, but $p_{2,2}$ and $p_{1,3}$ are not relative invariants of a characteristic system of $p_{4,0}=0$. Choosing one of other equations than $p_{4,0}=0$, we see that the similar statement is valid.

§7. The extension of Darboux's method.

Let us consider Cauchy's problem for an involutive system of partial differential equations. As already stated in §1, a system Φ can be represented by the differntial ideal $\Sigma(\Phi)$ associated with Φ. Cauchy's problem for Φ can be interpreted as follows: "Find the two-dimensional integral manifold $\mathscr{M}$ of $\Sigma(\Phi)$ satisfying $\dim (\rho_{-1}^m)_* T(\mathscr{M}) = 2$ passing through a given non-characteristic integral curve $\mathscr{J}$ of

$\Sigma(\Phi)$ satisfying $\dim(\rho^m_{-1})_* T(\mathscr{J})=1$. (We shall call such a curve an initial curve.)"

This problem has a unique solution for any given initial curve by the first existence theorem of Cartan-Kähler (E. Cartan [5], Chap. IV, part II; E. Kähler [9], p. 26). The problem to engage our attention is that of finding a method of solving Cauchy's problem by integrating a system of ordinary differential equations. The problem of this kind has been originated in the investigations of integrating a single partial differential equation of the second order by Monge, Ampère, Darboux et al. (cf. E. Goursat [8], A. R. Forsyth [7]).

Suppose that Φ is involutive at X_0. We shall denote by ν $m+1-r_m(\Phi; X)$ which remains constant around X_0 in $I\Phi$; ν is the degree of the characteristic polynomial of Φ at X_0 (§3). In treating the problem above proposed, we must distinguish the following three cases.

1°) *When $\nu=0$, Φ is completely integrable at X_0*; For any given integral point X of Φ sufficiently near X_0, there exists a unique solution of Φ passing through X. The solution can be obtained by integrating $\Sigma(\Phi)$ which is completely integrable at X and hence is solvable by integrating a system of ordinary differential equations.

2°) *When $\nu=1$, the solution of Cauchy's problem for Φ can be reduced to the integration of a system of ordinary differential equations.*

In fact, Φ is involutive at each point X in a neighbourhood $\mathscr{V}$ of X_0 in $I\Phi$, and $r_m(\Phi; X)$ is equal to m. Φ has only one characteristic direction c at X (Theorem II) and $\dim B(c)$ is equal to one (Proposition 5). The characteristic system $D^m(c)$ defined on $\mathscr{V}$ is of rank $(\dim \mathscr{V}-1)$ and hence it is considered to be a system of ordinary differential equations. Hence, for each point $X \in I\Phi$ near X_0, there passes a unique integral curve (in $I\Phi$) of $D^m(c)$ passing through X. By Proposition 7, these curves are Cauchy's characteristic curves of $\Sigma(\Phi)$: the manifold generated by the family of those curves, each of which passes a point of a given integral manifold of $\Sigma(\Phi)$, is also an integral manifold of $\Sigma(\Phi)$ (E. Cartan [4], §49; [5], §45). It is noticed that in this case $D^m(c)$ is essentially the characteristic system of the differential ideal $\Sigma(\Phi)$ in the sense of E. Cartan (cf. E. Cartan [5], §43). Hence, for any given initial curve passing sufficiently near X_0, the manifold generated by the one-parameter family of integral curves of $D^m(c)$, each of which passes a point of the initial curve, is a two-dimensional integral manifold of $\Sigma(\Phi)$. Thus we can obtain the solution of Cauchy's problem for Φ by integrating the characteristic system $D^m(c)$ which is in this case a system of ordinary differential equations.

3°) *When* $\nu>1$, Cauchy's problem for Φ cannot be solved, in general, by integrating a system of ordinary differential equations. However, for certain involutive systems of partial differential equations, we have a method of integration which is an extension of Darboux's method. Roughly speaking, this method is stated as follows. For a given initial curve, we construct a new involutive system containing Φ which has only one characteristic direction and which admits that curve as an initial curve. We can obtain the solution of Cauchy's problem passing through that curve by integrating the new involutive system by the method stated in 2°. If $\nu-1$ different characteristic systems of Φ have respectively two independent invariants which are independent of Φ, then for any given initial curve we can construct such an involutive system as stated above.

Let $\mathscr{J}$ be an initial curve. Since $\mathscr{J}$ is non-characteristic, $\mathscr{J}$ can be uniquely prolonged to a curve $\mathscr{J}^n$ in $J^n(M, N, \rho)$ $(n\geqq m)$ in such a manner that $\mathscr{J}^n$ is an integral curve of $p^{n-m}\Sigma(\Phi)$ and $\rho^n_m \mathscr{J}^n=\mathscr{J}$.

Our method of integration is stated rigorously as follows.

"*Suppose that* Φ *is involutive at* X_0 *and that* Φ *has* $\nu-1$ *different characteristic systems* $\mathbf{D}(c_1), \dots, \mathbf{D}(c_{\nu-1})$ *over a neighbourhood of* X_0 *in* $I\Phi$. *Let* $X^{n_1}_1, \dots, X^{n_{\nu-1}}_{\nu-1}$ *be a point of* $I(p^{n_1-m}\Phi), \dots,$ *a point of* $I(p^{n_{\nu-1}-m}\Phi)$ *respectively* $(n_1, \dots, n_{\nu-1}\geqq m)$ *such that* $\rho^{n_\alpha}_m X^{n_\alpha}_\alpha=X_0$ $(1\leqq\alpha\leqq\nu-1)$, $\rho^{n_\alpha}_{n_\beta}X^{n_\alpha}_\alpha=X^{n_\beta}_\beta$ *for any pair* $\{\alpha, \beta\}$ *satisfying* $n_\alpha\geqq n_\beta$, *and let* $\tilde{X}_0\in J^n(M, N, \rho)$ *denote such the point that* $\rho^n_{n_\alpha}\tilde{X}_0=X^{n_\alpha}_\alpha$ $(1\leqq\alpha\leqq\nu-1)$, *where* n *denotes* $\max\{n_\alpha;\ 1\leqq\alpha\leqq\nu-1\}$. *If each* $\mathbf{D}(c_\alpha)$ *has two independent invariants of order* n_α *at* $X^{n_\alpha}_\alpha$ *which are independent of* Φ *at* $X^{n_\alpha}_\alpha$ $(1\leqq\alpha\leqq\nu-1)$, *then for any initial curve* $\mathscr{J}$ *such that* $\mathscr{J}^n$ *passes sufficiently near* $\tilde{X}_0$, *Cauchy's problem having* $\mathscr{J}$ *as the initial curve can be solved by integrating a system of ordinary differential equations.*"

In fact, let u_α, v_α be such invariants of $\mathbf{D}(c_\alpha)$ $(1\leqq\alpha\leqq\nu-1)$ and $\mathscr{J}$ be such an initial curve as in the above statement. Then we can obtain an invariant $\chi_\alpha=\chi_\alpha(u_\alpha, v_\alpha)$ of order n_α of $\mathbf{D}(c_\alpha)$ such that χ_α vanishes on $\mathscr{J}^{n_\alpha}$ and is independent of Φ at a point on $\mathscr{J}^{n_\alpha}$ sufficiently near $X^{n_\alpha}_\alpha$ for each $\alpha=1, 2, \dots, \nu-1$. By Theorem IV, the system $\Phi^n(\chi_1, \dots, \chi_{\nu-1})$ is involutive at a point $X^n\in\mathscr{J}^n$ sufficiently near $\tilde{X}_0$ and the characteristic polynomial of $\Phi^n(\chi_1, \dots, \chi_{\nu-1})$ at X^n is of degree 1. Hence, by 2°, we can obtain the solution of $\Phi^n(\chi_1, \dots, \chi_{\nu-1})$ passing through $\mathscr{J}^n$ by integrating a system of ordinary differential equations. This solution gives the solution of Φ passing through $\mathscr{J}$. Thus the above statement has been proved.

Suppose further that there exist two independent invariants of order n_ν $(n_\nu\leqq n)$ u_ν, v_ν of $\mathbf{D}(c_\nu)$ at $X^{n_\nu}_\nu=\rho^n_{n_\nu}\tilde{X}_0$ which are independent of Φ at $X^{n_\nu}_\nu$, here $\mathbf{D}(c_\nu)$ is the remaining characteristic system of Φ which is assumed to be distinct from the above $\nu-1$ characteristic systems. The similar argument indicates that one can construct a completely integrable system $\Phi^n(\chi_1, \cdots, \chi_\nu)$ in such a manner that $\mathscr{J}^n$ is an integral curve of $\Sigma(\Phi^n(\chi_1, \cdots, \chi_\nu))$, where $\chi_\alpha=\chi_\alpha(u_\alpha, v_\alpha)$ $(1\leqq\alpha\leqq\nu)$.

We finally notice that it requires only algebraic operations including differentiations to see if there exist such invariants enough that the above method is successfully applied, for the invariants of each characteristic system are characterized as solutions of a system of linear homogeneous partial differential equations of the first order. It is also noticed that they can be obtained, if exist, by integrating systems of ordinary differential equations successively.

REMARK. M. Matsuda recently gave the following method of solving Cauchy's problem for Monge-Ampère's equations (cf. M. Matsuda [15], [16]): Consider Monge-Ampère's equation

(M-A) $$Hr+2Ks+Lt+M+N(rt-s^2)=0 \qquad (N\neq 0)\,,$$

where H, K, L, M, N are fuctions of x, y, z, p, q, and p, q, r, s, t denote $\partial z/\partial x$, $\partial z/\partial y$, $\partial^2 z/\partial x^2$, $\partial^2 z/\partial x\partial y$, $\partial^2 z/\partial y^2$ respectively. The equation has the two characteristics, each of which is a Pfaffian system generated by

(C) $$\theta=dx-pdx-qdy=0\,, \quad Ndp+Ldx+\lambda_1 dy=Ndq+\lambda_2 dx+Hdy=0\,,$$

where λ_1, λ_2 are the roots of the quadratic equation $\lambda^2+2K\lambda+HL-MN=0$. Let us fix one of them. Then a two-dimensional submanifold of the space (x, y, z, p, q) satisfying $\theta=0$ is a solution of (M-A) if and only if it is generated by one-parameter family of integral curves of the characteristic (C) (Monge-Ampère's theorem). Monge's method of integration is as follows; Find for a given initial curve $\mathscr{J}$ such an equation $V(x, y, z, p, q)=0$ of the first order that $dV=0$ along $\mathscr{J}$ and that the Lagrange-Charpit system

(L-C) $$\frac{dx}{V_p}=\frac{dy}{V_q}=\frac{dz}{pV_p+qV_q}=\frac{-dp}{V_x+pV_z}=\frac{-dq}{V_y+qV_z}$$

contains the characteristic (C). Suppose that such V can be found. Then the manifold generated by the one-parameter family of integral curves of (L-C) passing through $\mathscr{J}$ gives the solution of Cauchy's problem for (M-A), since (L-C) is the characteristic system of $\theta=dV=0$ (Cauchy-Darboux's theorem); If the initial curve satisfies $\theta=dV=0$, then the integral surface satisfies $\theta=dV=0$. Equation (M-A) is said to be Monge integrable if we can find such V for any initial curve $\mathscr{J}$. Monge's method was generalized by M. Matsuda as follows: Consider a Pfaffian system

$$\text{(S)} \qquad \frac{dx}{A}=\frac{dy}{B}=\frac{dz}{pA+qB}=\frac{-dp}{C}=\frac{-dq}{D},$$

where A, B, C, D are functions of x, y, z, p, q. Then it is called an integrable system if it is the characteristic system of $0=Cdx+Ddy+Adp+Bdq=0$. (L-C) is an example of an integrable system. Equation (M-A) is said to be solved by integrable systems of the first order if we can find for any initial curve $\mathscr{I}$ such an integrable system (S) that it contains the characteristic (C) and that $Cdx+Ddy+Adp+Bdq=0$ along $\mathscr{I}$. In this case, the manifold generated by the one-parameter family of integral curves of (S) passing $\mathscr{I}$ gives the solution of Cauchy's problem for (M-A). There exists such an example of (M-A) that it is solved by integrable systems of the first order and that it is not Monge integrable. This method of integration was extended by M. Matsuda himself to the method by integrable systems of higher order.

Let us compare Darboux's method of integration with Matsuda's one by integrable systems for solving Monge-Ampère's equations. Consider an equation

$$s+f(x, y, z, p, q)+g(x, y, z, p, q)t=0.$$

Then it is solved by integrable systems of order n along the characteristic containing $dx=0$ if and only if it is integrable by Darboux's method with respect to the other characteristic containing $dy-gdx=0$ of order $n+1$. In the case of Laplace linear equation $s+a(x, y)p+b(x, y)q+c(x, y)z=0$, the above statement can be proved by combining two results on Laplace transformations obtained by E. Goursat ([8], p. 178) and by M. Matsuda [15] respectively. This remark was given by M. Matsuda.

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