On certain irreducible representations for the real rank one classical groups

By Hisaichi MIDORIKAWA

(Communicated by N. Iwahori)

§ 1. Introduction.

In the representation theory of the semisimple Lie groups, one of the main problems is to construct all irreducible unitary representations of a given group. The purpose in this paper is to construct a certain series of the irreducible unitary representations which are irreducible components of reducible representations in the principal series for the following groups: Spin $(2l, 1)$, $SU(l, 1)$ and $Sp(l, 1)$ $(l \geqq 1)$.

These representations are an analogue of those which are constructed in the following results;

For the special linear group $G = SL(2, R)$ (which is isomorphic to $SU(1, 1)$ and Spin $(2, 1)$), V. Bargmann [1] constructed the representations $D_{1/2}^{\pm}$ defined by introducing an inner product in the representation space as the limit of the inner product in the space of $D_l^{\pm}$ $(l \geqq 1/2)$. It was proved by I. M. Gelfand, M. I. Graev and V. Ya. Vilenkin [4] and R. Takahashi [16] that $D_{1/2}^{+} \oplus D_{1/2}^{-}$ is equivalent to the principal series representation $C_{1/4}^{1/2}$ in the notation of [1].

When G is a real form of a simply connected complex simple Lie group and the corresponding symmetric space G/K is of a Hermitian type, A. W. Knapp and K. Okamoto [13] proved that any one of the limits of holomorphic discrete series representation is irreducible, and is equivalent to a proper subrepresentation of a certain representation in the principal series.

On the other hand, R. Takahashi showed the similar result for the group $Sp(1, 1)$ ([16]).

The contents of this paper are as follows.

After the preparations in § 2, we define the representation $U(\lambda, \tau)$ of a real rank one classical group G on the space $H(\lambda, \tau)$ where λ is an integral form on $\mathfrak{a}_0$ and τ is an irreducible unitary representation of a maximal compact subgroup K in G. In § 4, we study the asymptotic behaviour of the functions in the subspace $H^K(\lambda, \tau)$ of $H(\lambda, \tau)$. In § 5, we shall prove that if λ vanishes only on singular real roots, then the space $H(\lambda, \tau)$ with the given seminorm $\| \ \|$ is actually a Hilbert space and $U(\lambda, \tau)$ is a unitary representation. In §7, we prove Theorem 1 which plays an essential role in the proof of the irreducibility of $U(\lambda, \tau)$. Theorem 2 and Theorem 3 in §8 are the main results in this paper. Theorem 2 proves that $U(\lambda, \tau)$ is irreducible if $[\tau|M: \sigma_{\lambda_-}]=1$ where σ_{λ_-} is the irreducible representation of M with the highest weight $\lambda_-=\lambda-\rho_-$ in the notations of §6. Under the same conditions as in Theorem 2, Theorem 3 proves that the representation $U(\lambda, \tau)$ is equivalent to a proper subrepresentation of a principal series representation $V(\lambda_-, 0)$.

The most important tools in this paper are the theory of eigenfunctions of two sided G-invariant differential operators obtained by Harish-Chandra [10] and the asymptotic behaviour for the spherical functions of the representations in the principal series obtained by A. W. Knapp and E. M. Stein [14].

§2. Preliminaries.

Let G_c be a simply connected simple Lie group and G be a real form of G_c. Moreover we shall assume that (1) G is of real rank one and (2) G has a compact Cartan subgroup. Let $\mathfrak{g}_c$, $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebra of G_c, G, K respectively where K is a maximal compact subgroup of G. Then $\mathfrak{g}_c$ is a complexification of $\mathfrak{g}$. Throughout this paper $\mathfrak{l}_c$ means the complexification of the Lie algebra $\mathfrak{l}$ of $\mathfrak{g}$. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to Killing form B on $\mathfrak{g}_c$. Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{a}\cap\mathfrak{p}$ is a maximal abelian subspace of $\mathfrak{p}$. Then $\mathfrak{a}=(\mathfrak{a}\cap\mathfrak{k})+(\mathfrak{a}\cap\mathfrak{p})$. Put $\mathfrak{a}_I=\mathfrak{a}\cap\mathfrak{k}, \mathfrak{a}_R=\mathfrak{a}\cap\mathfrak{p}$. Then the assumption (1) means that $\dim \mathfrak{a}_R=1$. Let Σ be the root system of the pair $(\mathfrak{g}_c, \mathfrak{a}_c)$ and $\mathfrak{g}_c=\mathfrak{a}_c+\sum_{\alpha\in\Sigma}\mathfrak{g}_\alpha$ be the root space decomposition of $\mathfrak{g}_c$.

Let λ be a linear form on $\mathfrak{a}_c$. Then there exists a unique H_λ in $\mathfrak{a}_c$ such that $B(H, H_\lambda)=\lambda(H)$ for each $H\in\mathfrak{a}_c$. Put $(\lambda, \mu)=B(H_\lambda, H_\mu)$ for any two linear forms λ and μ on $\mathfrak{a}_c$. Choose $X_\alpha\in\mathfrak{g}_\alpha$ for each root α satisfying $B(X_\alpha, X_{-\alpha})=1$. Then $[X_\alpha, X_{-\alpha}]=H_\alpha$ for all α in Σ. Fix a singular real root $\alpha=\alpha_0$ on $\mathfrak{a}$ (i.e. $\alpha\equiv 0$ on $\mathfrak{a}_I$). Then $\mathfrak{a}_R=RH_{\alpha_0}$. As usual we shall fix a lexicographic ordering in Σ such that if $(\alpha, \alpha_0)>0$, then α is positive. Put

$$P_+=\{\alpha\in\Sigma|\alpha>0 \quad \text{and} \quad (\alpha, \alpha_0)\neq 0\},$$
$$P_-=\{\alpha\in\Sigma|\alpha>0 \quad \text{and} \quad (\alpha, \alpha_0)=0\}.$$

Define ρ_+ and ρ_- by $2\rho_+=\sum_{\alpha\in P_+}\alpha$ and $2\rho_-=\sum_{\alpha\in P_-}\alpha$.

Let θ be the Cartan involution corresponding to the decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ and

$\mathfrak{m}$ be the centralizer $\mathfrak{a}_R$ in $\mathfrak{k}$. Put $\mathfrak{m}_1=\mathfrak{m}+\mathfrak{a}_R$ and $\mathfrak{n}=(\sum_{\alpha\in P_+}\mathfrak{g}_\alpha)\cap\mathfrak{g}$. Then $\mathfrak{g}_c=\theta\mathfrak{n}_c+(\mathfrak{m}_1)_c+\mathfrak{n}_c$ and $\mathfrak{g}=\mathfrak{k}+\mathfrak{a}_R+\mathfrak{n}$. Let A_R and N be the analytic subgroup of G corresponding to $\mathfrak{a}_R$ and $\mathfrak{n}$ respectively. Then we have the Iwasawa decomposition $G=KA_RN$.

Let W be a finite dimensional vector space over the complex field $\boldsymbol{C}$ and $C^\infty(W,G)$ be the set of all W-valued C^∞-functions on G. When $W=\boldsymbol{C}$, we shall write $C^\infty(\boldsymbol{C},G)=C^\infty(G)$. Define the representations $R=R_X$ and $L=L_X$ of the Lie algebra $\mathfrak{g}$ $(X\in\mathfrak{g})$ on the space $C^\infty(W,G)$ as follows;

$$(R_Xf)(x)=\frac{d}{dt}f(x\exp tX)|_{t=0},\qquad (L_Xf)(x)=\frac{d}{dt}f(\exp -tXx)|_{t=0}.$$

Then R and L can be uniquely extend to the representation of the universal enveloping algebra $\mathfrak{u}$ of $\mathfrak{g}_c$. We denote $bf=R_bf$ and $b'f=L_bf$ for all $b\in\mathfrak{u}$ and $f\in C^\infty(W,G)$.

§ 3. The definition of $U(\lambda,\tau)$.

Let $d(p,q)$ $(p,q\in K\backslash G)$ be the G-invariant distance of the Riemannian symmetric space $K\backslash G$ defined by $e^{d(\exp Ho,o)}=e^{\rho+(H)}$ for all $H\in\mathfrak{a}_R$ where o is the origin of the space $K\backslash G$ (remark the rank $K\backslash G=1$). Let τ be an irreducible unitary representation of K on the vector space W. We put

$$C_\tau^\infty(G)=\{f\in C^\infty(W,G)\,|\,f(kx)=\tau(k)f(x)\ \text{for all}\ x\in G\ \text{and}\ k\in K\}.$$

Define the seminorm ν_b $(b\in\mathfrak{u})$ on $C_\tau^\infty(G)$ by

$$\nu_b(f)=\sup_{x\in G}|bf(x)|e^{d(xo,o)}$$

for each f in $C_\tau^\infty(G)$ where $|\ |$ is the norm in W. Let $\mathfrak{z}$ be the center of $\mathfrak{u}$ and the mapping $z\to\mu(z)$ be the Chevalley isomorphism of $\mathfrak{z}$ onto $I(\mathfrak{a}_c)$ where $I(\mathfrak{a}_c)$ is the set of all polynomials on $\mathfrak{a}_c$ invariant under the Weyl group of $(\mathfrak{g}_c,\mathfrak{a}_c)$ (see G. Warner [19], p. 168). Then for any integral form λ on $\mathfrak{a}_c$, we can define the infinitesimal character χ_λ of $\mathfrak{z}$ by $\chi_\lambda(z)=\lambda(\mu(z))$ for each z in $\mathfrak{z}$. Fix an integral form λ on $\mathfrak{a}_c$, and define $H_0(\lambda,\tau)$ as follows;

$$H_0(\lambda,\tau)=\{f\in C_\tau^\infty(G)\,|\,\nu_b(f)<+\infty\ \text{and}\ zf=\chi_\lambda(z)f\ \text{for all}\ b\in\mathfrak{u}\ \text{and}\ z\in\mathfrak{z}\}.$$

Then $H_0(\lambda,\tau)$ is a topological vector space with the family of seminorms $\{\nu_b\}_{b\in\mathfrak{u}}$. We put $\nu=\nu_1$ where 1 is the identity of $\mathfrak{u}$.

Let $R=R_y$ $(y\in G)$ be the right regular representation of G on $C_\tau^\infty(G)$ and $\mathscr{E}_K$ be the set of all equivalence classes of irreducible unitary representations of K. Define the projection operator $E(\delta)(\delta\in\mathscr{E}_K)$ by

$$(E(\delta)f)(x)=\deg\delta\int_K \operatorname{Trace}\delta(k^{-1})[R_K f](x)dk$$

where $f\in C_\tau^\infty(G)$ and dk is the Haar measure of K normarized as $\int_K dk=1$. Then we have the following lemma.

LEMMA 1. *Let δ be an element of $\mathscr{E}_K$.*

1) *Then for any $f\in H_0(\lambda,\tau)$, $R_y f$ is in $H_0(\lambda,\tau)$ for all $y\in G$.*

2) *$E(\delta)f$ belongs to $H(\lambda,\tau)$ for all $f\in H_0(\lambda,\tau)$.*

PROOF OF 1). Since

$$\begin{aligned}\sup_{x\in G}|R_y f(x)|e^{d(xo,o)}&\leq\sup_{x\in G}|f(xy)|e^{d(xyo,o)}e^{d(yo,o)}\\&\leq\nu(f)e^{d(y^{-1}o,o)}\end{aligned}$$

we have $\nu(R_y f)<+\infty$. By $\mathrm{Ad}(y^{-1})bf\in H_0(\lambda,\tau)$ and $b\circ R_y=R_y\circ\mathrm{Ad}(y^{-1})$, we have also $\nu_b(f)<+\infty$ for all $b\in\mathfrak{u}$. On the other hand by the condition $z\circ R_y=R_y\circ z$ for all z in $\mathfrak{z}$, we conclude that $R_y f$ belongs to $H_0(\lambda,\tau)$.

In the similar way, the assertion 2) in Lemma 1 is proved.

LEMMA 2. *In the topological vector space $H_0(\lambda,\tau)$, we have*

$$\lim_{n\to+\infty}\sum_{\delta\in\mathscr{E}_K,\ \deg\delta\leq n}E(\delta)f=f$$

for all $f\in H_0(\lambda,\tau)$.

PROOF. Let f be an element in $H_0(\lambda,\tau)$. Then by Peter-Weyl's theorem with respect to the compact group K, we have

$$f(x)=\sum_{\delta\in\mathscr{E}_K}[E(\delta)f](x)$$

for any fixed x in G. Let Ω_K be the Casimir operator of K. Then we have

$$E(\delta)\Omega_K^{-p}f=\Omega_K^{-p}E(\delta)f=\omega(\delta)^{-p}E(\delta)f$$

for all δ in $\mathscr{E}_K$ where $\omega(\delta)$ is the eigenvalue of the operator Ω_K corresponding to the eigenfunction $\delta=\delta(k)$ $(k\in K)$ and $p=0,1,2,\cdots$. Therefore we have

$$(f-\sum_{\deg\delta\leq m}E(\delta)f)=\sum_{\deg\delta>m}\omega(\delta)^{-p}|E(\delta)\Omega_K f|$$

for sufficiently large p. Hence we conclude that $\lim_{m\to+\infty}\nu(f-\sum_{\deg\delta\leq m}E(\delta)f)=0$. By the similar method, we have

$$\lim_{m\to+\infty} \nu_b(f - \sum_{\deg \delta \leq m} E(\delta)f) = 0$$

for all b in $\mathfrak{u}$. Let $H_0^K(\lambda, \tau)$ be the set of all K-finite functions in $H_0(\lambda, \tau)$, namely $H_0^K(\lambda, \tau)$ is the set of all f in $H_0(\lambda, \tau)$ satisfying $\dim \{R_k f | k \in K\} < +\infty$.

LEMMA 3. *Let e be the identity of G. Then we have followings;*

1) $\lim_{y\to e} R_y f = f$ *in $H_0(\lambda, \tau)$ for all $f \in H_0^K(\lambda, \tau)$,*

2) *the mapping R: $y \to R_y$ is a continuous representation of G on the space $H_0(\lambda, \tau)$.*

PROOF OF 1). For any fixed function f in $H_0^K(\lambda, \tau)$, there exists a C^∞-function β with the compact support such that

$$(f*\beta)(x) = \int_G \beta(w^{-1}) f(xw) dw = f(x)$$

for all x in G. (See [10], Theorem 1.) Let V_1 be a compact neighbourhood of the identity e of G and put $V_2 = \mathrm{supp}(f)$. Put $L = \sup_{x \in V_2 V_1} e^{d(xo, o)}$. Then we have

$$\nu_b(R_y f - f) \leq \sup_{x\in G} \int_G |b'\hat{\beta}(wy^{-1}) - b'\hat{\beta}(w)| |f(xw)| e^{d(xo,o)} dw$$

$$\leq L_v \nu(f) \int_G |b'\hat{\beta}(wy) - b'\hat{\beta}(w)| dw$$

where $\hat{\beta}(x) = \beta(x^{-1})$. Therefore we conclude that

$$\lim \nu_b(Rf - f) = 0 \quad \text{for all } b \in \mathfrak{u} \text{ and } f \in H_0^K(\lambda, \tau).$$

PROOF OF 2). Let V be a compact neighbourhood of $e \in G$. Then for any fixed b in $\mathfrak{u}$, there exist $b_1, b_2, \dots, b_p$ in $\mathfrak{u}$ and $L_v \geq 0$ satisfying

$$\nu_b(R_y f) \leq L_v\left(\sum_{i=1}^{p} b_i(f)\right)$$

for all $y \in V$ and $f \in H_0(\lambda, \tau)$. Therefore by the first assertion in this lemma, we have

$$\lim_{y\to e} \nu_b(R_y f - f) \leq L_v\left(\sum_{i=1}^{p} \nu_{b_i}(f - f')\right) + \nu_b(f - f')$$

for all $f' \in H_0^K(\lambda, \tau)$. Hence by Lemma 1, we conclude that $\lim_{y\to e} \nu_b(R_y f - f) = 0$. Q.E.D.

We define the seminorm $\| \ \|$ in $H(\lambda, \tau)$ by

$$\|f\|^2 = \lim_{\varepsilon\to+0} \varepsilon \int_G |f(x)|^2 e^{-\varepsilon d(xo,o)} dx$$

for each $f \in H_0(\lambda, \tau)$. With this seminorm, $H_0(\lambda, \tau)$ is a pre-Hilbert space.

Note that the seminorm $\| \ \|$ is not a norm in general. We shall prove that the seminorm $\| \ \|$ is actually the norm in $H_0(\lambda, \tau)$ for the special values of λ.

LEMMA 4. *There exists a constant C_0 such that $\|f\| \leqq C_0 \nu(f)$ for all f in $H_0(\lambda, \tau)$.*

PROOF. Normalizing suitably the Haar measure dx on G, we have the following integral formula (see S. Helgason [11])

$$\int_G f(x)dx = \int_0^\infty \int_K \int_K f(k_1 \exp tHk_2) \left| \prod_{\alpha \in P_+} \sinh \alpha(tH_0) \right| dt dk_1 dk_2$$

for all f in $C_c^\infty(G)$ where H_0 is the element in $\mathfrak{a}_R$ defined by $\rho_+(H)=1$ and $C_c^\infty(G)$ is the set of all C^∞-functions with the compact support. Then we have

$$\lim_{\varepsilon \to +0} \varepsilon \int_G |f(x)|^2 e^{-\varepsilon d(xo, o)} dx \leqq [\nu(f)]^2 \lim_{\varepsilon \to +0} \varepsilon \int_0^\infty e^{-\varepsilon t} dt.$$

Therefore we conclude our assertion.

Let $H(\lambda, \tau)$ be the completion of the space $H_0(\lambda, \tau)$ with respect to the seminorm $\| \ \|$, and define the representation $U(\lambda, \tau) \equiv U(\lambda, \tau; y)(y \in G)$ by

$$[U(\lambda, \tau)f](x) = f(xy) \qquad x, y \in G \quad \text{and} \quad f \in H(\lambda, \tau).$$

Put $U(\lambda, \tau)|K$ be the restriction of $U(\lambda, \tau)$ to K. And let $H^K(\lambda, \tau)$ be the subspace of $H(\lambda, \tau)$ consisting of all K-finite functions for the representation $U(\lambda, \tau)|K$.

LEMMA 5. *Let $H_0^K(\lambda, \tau)$ be the set of all K-finite functions in $H_0(\lambda, \tau)$. Then we have*

1) $H_0^K(\lambda, \tau) = H^K(\lambda, \tau)$

2) *$U(\lambda, \tau)$ is a continuous representation of G on the pre-Hilbert space $H(\lambda, \tau)$ and*

3) $\lim_{t \to 0} \left\{ \frac{1}{t} [U(\lambda, \tau; \exp tX)f](x) - f(x) \right\} = Xf(x)$ *for all $X \in \mathfrak{g}$ and $f \in H_0(\lambda, \tau)$.*

PROOF. By Lemma 2, 1) is obvious. Therefore we shall prove 2) and 3). Since

$$\begin{aligned} -\varepsilon[d(xo, o) + d(yo, o)] &\leqq -\varepsilon d(xyo, o) \\ &\leqq -\varepsilon[d(xo, o) - d(yo, o)] \end{aligned}$$

for all $x \in G$ and $y \in G$ satisfying $d(xo, o) \geqq d(yo, o)$ and for $\varepsilon > 0$, we have

$$(U(\lambda, \tau; y)f, g) = \lim_{\varepsilon \to +0} \varepsilon \int_G (f(xy), g(x)) e^{-\varepsilon d(xo, o)} dx$$

$$= \lim_{\varepsilon\to+0} \varepsilon \int_G (f(x), g(xy^{-1}))e^{-\varepsilon d(xy^{-1}o, o)}dx$$
$$= \lim_{\varepsilon\to+0} \varepsilon \int_{d(xo,o)\geqq d(yo,o)} (f(x), g(xy^{-1}))e^{-\varepsilon d(xy^{-1}o, o)}dx$$
$$= (f, U(\lambda, \tau; y^{-1})g)$$

for all f and g in $H_0(\lambda, \tau)$. Hence $U(\lambda, \tau; y)$ is a unitary operator. Next we shall prove that $y \to U(\lambda, \tau; y)f$ is continuous for any fixed f in $H(\lambda, \tau)$. By Lemma 4, we have

$$\lim_{y\to e} \| U(\lambda, \tau; y)f - f \| = 0 \quad \text{for all } f \in H^K(\lambda, \tau).$$

Since $H^K(\lambda, \tau)$ is dense in $H(\lambda, \tau)$, we have

$$\lim_{y\to e} \| U(\lambda, \tau; y)f - f \| = 0 \quad \text{for all } f \in H(\lambda, \tau).$$

Hence we conclude that the mapping $y \to U(\lambda, \tau; y)f$ is continuous.

Finally we shall prove the assertion 3). Let f be a fixed element in $H^K(\lambda, \tau)$ and β be an element in $C_c^\infty(G)$ satisfying $f*\beta = f$. Then we have

$$\frac{1}{t}\{[U(\lambda, \tau; \exp tX)f](x) - f(x)\} - Xf(x) = \int_G \left\{\frac{1}{t}[\hat{\beta}(\exp tXy) - \hat{\beta}(x)] - X'\hat{\beta}(y)\right\} f(xy)dy$$

for all $X \in \mathfrak{g}$. Therefore we conclude that

$$\lim_{t\to 0} \nu\left(\frac{1}{t}[U(\lambda, \tau; \exp tXf)](x) - f(x)\right) - Xf(x) = 0.$$

The similar results hold for seminorms ν_b $(b \in \mathfrak{u})$.

§4.

In this section we shall recall the results on the asymptotic behaviour of the certain eigenfunctions of $\mathfrak{z}$ obtained by Harish-Chandra. Define the function Ξ on G by

$$\Xi(x) = \int_K e^{\rho_+(H(x,k))}dk$$

where $xk = k_x \exp H(x, k)n(x, k)$, $k_x \in K$, $H(x, k) \in \mathfrak{a}_R$ and $n(x, k) \in N$. Therefore by Theorem 3 in [9] there exists an integer $p \geqq 0$ satisfying

$$e^{-\rho_+(H)} \leqq \Xi(\exp H) \leqq e^{-\rho_+(H)}(1 + |H|^2)^{-p}$$

for all $H \in \mathfrak{a}_R$ satisfying $\rho_+(H) \geqq 0$ where $|H|^2 = B(H, H)$.

LEMMA 6. *Let f be an element in $H_0^K(\lambda,\tau)$. Then we have*

1) $zf=\chi_\lambda(z)f$ *for all* $z\in\mathfrak{z}$ *and*

2) $|f(x)|\leq$ *const.* $\Xi(x)$ *for all* $x\in G$.

This lemma is a direct consequence of the definition of $H_0^K(\lambda,\tau)$ and the above estimation for $\Xi(x)$.

Let $\mathfrak{u}_1$ be the subalgebra generated by $(1,(\mathfrak{m}_1)_c)$ in $\mathfrak{u}$. And let $\mathfrak{z}_1$ be the center of $\mathfrak{u}_1$. Then for each z in $\mathfrak{z}$, there exists a unique $\mu_1(z)\in\mathfrak{z}_1$ such that

$$z-e^{-\rho_+}\circ\mu_1(z)\circ e^{\rho_+}\in(\theta\mathfrak{n}_c)\mathfrak{u}.$$

Let $z\to\mu'(z)$ be the Chevalley isomorphism $\mathfrak{z}_1$ onto $I_1(\mathfrak{a}_c)$ where $I_1(\mathfrak{a}_c)$ is the set of all polynomials on $\mathfrak{a}_c$ invariant under the Weyl group of $((\mathfrak{m}_1)_c,\mathfrak{a}_c)$. Then $\mu(z)=(\mu'\circ\mu_1)(z)$ for each z in $\mathfrak{z}$.

Let W' be a minimal invariant subspace ($W'\neq 0$) of $H(\lambda,\tau)$ with respect to the representation $U(\lambda,\tau)|K$. Then $U(\lambda,\tau)|K$ induces an irreducible representation τ' of K on the space W'. Let $f_1,f_2,\dots,f_{d'}$ ($d'=\deg\tau'$) be an orthonormal basis of W' and let $u_1,u_2,\dots,u_d$ ($d=\deg\tau$) be an orthonormal basis of W. Put $f_j(x)=\sum_{i=1}^{d}f_{ij}(x)u_i$ for each x in G, $1\leq j\leq d'$. Let $\mathfrak{l}(W',W)$ be the space of linear mappings from W' into W. Put $\boldsymbol{f}=(f_{ij})$. Then the linear mapping f in $\mathfrak{l}(W',W)$ satisfies following conditions;

A1) $\boldsymbol{f}(k_1xk_2)=\tau(k_1)\boldsymbol{f}(x)\tau'(k_2)$ for all $k_1,k_2\in K$,

A2) $z\boldsymbol{f}=\chi_\lambda(z)\boldsymbol{f}$ for all z in $\mathfrak{z}$ and

A3) $|\boldsymbol{f}(x)|\leq$ const. $\Xi(x)$ for all x in G.

Then we have the following lemma.

LEMMA 7.

1) *There exists a unique function* $\boldsymbol{f}^0$ *on* $M_1=A_RM$ *with values in* $\mathfrak{l}(W',W)$ *satisfying*

$$\lim_{\rho_+(H)\to 0}|e^{d(m\exp tH_0,0)}\boldsymbol{f}(m\exp H)-\boldsymbol{f}^0(m\exp H)|=0$$

and $\boldsymbol{f}^0(m_1mm_2)=\tau(m_1)\boldsymbol{f}^0(m)\tau(m_2)$ *for all* $m_1,m_2\in M$ *and for all* $m\in M_1$.

2) *If* $zf=0$ *for an element* z *in* $\mathfrak{z}$, *then* $\mu_1(z)\boldsymbol{f}^0=0$.

3) $|\boldsymbol{f}(\exp H)-e^{-\rho_+(H)}\boldsymbol{f}^0(\exp H)|\leq$ *const.* $\dfrac{1}{|H|}e^{-\rho_+(H)}$ *for all sufficiently large* $\rho_+(H)$, $H\in\mathfrak{a}_R$.

See P. C. Trombi and V. S. Varadarajan [18], Theorem 1 (6.1).

Let $\boldsymbol{f}^0$ be the function in Lemma 7 corresponding to $\boldsymbol{f}$. Define f_j^0 ($j=1,2,\dots,d'=\deg\tau'$) by

$$f_j^0(x)=\sum_{i=1}^{d} f_{ij}^0(x)u_i \qquad \text{where} \quad f^0=(f_{ij}^0).$$

COROLLARY TO LEMMA 7.

$$\|f_j\|^2=\text{const.}\lim_{\varepsilon\to+0}\varepsilon\int_0^\infty |f_j^0(\exp tH_0)|^2 e^{-\varepsilon t}dt.$$

PROOF. Define τ'_{ij} by $f_j(xk)=\sum_{i=1}^{d'}\tau'_{ij}(k)f_i(x)$ $(j=1,2,\dots,d')$ for each k in K. By the integral formula in the proof of Lemma 4, we have

$$\begin{aligned}\|f_i\|^2&=\lim_{\varepsilon\to+0}\varepsilon\int_0^\infty\int_K\int_K |f_i(k_1\exp tH_0)k_2)|^2 e^{-\varepsilon t}e^{2\rho+(H_0)}dtdk_1dk_2\\
&=\lim_{\varepsilon\to+0}\varepsilon\int_0^\infty\int_K |f_i(\exp tH_0k_2)|^2 e^{2\rho+(H)-\varepsilon t}dtdk_2\\
&=\lim_{\varepsilon\to+0}\varepsilon\int_0^\infty\int_K\sum_{m,n}\tau'_{mi}(k)\tau'_{ni}(k)(f_n(\exp tH_0),f_m(\exp tH_0))e^{2\rho+(tH_0)+\varepsilon t}dtdk\\
&=\text{const.}\lim_{\varepsilon\to+0}\varepsilon\int_0^\infty |f_i(\exp tH_0)|^2dt.\end{aligned}$$

Hence we have our assertion.

§ 5.

Put $\mathfrak{l}_0=\{\lambda\,|\,\lambda$ is an integral form on $\mathfrak{a}_c$ satisfying $\prod_{\alpha\neq\pm\alpha_0}(\lambda,\alpha)\neq 0, (\lambda,\alpha_0)=0$ and $(\lambda,\beta)>0$ for each $\beta\in P_-\}$. In the following we shall always assume that $\lambda\in\mathfrak{l}_0$. Let W', f and f^0 be the same as in § 4. Choosing suitable bases $f_1,f_2,\dots,f_{d'}$ of W' and $u_1,u_2,\dots,u_d$ of W, we can assume that

$$\tau(m)=\begin{bmatrix}\sigma_1(m) & & & \\ & \sigma_2(m) & 0 & \\ & 0 & \ddots & \\ & & & \sigma_r(m)\end{bmatrix},\qquad \tau'(m)=\begin{bmatrix}\sigma'_1(m) & & & \\ & \sigma'_2(m) & 0 & \\ & 0 & \ddots & \\ & & & \sigma'_s(m)\end{bmatrix}$$

for all m in M where σ_i and σ_j's are irreducible components of the representation $\tau|M$ and $\tau'|M$ respectively.

LEMMA 8. 1) f^0 *has the form*

$$f(m\exp tH)=\begin{bmatrix}C_{11}C_{12}\cdots C_{1s}\\ C_{21}C_{22}\cdots C_{2s}\\ \vdots \qquad\quad \vdots\\ C_{r1}C_{r2}\cdots C_{rs}\end{bmatrix}\begin{bmatrix}\sigma'_1(m) & & & \\ & \sigma'_2(m) & & \\ & 0 & \ddots & \\ & & & \sigma'_s(m)\end{bmatrix}$$

for all $m\in M$ *and* $H\in\mathfrak{a}_R$ *where* C_{ij} *is a constant matrix of type* $(\deg\sigma_i, \deg\sigma_j')$.

2) *If we put* $F(t)=f(\exp tH_0)-e^{-\rho+(tH_0)}f^0(\exp tH_0)$, *then there exists a constant* $C>0$ *such that*

$$|F(t)|\leqq C\frac{e^{-t}}{t} \qquad \text{for all sufficiently large } t>0.$$

PROOF. Since 2) is a direct consequence of 3) in Lemma 7, we give a proof of 1). Put $n_i=\deg\sigma_i$ $(i=1,2,\dots,d)$, $n_j'=\deg\sigma_j'$ $(j=1,2,\dots,d')$ and

$$f^0(\exp tH_0)=\begin{bmatrix} C_{11}(t)C_{12}(t)\cdots C_{1s}(t)\\ C_{21}(t)C_{22}(t)\cdots C_{2s}(t)\\ \vdots \qquad\qquad \vdots\\ C_{r1}(t)C_{r2}(t)\cdots C_{rs}(t)\end{bmatrix}$$

where C_{ij} is an (n_i, n_j')-matrix valued function of t. Remark that $f^0(m\exp tH_0)=f^0(\exp tH_0 m)$ for all $t\in R$ and $m\in M$. By Shur's lemma, C_{ij} has the form

$$C_{ij}(t)=\begin{cases}C_{ij}^*(t)I_{n_i} & \text{if } \sigma_i\cong\sigma_j'\\ 0 & \text{otherwise}\end{cases}$$

where I_{n_i} is the identity matrix with degree n_i. We shall prove that C_{ij}^*'s are constants. For this purpose, we shall fix a pair (i,j) such that $C_{ij}^*\neq 0$. Let Ω be the Casimir operator of G. Then Ω belongs to $\mathfrak{z}$. Therefore by 2) of Lemma 7, we have

$$\mu_1(\Omega)f^0=\chi_\lambda(\Omega)f^0(|\lambda+\rho|^2-|\rho|^2)f^0$$

where $2\rho=\sum_{\alpha>0}\alpha$. Therefore we have

$$\mu_1(\Omega)C_{ij}^*\sigma_j'=C_{ij}^*[|\lambda+\rho|^2-|\rho|^2]\sigma_j'.$$

Calculating $\mu_1(\Omega)$, C_{ij}^* has the following form

$$C_{ij}^*(t)=C_{ij}^+(t)e^{\beta_i t}+C_{ij}^-(t)e^{-\beta_i t}$$

for some polynomial functions C_{ij}^+, C_{ij}^- with the degree $\leqq 1$ and a complex constant β_i. On the other hand, it can be proved that

$$\lim_{\varepsilon\to+0}\varepsilon\int_0^\infty |f^0(\exp tH)|^2e^{-\varepsilon t}dt<+\infty$$

(see Corollary to Lemma 7). Therefore we conclude that $C_{ij}^\pm(t)$'s are constants and β_i must be a pure-imaginary or non-positive real number. Therefore by the uniqueness of f^0 in Lemma 7, we have

$$C^*_{ij}(t)=C^+_{ij}e^{\beta_i t}+C^-_{ij}e^{-\beta_i t} \tag{1}$$

where C_{ij} are constants and β_i is pure-imaginary.

Now we prove that $\beta_i=0$. Put $\tilde{\omega}=\prod_{\alpha>0}H_\alpha$. Then $\mu_1^{-1}(\tilde{\omega}^2)$ belongs to $\mathfrak{z}$. Therefore we have

$$\mu_1(\mu^{-1}(\tilde{\omega}^2))f^0=\chi_\lambda(\mu^{-1}(\tilde{\omega}^2))f^0=(\prod_{\alpha>0}(\lambda,\alpha))^2f^0\equiv 0. \tag{2}$$

Assume that β_i in (1) is not zero and $\mu_i-\rho_-$ be the highest weight of the representation σ'_i. Then we have

$$\mu_1(\mu^{-1}(\tilde{\omega}^2))C^*_{ij}\sigma'_i=\{\prod_{\alpha>0}(\sqrt{-1}\beta^*\alpha_0+\mu_i,\alpha)\}^2C^*_{ij}\sigma'_i$$

where β^* is a real number ($\neq 0$). By the condition (2) and by the fact

$$\prod_{\alpha>0}(\sqrt{-1}\beta^*\alpha_0+\mu_i,\alpha)\neq 0,$$

we have a contradiction. Hence β_i must be equal to 0. Therefore we have the assertion of 2) in Lemma 8.

LEMMA 9. *Let λ be an element in $\mathfrak{l}_0$ and τ be an irreducible unitary representation of K. Then we have the following assertion.*

If $\|f\|=0$ and $f\in H_0(\lambda,\tau)$, then $f\equiv 0$ on G.

PROOF. By Lemma 2, it is sufficient to prove this lemma for the function f in W' $(\subset H^K(\lambda,\tau))$ where W' is the same as in §4. Let $f\in W'$ and $\|f\|=0$. By Corollary to Lemma 7 and by Lemma 8, we have $e^{\rho+(tH_0)}f(\exp tH_0)\to 0$ as $t\to+\infty$. Hence f satisfies the assumption of Corollary to Lemma 68 in [10]. Therefore there exists a regular integral form λ' on $\mathfrak{a}_c$ (i.e. $\lambda'(\tilde{\omega})\neq 0$) such that $zf=\chi_{\lambda'}(z)f$ for each z in $\mathfrak{z}$. On the other hand $zf=\chi_\lambda(z)f$, $z\in\mathfrak{z}$ and $\lambda(\tilde{\omega})=0$. Therefore by $\chi_{\lambda'}(\mu^{-1}(\tilde{\omega}^2))\neq 0$ and by $\chi_\lambda(\mu^{-1}(\tilde{\omega}^2))=0$, we conclude that $f\equiv 0$ on G. Q.E.D.

§6.

In the following we shall assume that G is one of the groups $\mathrm{Spin}(2l,1)$, $SU(l,1)$, and $Sp(l,1)$ $(l\geqq 1)$. Let M_0 be the connected component of M containing the identity e. Then there exists an element $\gamma\in M$ such that $M=M_0\cup\gamma M_0$ and γ belongs to $K\cap\exp\sqrt{-1}\mathfrak{a}_R$. Let λ be an element in $\mathfrak{l}_0$ and put $\lambda_-=\lambda-\rho_-$. Then λ_- is a dominant integral form on $\mathfrak{a}_I\subset\mathfrak{m}$. We define the irreducible unitary representation σ_{λ_-} of M as follows;

1. Let $\sigma_{\lambda_-}|M_0$ be the restriction of σ_{λ_-} to M_0. Then λ_- is the highest weight of $\sigma_{\lambda_-}|M_0$.
2. If $\gamma \notin M$, then $\sigma_{\lambda_-}(\gamma)=(-1)\times$the identity operator.

REMARK. Except the group Spin $(2,1)=SU(1,1)$, $M=M_0$. For Spin $(2,1)$, $M=\{e,\gamma\}$. In this section, we shall introduce the spherical functions corresponding to the representations in the principal series. Let F be the representation space of σ_{λ_-}. Let $L_2^F(K)$ be the set of all square integrable F-valued functions on K with respect to the Haar measure dk of K. We define $\mathfrak{h}(\lambda_-)$ by

$$\mathfrak{h}(\lambda_-)=\{f\in L_2^F(K)\,|\,f(km)=\sigma_{\lambda_-}(m^{-1})f(k) \quad \text{for all } k\in K \text{ and } m\in M\}.$$

For each s in $\boldsymbol{C}$, we define the representation $V(\lambda_-,s;x)(x\in G)$ of G on the space $\mathfrak{h}(\lambda_-)$ by

$$[V(\lambda_-,s;x)f](k)=e^{-((1+s)\rho+(H(x^{-1},k))}f(k_x^{-1})$$

for all $f\in\mathfrak{h}(\lambda_-)$ and $x\in G$. Let ω be an irreducible unitary representation of K and let $[V(\lambda_-,s)|K:\omega]$ (resp. $[\omega|M:\sigma_{\lambda_-}]$) be the multiplicity of ω (resp. σ_{λ_-}) which occurs in $V(\lambda_-,s)|K$ (resp. σ_{λ_-} in $\omega|M$).

LEMMA 10. *Let ω be the same as above. Then we have*

$$[V(\lambda_-,s)|K:\omega]=[\omega|M:\sigma_{\lambda_-}]$$

for all s in $\boldsymbol{C}$.

PROOF. Since $V(\lambda_-,s)|K=\operatorname*{ind}_{M\uparrow K}\sigma_{\lambda_-}$, we have our conclusion (see G. W. Mackey [15]).

Let ω be the same as in Lemma 10 and assume that $[\omega|M:\sigma_{\lambda_-}]\neq 0$. Then by Lemm 10, there exists a finite dimensional subspace $\mathfrak{h}'$ in $\mathfrak{h}(\lambda_-)$ such that $V(\lambda_-,s)|K$ induces the irreducible unitary representation ω on $\mathfrak{h}'$. Choose an orthonormal basis $v_1,v_2,\dots,v_q$ ($q=\deg\omega$) and put

$$\phi_{ij}^s(x)=(V(\lambda_-,s;x)v_i,v_j) \qquad (1\leqq i,j\leqq q).$$

We define $\phi_j^\omega(s;x)$ $(j=1,2,\dots,q)$ by

$$\phi_j^\omega(s;x)=\sum_{i=1}^q\phi_{ij}^s(x)v_i.$$

Then by A. W. Knapp and E. M. Stein [14] (§§ 11, 12, 13), we have the following lemma.

LEMMA 11. *There exist the meromorphic functions $\Gamma_{ij}^{\pm}(s)$ $(i,j=1,2,\dots,q)$ on entire s-plane and the constants $c_1,c_2>0, 0<d_1<d_2$ such that*

1) $\left|\phi_j^\omega(s;\exp tH_0)-\sum_{i=1}^q[\Gamma_{ij}^+(s)e^{-(1+s)t}+\Gamma_{ij}^-(s)e^{-(1-s)t}]v_i\right|\leqq c_1e^{-(1+c_2)t}$

for all (s, t) *satisfying* $d_1 \leqq |\mathrm{Re}(s)| \leqq d_2$ *and* $t \geqq 0$

2) $\Gamma_{ij}^{\pm}(s)$ $(i, j=1, 2, \ldots, q)$ *are holomorphic on the domain* $\{s \in \boldsymbol{C} | \mathrm{Re}(s)| < d_2$ *and* $s \neq 0\}$

3) $V(\lambda_-, \sqrt{-1}\eta)$ $(\eta \in \boldsymbol{R}, \eta \neq 0)$ *is an irreducible unitary representation of* G.

Let $P_{\sigma_{\lambda_-}}(\sqrt{-1}\eta)$ *be the contribution to the Plancherel measure by the principal series representation* $V(\lambda_-, \sqrt{-1}\eta)$ $(\eta \in \boldsymbol{R}, \eta \neq 0$ *and* λ_- *is the highest weight of the representation* $\sigma_{\lambda_-})$. *Then*

4) $\sum_{i,j=1}^{q} |\Gamma_{ij}^{\pm}(\sqrt{-1}\eta)|^2 = const.\ (P_{\sigma_{\lambda_-}}(\sqrt{-1}\eta))^{-1}$

5) $P_{\sigma_{\lambda_-}}(0) \neq 0$ *and* $V(\lambda_-, 0)$ *is reducible.*

COROLLARY TO LEMMA 11.

1) $\Gamma_{ij}^{\pm}(s)$ *are holomorphic at* $s=0$.

2) $\left|\phi_j^{\omega}(0; \exp tH_0) - \left[\sum_{i=1}^{q} (\Gamma_{ij}^{+}(0) + \Gamma_{ij}^{-}(0)) e^{-t} v_i\right]\right| \leqq c_1 e^{-(1+c_2)t}$

for all $t \geqq 0$ *and* $j=1, 2, \ldots, q$.

PROOF. By 2), 4) and 5) in Lemma 11, we have the first assertion. Therefore by 1) in Lemma 11 and the maximum principle for the function of s

$$\phi_j^{\omega}(s; \exp tH) - \sum_{i=1}^{q} (\Gamma_{ij}^{+}(s) e^{-(1+s)t} + \Gamma_{ij}^{-}(s) e^{-(1-s)t}) v_i,$$

we have our conclusion 2) in this corollary.

LEMMA 12. *Put* $\lambda_s = \lambda + s\rho_+$ *for each* s *in* $\boldsymbol{C}$. *Then we have*

$$z\phi_j^{\omega}(s; x) = \chi_{\lambda_s}(z) \phi_j^{\omega}(s; x)$$

for all $z \in \mathfrak{z}$ *and* $j=1, 2, \ldots, q$.

PROOF. We define $V_f(\lambda_-, s) = \int_G f(x) V(\lambda_-, s; x) dx$ for each $s \in \boldsymbol{C}$ and $f \in C_c^{\infty}(G)$. Then the linear form $T^{\lambda_-, s}$: $f \to \mathrm{Trace}\, V_f(\lambda_-, s)$ is a distribution on G and satisfies $zT^{\lambda_-, s} = \chi_{\lambda_s}(z) T^{\lambda_-, s}$ for all z in $\mathfrak{z}$ (see Theorem 2 in [7] and Theorem 3 in [8]). On the other hand $V(\lambda_-, \sqrt{-1}\eta)$ $(\eta \neq 0, \eta \in R)$ is irreducible by 3) in Lemma 11. Therefore we have

$$z\phi_j^{\omega}(\sqrt{-1}; x) = \chi_{\sqrt{-1}\eta}(z) \phi_j^{\omega}(\sqrt{-1}\eta; x)$$

for all z in $\mathfrak{z}$ and j $(1 \leqq j \leqq q)$. Moreover $\phi_j^{\omega}(s; x)$, $z\phi_j^{\omega}(s; x)$ and $\chi_{\lambda_s}(z)$ $(z \in \mathfrak{z})$ are holomorphic on the whole s-plane. Hence by the analytic continuation, we conclude that $z\phi_j^{\omega}(s; x) = \chi_{\lambda_s}(z) \phi_j^{\omega}(s; x)$ for all $z \in \mathfrak{z}$, $s \in \boldsymbol{C}$ and $j=1, 2, \ldots, q$.

Let τ be an irreducible unitary representation of K on the space W. Let us suppose that $[\tau | M : \sigma_{\lambda_-}] \neq 0$. Then by Lemma 10, there exists a finite dimensional subspace $\mathfrak{h}'$ in $\mathfrak{h}(\lambda_-)$ (which is isomorphic to W) such that $V(\lambda_-, s) | K$ induces the representation τ on $\mathfrak{h}'$. We put $W = \mathfrak{h}'$. Let $u_1, u_2, \ldots, u_d$ $(d = \deg \tau)$ be the orthonormal basis in W. Put $\phi_{ij}(s;x)$ by

$$\phi_{ij}(s;x)=(V(\lambda_-,s;x)u_i,u_j)$$

for each $i,j=1,2,\ldots,d$. We define ϕ_j by

$$\phi_j(s;x)=\sum_{i=1}^{d}\phi_{ij}(s;x)u_i.$$

LEMMA 13. *Let λ be an element in $\mathfrak{l}_0$ and τ be an irreducible unitary representation of K on W. Assume that $[\tau|M:\sigma_{\lambda_-}]\neq 0$. Then ϕ_j $(\neq 0)$ belongs to $H(\lambda,\tau)$ for all $j=1,2,\ldots,d$.*

PROOF. Using Corollary to Lemma 11, we have $\nu(\phi_j)<+\infty$. On the other hand by Lemma 12, ϕ_j $(j=1,2,\ldots,d)$ are both K-finite and $\mathfrak{z}$-finite. Hence we can choose a function β in $C_c(G)$ satisfying $\phi_j*\beta=\phi_j$ for any fixed j. Thus we have

$$\begin{aligned}|b\phi_j(x)|e^{d(xo,o)}&\leqq\int_G|b'\hat{\beta}(y)||\phi_j(xy)|e^{d(xyo,o)+d(y^{-1}o,o)}dy\\&\leqq\nu(\phi_j)\int_G|b'\hat{\beta}(y)|e^{d(y^{-1}o,o)}dy<+\infty\end{aligned}$$

for all b in $\mathfrak{z}$. Hence we have $\nu_b(\phi_j)<+\infty$. Therefore we conclude that $0\neq\phi_j$ belongs to $H(\lambda,\tau)$ for all j $(1\leqq j\leqq d)$.

COROLLARY TO LEMMA 13. *Let λ and τ be the same as in above lemma. Then we have $[U(\lambda,\tau)|K:\tau]\neq 0$.*

PROOF. By Lemma 13, ϕ_j $(1\leqq j\leqq d)$ are in $H(\lambda,\tau)$ and $\phi_j\neq 0$. Since $E(\tau)\phi_j\neq 0$, we conclude that $[U(\lambda,\tau)|K:\tau]\neq 0$.

LEMMA 14. *Let us suppose that $[\tau|M:\sigma_{\lambda_-}]=1$. We define ϕ_τ by*

$$\phi_\tau(x)=\sum_{j=1}^{d}\phi_{jj}(0;x).$$

Then we have the followings.

1) $d^2\int_K\phi_\tau(xkyk^{-1})dk=\phi_\tau(x)\phi_\tau(y)$ *for all* $x,y\in G$.

2) *Put* $I(y)=\lim_{\varepsilon\to+0}\varepsilon\int_G\phi_\tau(x)\overline{\phi_\tau(xy)}e^{-\varepsilon d(xo,o)}dx$. *Then we have*

$$\begin{aligned}d^2I(y)&=I(e)\overline{\phi_\tau(y)}\\&=d\times\lim_{\varepsilon\to+0}\varepsilon\int_G\sum_{i,j}\phi_{ij}(x)\overline{\phi_{ij}(xy)}e^{-\varepsilon d(xo,o)}dx.\end{aligned}$$

3) $d^2\|\phi_j\|=I(e)$ *for all* $j=1,2,\ldots,d$.

4) $\phi_\tau(x)=d\sum_{i=1}^{d}(U(\lambda,\tau;x)\phi_i,\phi_i)\|\phi_i\|^{-2}$.

5) $\phi_{ii}(x)=(U(\lambda,\tau;x)\phi_i,\phi_i)\|\phi_i\|^{-2}$ *for all* $i=1,2,\ldots,d$.

PROOF OF 1). By Lemma 10 and by our assumption, we have

$$[V(\lambda_-, 0) | K : \tau] = [\tau | M : \sigma_{\lambda_-}] = 1.$$

Therefore 1) is proved by Theorem 10 in [6] (R. Godement).

PROOF OF 2). By 1) in this lemma, we have

$$\begin{aligned} d^2 I(y) &= d^2 \lim_{\varepsilon \to +0} \varepsilon \int_G \int_K \phi_\tau(x) \overline{\phi_\tau(xkyk^{-1})} e^{-\varepsilon d(x_0, o)} dk dx \\ &= \overline{\phi_\tau(y)} \lim_{\varepsilon \to +0} \varepsilon \int_G |\phi_\tau(x)|^2 e^{-\varepsilon d(x_0, o)} dx \\ &= I(e) \overline{\phi_\tau(y)}. \end{aligned}$$

On the other hand

$$\begin{aligned} I(y) &= d^2 \lim_{\varepsilon \to +0} \varepsilon \int_G \sum_{i,j} \phi_{ii}(x) \phi_{jj}(xy) e^{-\varepsilon d(x_0, o)} dx \\ &= d^2 \lim_{\varepsilon \to +0} \left\{ \int_K \sum_{i,j,m,n} \tau_{im}(k) \overline{\tau_{jn}(k)} dk \, \varepsilon \int_G \phi_{im}(x) \overline{\phi_{jn}(xy)} e^{-\varepsilon d(x_0, o)} dx \right\} \end{aligned}$$

where

$$\begin{aligned} (V(\lambda_-, 0; kx) u_i, u_i) &= (V(\lambda_-, 0; x) u_i, V(\lambda_-, 0; k^{-1}) u_i) \\ &= \sum_{j=1}^{d} \overline{\tau_{ji}(k^{-1})} (V(\lambda_-, 0; x) u_i, u_j) \\ &= \sum_j \tau_{ij}(k) \phi_{ij}(x). \end{aligned}$$

By Schur's orthogonality relations for the representation $\tau = (\tau_{ij})$ of the compact group K, we have our assertion.

PROOF OF 3). For any fixed j, we have

$$\begin{aligned} \|\phi_j\|^2 &= \lim_{\varepsilon \to +0} \varepsilon \int_G \sum_i |\phi_{ij}(x)|^2 e^{-\varepsilon d(x_0, o)} dx \\ &= \lim_{\varepsilon \to +0} \varepsilon \int_G \sum_i \left(\int_K \phi_{ij}(kx) \overline{\phi_{ij}(kx)} e^{-\varepsilon d(x_0, o)} dk \right) dx \\ &= \lim_{\varepsilon \to +0} \varepsilon \int_G \sum_{i,n,m} \left(\int_K \tau_{mj}(k) \overline{\tau_{nj}(k)} dk \right) \phi_{im}(x) \phi_{in}(x) e^{-\varepsilon d(x_0, o)} dx \\ &= \lim_{\varepsilon \to +0} \frac{\varepsilon}{d} \int_G \sum_{i,m} |\phi_{im}(x)|^2 e^{-\varepsilon d(x_0, o)} dx \\ &= d^{-2} I(e). \end{aligned}$$

PROOF OF 4). By 2) and 3) in this lemma, we have

$$\begin{aligned} \phi_\tau(y) &= \overline{(\overline{\phi_\tau(y)})} \\ &= \frac{1}{I(e)} I(y) \end{aligned}$$

$$=\frac{d^3}{I(e)}\lim_{\varepsilon\to+0}\int_G\sum_{i,j}\phi_{ij}(x)\phi_{ij}(xy)e^{-\varepsilon d(xo,o)}dx$$
$$=d\sum_i(U(\lambda,\tau;y)\phi_i,\phi_i)\|\phi_i\|^{-2}$$

PROOF OF 5). By 4) in this lemma, we have

$$\sum_i(V(\lambda_-,0;xk)u_i,u_i)=\sum_i(U(\lambda,\tau;xk)\phi_i,\phi_i)\|\phi_i\|^{-2}.$$

Hence we have

$$\begin{aligned}\phi_{ii}(x)&=(V(\lambda_-,0;x)u_i,u_i)\\&=\int_K\sum_{j=1}^d\overline{\tau_{ii}(k)}(V(\lambda_-,0;xk)u_j,u_j)dk\\&=\int_K\sum_{j=1}^d\overline{\tau_{ii}(k)}(U(\lambda,\tau;xk)\phi_j,\phi_j)\|\phi_j\|^{-2}dk\\&=(U(\lambda,\tau;x)\phi_i,\phi_i)\|\phi_i\|^{-2}.\end{aligned}$$

Q.E.D.

§ 7. A proof of Theorem 1.

Let τ be an irreducible unitary representation of K. Let us assume that $[\tau|M:\sigma_{\lambda_-}]=1$. Then by Corollary to Lemma 13, we have $[U(\lambda,\tau)|K:\tau]\neq0$.

Let W' be a subspace of $H(\lambda,\tau)$ on which $U(\lambda,\tau)|K$ induces the representation τ of K. Let f and f^0 be the same as in § 6 corresponding to W'. Then we have the following theorem.

THEOREM 1. *Assume $[\tau|M:\sigma_{\lambda_-}]=1$ and let f^0 be the same defined as above. Then there exists a constant C satisfying*

$$f^0(\exp Hm)=C\begin{bmatrix}0&0&0\\0&\sigma_{\lambda_-}(m)&0\\0&0&0\end{bmatrix}$$

for all $H\in\mathfrak{a}_R$ and $m\in M$.

REMARK. The existence of τ satisfying $[\tau|M:\sigma_{\lambda_-}]=1$ will be showed in § 9 for any fixed λ.

First we introduce the necessary notations. Let A be the Cartan subgroup of G corresponding to $\mathfrak{a}$. We define the function ξ_α $(\alpha\in\Sigma)$ on A by

$$\mathrm{Ad}(h)X_\alpha=\xi_\alpha(h)X_\alpha\qquad\text{for all } h \text{ in } A.$$

Let σ'_{ρ_-} be the irreducible unitary representation of M_0 with the highest weight ρ_-. We extend the representation σ'_{ρ_-} to M by $\sigma'_{\rho_-}(\gamma)=$the identity operator for $\gamma\notin M_0$. Define the function ξ'_{ρ_-} on A by

$$\mathrm{Ad}\,(h)v_{\rho_-} = \xi'_{\rho_-}(h)v_{\rho_-} \quad \text{for all } h \text{ in } A_I$$

where v_{ρ_-} is the highest vector of σ'_{ρ_-}. (If $M=\{e,\gamma\}$, we put $\xi'_{\rho_-}\equiv 1$.) We define the function ξ_{ρ_-} on A by

$$\xi_{\rho_-}(h_-h_+) = \xi_{\rho_-}(h_-) \quad \text{where} \quad h_- \in A_I \quad \text{and} \quad h_+ \in A_R.$$

Therefore we define the function Δ_- on A by

$$\Delta_-(h) = \xi_{\rho_-}(h) \prod_{\alpha \in P_-} (1 - \xi_\alpha(h^{-1}))$$

for all h in A.

PROOF OF THEOREM 1. By Lemma 11, $\boldsymbol{f}$ has the form $\boldsymbol{f}=(C_{ij}\sigma_j)$ where σ_j $(1\leqq j\leqq r)$ are the irreducible components of the representation $\tau|M$ and C_{ij} is a constant matrix with $\deg\sigma_i$ rows, $\deg\sigma_j$ columns for $i,j=1,2,\ldots,r$.

Suppose that $C_{ij}\neq 0$ for some i and j. Then C_{ij} is a scaler matrix with $\deg\sigma_i$. Put $\xi_i(h)=\Delta_-(h)\,\mathrm{Trace}\,\sigma_i(h)$, $h\in A$. Then by the conditions $\mu'\circ\mu_1(z)=\mu(z)$ and $\mu_1(z)\boldsymbol{f}^0=\chi_\lambda(z)\boldsymbol{f}^0$ $(z\in\mathfrak{z})$, we have

$$(*) \qquad \mu(z)\xi_i = \chi_\lambda(z)\xi_i$$

for all z in $\mathfrak{z}$. Therefore it is sufficient to prove that $\sigma_i=\sigma_{\lambda_-}$. For this purpose, we shall consider separately the groups $\mathrm{Spin}\,(2l,1)$, $SU(l,1)$ and $Sp(l,1)$.

(1) Let $G=\mathrm{Spin}\,(2l,1)$. Then $\dim\mathfrak{a}=l$. Let $e_1,e_2,\ldots,e_l$ be an orthonormal basis in the l-dimensional Euclidean space $\boldsymbol{R}^l\cong\sqrt{-1}\mathfrak{a}_I+\mathfrak{a}_R$. Then the root system Σ can be identified with the set $\{\pm(e_i\pm e_j)(1\leqq i<j\leqq l),\ \pm e_i\,(1\leqq i\leqq l)\}$. Let

$$P=\{e_i\pm e_j\ (1\leqq i<j\leqq l),\ \ e_i\ (1\leqq i\leqq l)\}$$

be the positive root system in Σ. Then we have $e_1>e_2>\cdots>e_l$ and $e_1=$the singular real positive root.

We define Q_d $(1\leqq d\leqq l)$ in $I(\mathfrak{a}_c)$ by

$$Q_d = \sum_{1\leqq i_1<i_2<\cdots<i_d\leqq l} (H_{i_1}H_{i_2}\cdots H_{i_d})^2$$

where $e_i(H_j)=\delta_{ij}$ and $H_j\in\boldsymbol{R}^l$. Put $\lambda=m_1e_1+m_2e_2+\cdots+m_le_l\,(\in\mathfrak{l}_0)$. Then $m_1=0$, $m_2>m_3>\cdots>m_l>0$ and all of m_i's are integers. Let $\mu_i-\rho_-$ be the highest weight of the representation σ_i of M. We put $\mu_i=m'_1e_1+m'_2e_2+\cdots+m'_le_l$. Then by $(*)$ we have the following;

$$\sum_{1\leqq i_1<i_2<\cdots<i_d\leqq l} (m_{i_1})^2(m_{i_2})^2\cdots(m_{i_d})^2 = \sum_{i\leqq i_1<i_2<\cdots<i_d\leqq l} (m'_{i_1})^2(m'_{i_2})^2\cdots(m'_{i_d})^2$$

for all d $(1\leqq d\leqq l)$. Therefore we have the identity

$$\prod_{i=1}^{l}(u-(m_i)^2)=\prod_{i=1}^{l}(u-(m_i')^2).$$

Hence we conclude that $m_i=m_i'(1\leqq i\leqq l)$ and $\sigma_i=\sigma_{\lambda_-}$.

(2) $G=SU(l,1)$. Then $\dim\mathfrak{a}=l$. Let $e_1,e_2,\dots,e_{l+1}$ be an orthonormal basis in the Euclidean space $\boldsymbol{R}^{l+1}$. Then the set of positive root system P is identified with the set $\{e_i\pm e_j\mid 1\leqq i<j\leqq l+1\}$. Therefore we define the polynomials Q_d $(1\leqq d\leqq l+1)$ invariant under the Weyl group of $(\mathfrak{a}_c,\mathfrak{a}_c)$ by

$$Q_d=\sum_{1\leqq i_1<i_2<\cdots<i_d\leqq l+1}H_{i_1}H_{i_2}\cdots H_{i_d}.$$

In this case e_1-e_{l+1}=the singular positive real root. By the same method as in Spin $(2l,1)$, we conclude that $\sigma_i=\sigma_{\lambda_-}$.

(3) $G=Sp(l,1)$. In this case $\dim\mathfrak{a}=l$, the set of all positive root $P=\{e_i\pm e_j\ (1\leqq i<j\leqq l),\ 2e_i\ (1\leqq i\leqq l)\}$ and the singular real root α_0 $(\in P)=e_1+e_2$. Using the same way as in the case (1), we have our conclusion. Q.E.D.

§8. Main theorems.

Let α_0 be the singular real root as in §2. Then we have the following theorem.

THEOREM 2. *Put* $\mathfrak{l}_0=\{\lambda\mid\lambda$ *is an integral form on* $\mathfrak{a}_c$ *and satisfies* $(\lambda,\alpha_0)=0$, $\prod_{\alpha\neq\pm\alpha_0}(\lambda,\alpha)\neq 0$, $(\lambda,\alpha)>0$ *for all* α *in* $P_-\}$. *Let* λ *be an element in* $\mathfrak{l}_0$ *and* τ *be an irreducible unitary representation of* K *on the space* W. *Let* σ_{λ_-} *be the irreducible representation of* M *with the highest weight* $\lambda_-=\lambda-\rho_-$ *satisfying* $\sigma_{\lambda_-}(\gamma)=-1$ *if* M *is not connected (remark* $\gamma\in K\cap\exp(\sqrt{-1}\mathfrak{a}_R)$). *Let us assume that* $[\tau\mid M:\sigma_{\lambda_-}]=1$. *Then the representation* $U(\lambda,\tau)$ *of* G *on the space* $H(\lambda,\tau)$ *is an irreducible unitary representation.*

PROOF. Let H' be a non-zero closed invariant subspace of $H(\lambda,\tau)$ and g be a K-finite element in H' satisfying $\|g\|\neq 0$. Then there exists x_0 in G such that $g(x_0)\neq 0$. Since $[U(\lambda,\tau;x_0)g](e)=g(x_0)\neq 0$ and $U(\lambda,\tau;x_0)g\in H'$, we can assume that $g\in H'$, $g(e)\neq 0$. Then by the fact $[E(\tau)g](e)=g(e)$ and by Lemma 9, we have $H_\tau\equiv E(\tau)H'\neq\{0\}$ where $E(\tau)$ is the projection operator defined as in p. 7. Let $\phi_j(x)=\sum_{i=1}^{d}\phi_{ij}(x)u_i$ $(1\leqq j\leqq d)$ be the same as in Lemma 12 where $u_1,u_2,\dots,u_d$ is the orthonormal basis in W. We put $\boldsymbol{\phi}(x)=(\phi_{ij}(x))$. Then by Lemma 8, there exists a unique matrix-valued function on M_1, $\boldsymbol{\phi}^0=(\phi_{ij}^0)$ corresponding to $\boldsymbol{\phi}$. Let $f_1,f_2,\dots,f_d$ be the orthonormal basis of an irreducible component in H_τ satisfying $U(\lambda,\tau;k)f_j=\sum_{i=1}^{d}\tau_{ij}(k)f_i$ for all k in K where τ_{ij} are defined by $\tau(k)u_j=\sum_{i=1}^{d}\tau_{ij}(k)u_i$. We put

$f(x)=(f_{ij}(x))$ where f_{ij} are defined by $f_j(x)=\sum_{i=1}^{d} f_{ij}(x)u_i$. Let $f^0=(f^0_{ij})$ be the function as in Lemma 8 corresponding to f. By Theorem 1, there exists a constant C such that

$$Cf^0(m \exp H)=\phi^0(m \exp H)$$

for all $m\in M$ and $H\in \mathfrak{a}_R$. Therefore by Corollary to Lemma 7, we have

$$\|Cf_j-\phi_j\|=0$$

for all $j=1,2,\dots,d$. Using Lemma 9, we conclude that $Cf_j=\phi_j$ for all j $(1\leqq j\leqq d)$. Hence we have $\phi_j\in H'$. Considering orthogonal complement of H' in $H(\lambda,\tau)$, we conclude that $H'=H(\lambda,\tau)$ and $U(\lambda,\tau)$ is irreducible.

THEOREM 3. *If λ and τ are the same as in Theorem 2, then $U(\lambda,\tau)$ is a proper subrepresentation of the principal series representation $V(\lambda_-,0)$.*

PROOF. By 5) in Lemma 14, there exist $u\in\mathfrak{h}(\lambda_-)$ and $v\in H(\lambda,\tau)$ $(u\neq 0, v\neq 0)$ such that $(U(\lambda,\tau;x)v,v)=(V(\lambda_-,0;x)u,u)$ for all x in G. Therefore $U(\lambda,\tau)$ is equivalent to a subrepresentation of $V(\lambda_-,0)$ (see [5]). Hence 5) in Lemma 11, and by Theorem 2, we have our assertion.

REMARK. Let W_G be the Weyl group of G. Then the order of W_G is equal to 2. By F. Bruhat ([3]), the cardinal number of the irreducible components occurring in the representation $V(\lambda_-,0)$ equals to 2 or 1.

§ 9. Appendix.

In this section, we shall state the existence*) of τ satisfying $[\tau|M:\sigma_{\lambda_-}]=1$ for any fixed λ in $\mathfrak{l}_0$.

Let $\mathfrak{b}$ be a Cartan subalgebra of $\mathfrak{g}$ satisfying $\mathfrak{a}_I\subset\mathfrak{b}\subset\mathfrak{k}$. Then there exists an element in G_c such that $\mathrm{Ad}(y)\mathfrak{b}_c=\mathfrak{a}_c$ and $\mathrm{Ad}(y)H=H$ for each H in $\mathfrak{a}_I$. We define α^y $(\alpha\in\Sigma)$ by $\alpha^y(H)=\alpha(\mathrm{Ad}(y)H)$ for all $H\in\mathfrak{b}_c$. Identifying α^y with α, Σ^y is identified with Σ. Let P_k be the set of all compact positive roots in Σ. Then we can assume that $P_-\subset P_k$. Let e_i's be the same as in the proof of Theorem 1. Then we can assume the followings;

$\mathfrak{b}=\sqrt{-1}R^l$ for the groups $\mathrm{Spin}(2l,1)$ and $Sp(l,1)$,

*) Those results have been obtained in [2] (Boerner), [20] (Lepowsky) for all real rank one semisimple Lie groups.

The results in [20] is pointed out to the author by A. Knapp and N. Wallach.

Using this, we can apply the arguments in this paper to the exceptional real rank one case (see a forthcoming paper).

$\mathfrak{b}=\{H\in\sqrt{-1}\boldsymbol{R}^{l+1}\mid e_1(H)+e_2(H)+\cdots+e_l(H)+e_{l+1}(H)=0\}$ for the group $SU(l,1)$.

(1) When $G=\mathrm{Spin}(2l,1)$, $\mathfrak{a}_I=\boldsymbol{R}\sqrt{-1}H_1+\boldsymbol{R}\sqrt{-1}H_2+\cdots+\boldsymbol{R}\sqrt{-1}H$ where H_i $(1\leqq i\leqq l)$ are defined by $e_j(H_i)=\delta_{ji}$ $1\leqq j\leqq l$, $H_i\in\boldsymbol{R}^l$. Fix an element λ in $\mathfrak{l}_0$ and put $\lambda_-=\lambda-\rho_-=\sum_{i=1}^{l}m_ie_i$. Then $m_1=0, m_2\geqq m_3\geqq\cdots\geqq m_l\geqq 0$ and all of m_i's are strictly half integers. Let $\mu=\sum_{i=1}^{l}\mu_ie_i$ be an integral form on $\mathfrak{b}$ satisfying $\mu_1\geqq m_2\geqq \mu_2\geqq m_3\geqq\cdots\geqq m_l\geqq\mu_l=\frac{1}{2}$. We put $\mu_+=\mu$ and $\mu_-=\mu_1e_1+\mu_2e_2+\cdots+\mu_{l-1}e_{l-1}-\frac{1}{2}e_l$.

LEMMA 15. *Let τ_+ and τ_- be the irreducible representations of K with the highest weight μ_+ and μ_- respectively. Then we have $[\tau_+|M:\sigma_{\lambda_-}]=[\tau_-|M\sigma_{\lambda_-}]=1$.*

For a proof of this lemma, see [2] (Boerner).

PROPOSITION. *Let τ_+ and τ_- be the same as in Lemma 15. Then we have $V(\lambda_-,0)=U(\lambda,\tau_+)\oplus U(\lambda,\tau_-)$ (direct sum).*

PROOF. By the remark after Theorem 3 and by 5) in Lemma 11, the number of the irreducible components in $V(\lambda_-,0)$ is equal to 2. On the other hand by Theorem 3, $U(\lambda,\tau_+)$ and $U(\lambda,\tau_-)$ are contained in $V(\lambda_-,0)$. By T. Hirai [12], $[U(\lambda,\tau_+)|K:\tau_-]=0$, $[U(\lambda,\tau_-)|K:\tau_+]=0$. Hence $U(\lambda,\tau_+)$ and $U(\lambda,\tau_-)$ are inequivalent to each other. Therefore we conclude that $V(\lambda_-,0)=U(\lambda,\tau_+)\oplus U(\lambda,\tau_-)$.

(2) Let $G=SU(l,1)$. In this case $\mathfrak{a}_I=\{H\in\mathfrak{b}\mid e_1(H)=e_{l+1}(H)\}$, $P_k=\{e_i-e_j\mid 2\leqq i<j\leqq l+1\}$ and $P_-=\{e_i-e_j\mid 2\leqq i<j\leqq l\}$. Let $\mu=\sum_{i=1}^{l}\mu_ie_i$ be an integral form on $\mathfrak{b}$ satisfying $\mu_2>\mu_3>\cdots>\mu_l>0$. And let $\pi_{\mu-\rho_k}$ be the irreducible representation of K with the highest weight $\mu-\rho_k$ where $\rho_k=\frac{1}{2}\sum_{\alpha\in P_k}\alpha$. We put

$$A(\mu)=\{\nu=(\nu_2,\nu_3,\dots,\nu_l)\mid \mu_2\geqq\nu_2\geqq\cdots\geqq\nu_l>0 \text{ and all of } \nu_i\text{'s are integers}\}$$

and

$$\nu_0=\frac{1}{2}\left\{(\mu_1+\mu_2+\cdots+\mu_l)-(\nu_2+\nu_3+\cdots+\nu_l)-2\nu_l+\frac{1}{2}(l+1)\right\}(e_1+e_{l+1})$$
$$+(\nu_2-\nu_l)e_2+(\nu_3-\nu_l)e_3+\cdots+(\nu_{l-1}-\nu_l)e_{l-1}.$$

LEMMA 16. *Let $G=SU(l,1)$ and $l\geqq 2$. Then we have*

$$\mathrm{Trace}\,\pi_{\mu-\rho_k}(\exp H)=\sum_{\nu_0\in A(\mu)}\mathrm{Trace}\,\sigma_{\nu_0-\rho_-}(\exp H)$$

for all H in $\mathfrak{a}_I$ where $\sigma_{\nu_0-\rho_-}$ is the irreducible representation of M with the highest weight $\nu_0-\rho_-$.

COROLLARY TO LEMMA 16. *Let λ be an element in $\mathfrak{l}_0$. We put $\lambda=\sum_{i=1}^{l}m_ie_i$.*

Then $m_1=0, m_2>m_3>\cdots>m_l$. *Let* p_i $(2\leqq i\leqq l)$ *be* $l-1$ *integers satisfying*

$$m_2+p_2\geqq m_2+p_l>m_3+p_3\geqq m_3+p_l>\cdots>m_{l-1}+p_{l-1}\geqq m_{l-1}+p_l>m_l+p_l>0.$$

Put $\mu=\sum_{i=1}^{l}\mu_i e_i$ *where* $\mu_l=m_l+p_l,\ \mu_{l-1}=m_{l-1}+p_{l-1},\ \ldots,\ \mu_2=m_2+p_2,$

$$\mu_1=-\frac{1}{2}(l+1)+2p_l-[(p_2-p_l)+(p_3-p_l)+\cdots+(p_{l-1}-p_l)].$$

Then we have $[\pi_{\mu-\rho_k}|M:\sigma_{\lambda\_}]=1$ *where* $\pi_{\mu-\rho_k}$ *is the irreducible representation of* M *with the highest weight* $\mu-\rho_k$.

REMARK. Let $\lambda=\sum_{i=1}^{l}m_i e_i$ be an element in $\mathfrak{t}_0$ satisfying $m_2>m_3>\cdots>m_l>0$. We put $\mu=\sum_{i=1}^{l}\mu_i e_i$ where $\mu_1=-\frac{1}{2}(l+1), \mu_2=m_2, \mu_3=m_3\cdots, \mu_l=m_l$. Then $U(\lambda, \pi_{\mu-\rho_k})$ is equivalent to a representation in the "limits of holomorphic discrete series" which is constructed by A. W. Knapp and K. Okamoto in [13];

Put $\Lambda=\mu-\rho$ where $\rho=\frac{1}{2}\sum_{\alpha>0}\alpha$. Then Λ is a dominant integral form on $\mathfrak{b}$. Moreover Λ satisfies that $(\Lambda+\rho, \alpha_0)=0$. Let π_Λ be the irreducible representation of K with the highest weight Λ. Since

$$\begin{aligned}\Lambda+\rho_k&=\lambda-(\rho-\rho_k)\\&=m_2e_2+m_3e_3+\cdots+m_le_l-\frac{1}{2}(l+1)e_l\\&=\mu\end{aligned}$$

we have $[\pi_\Lambda|K:\sigma_{\lambda\_}]=1$ (see Corollary to Lemma 16). Therefore $U(\lambda, \pi_\Lambda)$ $(=U(\lambda, \pi_{\mu-\rho_k}))$ is a unitary representation which is equivalent to the representation U_Λ constructed in [13].

(3) Let $G=Sp(l, 1)$. Then $\mathfrak{a}_I=\{H\in\mathfrak{b}|e_1(H)+e_2(H)=0\}$,

$$P_k=\{e_i\pm e_j\ (2\leqq i<j\leqq l),\ 2e_i\ (1\leqq i\leqq l)\}$$

and

$$P_-=\{e_i\pm e_j\ (3\leqq i<j\leqq l),\ e_1-e_2,\ 2e_i\ (3\leqq i\leqq l)\}.$$

Let $\mu=\sum_{i=1}^{l}\mu_i e_i$ be an integral form on $\mathfrak{b}$ satisfying $\mu_2>\mu_3>\cdots>\mu_l>0$. Then all of μ_i's are integers. We put $2\rho_k=\sum_{\alpha>0}\alpha$. Then we have the following lemma.

LEMMA 17. *Let* Z *be the set of all integers and* $\pi_{\mu-\rho_k}$ *be the irreducible representation of* K *with the highest weight* $\mu-\rho_k$. *Then we have*

$$\begin{aligned}&\Delta_-(\exp H)\ \mathrm{Trace}\ \pi_{\mu-\rho_k}(\exp H)\\&=\sum_{\substack{q_2,q_3,\ldots,q_{l-1}\\p_2,p_3,\ldots,p_{l-1},p_l}}{}' \ \mathrm{sign}\ (p_2-\mu_4)(p_3-\mu_5)\cdots(p_{l-2}-\mu_l)[(\lambda_1\lambda_2^{-1})^{p_l}-(\lambda_1\lambda_2^{-1})^{-p_l}]\end{aligned}$$

$$\times\begin{vmatrix} \lambda_3^{q_2}-\lambda_3^{-q_2} & \lambda_4^{q_2}-\lambda_4^{-q_2} & \cdots & \lambda_l^{q_2}-\lambda_l^{-q_2} \\ \lambda_3^{q_3}-\lambda_3^{-q_3} & & & \\ \vdots & & & \\ \lambda_3^{q_{l-1}}-\lambda_3^{-q_{l-1}} & \lambda_4^{q_{l-1}}-\lambda_4^{-q_{l-1}} & \cdots & \lambda_l^{q_{l-1}}-\lambda_l^{-q_{l-1}} \end{vmatrix}$$

($\lambda_i = \exp e_i(H), i=1,2,\dots,l$) *for all* H *in* $\mathfrak{a}_I$ *where* Σ' *is the summation over the set of all elements* $(p_2, p_3, \dots, p_l, q_2, q_3, \dots, q_{l-1}) \in \boldsymbol{Z}^{2l-1}$ *satisfying*

$$\begin{aligned} &|p_2-q_2| \leqq \mu_2-\mu_3-1 < p_2+q_2 \leqq \mu_2+\mu_3-1, \\ &|p_3-q_3| \leqq |p_2-\mu_4|-1 < p_3+q_3 \leqq p_2+\mu_4-1, \dots, \\ &|p_{l-1}-q_{l-1}| \leqq |p_{l-2}-\mu_l|-1 < p_{l-1}+q_{l-1} \leqq p_{l-2}+\mu_l-1, \\ &|p_{l-1}-\mu_1| < p_l \leqq p_{l-1}+\mu_1-1. \end{aligned}$$

PROOF. Let η be an integral form on $\mathfrak{b}$. We define the function on A' (which is corresponding to $\mathfrak{b}$) by $\xi_\eta(\exp H) = \exp \eta(H)$ for all H in $\mathfrak{b}$. Put Δ_k on A' by

$$\Delta_k(b) = \xi_{\rho_k}(b) \prod_{\alpha \in P_k} (1-\xi_\alpha(b^{-1})), \qquad b \in A'.$$

Then by Weyl's character formula, we have $\Delta_k(b)$ Trace $\pi_{\mu-\rho_k}(b) = \sum_{s \in W_K} \varepsilon(s)\xi_{s\mu}(b)$ where W_K is the Weyl group of the pair $(\mathfrak{k}_c, \mathfrak{b}_c)$ and $\varepsilon(s) = \text{sign}(s)$.

Since

$$\mathfrak{a}_I = \{H = \sqrt{-1}\theta_2(e_1-e_2) + \sqrt{-1}\theta_3 e_3 + \cdots + \sqrt{-1}\theta_l e_l \,|\, \theta_i \in \boldsymbol{R},\ 2 \leqq i \leqq l\},$$

$$\Delta_k(\exp H) = (\lambda_2^{-1}-\lambda_2) \prod_{i=3}^{l} (\lambda_2+\lambda_2^{-1}-\lambda_i-\lambda_i^{-1}) \Delta_-(\exp H)$$

for all H in $\mathfrak{a}_I$ where $\lambda_i = \exp e_i(H)$ $(1 \leqq i \leqq l)$.

Therefore if we prove the following lemma, we have our conclusion.

LEMMA 17'. *Let* $n_2, n_3, \dots, n_l$ *be* $l-1$ *positive integers. Then we have*

$$D' = \begin{vmatrix} \lambda_2^{n_2}-\lambda_2^{-n_2} & \lambda_3^{n_2}-\lambda_3^{-n_2} & \cdots & \lambda_l^{n_2}-\lambda_l^{-n_2} \\ \lambda_2^{n_3}-\lambda_2^{-n_3} & \lambda_3^{n_3}-\lambda_3^{-n_3} & & \\ \vdots & \vdots & & \\ \lambda_2^{n_l}-\lambda_2^{-n_l} & \lambda_3^{n_l}-\lambda_2^{-n_l} & \cdots & \lambda_l^{n_l}-\lambda_l^{-n_l} \end{vmatrix}$$

$$= \sum_{\substack{p_2, p_3, \dots, p_{l-1} \\ q_2, q_3, \dots, q_{l-1}}}' \text{sign}\,(n_2-n_3)(p_2-n_4) \cdots (p_{l-2}-n_l) \times$$

$$\times \prod_{j=3}^{l} (\lambda_2+\lambda_2^{-1}-\lambda_j-\lambda_j^{-1})(\lambda_2^{p_{l-1}}\lambda_2^{p_{l-1}}) \begin{vmatrix} \lambda_3^{q_2}-\lambda_3^{-q_2} & \cdots & \lambda_l^{q_2}-\lambda_l^{-q_2} \\ \lambda_3^{q_3}-\lambda_3^{-q_3} & & \\ \vdots & & \\ \lambda_3^{q_{l-1}}-\lambda_3^{-q_{l-1}} & \cdots & \lambda_l^{q_{l-1}}-\lambda_l^{q_{l-1}} \end{vmatrix}$$

where the summation Σ' runs over the set of all $(p_2, p_3, \dots, p_{l-1}, q_2, q_3, \dots, q_{l-1}) \in \boldsymbol{Z}^{2l-2}$ satisfying

$$|p_2-q_2| \leqq |n_2-n_3|-1 < p_2+q_2 \leqq n_2+n_3-1, \dots,$$
$$|p_{l-1}-q_{l-1}| \leqq |p_{l-2}-n_l|-1 < p_{l-1}+q_{l-1} \leqq p_{l-2}+n_l-1.$$

We shall prove Lemma 17′ by using an induction for l. If $l=3$, then our assertion is easily proved. Let us assume that Lemma 17′ is satisfied for $l-1$.

We define $D(j_2, j_3)$ and $D^*(j_2, j_3)$ $(2 \leqq j_2 < j_3 \leqq l)$ by

$$D(j_2, j_3) = \begin{vmatrix} \lambda_{j_2}^{n_2}-\lambda_{j_2}^{-n_2} & \lambda_{j_3}^{n_2}-\lambda_{j_3}^{-n_2} \\ \lambda_{j_2}^{n_3}-\lambda_{j_2}^{-n_3} & \lambda_{j_3}^{n_3}-\lambda_{j_3}^{n_3} \end{vmatrix}$$

$$D^*(j_2, j_3) = \begin{vmatrix} \lambda_{j_4}^{n_4}-\lambda_{j_4}^{-n_4} & \lambda_{j_5}^{n_4}-\lambda_{j_5}^{-n_4} & \cdots & \lambda_{j_l}^{n_4}-\lambda_{j_l}^{-n_4} \\ \lambda_{j_4}^{n_5}-\lambda_{j_4}^{-n_5} & \lambda_{j_5}^{n_5}-\lambda_{j_5}^{-n_5} & \cdots & \lambda_{j_l}^{n_5}-\lambda_{j_l}^{-n_5} \\ \vdots & & & \vdots \\ \lambda_{j_4}^{n_l}-\lambda_{j_4}^{-n_l} & & & \lambda_{j_l}^{n_l}-\lambda_{j_l}^{-n_l} \end{vmatrix}$$

where $2 \leqq j_4 < j_5 < \cdots < j_l \leqq l$ and any one of $j_4, j_5, \dots, j_l$ are different from j_2 and j_3.

Then we have

$$\begin{aligned}
D' &= \sum_{2 \leqq j_2 < j_3 \leqq l} (-1)^{1+2+j_2+j_3} D(j_2 j_3) D^*(j_2, j_3) \\
&= \sum_{2 \leqq j_2 < j_3 \leqq l} (-1)^{1+2+j_2+j_3} (\lambda_{j_2}^{p_2}-\lambda_{j_2}^{-p_2})(\lambda_{j_3}^{q}-\lambda_{j_3}^{-q_2})(\lambda_{j_2}+(\lambda_{j_2})^{-1}-\lambda_{j_3}-(\lambda_{j_3})^{-1}) \\
&\quad \times \operatorname{sign}(n_2-n_3) D^*(j_2, j_3) \\
&= \sum_{p_2, q_2}{}' \Big\{ \sum_{j=3}^{l} (\lambda_2+\lambda_2^{-1}-\lambda_j-\lambda_j^{-1}) \operatorname{sign}(n_2-n_3) \\
&\quad \times [\sum_{2 \leqq t < j} (\lambda_j^{p_2}-\lambda_j^{-p_2})(\lambda_t^{q_2}-\lambda_t^{-q_2}) D^*(j, t)(-1)^{1+2+j+t} \\
&\quad - \sum_{l \geqq t > j} (\lambda_j^{p_2}-\lambda_j^{-p_2})(\lambda_t^{q_2}-\lambda_t^{-q_2}) D^*(j, t)(-1)^{1+2+j+t}] \Big\} \\
&= \sum_{p_2, q_2}{}' \sum_{j=3}^{l} (\lambda_j^{q_2}-\lambda_j^{-q_2})(-1)^{j+1} \operatorname{sign}(n_2-n_3)(\lambda_2+(\lambda_2)^{-1}-\lambda_j-(\lambda_j)^{-1}) \\
&\quad \times \begin{vmatrix} \lambda_2^{p_2}-\lambda_2^{-p_2} & \cdots & \lambda_{j-1}^{p_2}-\lambda_{j-1}^{-p_2} & \lambda_{j+1}^{p_2}-\lambda_{j+1}^{-p_2} & \cdots & \lambda_l^{p_2}-\lambda_l^{-p_2} \\ \lambda_2^{n_4}-\lambda_2^{-n_4} & \cdots & \lambda_{j-1}^{n_4}-\lambda_{j-1}^{-n_4} & \lambda_{j+1}^{n_4}-\lambda_{j+1}^{-n_4} & \cdots & \lambda_l^{n_4}-\lambda_l^{-n_4} \\ \vdots & & & & & \\ \lambda_2^{n_l}-\lambda_2^{-n_l} & & & & & \lambda_l^{n_l}-\lambda_l^{-n_l} \end{vmatrix}
\end{aligned}$$

where Σ' is the summation over the set of all $(p_2, q_2) \in \boldsymbol{Z}^2$ satisfying

$$|p_2-q_2| \leqq |n_2-n_3|-1<p_2+q_2 \leqq n_2+n_3-1.$$

Hence by the assumption of the induction, we have

$$\begin{aligned}
D' = & \sum_{p_2,q_2}' \sum_{j=3}^{l} (\lambda_j^{q_2}-\lambda_j^{-q_2})(-1)^{j+1} \operatorname{sign}(n_2-n_3)(\lambda_2+\lambda_2^{-1}-\lambda_j-\lambda_j^{-1}) \\
& \times \Bigg(\sum_{\substack{p_2,p_3,\dots,p_{l-1} \\ q_2,q_3,\dots,q_{l-1}}} \prod_{\substack{j \neq i \\ 3 \leqq i \leqq l}} (\lambda_2+\lambda_2^{-1}-\lambda_i-\lambda_i^{-1}) \operatorname{sign}(p_2-n_4)(p_3-n_5) \cdots (p_{l-2}-n_l) \\
& \times \begin{vmatrix} \lambda_3^{q_3}-\lambda_3^{-q_3} & \cdots & \lambda_{j-1}^{q_3}-\lambda_{j-1}^{-q_3} & \lambda_{j+1}^{q_3}-\lambda_{j+1}^{-q_3} & \cdots & \lambda_l^{q_3}-\lambda_l^{-q_3} \\ \vdots & & \vdots & \vdots & & \vdots \\ \lambda_3^{q_{l-1}}-\lambda_3^{-q_{l-1}} & & & & & \lambda_l^{q_{l-1}}-\lambda_l^{-q_{l-1}} \end{vmatrix} \Bigg) (\lambda_2^{p_{l-1}}-\lambda_2^{-p_{l-1}}) \\
= & \sum_{\substack{p_2,p_3,\dots,p_{l-1} \\ q_2,q_3,\dots,q_{l-1}}}' \prod_{j=3}^{l} (\lambda_2+\lambda_2^{-1}-\lambda_j-\lambda_j^{-1}) \operatorname{sign}(n_2-n_3)(p_2-n_4) \cdots (p_{l-2}-n_l) \\
& \times \begin{vmatrix} \lambda_3^{q_2}-\lambda_3^{-q_2} & \lambda_4^{q_2}-\lambda_4^{-q_2} & \cdots & \lambda_l^{q_2}-\lambda_l^{-q_2} \\ \lambda_3^{q_3}-\lambda_3^{-q_3} & & & \\ \vdots & & & \\ \lambda_3^{q_{l-1}}-\lambda_3^{-q_{l-1}} & & & \lambda_l^{q_{l-1}}-\lambda_l^{-q_{l-1}} \end{vmatrix} (\lambda_2^{p_{l-1}}-\lambda_2^{-p_{l-1}}).
\end{aligned}$$

Hence we have our assertion.

COROLLARY TO LEMMA 17. *Let λ be an element in $\mathfrak{l}_0$. Then λ has the form $\lambda = m(e_1-e_2)+m_3e_3+m_4e_4+\cdots+m_le_l$, $m>0$ and $m_3>m_4>\cdots>m_l>0$, $m \neq m_j$ $(3 \leqq j \leqq l)$, m and m_j $(3 \leqq j \leqq l)$ are integers. We put $m_2=+\infty$ and $m_{l+1}=0$. Then there exists a unique k $(l+1 \geqq k \geqq 3)$ such that $m_k<m+l-k+1<m_{k-1}$.*

Put $\mu_2=m_3+1$, $\mu_3=m_4+1, \dots, \mu_{k-2}=m_{k-1}+1$, $\mu_{k-1}=m+l-k+1$, $\mu_k=m_k$, $\mu_{k+1}=m_{k+1}, \dots, \mu_l=m_l$. Let $\mu=\sum_{i=2}^{l} \mu_i e_i+e_1$ and let $\tau_{\mu-\rho_k}$ be the irreducible representation of K with the highest weight $\mu-\rho_k$. Then we have $[\tau_{\mu-\rho_k}|M: \sigma_{\lambda\_}]=1$. By Lemma 17, we have our conclusion.

References

[1] Bargmann, V., Irreducible unitary representation of the Lorentz group, Ann. of Math., **48** (1947), 568-640.

[2] Boerner, H., Darstellungen von Gruppen, Springer-Verlag, Berlin, 1955.

[3] Bruhat, F., Sur les représentations induites des groupes de Lie, Bull. Soc. Math. France, **90** (1962), 97-205.

[4] Graev M. I. Gelfand, I. M., and N. Ya. Vilenkin, Generalized Functions V, Acad.emic Press, New York., 1966.
[5] Godement, R., Les fonctions de type positif et la théorie des groupes, Trans. Amer. Math. Soc., **63** (1948), 1-84.
[6] Godement, R., A theory of spherical functions I, Trans. Amer. Math. Soc., **73** (1952), 496-556.
[7] Harish-Chandra, The Plancherel formula for complex semisimple Lie groups, Trans. Amer. Math. Soc., **76** (1954), 485-524.
[8] Harish-Chandra, A formula for semisimple Lie groups, Amer. J. Math., **79** (1957), 773-760.
[9] Harish-Chandra, Spherical functions on semisimple Lie groups I, Amer. J. Math., **80** (1958), 241-310.
[10] Harish-Chandra, Discrete series for semisimple Lie group II, Acta Math., **116** (1966), 1-111.
[11] Helgason, S., Differential Geometry and Symmetric Space, Acad. Press, New York, 1962.
[12] Hirai, T., On irreducible representations of Lorentz group of n-th order, Proc. Japan Acad., **38** (1962), 258-262.
[13] Knapp, A. W. and K. Okamoto, Limits of holomorphic discrete series, J. Functional Anal., **9** (1972), 375-409.
[14] Knapp, A. W. and E. M. Stein, Intertwining operators for semi-simple groups, Ann. of Math., **93** (1971), 489-578.
[15] Mackey, G. W., On induced representations of groups, Amer. J. Math., **73** (1951), 576-592.
[16] Takahashi, R., Sur les fonctions sphérique et la fomule de Plancherel dans le groupe hyperbolique, Japan, J. Math., **31** (1961), 55-90.
[17] Takahashi, R., Sur les représentations unitaires des groupes de Lorentz généralisés, Bull. Soc. Math. France, **91** (1963), 289-433.
[18] Trombi, R. C. and V. S. Varadarajan, Asymptotic behaviour of eigenfunctions on a semi-simple Lie group: The discrete spectrum, to appear.
[19] Warner, G., Harmonic analysis on semi-simple Lie group II, Springer-Verlag, 1972.
[20] Lepowsky, J., Multiplicity formulas for certain semisimple Lie groups, Bull. Amer. Math. Soc., **77** (1971), 601-605.

(Received May 13, 1974)

Department of Mathematics
Tsuda College
Kodaira, Tokyo
187 Japan