*A coincidence theorem for involutions on mod 2 homology spheres**

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1. Statement of results

The purpose of the present note is to give a simple proof of the following

THEOREM 1. *Let M be a closed topological manifold such that $H_*(M; Z_2)$ is isomorphic to $H_*(S^n; Z_2)$. If σ is a nontrivial involution on M and τ is a free involution on M, then there exists $x \in M$ such that $\sigma x = \tau x$.*

In [2] A. Dress pointed out that the theorem can be proved using Milnor's method in [4] if the involution σ is free, and conjectured that the theorem might hold in its full generality. Indeed, investigating Milnor's method one can see that the crucial point lies in the following. Let M^* be the orbit space of M by the involution σ and $\pi: M \to M^*$ the canonical projection. Then the homomorphism $\pi_*: H_n(M; Z_2) \to H_n(M^*; Z_2)$ is known to be trivial if σ is a free involution while, surprisingly, it seems unknown to be true when σ is nontrivial but not necessarily free.

In this note we shall give a proof of the following

THEOREM 2. *Let M be a connected, closed topological n-manifold. Suppose that the cyclic group Z_p of prime order p acts non-trivially on M. Then the induced homomorphism $\pi^*: \check{H}^n(M/Z_p; Z_p) \to \check{H}^n(M; Z_p)$ is trivial where $\check{H}^*$ denotes the Čech cohomology.*

2. Proof of theorems

PROOF OF THEOREM 2. Let F be the fixed point set. By Newman's theorem [5] F is nowhere dense, i.e. the interior of F is empty. Let U be a closed subset of M which is homeomorphic to n-disk D^n, then inddim $F \cap U \leq n-1$ by Theorem IV. 3 of [3]. By the Sum Theorem of [3] inddim $F \leq n-1$. Thus by Theorem V. 1 of [3] the covering dimension of F is not greater than $n-1$. Hence $\check{H}^n(F)=0$.

* The contents of the present note consist of a part of the author's master's thesis at the University of Tokyo.

Since the covering dimension of M is equal to n, it follows that $\check{H}^{n+1}_\tau(M)=0$ (see, e.g. Bredon [1] Theorem III 7.9).

Consider the diagram

$$\begin{array}{ccccccc}
\longrightarrow \check{H}^n_\tau(M) & \xrightarrow{\tau^*} & \check{H}^n(M) & \xrightarrow{i^*} & \check{H}^n_\sigma(M)\oplus\check{H}^n(F) & \longrightarrow & \check{H}^{n+1}_\tau(M) \\
 & & \downarrow \phi^* & & \downarrow \phi^* & & \\
\check{H}^n(F) & \longleftarrow & \check{H}^n(M/Z_p) & \xleftarrow{j^*} & \check{H}^n(M/Z_p, F) & \longleftarrow &
\end{array}$$

where the upper sequence is the Smith sequence and ϕ^* is the transfer and ϕ^*: $\check{H}^n_\sigma(M)\to H^n(M/Z_p, F)$ is an isomorphism. It is clear that the diagram is commutative. Since $\check{H}^{n+1}_\tau(M)=0$ and $\check{H}^n(F)=0$, i^* and j^* are onto. Therefore $\phi^*: H^n(M)\to \check{H}^n(M/Z_p)$ is onto. Moreover $\check{H}^n(M)\cong Z_p$ and $\phi^*\pi^*=0$ as is well known. Hence $\pi^*=0$. Q.E.D.

PROOF OF THEOREM 1. Let A be the set of all points $(x, \tau x)$ in $M\times M$, $M*M$ be the symmetric product, and let A' be the set of all points $\{x, \tau x\}$ in $M*M$. Suppose $\sigma x\neq\tau x$ for all $x\in M$, then there is a commutative diagram

$$\begin{array}{ccc}
 & \overset{1}{\nearrow} & M \\
 & & \uparrow p_1 \\
M & \xrightarrow{f_1} & M\times M-A \\
\downarrow \pi_1 & & \downarrow \pi_2 \\
M/\sigma & \longrightarrow & M*M-A'
\end{array}$$

where π_1, π_2 are the canonical projections and where $f_1(x)=(x, \sigma x)$, $f_2(\{x, \sigma x\})=\{x, \sigma x\}$, $p_1(x, y)=x$.

Let the coefficient group of cohomology be Z_2. By Lemma 1 of [4] the homomorphism $\pi_2^*: \check{H}^n(M*M-A)\to\check{H}^n(M\times M-A)$ is an isomorphism, since τ is free.

Consider the commutative diagram

$$\begin{array}{ccc}
 & \overset{1^*}{\swarrow} & \check{H}^n(M) \\
 & & \downarrow p_1^* \\
\check{H}^n(M) & \xleftarrow{f_1^*} & \check{H}^n(M\times M-A) \\
\uparrow \pi_1^* & & \downarrow (\pi_2^*)^{-1} \\
H^n(M/Z_p) & \xleftarrow[f_2^*]{} & \check{H}^n(M*M-A).
\end{array}$$

By Theorem 2 π_1^* is trivial. Therefore 1^* is trivial. This is a contradiction. Thus there exists a point $x\in M$ such that $\sigma x=\tau x$. Q.E.D.

References

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