The conjugacy classes of Chevalley groups of type (F_4) *over finite fields of characteristic 2*

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In this paper, we will determine the conjugacy classes of Chevalley groups of type (F_4) over finite fields of characteristic 2. After some preliminaries in § 1, we shall deal with the unipotent classes in § 2. The main tools used here are Bruhat decomposition and elementary properties of the Weyl group of type (F_4). As a result, we get the following: there are 35 unipotent classes including the class of the identity element (Theorem 2.1), especially 4 classes of involutions in F_4 and 2 classes of involutions in ${}^2F_4$. Finally in §3, we shall determine the semisimple classes and the general classes and we get

THEOREM 3.3. *The number of conjugacy classes in* $F_4(q)$ *is given by* $q^4+2q^3+6q^2+10q+19$.

Since T. Shoji has determined the conjugacy classes of Chevalley groups of type (F_4) over finite fields of odd characteristic, the determination of conjugacy classes of Chevalley groups of type (F_4) is thus complete.

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§ 1. Preliminaries

(1.0) *Notations*

We shall use the contents and notations in Lecture Notes [2], in particular part A by A. Borel, part E by T. A. Springer and R. Steinberg, part F by N. Iwahori and part G by R. Carter.

$k=\boldsymbol{F}_q$ is a finite field with $q=p^n$ elements, p its characteristic, and $\bar{k}$ its algebraic closure.

Let $\bar{G}=G(F_4)_{\bar{k}}$ be a Chevalley group associated with the pair consisting of the complex Lie algebra of type (F_4) and $\bar{k}$. Then $\bar{G}$ is a simply connected semisimple algebraic group defined over the prime field $\boldsymbol{F}_p$. Let $\bar{H}$ denote a maximal torus of $\bar{G}$ splitting over $\boldsymbol{F}_p, \Phi=\Phi(\bar{H},\bar{G})$ the set of roots relative to $\bar{H}, \bar{B}$ a Borel subgroup containing $\bar{H}, \bar{U}$ the unipotent radical of $\bar{B}$ and $\bar{N}$ the normalizer of $\bar{H}$ in

$\bar{G}$. Since $\bar{G}, \bar{H}, \bar{B}, \bar{U}, \bar{N}$ are all defined over $\boldsymbol{F}_p$, we write their k-rational points as G, H, B, U, N respectively. If we put $P_r=\sum_{\alpha\in\Phi}\boldsymbol{Z}\alpha$, $\bar{H}$ and H may be identified with $\mathrm{Hom}(P_r, \bar{k}^*)$ and $\mathrm{Hom}(P_r, k^*)$ respectively under the usual correspondence by $\chi\mapsto h(\chi)$, where $\bar{k}^*$ (resp. k^*) denotes the multiplicative group of $\bar{k}$ (resp. k). $N/H=W$ is the Weyl group of type (F_4). We choose a representative system $\{n_w\}_{w\in W}$ of W in N and fix them throughout this paper. $\bar{B}$ determines an ordering of Φ. Φ^+ denotes the set of positive roots with respect to this ordering and $\Phi_w^-=\{\alpha\in\Phi^+ \mid w(\alpha)<0\}$ where $w\in W$. $\bar{U}_\alpha$ is the one parameter subgroup corresponding to the root α and x_α is an isomorphism from $\bar{k}$ to $\bar{U}_\alpha$ such that

$$hx_\alpha(t)h^{-1}=x_\alpha(\alpha(h)t)=x_\alpha(\chi(\alpha)t) \tag{1.0.1}$$

for every $h=h(\chi)\in\bar{H}$. Further we put $U_\alpha=\{x_\alpha(t)\mid t\in k\}$, $\bar{U}_w=\langle\bar{U}_\alpha\mid\alpha\in\Phi_w^-\rangle$ and $U_w=\langle U_\alpha\mid\alpha\in\Phi_w^-\rangle$, where $L=\langle X\rangle$ means that L is the abstract group generated by X.

Finally we list the notations we will often use. $Z_X(Y)$ means the centralizer of Y in X, $N_X(Y)$ the normalizer of Y in X and $|X|$ the cardinality of X. For $x, y\in G$ and $X\subset G$, $x\sim y$ (resp. $x\underset{X}{\sim}y$) means that x is conjugate to y (resp. conjugate to y by an element of X) and $x\not\sim y$ means that x is not conjugate to y.

The following facts are well known.

(1.1) THEOREM (Lang-Steinberg, [6], 10.1). *Let A be a connected linear algebraic group and σ its endomorphism of A onto A such that $A_\sigma=\{a\in A\mid\sigma(a)=a\}$ is finite. Then the map f: $a\mapsto a\sigma(a)^{-1}$ of A into A is surjective.*

(1.2) PROPOSITION ([2], E. I, 3.4). *Let A, σ be as in (1.1) and C a conjugate class of A fixed by σ. Then*

(a) *C contains an element x fixed by σ.*

(b) *Let x be such an element, then conjugacy classes of A_σ in $A_\sigma\cap C$ are in one to one correspondence with the elements of $H^1(\sigma, Z_A(x)/Z_A(x)_0)$, where $Z_A(x)_0$ means the connected component of the identity element in $Z_A(x)$. Hence if $Z_A(x)$ is connected, then for $x, y\in A_\sigma$, $x\underset{A}{\sim}y$ if and only if $x\underset{A_\sigma}{\sim}y$.*

(1.3) (*Bruhat decomposition*)

$$\bar{G}=\bigcup_{w\in W}\bar{U}\bar{H}n_w\bar{U}_w$$

$$G=\bigcup_{w\in W}UHn_wU_w.$$

Every element g in $\bar{G}$ (resp. G) can be written uniquely in the form $g=uhn_wv$, where $u\in\bar{U}$ (resp. U), $h\in\bar{H}$ (resp. H) and $v\in\bar{U}_w$ (resp. U_w).

(1.4) Every element u of $\bar{U}$ (resp. U) can be written uniquely in the form

$$u=\prod_{\alpha\in\Phi^+} x_\alpha(t_\alpha), \qquad t_\alpha\in\bar{k} \text{ (resp. } k)$$

where the product is taken in any fixed order.

(1.5) The element of Φ^+ are as follows:

$$\varepsilon_i\pm\varepsilon_j\ (1\leqq i<j\leqq 4), \qquad \varepsilon_i\ (i=1,2,3,4), \quad \frac{1}{2}(\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4).$$

$\Phi=\Phi^+\cup(-\Phi^+)$. The set of simple roots Δ consists of

$$\varepsilon_2-\varepsilon_3,\ \varepsilon_3-\varepsilon_4,\ \varepsilon_4 \text{ and } \frac{1}{2}(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4).$$

Φ contains the following root systems denoted by $\Phi(B_3)$ and $\Phi(B_4)$ henceforth:

$$\Phi(B_3)=\{\pm(\varepsilon_i\pm\varepsilon_j),\ (2\leqq i<j\leqq 4),\ \pm\varepsilon_i(i=2,3,4)\},$$
$$\Phi(B_4)=\{\pm(\varepsilon_i\pm\varepsilon_j),\ (1\leqq i<j\leqq 4),\ \pm\varepsilon_i(i=1,2,3,4)\}.$$

(1.6.1) $$n_w\bar{U}_\alpha n_w^{-1}=\bar{U}_{w(\alpha)} \text{ and } n_w U_\alpha n_w^{-1}=U_{w(\alpha)}.$$

(1.6.2) $$n_w h(\chi) n_w^{-1}=h({}^w\chi), \text{ where } {}^w\chi=\chi\cdot w^{-1}.$$

(1.7) $$|G|=q^{24}(q^2-1)(q^6-1)(q^8-1)(q^{12}-1),$$
$$|U|=q^{24},\quad |H|=(q-1)^4 \text{ and } |W|=1{,}152.$$

The number of unipotent elements of G is q^{48} (see Steinberg [6], 15.3).

(1.8) In the case where there is no afraid of confusion, we shall write simply, $i-j, i$ or $1-2-3-4$, etc. to denote the roots $\varepsilon_i-\varepsilon_j, \varepsilon_i$ or $(1/2)(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4)$ respectively. Similarly we apply this expression of the root α, to r_α and $x_\alpha(t)$; for example, r_{i-j} and $x_i(t)$ stands for $r_{\varepsilon_i-\varepsilon_j}$ and $x_{\varepsilon_i}(t)$ respectively, where r_α denote the reflection in W with respect to the root α.

§ 2. Unipotent classes

Henceforth we assume that the characteristic of k is 2.

(2.1) The following commutator relations hold, where $[x,y]=xyx^{-1}y^{-1}$ and 1 denotes the identity element of G.

$$[x_i(t), x_j(u)]=1,$$
$$[x_i(t), x_{-i+j}(u)]=x_j(tu)x_{i+j}(t^2u),$$
$$[x_{i+j}(t), x_{-j+k}(u)]=x_{i+k}(tu),$$

$$[x_i(t), x_{-i+j+k+1}(u)]=x_{i+j+k+1}(tu),$$
$$[x_{i+j}(t), x_{-i-j+k+1}(u)]=x_{i+j+k+1}(tu)x_{k+1}(tu^2),$$
$$[x_{i+j+k+1}(t), x_{i-j-k-1}(u)]=x_i(tu),$$
$$[x_{i+j+k+1}(t), x_{i+j-k-1}(u)]=1,$$

where all the other commutators of the form $[x_\alpha(t), x_\beta(u)]$ $(\alpha\neq-\beta)$ are 1.

(2.2) Let $k[X]$ be a polynomial ring over k in the indeterminate X and f be an additive homomorphism of k into k defined by $f(t)=t^2+t$. Then $\mathrm{Ker} f=\{0,1\}$, so we can choose $\eta\in k-f(k)$. Clearly $X^2+X+\eta$ is an irreducible polynomial in $k[X]$. Moreover the mapping f': $t\to t^3+t$ of k into k is not bijective, hence we can choose $\zeta\in k$ such that $X^3+X+\zeta$ is irreducible in $k[X]$. We fix such η and ζ throughout this paper. Then we have the following lemma.

LEMMA 2.1 ([4], Lemma 2.2). *$X^2+\zeta X+\zeta^2+1$ is a reducible polynomial.*

Next we prove a lemma used for the calculation of the unipotent classes.

LEMMA 2.2. *Let K be a commutative field, G be a Chevalley group of any type over K, B, H, U be the usual subgroups of G* (see §1), *and Φ be the set of roots, Δ the set of simple roots with respect to the ordering defined by B. For a subset π of Δ, we put*

$$\Psi=\Phi\cap(\sum_{\alpha\in\pi} Z\alpha), \qquad \Psi^+=\Phi^+\cap\Psi$$

and

$$U_0=\{\prod_{\alpha\in\Phi^+-\Psi^+} x_\alpha(t_\alpha)\,|\,t_\alpha\in K\}.$$

Let $P(\pi)$ be a parabolic subgroup corresponding to π (i.e. $P(\pi)=\bigcup_{w\in W(\pi)} Bn_wU_w$, where $W(\pi)=\langle r_\alpha|\alpha\in\pi\rangle$), then U_0 is a subgroup of G and $P(\pi)$ is contained in $N_G(U_0)$.

PROOF. Put $\Gamma=\Phi^+-\Psi^+$, then Γ is closed, i.e. if $\alpha,\beta\in\Gamma$ and $\alpha+\beta\in\Phi$, then $\alpha+\beta\in\Gamma$. The following commutator relation holds in general:

$$(2.2.1)\qquad [x_\alpha(t), x_\beta(s)]=\prod_{i,j\in Z^+} x_{i\alpha+j\beta}(c_{i,j,\alpha,\beta}t^is^j),$$

where

$$c_{i,j,\alpha,\beta}\in K.$$

So we can see easily that U_0 is a subgroup.

Next let g be an element in $P(\pi)$, then we can write $g=uhn_wv$ where $w\in W(\pi)$. $N_G(U_0)$ contains u, v and h, because of (2.2.1) and (1.0.1); if w is an element of $W(\pi)$, then w induces a permutation on Γ and hence from (1.6.1) $N_G(U_0)$ also contains n_w. Therefore $N_G(U_0)$ contains g.

(2.3) We define the subgroups $G(B_3)$ and $G(B_4)$ of G as follows:

$$G(B_3)=\langle H', U_\alpha \mid \alpha\in\Phi(B_3)\rangle,$$

where

$$H'=\{h(\chi)\in H \mid \chi(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4)=1\}$$

and

$$G(B_4)=\langle H, U_\alpha \mid \alpha\in\Phi(B_4)\rangle.$$

Using Lemma 2.2, first we can calculate the unipotent classes of $G(B_3)$, then $G(B_4)$, and we get the following Proposition 2.1 and Proposition 2.2; we omit the proof.

PROPOSITION 2.1. *$G(B_3)$ has 12 unipotent classes including the identity element. Their representatives and the orders of their centralizers are as follows:*

$z_0=1$	$q^9(q^2-1)(q^4-1)(q^6-1)$
$z_1=x_2(1)$	$q^9(q^2-1)(q^4-1)$
$z_2=x_{2+3}(1)$	$q^9(q^2-1)^2$
$z_3=x_2(1)x_{2+3}(1)$	$q^9(q^2-1)$
$z_4=x_2(1)x_{3+4}(1)$	$q^7(q^2-1)$
$z_5=x_{2-4}(1)x_3(1)x_{3+4}(1)$	$2q^6(q-1)$
$z_6=x_{2-4}(1)x_3(1)x_{3+4}(1)x_{2+4}(\eta)$	$2q^6(q+1)$
$z_7=x_4(1)x_{2-4}(1)$	$2q^5(q^2-1)$
$z_8=x_4(1)x_{2-4}(1)x_{2+4}(\eta)$	$2q^5(q^2-1)$
$z_9=x_4(1)x_{2-4}(1)x_{3+4}(1)$	q^5
$z_{10}=x_{2-3}(1)x_{3-4}(1)x_4(1)$	$2q^3$
$z_{11}=x_{2-3}(1)x_{3-4}(1)x_4(1)x_{3+4}(\eta)$	$2q^3$

PROPOSITION 2.2. *In $G(B_4)$ there are 25 unipotent classes including a class of the identity element. Their representatives and the orders of their centralizers are as follows:*

$y_0=1$	$q^{16}(q^2-1)(q^4-1)(q^6-1)(q^8-1)$
$y_1=x_1(1)$	$q^{16}(q^2-1)(q^4-1)(q^6-1)$
$y_2=x_{1+2}(1)$	$q^{16}(q^2-1)^2(q^4-1)$
$y_3=x_1(1)x_{1+2}(1)$	$q^{16}(q^2-1)(q^4-1)$
$y_4=x_2(1)x_{1-2}(1)$	$2q^{10}(q^2-1)(q^4-1)$
$y_5=x_2(1)x_{1-2}(1)x_{1+2}(\eta)$	$2q^{10}(q^2-1)(q^4-1)$
$y_6=x_{2+3}(1)x_1(1)$	$q^{14}(q^2-1)^2$
$y_7=x_{2+3}(1)x_{1+4}(1)$	$q^{14}(q^2-1)(q^4-1)$
$y_8=x_{2+3}(1)x_1(1)x_{1+4}(1)$	$q^{14}(q^2-1)$
$y_9=x_2(1)x_{2+3}(1)x_{1-3}(1)$	$2q^{11}(q-1)(q^2-1)$

$y_{10}=x_2(1)x_{2+3}(1)x_{1-3}(1)x_{1+3}(\eta)$	$2q^{11}(q+1)(q^2-1)$
$y_{11}=x_2(1)x_{2+3}(1)x_{1-2}(1)$	$q^{10}(q^2-1)$
$y_{12}=x_2(1)x_{3+4}(1)x_{1-4}(1)$	$q^{10}(q^2-1)$
$y_{13}=x_2(1)x_{3+4}(1)x_{1-2}(1)$	$2q^{10}(q^2-1)$
$y_{14}=x_2(1)x_{3+4}(1)x_{1-2}(1)x_{1+2}(\eta)$	$2q^{10}(q^2-1)$
$y_{15}=x_4(1)x_{2-4}(1)x_{3+4}(1)x_1(1)$	q^{10}
$y_{16}=x_{2-4}(1)x_{3+4}(1)x_{1-3}(1)$	$q^8(q^2-1)$
$y_{17}=x_{2-3}(1)x_{3-4}(1)x_4(1)$	$2q^6(q^2-1)$
$y_{18}=x_{2-3}(1)x_{3-4}(1)x_4(1)x_{3+4}(\eta)$	$2q^6(q^2-1)$
$y_{19}=x_{2-4}(1)x_3(1)x_{3+4}(1)x_{1-3}(1)$	$2q^8$
$y_{20}=x_{2-4}(1)x_3(1)x_{3+4}(1)x_{2+4}(\eta)x_{1-3}(1)$	$2q^8$
$y_{21}=x_4(1)x_{2-4}(1)x_{3+4}(1)x_{1-2}(1)$	$2q^6$
$y_{22}=x_4(1)x_{2-4}(1)x_{3+4}(1)x_{2+4}(\eta)x_{1-2}(1)$	$2q^6$
$y_{23}=x_{1-2}(1)x_{2-3}(1)x_{3-4}(1)x_4(1)$	$2q^4$
$y_{24}=x_{1-2}(1)x_{2-3}(1)x_{3-4}(1)x_4(1)x_{3+4}(\eta)$	$2q^4$

(2.4) *Unipotent classes of G*

Now, using the results we have obtained so far, we will show how to settle the case of the unipotent classes of G.

Every unipotent element in G is conjugate to some element of U and an element u of U can be written uniquely in the form $u=\prod_{\alpha\in\Phi(B_3)^+} x_\alpha(t_\alpha)\prod_{\beta\in\Gamma} x_\beta(t_\beta)$ where $\Gamma=\Phi^+-\Phi(B_3)^+$ (see (1.4)). Set $U_0=\{\prod_{\beta\in\Gamma} x_\beta(t_\beta)\mid t_\beta\in k\}$, then by Lemma 2.2 we have $G(B_3)\subset P(\pi)\subset N_G(U_0)$, where $\pi=\{\varepsilon_2-\varepsilon_3, \varepsilon_3-\varepsilon_4, \varepsilon_4\}$. On the other hand $\prod_{\alpha\in\Phi(B_3)^+} x_\alpha(t_\alpha)$ is contained in $G(B_3)$, so u is conjugate by an element of $G(B_3)$ to some element of $\bigcup_{i=0}^{11} z_iU_0$, where z_i is the element defined in Proposition 2.1. That is to say, we can choose the representative elements of unipotent classes from the elements in $\bigcup_{i=0}^{11} z_iU_0$. Furthermore if $z_iu_0(u_0\in U_0)$ is contained in $G(B_4)$, then z_iu_0 is conjugate to some y_j defined in Proposition 2.2. Therefore, in the first place we find a representative elements $\{x'_j\}_{j\in J}$ among the elements in $\bigcup_{i=0}^{11} z_iU_0-G(B_4)$, classifing those elements by conjugation, and finally we can choose a representative system $\{x_i\}_{i\in I}$ from $\{x'_j\}_{j\in J}\cup\{y_i\}$. To prove they are really a representative system, it is sufficient to show $\sum_{i\in I}|G|/|Z_G(x_i)|=q^{48}$. Thus we can obtain the following theorem.

THEOREM 2.1. *G has 35 conjugacy classes of unipotent elements containing the class of the identity element. Their representatives and the orders of their centralizers are as follows:*

$x_0=1$	$q^{24}(q^2-1)(q^6-1)(q^8-1)(q^{12}-1)$
$x_1=x_1(1)$	$q^{24}(q^2-1)(q^4-1)(q^6-1)$
$x_2=x_{1+2}(1)$	$q^{24}(q^2-1)(q^4-1)(q^6-1)$
$x_3=x_1(1)x_{1+2}(1)$	$q^{24}(q^2-1)(q^4-1)$
$x_4=x_{2+3}(1)x_1(1)$	$q^{20}(q^2-1)^2$
$x_5=x_2(1)x_{2+3}(1)x_{1-3}(1)$	$2q^{17}(q^2-1)(q^3-1)$
$x_6=x_2(1)x_{2+3}(1)x_{1-3}(1)x_{1+3}(\eta)$	$2q^{17}(q^2-1)(q^3+1)$
$x_7=x_2(1)x_{2+3}(1)x_{1-2+3+4}(1)$	$2q^{17}(q^2-1)(q^3-1)$
$x_8=x_2(1)x_{2+3}(1)x_{1-2+3+4}(1)x_{1+4}(\eta)$	$2q^{17}(q^2-1)(q^3+1)$
$x_9=x_2(1)x_{1-2}(1)$	$2q^{14}(q^2-1)(q^4-1)$
$x_{10}=x_2(1)x_{1-2}(1)x_{1+2}(\eta)$	$2q^{14}(q^2-1)(q^4-1)$
$x_{11}=x_2(1)x_{3+4}(1)x_{1-4}(1)$	$q^{16}(q^2-1)$
$x_{12}=x_2(1)x_{1-2+3+4}(1)x_{1-4}(1)$	$q^{16}(q^2-1)$
$x_{13}=x_2(1)x_{2+3}(1)x_{1-2}(1)$	$q^{14}(q^2-1)$
$x_{14}=x_2(1)x_{3+4}(1)x_{1-2}(1)$	$q^{14}(q^2-1)$
$x_{15}=x_2(1)x_{2+3}(1)x_{1-2+3+4}(1)x_{1-3}(1)$	q^{16}
$x_{16}=x_2(1)x_{2+3}(1)x_{1-2+3+4}(1)x_{1-2}(1)$	q^{14}
$x_{17}=x_2(1)x_{2+3}(1)x_{1-2-3+4}(1)x_{1-2}(1)$	$6q^{12}$
$x_{18}=x_2(1)x_{2+3}(1)x_{1-2-3+4}(1)x_{1-2}(1)x_{1-4}(\eta)$	$2q^{12}$
$x_{19}=x_2(1)x_{3+4}(1)x_{1-2+3-4}(1)x_{1-2}(1)x_{1-3}(\zeta)$	$3q^{12}$
$x_{20}=x_{1-2}(1)x_{2-3}(1)x_3(1)$	$2q^8(q^2-1)$
$x_{21}=x_{1-2}(1)x_{2-3}(1)x_3(1)x_{2+3}(\eta)$	$2q^8(q^2-1)$
$x_{22}=x_4(1)x_{2-4}(1)x_{1-2+3-4}(1)$	$2q^8(q^2-1)$
$x_{23}=x_4(1)x_{2-4}(1)x_{2+4}(\eta)x_{1-2+3-4}(1)$	$2q^8(q^2-1)$
$x_{24}=x_{2-4}(1)x_{3+4}(1)x_{1-2-3-4}(1)x_{1-2-3+4}(1)$	$8q^8$
$x_{25}=x_{2-4}(1)x_{3+4}(1)x_{1-2-3-4}(1)x_{1-2-3+4}(1)x_{1-2}(\eta)$	$4q^8$
$x_{26}=x_{2-4}(1)x_{3+4}(1)x_{1-2-3-4}(1)x_{1-2-3+4}(1)x_{1-2}(\eta)x_{1-3}(\eta)$	$8q^8$
$x_{27}=x_{2-4}(1)x_3(1)x_{3+4}(1)x_{2+4}(\eta)x_{1-2-3+4}(1)$	$4q^8$
$x_{28}=x_{2-4}(1)x_3(1)x_{3+4}(1)x_{2+4}(\eta)x_{1-2-3+4}(1)x_{1-2}(\eta)$	$4q^8$
$x_{29}=x_{1-2}(1)x_{2-3}(1)x_{3-4}(1)x_4(1)$	$2q^6$
$x_{30}=x_{1-2}(1)x_{2-3}(1)x_{3-4}(1)x_4(1)x_{3+4}(\eta)$	$2q^6$
$x_{31}=x_{2-3}(1)x_{3-4}(1)x_4(1)x_{1-2-3-4}(1)$	$4q^4$
$x_{32}=x_{2-3}(1)x_{3-4}(1)x_4(1)x_{1-2-3-4}(1)x_{3+4}(\eta)$	$4q^4$
$x_{33}=x_{2-3}(1)x_{3-4}(1)x_4(1)x_{1-2-3-4}(1)x_{1-2}(\eta)$	$4q^4$
$x_{34}=x_{2-3}(1)x_{3-4}(1)x_4(1)x_{1-2-3-4}(1)x_{3+4}(\eta)x_{1-2}(\eta)$	$4q^4$

The elements $x_i, x_{i+1}, \ldots$, which are put together by the symbol {, *are conjugate in* $\bar{G}$. *Hence* $\bar{G}$ *has 20 unipotent classes.*

In order to prove Theorem 2.1 directly, it is sufficient to calculate and to show the following three facts:

(i) the order of $Z_G(x_i)$, for $i=0,1,2,\ldots,34$,

(ii) if $|Z_G(x_i)|=|Z_G(x_j)|$ $(i\neq j)$, then $x_i \not\sim x_j$,

(iii) $\sum_{i=0}^{34} |G|/|Z_G(x_i)|=q^{48}$.

(ii) and (iii) are easy, and (i) is rather straightforward but very long calculations, so we do not prove Theorem 2.1; we only mention some facts about (i).

The calculations of $|Z_G(x_i)|$ depend on counting the number of solutions of the equation

$$(*)\qquad gx_ig^{-1}=x_i.$$

Except for $i=5,6$ or 19, to count the number of solutions of (*) is straightforward and not so difficult. For $i=5,6$ the calculations are complicated, but can be completed by combinatorial method; for $i=19$, the following equations appear from (*),

$$\begin{aligned} &a^3+a+\zeta+\zeta^3=0,\\ &b^3+b+\zeta+\zeta^3=0,\\ &\zeta a=1+b^2,\\ &\zeta b=1+a^2, \qquad \text{where} \quad a,b\in k, \end{aligned}$$

which are solved essentially in ([4], p. 504) using Lemma 2.1.

The conjugacy classes in $\bar{G}$ can be obtained immediately from the process of calculations, and the following corollaries are also immediate consequences of theorem 2.1.

COROLLARY 1. *G has 4 classes of involutions and so does $\bar{G}$. Their representatives are*

$$x_1=x_1(1), \qquad x_2=x_{1+2}(1),$$
$$x_3=x_1(1)x_{1+2}(1), \qquad x_4=x_{2+3}(1)x_1(1).$$

And $Z_{\bar{G}}(x_i)$ $(i=1,2,3,4)$ is connected.

COROLLARY 2. *${}^2F_4(q)$ has two classes of involutions. Their representatives are as follows:*

$$x_1(1)x_{1+2}(1)=x_3,$$
$$x_{1+2+3+4}(1)x_{1+4}(1)\underset{\bar{G}}{\sim}x_4.$$

§ 3. Semisimple classes and the general conjugacy classes

General theory for semisimple classes is much more developed than that of unipotent classes. For example, see ([2], E. II, 3). The method for determining the semisimple classes we use here is deduced easily from that and it is the same as that used in [3] or [4]. But for the sake of convenience, we explain here the method without proof. For detailes, see those papers above.

(3.1) Let x be a semisimple element of G. Then x is conjugate to some element h of $\bar{H}$ in $\bar{G}$. Conversely if $h(\in \bar{H})$ is conjugate to an element of G, then $h \underset{\bar{G}}{\sim} h^{(q)}$, where $y \mapsto y^{(q)}$ is the Frobenius endomorphism of $\bar{G}$. Hence by the uniqueness of Bruhat decomposition, $h \underset{W}{\sim} h^{(q)}$. So if we put, for $w \in W$,

$$H(w) = \{h \in \bar{H} \mid h^{(q)} = n_w h n_w^{-1}\},$$

then every semisimple element of G is conjugate to some element of $\bigcup_{w \in W} H(w)$; and conversely, every element of $\bigcup_{w \in W} H(w)$ is conjugate to a semisimple element of G.

THEOREM ([6], 8.1). *Let $\bar{G}$ be a semisimple simply connected algebraic group. Then for any semisimple element x of $\bar{G}$, $Z_{\bar{G}}(x)$ is a connected reductive group.*

Using above theorem and (1.2), we know that there is one to one correspondence between the semisimple classes of G and $\bigcup_{w \in W} H(w)/W$. Moreover for $w, w' \in W$ we get

$$w'H(w)w'^{-1} = H(w'ww'^{-1}).$$

Hence we can choose a representative system of semisimple classes from $\bigcup_{i \in I} H(w_i)$, where $\{w_i\}_{i \in I}$ is a representative system of conjugacy classes of W.

(3.2) We choose and fix a representative system of conjugacy classes of W as in Table I. In Table I we represent the elements of W in $GL(V)$ or $O(V)$, where $V = \sum_{\alpha \in \Phi} \boldsymbol{R}\alpha$, and the matrix representation is written with respect to the basis $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$. The symbols on the left side are used by R. Carter in [2] part G.

We shall express the elements of $\bar{H}$ as follows:

$$h(\chi) = (z_1, z_2, z_3, z_4, s)$$

where $\chi(\varepsilon_i) = z_i$ and $s^2 = z_1 z_2 z_3 z_4$, and this expression is valid for a field of any characteristic.

For this expression, W acts on $\bar{H}$ as follows. If we put

$$W(B_4) = \langle r_{1-2}, r_{2-3}, r_{3-4}, r_4 \rangle,$$

Table I. The Representative Elements of Conjugacy Classes in W.

type	representative elements w_i	$\det(xI-w_i)$
$\varnothing$	$w_1=\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$	$(x-1)^4$
A_1	$w_2=\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{pmatrix}$	$(x-1)^3(x+1)$
$\tilde{A}_1$	$w_3=\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$	$(x-1)^3(x+1)$
$2A_1$	$w_4=\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$	$(x-1)^2(x+1)^2$
$A_1+\tilde{A}_1$	$w_5=\begin{pmatrix} & 1 & & \\ 1 & & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$	$(x-1)^2(x+1)^2$
A_2	$w_6=\begin{pmatrix} 1 & & & \\ & & 1 & \\ & & & 1 \\ & 1 & & \end{pmatrix}$	$(x-1)^2(x^2+x+1)$
$\tilde{A}_2$	$w_7=\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix}$	$(x-1)^2(x^2+x+1)$
B_2	$w_8=\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & -1 & \end{pmatrix}$	$(x-1)^2(x^2+1)$
$3A_1$	$w_9=\begin{pmatrix} -1 & & & \\ & -1 & & \\ & & & 1 \\ & & 1 & \end{pmatrix}$	$(x-1)(x+1)^3$

Table I (continued)

type	representative elements w_i	$\det(xI-w_i)$
$2A_1+\tilde{A}_1$	$w_{10}=\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$	$(x-1)(x+1)^3$
A_3	$w_{11}=\begin{pmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{pmatrix}$	$(x-1)(x^3+x^2+x+1)$
B_2+A_1	$w_{12}=\begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & -1 & \end{pmatrix}$	$(x-1)(x+1)(x^2+1)$
C_3	$w_{13}=\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \end{pmatrix}$	$(x-1)(x^3+1)$
B_3	$w_{14}=\begin{pmatrix} 1 & & & \\ & & 1 & \\ & & & 1 \\ & -1 & & \end{pmatrix}$	$(x-1)(x^3+1)$
$\tilde{A}_2+A_1$	$w_{15}=\frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & -1 \end{pmatrix}$	$(x-1)(x+1)(x^2+x+1)$
$A_2+\tilde{A}_1$	$w_{16}=\begin{pmatrix} -1 & & & \\ & & 1 & \\ & & & 1 \\ & 1 & & \end{pmatrix}$	$(x-1)(x+1)(x^2+x+1)$
$4A_1$	$w_{17}=\begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$	$(x+1)^4$
$A_2+\tilde{A}_2$	$w_{18}=\frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 \end{pmatrix}$	$(x^2+x+1)^2$

Table I (continued)

type	representative elements w_i	$\det(xI-w_i)$
$A_3+\tilde{A}_1$	$w_{19}=\begin{pmatrix} -1 & & & \\ & -1 & & \\ & & & 1 \\ & & -1 & \end{pmatrix}$	$(x+1)(x^3+x^2+x+1)$
C_3+A_1	$w_{20}=\frac{1}{2}\begin{pmatrix} -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \end{pmatrix}$	$(x+1)(x^3+1)$
D_4	$w_{21}=\begin{pmatrix} & 1 & & \\ & & 1 & \\ -1 & & & \\ & & & -1 \end{pmatrix}$	$(x+1)(x^3+1)$
$D_4\ (a_1)$	$w_{22}=\begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & 1 \\ & & -1 & \end{pmatrix}$	$(x^2+1)^2$
B_4	$w_{23}=\begin{pmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ -1 & & & \end{pmatrix}$	x^4+1
F_4	$w_{24}=\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$	x^4-x^2+1
$F_4\ (a_1)$	$w_{25}=\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$	$(x^2-x+1)^2$

then

$$W=W(B_4)\cup W(B_4)r_{1-2-3-4}\cup W(B_4)r_{1-2-3+4}.$$

The action of $W(B_4)$ is given by

$$(z_1, z_2, z_3, z_4, s)\longmapsto(z_{\tau(1)}^{\epsilon_1}, z_{\tau(2)}^{\epsilon_2}, z_{\tau(3)}^{\epsilon_3}, z_{\tau(4)}^{\epsilon_4}, s'),$$

where $\tau\in\mathfrak{S}_4$=the symmetric group on four letters $\{1, 2, 3, 4\}$ and

$$\epsilon_i=\pm1, \qquad s'=s\prod z_{\tau(i)}^{(\epsilon_i-1)/2}.$$

And

$$^{r_{1-2-3-4}}(z_1, z_2, z_3, z_4, s) = (s, sz_3^{-1}z_4^{-1}, sz_2^{-1}z_4^{-1}, sz_2^{-1}z_3^{-1}, z_1),$$
$$^{r_{1-2-3+4}}(z_1, z_2, z_3, z_4, s) = (sz_4^{-1}, sz_3^{-1}, sz_2^{-1}, sz_1^{-1}, s)$$

where $^w h(\chi)$ denotes $n_w h(\chi) n_w^{-1}$ (cf. [7], p. 337).

Now we turn to the case of characteristic 2. Then from $s^2 = z_1 z_2 z_3 z_4$, s is determined uniquely. Hence we can write

$$h(\chi) = (z_1, z_2, z_3, z_4), \qquad \chi(\varepsilon_i) = z_i \in \bar{k}^*.$$

The action of W is the same as above. We write $H(w_i) = H(i)$, and we express their elements explicitly. Besides, it is known that

$$|H(i)| = \det (q.I - w_i). \qquad \text{(cf. [2], E. II, 1.7)}$$

$$H(1) = \{(z_1, z_2, z_3, z_4) \mid z_i \in k^*\}$$
$$H(2) = \{(z_1, z_2, z, z^q) \mid z_i \in k^*, z^{q^2-1} = 1\}$$
$$H(3) = \{(z_1, z_2, z_3, t) \mid z_i \in k^*, t^{q+1} = 1\}$$
$$H(4) = \{(z_1, z_2, t_1, t_2) \mid z_i \in k^*, t_i^{q+1} = 1\}$$
$$H(5) = \{(z, z^q, z_1, t) \mid z_1 \in k^*, z^{q^2-1} = 1, t^{q+1} = 1\}$$
$$H(6) = \{(z_1, z, z^q, z^{q^2}) \mid z_1 \in k^*, z^{q^3-1} = 1\}$$
$$H(7) = \{(z_1 z, z_1 z^{-1}, z^{q+q^2}, z^{q-q^2}) \mid z_1 \in k^*, z^{q^3-1} = 1\}$$
$$H(8) = \{(z_1, z_2, z, z^q) \mid z_i \in k^*, z^{q^2+1} = 1\}$$
$$H(9) = \{(t_1, t_2, z, z^q) \mid t_i^{q+1} = 1, z^{q^2-1} = 1\}$$
$$H(10) = \{(z, t_1, t_2, t_3) \mid z \in k^*, t_i^{q+1} = 1\}$$
$$H(11) = \{(z, z^q, z^{q^2}, z^{q^3}) \mid z^{q^4-1} = 1\}$$
$$H(12) = \{(z, z^q, t, t^q) \mid z^{q^2-1} = 1, t^{q^2+1} = 1\}$$
$$H(13) = \{(z_1 z, z_1 z^{-1}, z^{q^2+q}, z^{q^2-q}) \mid z_1 \in k^*, z^{q^3+1} = 1\}$$
$$H(14) = \{(z_1, z, z^q, z^{q^2}) \mid z_1 \in k^*, z^{q^3+1} = 1\}$$
$$H(15) = \{(tz, t^{-1}z, z^{q+q^2}, z^{q-q^2}) \mid t^{q+1} = 1, z^{q^3-1} = 1\}$$
$$H(16) = \{(t, z, z^q, z^{q^2}) \mid t^{q+1} = 1, z^{q^3-1} = 1\}$$
$$H(17) = \{(t_1, t_2, t_3, t_4) \mid t_i^{q+1} = 1\}$$
$$H(18) = \{(tz, tz^{-1}, t^{q-q^2}z, t^{-1}z^{q-q^2}) \mid t^{q^2+q+1} = 1, z^{q^2+q+1} = 1\}$$
$$H(19) = \{(t_1, t_2, z, z^q) \mid t_i^{q+1} = 1, z^{q^2+1} = 1\}$$
$$H(20) = \{(tz, tz^{-1}, z^{q+q^2}, z^{q^2-q}) \mid t^{q+1} = 1, z^{q^3+1} = 1\}$$
$$H(21) = \{(z, z^q, z^{q^2}, t) \mid z^{q^3+1} = 1, t^{q+1} = 1\}$$
$$H(22) = \{(z_1, z_1^q, z_2, z_2^q) \mid z_i^{q^2+1} = 1\}$$
$$H(23) = \{(z, z^q, z^{q^2}, z^{q^3}) \mid z^{q^4+1} = 1\}$$
$$H(24) = \{(t^{-q^3+2q+1}, t^{q^3+2q^2-1}, t^{q^3+1}, t^{1-q^3}) \mid t^{q^4-q^2+1} = 1\}$$
$$H(25) = \{(zt, zt^{-1}, z^{q^2+q}t, z^{-1}t^{q^2+q}) \mid t^{q^2-q+1} = 1, z^{q^2-q+1} = 1\}$$

Table II. Semisimple classes of G

	representative elements	number of classes	
		$q\equiv 1 \pmod 3$	$q\equiv -1 \pmod 3$
$H(1)$	$h_0=(1,1,1,1)$	1	
	$h_1=(1,1,1,z)$	$\frac{1}{2}(q-2)$	
	$h_2=(1,1,z,z)$	$\frac{1}{2}(q-2)$	
	$h_3=(1,\omega,\omega,\omega)$ $\omega^3=1, \omega\neq 1$	1	0
	$h_4=(1,z,z,z)$ $z^3\neq 1$	$\frac{1}{2}(q-4)$	$\frac{1}{2}(q-2)$
	$h_5=(1,z^2,z,z)$ $z^3\neq 1$	$\frac{1}{2}(q-4)$	$\frac{1}{2}(q-2)$
	$h_6=(1,1,z_1,z_2)$	$\frac{1}{8}(q-2)(q-4)$	
	$h_7=(1,z_1,z_2,z_2)$ $z_1^{\pm 1}\neq z_2^2$	$\frac{1}{4}(q-4)^2$	$\frac{1}{4}(q-2)(q-6)$
	$h_8=(z_1,z_2,z_2,z_2)$ $z_1^{\pm 1}\neq z_2^3$	$\frac{1}{12}(q-4)^2$	$\frac{1}{12}(q-2)(q-6)$
	$h_9=(1,z_1,z_2,z_3)$ $z_1z_2z_3=1$	$\frac{1}{12}(q-4)^2$	$\frac{1}{12}(q-2)(q-6)$
	$h_{10}=(1,z_1,z_2,z_3)$ $z_1^{\pm 1}z_2^{\pm 1}z_3^{\pm 1}\neq 1$	$\frac{1}{48}(q-4)(q^2-12q+28)$	$\frac{1}{48}(q-2)(q-6)(q-8)$
	$h_{11}=(z_1,z_2,z_3,z_3)$ $z_1z_2^{\pm 1}z_3^{\pm 2}\neq 1$	$\frac{1}{48}(q-4)(q^2-12q+28)$	$\frac{1}{48}(q-2)(q-6)(q-8)$
	$h_{12}=(z_1,z_2,z_3,z_4)$ $z_1^{\pm 1}z_2^{\pm 1}z_3^{\pm 1}z_4^{\pm 1}\neq 1$	$\frac{1}{1152}(q-4)(q^3-24q^2+172q-320)$	$\frac{1}{1152}(q-2)(q-6)(q-8)(q-12)$
$H(17)$	$h_{13}=(1,1,1,z)$ $z^{q+1}=1$	$\frac{1}{2}q$	
	$h_{14}=(1,1,z,z)$	$\frac{1}{2}q$	
	$h_{15}=(1,\omega,\omega,\omega)$ $\omega^3=1, \omega\neq 1$	0	1
	$h_{16}=(1,z,z,z)$ $z^3\neq 1$	$\frac{1}{2}q$	$\frac{1}{2}(q-2)$
	$h_{17}=(1,z^2,z,z)$ $z^3\neq 1$	$\frac{1}{2}q$	$\frac{1}{2}(q-2)$
	$h_{18}=(1,1,z_1,z_2)$	$\frac{1}{8}q(q-2)$	
	$h_{19}=(1,z_1,z_2,z_2)$ $z_1^{\pm 1}\neq z_2^2$	$\frac{1}{4}q(q-4)$	$\frac{1}{4}(q-2)^2$

Table II (continued)

	representative elements	number of classes	
		$q\equiv 1$ (mod. 3)	$q\equiv -1$ (mod. 3)
$H(17)$	$h_{20}=(z_1, z_2, z_2, z_2)$ $z_1^{\pm 1}\neq z_2^3$	$\frac{1}{12}q(q-4)$	$\frac{1}{12}(q-2)^2$
	$h_{21}=(1, z_1, z_2, z_3)$ $z_1z_2z_3=1$	$\frac{1}{12}q(q-4)$	$\frac{1}{12}(q-2)^2$
	$h_{22}=(1, z_1, z_2, z_3)$ $z_1^{\pm 1}z_2^{\pm 1}z_3^{\pm 1}\neq 1$	$\frac{1}{48}q(q-4)(q-6)$	$\frac{1}{48}(q-2)(q^2-8q+8)$
	$h_{23}=(z_1, z_2, z_3, z_3)$ $z_1^{\pm 1}z_2^{\pm 1}z_3^{\pm 2}\neq 1$	$\frac{1}{48}q(q-4)(q-6)$	$\frac{1}{48}(q-2)(q^2-8q+8)$
	$h_{24}=(z_1, z_2, z_3, z_4)$ $z_1^{\pm 1}z_2^{\pm 1}z_3^{\pm 1}z_4^{\pm 1}\neq 1$	$\frac{1}{1152}q(q-4)(q-6)(q-10)$	$\frac{1}{1152}(q-2)(q^3-18q^2+88q-64)$
$H(2)$	$h_{25}=(z_1, 1, z, z^q)$ $z_1^{q-1}=1, z^{q+1}=1$	$\frac{1}{4}q(q-2)$	
	$h_{26}=(z_1, z_1, z, z^q)$	$\frac{1}{4}q(q-2)$	
	$h_{27}=(z_1, z_2, z, z^q)$	$\frac{1}{16}q(q-2)(q-4)$	
	$h_{28}=(z^{q+1}, 1, z, z^q)$ $z^{q+1}\neq 1$	$\frac{1}{4}q(q-2)$	
	$h_{29}=(z_1, 1, z, z^q)$ z^{q+1} and $z_1^{\pm 1}\neq z^{q+1}$	$\frac{1}{8}q(q-2)(q-4)$	
	$h_{30}=(z_1, z_2, z, z^q)$ $z_1^{\pm 1}z_2^{\pm 1}z^{\pm(q+1)}\neq 1$	$\frac{1}{96}q(q-2)(q-4)(q-6)$	
$H(3)$	$h_{31}=(1, 1, z_1, z)$ $z_1^{q-1}=1$, $z^{q+1}=1$	$\frac{1}{4}q(q-2)$	
	$h_{32}=(1, z_1, z_2, z)$	$\frac{1}{4}q(q-2)$	
	$h_{33}=(1, z_1, z_2, z)$	$\frac{1}{16}q(q-2)(q-4)$	
	$h_{34}=(z_1, z_1, z_1, z)$	$\frac{1}{4}q(q-2)$	
	$h_{35}=(z_1, z_2, z_2, z)$	$\frac{1}{8}q(q-2)(q-4)$	
	$h_{36}=(z_1, z_2, z_3, z)$	$\frac{1}{96}q(q-2)(q-4)(q-6)$	
$H(4)$	$h_{37}=(1, z_1, t_1, t_2)$ $z_1^{q-1}=1, t_i^{q+1}=1$	$\frac{1}{16}q(q-2)^2$	

Table II (continued)

	representative elements	number of classes	
		$q\equiv 1$ (mod. 3)	$q\equiv -1$ (mod. 3)
$H(4)$	$h_{38}=(z_1, z_1, t_1, t_2)$	$\frac{1}{16}q(q-2)^2$	
	$h_{39}=(z_1, z_2, t_1, t_2)$	$\frac{1}{64}q(q-2)^2(q-4)$	
$H(5)$	$h_{40}=(z, z^q, z_1, z)$ $z^{q+1}=1, z_1^{q-1}=1$	$\frac{1}{4}q(q-2)$	
	$h_{41}=(z, z^q, z_1, t)$ $z^{q+1}=1, t^{\pm 1}\neq z$	$\frac{1}{8}q(q-2)^2$	
	$h_{42}=(z, z^q, 1, z^{q-1})$ $z^{q+1}\neq 1$	$\frac{1}{4}q(q-2)$	
	$h_{43}=(z, z^q, 1, t)$ $t^{\pm 1}\neq z^{q-1}$	$\frac{1}{8}q(q-2)^2$	
	$h_{44}=(z, z^q, z_1, t)$	$\frac{1}{16}q^2(q-2)^2$	
$H(6)$	$h_{45}=(1, z, z^q, z^{q^2})$ $z^{q^2+q+1}=1$	$\frac{1}{6}(q^2+q-2)$	$\frac{1}{6}(q^2+q)$
	$h_{46}=(1, z, z^q, z^{q^2})$ $z^{q^2+q+1}\neq 1$	$\frac{1}{6}(q-1)(q^2-2)$	$\frac{1}{6}q(q+1)(q-2)$
	$h_{47}=(z_1, z, z^q, z^{q^2})$ $z_1^{\pm 1}\neq z^{q^2+q+1}$	$\frac{1}{36}(q-1)^2(q^2-2q-4)$	$\frac{1}{36}q(q+1)(q-2)(q-3)$
$H(21)$	$h_{48}=(1, z, z^q, z^{q^2})$ $z^{q^2-q+1}=1$	$\frac{1}{6}(q^2-q)$	$\frac{1}{6}(q^2-q-2)$
	$h_{49}=(1, z, z^q, z^{q^2})$ $z^{q^2-q+1}\neq 1$	$\frac{1}{6}q^2(q-1)$	$\frac{1}{6}(q^3-q^2+2)$
	$h_{50}=(t, z, z^q, z^{q^2})$ $t^{\pm 1}\neq z^{q^2-q+1}$	$\frac{1}{36}q^2(q-1)^2$	$\frac{1}{36}(q-2)(q+1)(q^2-q+2)$
$H(14)$	$h_{51}=(z, z, z^q, z^{q^2})$ $z_1^{q-1}=1, z^{q^3+1}=1$	$\frac{1}{12}q(q-2)(q^2-1)$	
$H(16)$	$h_{52}=(t, z, z^q, z^{q^2})$ $t^{q+1}=1, z^{q^3-1}=1$	$\frac{1}{12}q^2(q^2-1)$	
$H(8)$	$h_{53}=(1, 1, z, z^q)$ $z^{q^2+1}=1$	$\frac{1}{4}q^2$	
	$h_{54}=(1, z_1, z, z^q)$	$\frac{1}{8}q^2(q-2)$	
	$h_{55}=(z_1, z_1, z, z^q)$	$\frac{1}{8}q^2(q-2)$	

Table II (continued)

	representative elements	number of classes	
		$q\equiv 1 \pmod 3$	$q\equiv -1 \pmod 3$
$H(8)$	$h_{56}=(z_1, z_2, z, z^q)$	$\frac{1}{32}q^2(q-2)(q-4)$	
$H(9)$	$h_{57}=(t_1, t_2, z, z^q)\ \ z^{q+1}\neq 1,$ $t_1{}^{\pm 1}t_2{}^{\pm 1}z^{\pm(q-1)}\neq 1$	$\frac{1}{96}q(q-2)^2(q-4)$	
$H(10)$	$h_{58}=(z, t_1, t_2, t_3)\ \ z^{q-1}=1,$ $t_i{}^{q+1}=1$	$\frac{1}{96}q(q-2)^2(q-4)$	
$H(11)$	$h_{59}=(z, z^q, z^{q^2}, z^{q^3})$ $z^{q^2\pm 1}\neq 1,\ z^{q^3+q^2+q+1}=1$	$\frac{1}{8}q^3$	
	$h_{60}=(z, z^q, z^{q^2}, z^{q^3})$ $z^{q^3\pm q^2+q\pm 1}\neq 1$	$\frac{1}{16}q^3(q-2)$	
$H(12)$	$h_{61}=(z_1, z_1{}^q, z_2, z_2{}^q)$ $z_1{}^{q+1}=1$	$\frac{1}{8}q^3$	
	$h_{62}=(z_1, z_1{}^q, z_2, z_2{}^q)$ $z_1{}^{q+1}\neq 1$	$\frac{1}{16}q^3(q-2)$	
$H(22)$	$h_{63}=(z_1, z_1{}^q, z_2, z_2{}^q)$ $z_i{}^{q^2+1}=1$	$\frac{1}{96}q^2(q^2-4)$	
$H(19)$	$h_{64}=(t_1, t_2, z, z^q)\ \ z^{q^2+1}=1$	$\frac{1}{32}q^3(q-2)$	
$H(23)$	$h_{65}=(z, z^q, z^{q^2}, z^{q^3})$ $z^{q^4+1}=1$	$\frac{1}{8}q^4$	
$H(7)$	$h_{66}=(t, t^{-1}, t^{-1}, t^{q-q^2})$ $t^{q^2+q+1}=1$	$\frac{1}{6}(q^2+q-2)$	$\frac{1}{6}(q^2+q)$
	$h_{67}=(t_1t, t_1t^{-1}, t_1t^{-1}, t^{q-q^2})$ $t_1=t^{q^2+q+1}$	$\frac{1}{6}(q-1)(q^2-2)$	$\frac{1}{6}q(q-2)(q+1)$
	$h_{68}=(t_1t, t_1t^{-1}, t^{q+q^2}, t^{q-q^2})$ $t_1{}^{\pm 1}\neq t^{q^2+q+1}$	$\frac{1}{36}(q-1)^2(q^2-2q-4)$	$\frac{1}{36}q(q+1)(q-2)(q-3)$
$H(20)$	$h_{69}=(t, t^{-1}, t^{2q-1}, t^{-1})$ $t^{q^2-q+1}=1$	$\frac{1}{6}(q^2-q)$	$\frac{1}{6}(q^2-q-2)$
	$h_{70}=(tt_1, t^{-1}t_1, t^{q^2+q}, t^{q^2-q})$ $t_1=t^{q^2-q+1}$	$\frac{1}{6}(q^3-q^2)$	$\frac{1}{6}(q^3-q^2+2)$
	$h_{71}=(tt_1, t^{-1}t_1, t^{q^2+q}, t^{q^2-q})$ $t_1{}^{\pm 1}\neq t^{q^2-q+1}$	$\frac{1}{36}q^2(q-1)^2$	$\frac{1}{36}(q+1)(q-2)(q^2-q+2)$
$H(13)$	$h_{72}=(t_1t, t_1t^{-1}, t^{q^2+q}, t^{q^2-q})$	$\frac{1}{12}(q-2)(q^3-q)$	
$H(15)$	$h_{73}=(t_1t, t^{-1}t, t^{q^2+q}, t^{q^2-q})$	$\frac{1}{12}q^2(q^2-1)$	

Table II (continued)

	representative elements	number of classes	
		$q\equiv 1$ (mod. 3)	$q\equiv -1$ (mod. 3)
$H(18)$	$h_{74}=(ts, ts^{-1}, t^{-q^2+q}s, t^{-1}s^{-q^2+q})$.	$\frac{1}{72}(q^2+q-4)(q+2)(q-1)$	$\frac{1}{72}q(q+1)(q+3)(q-2)$
$H(25)$	$h_{75}=(ts, t^{-1}s, s^{q^2+q}t, s^{-1}t^{q^2+q})$	$\frac{1}{72}q(q-1)(q-3)(q+2)$	$\frac{1}{72}(q^2-q-4)(q-2)(q+1)$
$H(24)$	$h_{76}=(t^{-q^3+2q+1}, t^{q^3+2q^2-1}, t^{q^3+1}, t^{1-q^3})$	$\frac{1}{12}q^2(q^2-1)$	

(3.3) As we showed in (3.1) we can choose the representative elements from $\bigcup_{i=1}^{25} H(i)$. To do this we use the following method:

(i) first we search for the representative elements $\{h(\chi_j)\}_{j\in J}$ of the equivalence classes by the action of $W(B_4)$,

(ii) next, we calculate ${}^{r_{1-2-3-4}}h(\chi_j)$ and ${}^{r_{1-2-3+4}}h(\chi_j)$, then together with the result obtained in (i) we can choose a representative system $\{h(\chi_{j_0})\}_{j_0\in J_0}$.

Calculation along this program is not so difficult but very long and needs much patience. Omitting the detailes, we give only the final results.

THEOREM 3.1. *A representative system of semisimple classes of G is given in Table II.*

In Table II, t_i or z_j always differs from 1; and if $i\neq j$, then always $z_i^{\pm 1}\neq z_j$ (resp. $t_i^{\pm 1}\neq t_j$) holds for z_i, z_j (resp. t_i, t_j), which appear in the expression of the same h. Where the column of the number of classes in the case of $q\equiv -1$ is empty, the same number of $q\equiv 1$ is to be written.

REMARK. In general, the following theorem is known.

THEOREM ([6], 14.11). *Let $\bar{G}$ be a connected semisimple algebraic group defined over k, and r be the rank of $\bar{G}$. If $G=\bar{G}(k)$, then the number of semisimple conjugacy classes of G is equal to q^r.*

Using this theorem, we can check the results obtained above. If we add the number of classes in the case of $q\equiv 1$ or $q\equiv -1$ respectively, we obtain q^4 as the total number; and thus the results in Table II are assured.

(3.4) Now let us calculate the order of the centralizers of h_i defined in Table II. If x is a semisimple element of G and conjugate to $h\in H(w)$, then

(3.4.1) $$Z_G(x)\cong Z_{\bar{G}}(h)\cap G(w),$$

where $G(w)=\{g\in\bar{G}\mid g^{(q)}=n_w g n_w^{-1}\}$; for the proof see [3] or [4].

For $h=h(\chi)$, we put

$$\begin{aligned}
&\Phi_\chi=\{\alpha\in\Phi \mid \chi(\alpha)=1\}\\
&W_\chi=\langle w\in W \mid {}^w\chi=\chi\rangle \quad \text{where} \quad {}^w\chi=\chi\cdot w^{-1}\\
&\bar U^\chi=\langle \bar U_\alpha \mid \alpha\in\Phi^+\cap\Phi_\chi\rangle\\
&\bar U_w{}^\chi=\langle \bar U_\alpha \mid \alpha\in\Phi_w^-\cap\Phi_\chi\rangle.
\end{aligned}$$

Then the following equation holds.

(3.4.2) $$Z_{\bar G}(h(\chi))=\bigcup_{w\in W_\chi}\bar U^\chi \bar H n_w \bar U_w{}^\chi$$ ([2], F-3, Proposition 1).

Using (3.4.1) and (3.4.2), we can calculate the orders of centralizers rather easily.

Next we consider a general element x of G. Since $G=\bar G(k)$ and k is perfect, we can decompose x uniquely as follows: (Jordan decomposition) $x=x_s\cdot x_u=x_ux_s$, $x_u, x_s\in G$, where x_u is unipotent and x_s is semisimple.

Similarly let for $x'\in G, x'=x'_s\cdot x'_u$ be the Jordan decomposition of x'. If $x\sim x'$, then by the uniqueness of the Jordan decomposition, we get $x_s\sim x'_s$ and $x_u\sim x'_u$. In particular, if $x_s=x'_s$ and $gxg^{-1}=x'$ $(g\in G)$, then $g\in Z_G(x_s)$ and $gx_ug^{-1}=x'_u$. Hence if $x_s\sim h\in H(w)$ and $\{u_j\}_{j\in J}$ is a representative system of unipotent classes of $Z_{G(w)}(h)$, then there is one to one correspondence between the conjugacy classes of G whose semisimple parts are conjugate to x_s, and $\{hu_j\}_{j\in J}$. Moreover $Z_G(x)=Z_{Z_G(x_s)}(x_u)\cong Z_{Z_{G(w)}(h)}(u_j)$ for some u_j. Therefore the problem to determine all the conjugacy classes is reduced to the problem to determine the unipotent classes of $Z_{G(w)}(h_i)$ $(i=0, 1, \ldots, 76)$, where $h_i\in H(w)$.

$Z_{G(w)}(h_i)$ is the product of $H(w)$ and L, where L is a normal subgroup of $Z_{G(w)}(h_i)$ and isomorphic to one of the following groups:

$$\begin{aligned}
&F_4(q),\ G(B_3),\ G(B_2),\ SL(2,q)\times SL(3,q),\\
&SL(2,q)\times SU(3,q^2),\ SL(2,q)\times SL(2,q),\ SL(3,q),\\
&SU(3,q^2),\ SL(2,q)
\end{aligned}$$

and in the case of $i=3$ or 15, i.e. $h=h(1,\omega,\omega,\omega)$, L is isomorphic to

$$G_1\cdot G_2$$

where G_1 and G_2 are isomorphic to $SL(3,q)$ in the case of $q\equiv 1$ (mod. 3) and isomorphic to $SU(3,q^2)$ in the case of $q\equiv -1$ (mod. 3); G_1 and G_2 are mutually commutative and $G_1\cap G_2=\{1,h,h^2\}$.

Unipotent classes of the groups above are easily determined and as for $F_4(q)$ and $G(B_3)$ we can use the results in § 2. Thus we obtain the following theorem.

Table III. The Centralizer Z_i of h_i.

h_i	type of Φ_{χ_i}	m_i	$\|Z_i\|$
h_0	F_4	35	$q^{24}(q^2-1)(q^6-1)(q^8-1)(q^{12}-1)$
h_1	B_3	12	$q^9(q-1)(q^2-1)(q^4-1)(q^6-1)$
h_2	C_3	12	$q^9(q-1)(q^2-1)(q^4-1)(q^6-1)$
h_3	$\tilde{A}_2\times A_2$	11	$q^6(q^2-1)^2(q^3-1)^2$
h_4	$\tilde{A}_1\times A_2$	6	$q^4(q-1)(q^2-1)^2(q^3-1)$
h_5	$A_1\times \tilde{A}_2$	6	$q^4(q-1)(q^2-1)^2(q^3-1)$
h_6	B_2	6	$q^4(q-1)^2(q^2-1)(q^4-1)$
h_7	$\tilde{A}_1\times A_1$	4	$q^2(q-1)^2(q^2-1)^2$
h_8	A_2	3	$q^3(q-1)^2(q^2-1)(q^3-1)$
h_9	$\tilde{A}_2$	3	$q^3(q-1)^2(q^2-1)(q^3-1)$
h_{10}	$\tilde{A}_1$	2	$q(q-1)^3(q^2-1)$
h_{11}	A_1	2	$q(q-1)^3(q^2-1)$
h_{12}	$\emptyset$	1	$(q-1)^4$
h_{13}	B_3	12	$q^9(q+1)(q^2-1)(q^4-1)(q^6-1)$
h_{14}	C_3	12	$q^9(q+1)(q^2-1)(q^4-1)(q^6-1)$
h_{15}	$\tilde{A}_2\times A_2$	11	$q^6(q^2-1)^2(q^3+1)^2$
h_{16}	$\tilde{A}_1\times A_2$	6	$q^4(q+1)(q^2-1)^2(q^3+1)$
h_{17}	$A_1\times \tilde{A}_2$	6	$q^4(q+1)(q^2-1)^2(q^3+1)$
h_{18}	B_2	6	$q^4(q+1)^2(q^2-1)(q^4-1)$
h_{19}	$\tilde{A}_1\times A_1$	4	$q^2(q+1)^2(q^2-1)^2$
h_{20}	A_2	3	$q^3(q+1)^2(q^2-1)(q^3+1)$
h_{21}	$\tilde{A}_2$	3	$q^3(q+1)^2(q^2-1))q^3+1)$
h_{22}	$\tilde{A}_1$	2	$q(q+1)^3(q^2-1)$
h_{23}	A_1	2	$q(q+1)^3(q^2-1)$
h_{24}	$\emptyset$	1	$(q+1)^4$
h_{25}	$\tilde{A}_1\times A_1$	4	$q^2(q^2-1)^3$
h_{26}	B_2	6	$q^4(q^2-1)^2(q^4-1)$
h_{27}	A_1	2	$q(q-1)(q^2-1)^2$
h_{28}	$\tilde{A}_2$	3	$q^3(q^2-1)(q^3-1)$
h_{29}	$\tilde{A}_1$	2	$q(q-1)(q^2-1)^2$
h_{30}	$\emptyset$	1	$(q-1)^2(q^2-1)$
h_{31}	B_2	6	$q^4(q^2-1)^2(q^4-1)$
h_{32}	$\tilde{A}_1\times A_1$	4	$q^2(q^2-1)^3$
h_{33}	$\tilde{A}_1$	2	$q(q-1)(q^2-1)^2$
h_{34}	A_2	3	$q^3(q^2-1)^2(q^3-1)$
h_{35}	A_1	2	$q(q-1)(q^2-1)^2$
h_{36}	$\emptyset$	1	$(q-1)^2(q^2-1)$
h_{37}	$\tilde{A}_1$	2	$q(q+1)(q^2-1)^2$
h_{38}	A_1	2	$q(q+1)(q^2-1)^2$

Table III (continued)

h_i	type of Φ_{x_i}	m_i	$\|Z_i\|$
h_{39}	$\emptyset$	1	$(q^2-1)^2$
h_{40}	A_2	3	$q^3(q^2-1)^2(q^3+1)$
h_{41}	A_1	2	$q(q+1)(q^2-1)^2$
h_{42}	$\tilde{A}_2$	3	$q^3(q^2-1)^2(q^3+1)$
h_{43}	$\tilde{A}_1$	2	$q(q+1)(q^2-1)^2$
h_{44}	$\emptyset$	1	$(q^2-1)^2$
h_{45}	$\tilde{A}_2$	3	$q^3(q^2+q+1)(q^2-1)(q^3-1)$
h_{46}	$\tilde{A}_1$	2	$q(q^3-1)(q^2-1)$
h_{47}	$\emptyset$	1	$(q-1)(q^3-1)$
h_{48}	$\tilde{A}_2$	3	$q^3(q^2-q+1)(q^2-1)(q^3+1)$
h_{49}	$\tilde{A}_1$	2	$q(q^3+1)(q^2-1)$
h_{50}	$\emptyset$	1	$(q+1)(q^3+1)$
h_{51}	$\emptyset$	1	$(q-1)(q^3+1)$
h_{52}	$\emptyset$	1	$(q+1)(q^3-1)$
h_{53}	B_2	6	$q^4(q^2+1)(q^2-1)(q^4-1)$
h_{54}	$\tilde{A}_1$	2	$q(q-1)(q^2+1)(q^2-1)$
h_{55}	A_1	2	$q(q-1)(q^2+1)(q^2-1)$
h_{56}	$\emptyset$	1	$(q-1)^2(q^2+1)$
h_{57}	$\emptyset$	1	$(q^2-1)(q+1)^2$
h_{58}	$\emptyset$	1	$(q-1)(q+1)^3$
h_{59}	$\tilde{A}_1$	2	$q(q^4-1)(q+1)$
h_{60}	$\emptyset$	1	q^4-1
h_{61}	A_1	2	$q(q^4-1)(q+1)$
h_{62}	$\emptyset$	1	q^4-1
h_{63}	$\emptyset$	1	$(q^2+1)^2$
h_{64}	$\emptyset$	1	$(q+1)^2(q^2+1)$
h_{65}	$\emptyset$	1	q^4+1
h_{66}	A_2	3	$q^3(q+1)(q^3-1)^2$
h_{67}	A_1	2	$q(q^2-1)(q^3-1)$
h_{68}	$\emptyset$	1	$(q-1)(q^3-1)$
h_{69}	A_2	3	$q^3(q-1)(q^3+1)^2$
h_{70}	A_1	2	$q(q^2-1)(q^3+1)$
h_{71}	$\emptyset$	1	$(q+1)(q^3+1)$
h_{72}	$\emptyset$	1	$(q-1)(q^3+1)$
h_{73}	$\emptyset$	1	$(q+1)(q^3-1)$
h_{74}	$\emptyset$	1	$(q^2+q+1)^2$
h_{75}	$\emptyset$	1	$(q^2-q+1)^2$
h_{76}	$\emptyset$	1	q^4-q^2+1

$\tilde{A}_i$ means that Φ_χ is type A_i and consists of short roots.

Table IV. General classes.

class representative	order of centralizer
$h_1x_1(1)$	$q^9(q-1)(q^2-1)(q^4-1)$
$h_1x_{1+2}(1)$	$q^9(q-1)(q^2-1)^2$
$h_1x_1(1)x_{1+2}(1)$	$q^9(q^2-1)(q-1)$
$h_1x_1(1)x_{2+3}(1)$	$q^7(q-1)(q^2-1)$
$h_1x_{1-3}(1)x_2(1)x_{2+3}(1)$	$2q^6(q-1)^2$
$h_1x_{1-3}(1)x_2(1)x_{2+3}(1)x_{1+3}(\eta)$	$2q^6(q^2-1)$
$h_1x_3(1)x_{1-3}(1)$	$2q^5(q-1)(q^2-1)$
$h_1x_3(1)x_{1-3}(1)x_{1+3}(\eta)$	$2q^5(q-1)(q^2-1)$
$h_1x_3(1)x_{1-3}(1)x_{2+3}(1)$	$q^5(q-1)$
$h_1x_{1-2}(1)x_{2-3}(1)x_3(1)$	$2q^3(q-1)$
$h_1x_{1-2}(1)x_{2-3}(1)x_3(1)x_{2+3}(\eta)$	$2q^3(q-1)$
$h_2x_{1+2}(1)$	$q^9(q-1)(q^2-1)(q^4-1)$
$h_2x_1(1)$	$q^9(q-1)(q^2-1)^2$
$h_2x_1(1)x_{1+2}(1)$	$q^9(q-1)(q^2-1)$
$h_2x_{1+2}(1)x_{1-2+3+4}(1)$	$q^7(q-1)(q^2-1)$
$h_2x_{1+2-3-4}(1)x_{1-2}(1)x_{1-2+3+4}(1)$	$2q^6(q-1)^2$
$h_2x_{1+2-3-4}(1)x_{1-2}(1)x_{1-2+3+4}(1)x_{1+2+3+4}(\eta)$	$2q^6(q^2-1)$
$h_2x_{3+4}(1)x_{1+2-3-4}(1)$	$2q^5(q-1)(q^2-1)$
$h_2x_{3+4}(1)x_{1+2-3-4}(1)x_{1+2+3+4}(\eta)$	$2q^5(q-1)(q^2-1)$
$h_2x_{3+4}(1)x_{1+2-3-4}(1)x_{1-2+3+4}(1)$	$q^5(q-1)$
$h_2x_2(1)x_{1-2-3-4}(1)x_{3+4}(1)$	$2q^3(q-1)$
$h_2x_2(1)x_{1-2-3-4}(1)x_{3+4}(1)x_{1-2+3+4}(\eta)$	$2q^3(q-1)$
$h_3x_1(1)$	$q^6(q-1)(q^2-1)(q^3-1)$
$h_3x_{2-4}(1)$	$q^9(q-1)(q^2-1)(q^3-1)$
$h_3x_1(1)x_{2-4}(1)$	$q^6(q-1)^2$
$h_3x_{1-2-3-4}(1)x_{1+2+3+4}(1)$	$q^5(q^2-1)(q^3-1)$
$h_3x_{2-3}(1)x_{3-4}(1)$	$q^5(q^2-1)(q^3-1)$
$h_3x_{1-2-3-4}(1)x_{1+2+3+4}(1)x_{2-4}(1)$	$q^5(q-1)$
$h_3x_{2-3}(1)x_{3-4}(1)x_1(1)$	$q^5(q-1)$
$h_3x_{1-2-3-4}(1)x_{1+2+3+4}(1)x_{2-3}(1)x_{3-4}(1)$	$3q^4$
$h_3x_{1-2-3-4}(1)x_{1+2+3+4}(\rho^*)x_{2-3}(1)x_{3-4}(1)$	$3q^4$
$h_3x_{1-2-3-4}(1)x_{1+2+3+4}(\rho^2)x_{2-3}(1)x_{3-4}$	$3q^4$
$h_4x_1(1)$	$q^4(q-1)(q^2-1)(q^3-1)$
$h_4x_{2-4}(1)$	$q^4(q-1)^2(q^2-1)$
$h_4x_1(1)x_{2-4}(1)$	$q^4(q-1)^2$
$h_4x_{2-3}(1)x_{3-4}(1)$	$q^3(q-1)(q^2-1)$
$h_4x_1(1)x_{2-3}(1)x_{3-4}(1)$	$q^3(q-1)$
$h_5x_{3-4}(1)$	$q^4(q-1)(q^2-1)(q^3-1)$

*) ρ is a generator of F_q^*.

Table IV (continued)

class representative	order of centralizer
$h_5x_1(1)$	$q^4(q-1)^2(q^2-1)$
$h_5x_{3-4}(1)x_1(1)$	$q^4(q-1)^2$
$h_5x_{1+2-3-4}(1)x_{1-2+3+4}(1)$	$q^3(q-1)(q^2-1)$
$h_5x_{3-4}(1)x_{1+2-3-4}(1)x_{1-2+3+4}(1)$	$q^3(q-1)$
$h_6x_1(1)$	$q^4(q-1)^2(q^2-1)$
$h_6x_{1+2}(1)$	$q^4(q-1)^2(q^2-1)$
$h_6x_1(1)x_{1+2}(1)$	$q^4(q-1)^2$
$h_6x_{1-2}(1)x_2(1)$	$2q^2(q-1)^2$
$h_6x_{1-2}(1)x_2(1)x_{1+2}(\eta)$	$2q^2(q-1)^2$
$h_7x_1(1)$	$q^2(q-1)^2(q^2-1)$
$h_7x_{3-4}(1)$	$q^2(q-1)^2(q^2-1)$
$h_7x_1(1)x_{3-4}(1)$	$q^2(q-1)^2$
$h_8x_{2-4}(1)$	$q^3(q-1)^3$
$h_8x_{2-3}(1)x_{3-4}(1)$	$q^2(q-1)^2$
$h_9x_1(1)$	$q^3(q-1)^3$
$h_9x_{1-2-3-4}(1)x_{1+2+3+4}(1)$	$q^2(q-1)^2$
$h_{10}x_1(1)$	$q(q-1)^3$
$h_{11}x_{3-4}(1)$	$q(q-1)^3$
$h_{13}x_1(1)$	$q^9(q+1)(q^2-1)(q^4-1)$
$h_{13}x_{1+2}(1)$	$q^9(q+1)(q^2-1)^2$
$h_{13}x_1(1)x_{1+2}(1)$	$q^9(q+1)(q^2-1)$
$h_{13}x_1(1)x_{2+3}(1)$	$q^7(q+1)(q^2-1)$
$h_{13}x_{1-3}(1)x_2(1)x_{2+3}(1)$	$2q^6(q^2-1)$
$h_{13}x_{1-3}(1)x_2(1)x_{2+3}(1)x_{1+3}(\eta)$	$2q^6(q+1)^2$
$h_{13}x_3(1)x_{1-3}(1)$	$2q^5(q+1)(q^2-1)$
$h_{13}x_3(1)x_{1-3}(1)x_{1+3}(\eta)$	$2q^5(q+1)(q^2-1)$
$h_{13}x_3(1)x_{1-3}(1)x_{2+3}(1)$	$q^5(q+1)$
$h_{13}x_{1-2}(1)x_{2-3}(1)x_3(1)$	$2q^3(q+1)$
$h_{13}x_{1-2}(1)x_{2-3}(1)x_3(1)x_{2+3}(\eta)$	$2q^3(q+1)$
$h_{14}x_{1+2}(1)$	$q^9(q+1)(q^2-1)(q^4-1)$
$h_{14}x_1(1)$	$q^9(q+1)(q^2-1)^2$
$h_{14}x_1(1)x_{1+2}(1)$	$q^9(q+1)(q^2-1)$
$h_{14}x_{1+2}(1)x_{1-2+3+4}(1)$	$q^7(q+1)(q^2-1)$
$h_{14}x_{1+2-3-4}(1)x_{1-2}(1)x_{1-2+3+4}(1)$	$2q^6(q^2-1)$
$h_{14}x_{1+2-3-4}(1)x_{1-2}(1)_{1-2+3+4}(1)x_{1+2+3+4}(\eta)$	$2q^6(q+1)^2$
$h_{14}x_{3+4}(1)x_{1+2-3-4}(1)$	$2q^5(q+1)(q^2-1)$
$h_{14}x_{3+4}(1)x_{1+2-3-4}(1)x_{1+2+3+4}(\eta)$	$2q^5(q+1)(q^2-1)$
$h_{14}x_{3+4}(1)x_{1+2-3-4}(1)x_{1+2+3+4}(1)$	$q^5(q+1)$
$h_{14}x_2(1)x_{1-2-3-4}(1)x_{3+4}(1)$	$2q^3(q+1)$

Table IV (continued)

class representative	order of centralizer
$h_{14}x_2(1)x_{1-2-3-4}(1)x_{3+4}(1)x_{1-2+3+4}(\eta)$	$2q^3(q+1)$
$h_{15}x_1(1)$	$q^6(q+1)(q^2-1)(q^3+1)$
$h_{15}x_{2+3}(1)$	$q^6(q+1)(q^2-1)(q^3+1)$
$h_{15}x_1(1)x_{2+3}(1)$	$q^6(q+1)^2$
$h_{15}x_{3+4}(1)x_{2-4}(1)x_{2+3}(\tau^*)$	$q^5(q^2-1)(q^3+1)$
$h_{15}x_{1-2+3-4}(1)x_{1+2-3+4}(1)x_1(\tau)$	$q^5(q^2-1)(q^3+1)$
$h_{15}x_1(1)x_{3+4}(1)x_{2-4}(1)x_{2+3}(\tau)$	$q^5(q+1)$
$h_{15}x_{2+3}(1)x_{1-2+3-4}(1)x_{1+2-3+4}(1)x_1(\tau)$	$q^5(q+1)$
$h_{15}x_{3+4}(1)x_{2-4}(1)x_{2+3}(\tau)x_{1-2+3-4}(1)x_{1+2-3+4}(1)x_1(\tau)$	$3q^4$
$h_{15}x_{3+4}(1)x_{2-4}(1)x_{2+3}(\tau)x_{1-2+3-4}(\lambda^*)x_{1+2-3+4}(\lambda^q)x_1(\tau_1)$*	$3q^4$
$h_{15}x_{3+4}(1)x_{2-4}(1)x_{2+3}(\tau)x_{1-2+3-4}(\lambda^{-1})x_{1+2-3+4}(\lambda^{-q})x_1(\tau_2)$*	$3q^4$
$h_{16}x_1(1)$	$q^4(q+1)(q^2-1)(q^3+1)$
$h_{16}x_{2+3}(1)$	$q^4(q+1)^2(q^2-1)$
$h_{16}x_1(1)x_{2+3}(1)$	$q^4(q+1)^2$
$h_{16}x_{3+4}(1)x_{2-4}(1)x_{2+3}(\tau)$	$q^3(q+1)(q^2-1)$
$h_{16}x_1(1)x_{3+4}(1)x_{2-4}(1)x_{2+3}(\tau)$	$q^3(q+1)$
$h_{17}x_{3-4}(1)$	$q^4(q+1)(q^2-1)(q^3+1)$
$h_{17}x_1(1)$	$q^4(q+1)^2(q^2-1)$
$h_{17}x_{3-4}(1)x_1(1)$	$q^4(q+1)^2$
$h_{17}x_{1+2-3-4}(1)x_{1-2+3+4}(1)x_1(\tau)$	$q^3(q+1)(q^2-1)$
$h_{17}x_{3-4}(1)x_{1+2-3-4}(1)x_{1-2+3+4}(1)x_1(\tau)$	$q^3(q+1)$
$h_{18}x_1(1)$	$q^4(q+1)^2(q^2-1)$
$h_{18}x_{1+2}(1)$	$q^4(q+1)^2(q^2-1)$
$h_{18}x_1(1)x_{1+2}(1)$	$q^4(q+1)^2$
$h_{18}x_{1-2}(1)x_2(1)$	$2q^2(q+1)^2$
$h_{18}x_{1-2}(1)x_2(1)x_{1+2}(\eta)$	$2q^2(q+1)^2$
$h_{19}x_1(1)$	$q^2(q+1)^2(q^2-1)$
$h_{19}x_{3-4}(1)$	$q^2(q+1)^2(q^2-1)$
$h_{19}x_1(1)x_{3-4}(1)$	$q^2(q+1)^2$
$h_{20}x_{2-4}(1)$	$q^3(q+1)^3$
$h_{20}x_{2-3}(1)x_{3-4}(1)x_{2-4}(\tau)$	$q^2(q+1)^2$
$h_{21}x_1(1)$	$q^3(q+1)^3$
$h_{21}x_{1-2-3-4}(1)x_{1+2+3+4}(1)x_1(\tau)$	$q^2(q+1)^2$
$h_{22}x_1(1)$	$q(q+1)^3$
$h_{23}x_{3-4}(1)$	$q(q+1)^3$
$h_{25}x_2(1)$	$q^2(q^2-1)^2$
$h_{25}x_{3+4}(1)$	$q^2(q^2-1)^2$
$h_{25}x_2(1)x_{3+4}(1)$	$q^2(q^2-1)$

*) λ is a generator of $F_{q^2}^*$. $\tau, \tau_1, \tau_2 \in F_{q^2}$ and $\tau^q+\tau+1=0$. $\tau_1^q+\tau_1+\lambda^{q+1}=0$, $\tau_2^q+\tau_2+\lambda^{-q-1}=0$.

Table IV (continued)

class representative	order of centralizer
$h_{26}x_{1-2+3+4}(1)$	$q^4(q^2-1)^2$
$h_{26}x_{1-2}(1)$	$q^4(q^2-1)^2$
$h_{26}x_{1-2+3+4}(1)x_{1-2}(1)$	$q^4(q^2-1)$
$h_{26}x_{3+4}(1)x_{1-2-3-4}(1)$	$2q^2(q^2-1)$
$h_{26}x_{3+4}(1)x_{1-2-3-4}(1)x_{1-2}(\eta)$	$2q^2(q^2-1)$
$h_{27}x_{3+4}(1)$	$q(q-1)(q^2-1)$
$h_{28}x_{1+2-3-4}(1)$	$q^3(q-1)(q^2-1)$
$h_{28}x_2(1)x_{1-2-3-4}(1)$	$q^2(q^2-1)$
$h_{29}x_2(1)$	$q(q-1)(q^2-1)$
$h_{31}x_1(1)$	$q^4(q^2-1)^2$
$h_{31}x_{1+2}(1)$	$q^4(q^2-1)^2$
$h_{31}x_1(1)x_{1+2}(1)$	$q^4(q^2-1)$
$h_{31}x_{1-2}(1)x_2(1)$	$2q^2(q^2-1)$
$h_{31}x_{1-2}(1)x_2(1)x_{1+2}(\eta)$	$2q^2(q^2-1)$
$h_{32}x_1(1)$	$q^2(q^2-1)^2$
$h_{32}x_{2-3}(1)$	$q^2(q^2-1)^2$
$h_{32}x_1(1)x_{2-3}(1)$	$q^2(q^2-1)$
$h_{33}x_1(1)$	$q(q-1)(q^2-1)$
$h_{34}x_{1-3}(1)$	$q^3(q-1)(q^2-1)$
$h_{34}x_{1-2}(1)x_{2-3}(1)$	$q^2(q^2-1)$
$h_{35}x_{2-3}(1)$	$q(q-1)(q^2-1)$
$h_{37}x_1(1)$	$q(q+1)(q^2-1)$
$h_{38}x_{1-2}(1)$	$q(q+1)(q^2-1)$
$h_{40}x_{1+2}(1)$	$q^3(q+1)(q^2-1)$
$h_{40}x_{1-4}(1)x_{2+4}(1)x_{1+2}(\tau)$	$q^2(q^2-1)$
$h_{41}x_{1+2}(1)$	$q(q+1)(q^2-1)$
*) $h'_{42}x_1(1)$	$q^3(q+1)(q^2-1)$
$h'_{42}x_{1-2+3-4}(1)x_{1+2-3+4}(1)x_1(\tau)$	$q^2(q^2-1)$
$h_{43}x_3(1)$	$q(q+1)(q^2-1)$
$h_{45}x_1(1)$	$q^3(q^3-1)$
$h_{45}x_{1-2-3-4}(1)x_{1+2+3+4}(1)$	$q^2(q^2+q+1)$
$h_{46}x_1(1)$	$q(q^3-1)$
$h_{48}x_1(1)$	$q^3(q^3+1)$
$h_{48}x_{1-2+3-4}(1)x_{1+2-3+4}(1)x_1(\tau)$	$q^2(q^2-q+1)$
$h_{49}x_1(1)$	$q(q^3+1)$
$h_{53}x_1(1)$	$q^4(q^2+1)(q^2-1)$
$h_{53}x_{1+2}(1)$	$q^4(q^2+1)(q^2-1)$
$h_{53}x_1(1)x_{1+2}(1)$	$q^4(q^2+1)$

*) $h'_{42}=(1, z^{1-q}, z, z^q)$ $(z^{q^2-1}=1, z^{q-1}\neq 1)\sim h_{42}$

Table IV (continued)

class representative	order of centralizer
$h_{53}x_{1-2}(1)x_2(1)$	$2q^2(q^2+1)$
$h_{53}x_{1-2}(1)x_2(1)x_{1+2}(\eta)$	$2q^2(q^2+1)$
$h_{54}x_1(1)$	$q(q-1)(q^2+1)$
$h_{55}x_{1-2}(1)$	$q(q-1)(q^2+1)$
$h_{59}x_{1+2+3+4}(1)$	$q(q^3+q^2+q+1)$
$h_{61}x_{1+2}(1)$	$q(q+1)(q^2+1)$
$h_{66}x_{1+2}(1)$	$q^3(q^3-1)$
$h_{66}x_{1+3}(1)x_{2-3}(1)$	$q^2(q^2+q+1)$
$h_{67}x_{2-3}(1)$	$q(q^3-1)$
$h_{69}x_{1+2}(1)$	$q^3(q^3+1)$
$h_{69}x_{1+4}(1)x_{2-4}(1)x_{1+2}(\tau)$	$q^2(q^2-q+1)$
$h_{70}x_{2-4}(1)$	$q(q^3+1)$

The number of classes (or "frequency") of h_ix is equal to the number of h_i.

THEOREM 3.2. *For $h_i=h(\chi_i)$ $(i=0,1,\dots,76)$ defined in Table II, the order of $Z_i=Z_{G(w)}(h_i)$ (where $h_i\in H(w)$), the number of unipotent classes of $Z_{G(w)}(h_i)$ (this number will be denoted by m_i) and the type of Φ_{χ_i} are given in Table III. The representatives of the general classes and the orders of their centralizers are given in Table IV.*

REMARK. In general the following theorem is known.

THEOREM (Steinberg [6], 15.3). *Let $\bar{G}$ be a connected linear algebraic group defined over k. Let n be the dimmension of $\bar{G}$ and r the rank of $\bar{G}$. Then the number of unipotent elements of $G=\bar{G}(k)$ is q^{n-r}.*

Hence we see that the number of unipotent elements of Z_i $(=Z_{G(w)}(h_i))$ is $q^{|\Phi_{\chi_i}|}$. Therefore if $\{u_j^i\}_{j\in J_i}$ is a representative system of unipotent classes of Z_i, then we have the following: the number of elements of G whose semisimple parts are conjugate to h_i is equal to

$$\sum_{j\in J_i}|G|/|Z_{Z_i}(u_j^i)|=|G|/|Z_i|\sum_{j\in J_i}|Z_i|/|Z_{Z_i}(u_j^i)|=q^{|\Phi_{\chi_i}|}|G|/|Z_i|.$$

Let n_i be the number of classes of h_i in Table II, then we must have:

$$(*)\qquad |G|=\sum_{i=0}^{76}n_iq^{|\Phi_{\chi_i}|}|G|/|Z_i|.$$

In fact the validity of $(*)$ was verified with the aid of computer.

(3.5) The whole number of conjugacy classes of G is given by $\sum_{i=0}^{76}n_im_i$. Whence we have finally the following theorem.

THEOREM 3.3. *In G there are* $q^4+2q^3+6q^2+10q+19$ *conjugacy classes. Their representatives are given in Theorem* 2.1, *Theorems* 3.1 *and* 3.2.

References

[1] Borel, A., Linear Algebraic Groups, Benjamin lecture note series, New York, 1969.
[2] Borel, A. et al., Seminar on Algebraic Groups and Related Finite Groups, Springer lecture note, 131, 1970.
[3] Chang, B., The conjugate classes of Chevalley groups of type (G_2), J. Algebra **9** (1968), 190-211.
[4] Enomoto, H., The conjugacy classes of Chevalley groups of type (G_2) over finite fields of characteristic 2 or 3, J. Fac. Sci. Univ. Tokyo, Sec. I **16** (1970), 497-512.
[5] Ree, R., A family of simple groups associated with the simple Lie algebra of type (F_4), Amer. J. Math. **83** (1961), 401-420.
[6] Steinberg, R., Endomorphisms of linear algebraic groups, A.M.S. Memoirs, 80, 1968.
[7] Veldkamp, F. S., Representation of algebraic groups of type (F_4) in characteristic 2, J. Algebra **16** (1970), 326-339.

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