A coincidence theorem for involutions on mod 2 homology spheres*

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(Communicated by A. Hattori)

1. Statement of results

The purpose of the present note is to give a simple proof of the following THEOREM 1. Let M be a closed topological manifold such that $H_*(M; \mathbb{Z}_2)$ is isomorphic to $H_*(S^n; \mathbb{Z}_2)$. If σ is a nontrivial involution on M and τ is a free involution on M, then there exists $x \in M$ such that $\sigma x = \tau x$.

In [2] A. Dress pointed out that the theorem can be proved using Milnor's method in [4] if the involution σ is free, and conjectured that the theorem might hold in its full generality. Indeed, investigating Milnor's method one can see that the crucial point lies in the following. Let M^* be the orbit space of M by the involution σ and $\pi: M \rightarrow M^*$ the canonical projection. Then the homomorphism $\pi_*: H_n(M; \mathbb{Z}_2) \rightarrow H_n(M^*; \mathbb{Z}_2)$ is known to be trivial if σ is a free involution while, surprisingly, it seems unknown to be true when σ is nontrivial but not necessarily free.

In this note we shall give a proof of the following

THEOREM 2. Let M be a connected, closed topological n-manifold. Suppose that the cyclic group Z_p of prime order p acts non-trivially on M. Then the induced homomorphism $\pi^* \colon \check{H}^n(M/Z_p; Z_p) \to \check{H}^n(M; Z_p)$ is trivial where \check{H}^* denotes the Čech cohomology.

2. Proof of theorems

PROOF OF THEOREM 2. Let F be the fixed point set. By Newman's theorem [5] F is nowhere dense, i.e. the interior of F is empty. Let U be a closed subset of M which is homeomorphic to n-disk D^n , then inddim $F \cap U \le n-1$ by Theorem IV.3 of [3]. By the Sum Theorem of [3] inddim $F \le n-1$. Thus by Theorem V.1 of [3] the covering dimension of F is not greater than n-1. Hence $\check{H}^n(F) = 0$.

^{*} The contents of the present note consist of a part of the author's master's thesis at the University of Tokyo.

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Since the covering dimension of M is equal to n, it follows that $\check{H}_{\tau}^{n+1}(M)=0$ (see, e.g. Bredon [1] Theorem III 7.9).

Consider the diagram

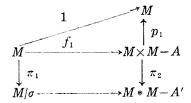
$$\xrightarrow{\tau^*} \check{H}^n(M) \xrightarrow{\tau^*} \check{H}^n(M) \xrightarrow{i^*} \check{H}^n(M) \oplus \check{H}^n(F) \xrightarrow{} \check{H}^{n+1}_{\varepsilon}(M)$$

$$\downarrow \phi^* \qquad \qquad \downarrow \phi^* \qquad \qquad \downarrow \phi^* \qquad \qquad \downarrow \phi^*$$

$$\check{H}^n(F) \longleftarrow \check{H}^n(M/Z_p) \longleftarrow \check{H}^n(M/Z_p, F) \longleftarrow$$

where the upper sequence is the Smith sequence and ϕ^* is the transfer and ϕ^* : $\check{H}^n_\sigma(M) \to H^n(M/\mathbb{Z}_p, F)$ is an isomorphism. It is clear that the diagram is commutative. Since $\check{H}^{n+1}_\tau(M) = 0$ and $\check{H}^n(F) = 0$, i^* and j^* are onto. Therefore $\phi^*: H^n(M) \to \check{H}^n(M/\mathbb{Z}_p)$ is onto. Moreover $\check{H}^n(M) \cong \mathbb{Z}_p$ and $\phi^*\pi^* = 0$ as is well known. Hence $\pi^* = 0$.

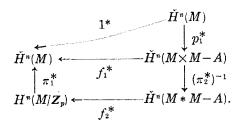
PROOF OF THEOREM 1. Let A be the set of all points $(x, \tau x)$ in $M \times M$, M * M be the symmetric product, and let A' be the set of all points $\{x, \tau x\}$ in M * M. Suppose $\sigma x \neq \tau x$ for all $x \in M$, then there is a commutative diagram



where π_1, π_2 are the canonical projections and where $f_1(x) = (x, \sigma x), f_2(\{x, \sigma x\}) = \{x, \sigma x\}, p_1(x, y) = x.$

Let the coefficient group of cohomology be \mathbb{Z}_2 . By Lemma 1 of [4] the homomorphism $\pi_2^*: \check{H}^n(M*M-A) \to \check{H}^n(M\times M-A)$ is an isomorphism, since τ is free.

Consider the commutative diagram



By Theorem 2 π_1^* is trivial. Therefore 1* is trivial. This is a contradiction. Thus there exists a point $x \in M$ such that $\sigma x = \tau x$. Q.E.D.

References

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(Received May 27, 1974)

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