On the analyticity in time of solutions of initial boundary value problems for semi-linear parabolic differential equations with monotone nonlinearity

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§ 0. Introduction.

Recently a number of authors investigated the initial boundary value problem

$$
(0.1) \qquad \begin{aligned} &\frac{\partial u}{\partial t}=\Delta u+f(u)\ , \\ &u(t, x)|_{\partial\Omega}=0\ , \\ &u(0, x)=a(x)\ , \end{aligned}
$$

where Ω is a bounded domain in R^n with the smooth boundary $\partial\Omega$, assuming that the nonlinear term $f(u)$ is a monotone nonincreasing function. (For example, see Brezis [3], Brezis, Crandall and Pazy [4], Brezis and Strauss [5] and Konishi [14], [15], [16].) They studied the equation (0.1) in connection with the theory of non-linear semi-groups developed by Kōmura [12], [13], Kato [11], Crandall and Liggett [6] and others. Their result is the following:

The unique solution of the equation (0.1) exists globally in time and the mapping $S_t a=u(t, \cdot)$ has the semi-group property

$$
(0.2) \qquad \begin{aligned} &S_t S_s a=S_{t+s} a \quad (t, s\geqq 0) \\ &S_0 a=a \\ &\lim_{t\to s} S_t a=S_s a \quad \text{in} \quad X\ , \end{aligned}
$$

and contraction property

$$
\|S_t a-S_t b\|_X\leqq \|a-b\|_X\ ,
$$

where X is a suitable real Banach space with the norm $\|\cdot\|_X$, for example, $L^p(\Omega)$ $(1\leqq p<\infty)$ or, $C_0(\bar{\Omega})$ consisting of the totality of continuous functions vanishing at the boundary $\partial\Omega$ with the norm $\|g\|=\sup_{x\in\Omega}|g(x)|$.

On the other hand, for the solution of the initial boundary value problem of the linear heat equation (diffusion equation)

$$
(0.3) \qquad \begin{aligned} &\frac{\partial v}{\partial t}=\Delta v\,, \\ &v(t,x)|_{\partial\Omega}=0\,, \\ &v(0,x)=a(x)\,, \end{aligned}
$$

the following fact is well-known:

The solution $v(t,x)$ can be extended analytically in t to the sector $\Sigma_{\pi/2}=\left\{z=t+it'\,;\ |\arg z|<\frac{\pi}{2}\right\}$ in the complex domain, and the linear operator $T_z a=v(z,\cdot)$ $(z\in\Sigma_{\pi/2})$ is a holomorphic semi-group in complex Banach space $L^p(\Omega)$ $(1<p<\infty)$ or $C_0(\bar{\Omega})$ (cf. Krein [17], Yosida [20], [21]).

Now a question arises: Has the solution of the semi-linear equation (0.1) a regularity property similar to that of the solution of the linear equation (0.3)?

The purpose of this paper is to show that solutions of the initial boundary value problems for semi-linear parabolic differential equations like (0.1) are extensible holomorphically in t to a sector Σ_θ in the complex domain which does not depend on initial values, if the nonlinear term $f(u)$ is a monotone decreasing polynomial.

The content is the following:

§1. Notations and summary.

Let us now introduce some basic notations and definitions in order to state a summary of the present paper. Let Ω be a bounded domain in R^n with the smooth boundary $\partial\Omega$. We shall deal with the initial boundary value problem for $u(t,x)$ $(t\geqq 0,\ x\in\Omega)$:

$$
(1.1) \qquad \frac{\partial u}{\partial t}=Au+f(u) \quad (t>0)\,,
$$

$$
(1.2) \qquad Bu|_{\partial\Omega}=0\,,
$$

$$
(1.3) \qquad u(0,x)=a(x)\,.
$$

A is a second order elliptic differential operator of the form

$$
(1.4) \qquad Au=\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}u+\sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i}u+c(x)u\,.
$$

We assume that $a_{ij}(x)$, $b_i(x)$, $c(x)$ are real valued functions, $\in C^\alpha(\bar{\Omega})$ $(0<\alpha<1)$ and $c(x)\leqq 0$. The $a_{ij}(x)$ are symmetric, i.e. $a_{ij}(x)=a_{ji}(x)$. A is uniformly elliptic;

(1.5) $$\delta_1|\xi|^2\leqq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j\leqq\delta_2|\xi|^2 \quad (\delta_1, \delta_2>0, \xi\in R^n) ,$$

and B is one of the following boundary operators;

(1.6)

(i) Diriclet condition $Bu=u$,

(ii) Neumann condition $Bu=\dfrac{\partial u}{\partial \nu}$,

(iii) The third boundary condition $Bu=\dfrac{\partial u}{\partial \nu}+\sigma(x)u$.

Here $\partial/\partial\nu$ denotes the outward conormal derivative, and we assume that $\sigma(x)\geqq 0$ and smooth.

The nonlinear term $f(u)$ in (1.1) is a polynomial with real coefficients. The initial boundary value problem (1.1), (1.2), (1.3) is denoted by (IBVP).

REMARK 1.1. The comparison theorem for parabolic differential inequalities holds for (IBVP) under the conditions described above concerning elliptic operator A, boundary condition B and $f(u)$. More precisely, if

$$\frac{\partial u}{\partial t}\geqq Au+f(u) \quad (t>0, x\in\Omega) ,$$

$$\frac{\partial v}{\partial t}\leqq Av+f(v) \quad (t>0, x\in\Omega) ,$$

$$B(u-v)\geqq 0 \quad \text{on} \quad \partial\Omega ,$$

and $$u(0, x)\geqq v(0, x) ,$$

then $u(t, x)\geqq v(t, x)$ for $t\geqq 0$, $x\in\bar{\Omega}$ (cf. Friedman [8]).

Some definitions concerning $f(u)$ will be given.

DEFINITION 1.2. A polynomial with real coefficients $f(u)$ is said to be monotone or to satisfy condition (M), if $f(0)=0$ and $f'(u)\leqq 0$ for $-\infty<u<+\infty$.

Examples. $f(u)=-u^{2p+1}$, $-u-u^3$, $-u^3-u^5$.

DEFINITION 1.3. A polynomial with real coefficients $f(u)$ is said to be monotone on $R_+=[0,\infty)$ or to satisfy condition (M_+), if $f(u)=0$ and $f'(u)\leqq 0$ for $0\leqq u<\infty$.

Examples. $f(u)=-u^{2p}$, $-u-u^4$, $-u^2-u^6$.

Let $U(t, x, y)$ be the Green's function of the linear initial boundary value problem

$$\frac{\partial u}{\partial t}=Au \qquad t>0,\ x\in\Omega\ , \tag{1.7}$$

$$Bu=0 \qquad t>0,\ x\in\partial\Omega\ , \tag{1.8}$$

$$u(0,x)=\delta(x-y) \quad (\delta\text{-measure at } x=y)\ . \tag{1.9}$$

Under our standing assumptions $U(t, x, y)$ exists. The following fact holds for the linear system of equations (1.7), (1.8), (1.9): Set $z=t+it'$.

(1.10) (i) $U(t, x, y)$ can be extended holomorphically in t to the domain $\Sigma_{\pi/2}=\{z;\ \mathrm{Re}\, z>0\}$.

We denote the holomorphic extension of $U(t, x, y)$ by $U(z, x, y)$.

(ii) Let $h(z, x)$ be a boundedly continuous function defined on $\Sigma_\theta=\left\{z;\ |\arg z|<\theta<\frac{\pi}{2}\right\}\times\Omega$, holomorphic in z ($z\in\Sigma_\theta$) and locally Hölder continuous in x uniformly on any compact set in Σ_θ, and let $g(x)\in L^p(\Omega)$ $(1<p<\infty)$. Then

$$u(z,x)=\int_\Omega U(z,x,y)g(y)dy+\int_0^z d\tau\int_\Omega U(\tau,x,y)h(z,y)dy \tag{1.11}$$

satisfies

$$\frac{\partial u}{\partial z}=Au+h(z,x) \qquad z\in\Sigma_\theta,\ x\in\Omega\ , \tag{1.12}$$

$$Bu|_{\partial\Omega}=0 \qquad z\in\Sigma_\theta,\ x\in\partial\Omega\ , \tag{1.13}$$

$$\lim_{t\to+0}\|u(t,\cdot)-g\|_{L^p(\Omega)}=0 \tag{1.14}$$

(cf. Friedman [8], Krein [17], Tanabe [19] and Yosida [20]).

DEFINITION 1.4. The Green's function of the linear system of equations (1.7), (1.8), (1.9) has property (U.B.) or said to be uniformly bounded, if for every boundedly continuous function $g(x)$, there exist constants $K>0$ and $0<\theta<\frac{\pi}{2}$ which are independent of $g(x)$ such that

$$\left|\int_\Omega U(z,x,y)g(y)dy\right|\leqq K\|g\|_\infty$$

for $z\in\Sigma_\theta=\{z;\ |\arg z|<\theta\}$, where $\|g\|_\infty=\sup_{x\in\Omega}|g(x)|$.

DEFINITION 1.5. $u(t, x)$ is said to be a solution of (IBVP) in $0\leqq t<T$ $(T>0)$, if the following conditions are satisfied:

(i) u is continuous and bounded in $(0, T)\times\bar{\Omega}$. At $t=0$, $\lim_{t\to+0}u(t,x)=a(x)$, $x\in\Omega$.

(ii) $\frac{\partial u}{\partial x_i}$, $i=1,\cdots,n$, is continuous in $(0, T)\times\bar{\Omega}$ and the boundary condition (1.2) is satisfied on $\partial\Omega$.

(iii) $u(t, x)$ is once continuously differentiable in t and twice continuously differentiable in x in $(0, T)\times\Omega$ and the equation (1.1) is satisfied there.

THEOREM 1.6. *Suppose that the nonlinear term $f(u)$ in (IBVP) satisfies condition (M) and the initial value $a=a(x)$ is real valued and boundedly continuous in Ω, and that the Green's function $U(z, x, y)$ of the linear system of equations (1.7), (1.8), (1.9) has property (U.B.). Then there exists a sector $\Sigma_{\theta_0}=\{z\,;\ |\arg z|<\theta_0\}$ in the complex domain which is independent of $a(x)$ such that the solution $u(t, x)$ of (IBVP) is analytically extensible in t to the sector Σ_{θ_0}.*

THEOREM 1.7. *Suppose that the nonlinear term $f(u)$ in (IBVP) satisfies condition (M_+) and the initial value $a=a(x)$ is a nonnegative and boundedly continuous function in x, and that the Green's function $U(z, x, y)$ of linear system of equations (1.7), (1.8), (1.9) has property (U.B.). Then there is a sector $\Sigma_{\theta_0}=\{z\,;\ |\arg z|<\theta_0\}$ in the complex domain which does not depend on $a(x)$ such that the nonnegative solution $u(t, x)$ of (IBVP) is analytically extensible in t to the sector Σ_{θ_0}.*

§ 2. Monotone polynomials.

In this section we shall give lemmas concerning polynomials with real coefficients satisfying condition (M) or (M_+) as a preparation for the next section.

LEMMA 2.1. *Let a polynomial $f(u)$ satisfy condition (M) and $f(u)\not\equiv 0$. Then the followings hold:*

(i) *There exist nonnegative integers k, h and a positive constant K such that*

$$|f(u)|\leqq K|u|^{2k+1}(1+|u|^{2h}) \quad \textit{for all } u\in \boldsymbol{C}.$$

(ii) *There is a positive constant L such that if $u\geqq 0$, then $f(u)\leqq -Lu^{2k+1}\times(1+u^{2h})$, and if $u\leqq 0$, then $f(u)\geqq -Lu^{2k+1}(1+u^{2h})$.*

PROOF. Since $f(0)=0$ and $f(u)\not\equiv 0$, we can put

$$f(u)=-u^s(b_0+b_1u+\cdots+b_tu^t)\,,$$

where $s\geqq 1$ and b_t are all real, $b_0\neq 0$, $b_t\neq 0$. Set

$$g(u)=b_0+b_1u+\cdots+b_tu^t\,.$$

Since $f(0)=0$ and $f'(u)\leqq 0$ for $u\in(-\infty, +\infty)$, we have

$$f(u)<0 \text{ for } u>0 \text{ and } \lim_{u\to\infty} f(u)=-\infty\,, \tag{2.1}$$

$$f(u)>0 \text{ for } u<0 \text{ and } \lim_{u\to-\infty} f(u)=+\infty\,, \tag{2.2}$$

$$g(u) \neq 0 \text{ for } -\infty < u < +\infty . \tag{2.3}$$

Hence from (2.1), (2.2), (2.3) it follows immediately that $b_t > 0$ and $g(u) > 0$ for $-\infty < u < +\infty$, so $t = 2h$ (an even integer) and $s = 2k+1$ (an odd integer). Thus we see

$$f(u) = -u^{2k+1}(b_0 + b_1 u + \cdots + b_{2h} u^{2h}) \tag{2.4}$$

and

$$g(u) = b_0 + b_1 u + \cdots + b_{2h} u^{2h} , \tag{2.5}$$

where b_0, $b_{2h} > 0$. From (2.4) $|f(u)| \leqq K|u|^{2k+1}(1+|u|^{2h})$ for some constant K. Since $g(u) > 0$ for $-\infty < u < +\infty$, it follows easily from (2.5) that there exists a constant L such that $g(u) \geqq L(1+u^{2h})$ for $-\infty < u < +\infty$. This implies (ii).

LEMMA 2.2. *Let a polynomial $f(u)$ satisfy condition (M_+) and $f(u) \not\equiv 0$. Then followings hold:*

(i) *There exist nonnegative integers k, h and a positive constant K such that*

$$|f(u)| \leqq K|u|^{k+1}(1+|u|^h) \textit{ for all } u \in \boldsymbol{C} .$$

(ii) *There exists a positive constant L such that if $u \geqq 0$, then $f(u) \leqq -Lu^{k+1}(1+u^h)$.*

PROOF. From the assumptions we can put

$$f(u) = -u^{k+1}(b_0 + b_1 u + \cdots + b_h u^h) , \tag{2.6}$$

where b_0, $b_h \neq 0$ and k, h are nonnegative integers. Set $g(u) = b_0 + b_1 u + \cdots + b_h u^h$. We can easily show that $g(u) > 0$, if $u \geqq 0$. Lemma 2.2 follows immediately from this fact.

§3. Ordinary differential equations with monotone nonlinearity.

The purpose of this section is to obtain estimates of the solution of the initial value problem for the ordinary differential equation;

$$\begin{aligned} &\frac{du}{dt} = f(u) \quad (t > 0) , \\ &u(0) = a \quad (a \in \boldsymbol{R}) , \end{aligned} \tag{3.1}$$

where $f(u)$ is a polynomial satisfying condition (M) or (M_+).

First we assume that $f(u)$ $(\not\equiv 0)$ in (3.1) satisfies condition (M). Then from Lemma 2. 1

$$|f(u)| \leqq K|u|^{2k+1}(1+|u|^{2h}) \quad \text{for all} \quad u \in \boldsymbol{C} ,$$
$$f(u) \leqq -Lu^{2k+1}(1+u^{2h}) \quad \text{for} \quad u \geqq 0 , \tag{3.2}$$
$$f(u) \geqq -Lu^{2k+1}(1+u^{2h}) \quad \text{for} \quad u \leqq 0 .$$

PROPOSITION 3.1. *Suppose that $f(u)$ satisfies condition (M) and is not a linear function. Then for the solution $u(t)$ of (3.1) the following estimate holds:*

(i) *if* $f'(0)=0$ $(k \geqq 1)$,

$$|u(t)| \leqq C \min (t^{-1/2k}, t^{-1/2(k+h)}) , \tag{3.3}$$

(ii) *if* $f'(0) \neq 0$ $(k=0,\ h \geqq 1)$,

$$|u(t)| \leqq \min (e^{-Lt}|a|, Ct^{-1/2h}) ,$$

where C is independent of a.

PROOF. Let $f'(0)=0$ $(k \geqq 1)$. First we assume $a \geqq 0$. Then $u(t) \geqq 0$, because $f(0)=0$. By (3.2), if $u \geqq 0$,

$$f(u) \leqq -Lu^{2k+1}(1+u^{2h}) \leqq -Lu^{2k+1} . \tag{3.4}$$

Now consider the equation

$$\frac{dv}{dt} = -Lv^{2k+1} , \quad v(0)=a . \tag{3.5}$$

The solution of (3.5) is $v(t)=a(1+a^{2k}2kLt)^{-1/2k}$. From (3.4) and the comparison theorem for ordinary differential equations we have

$$0 \leqq u(t) \leqq v(t) . \tag{3.6}$$

Since $v(t)=a(1+a^{2k}2kLt)^{-1/2k}<(2kL)^{-1/2k}t^{-1/2k}$, by (3.6) $u(t)<(2kL)^{-1/2k}t^{-1/2k}$.

Next let $a<0$. Then $u(t)<0$. From (3.2), if $u<0$,

$$f(u) \geqq -Lu^{2k+1}(1+u^{2h}) \geqq -Lu^{2k+1} . \tag{3.7}$$

Hence comparing with $v(t)$,

$$0 \geqq u(t) \geqq a(1+a^{2k}2kLt)^{-1/2k} > -(2kL)^{-1/2k}t^{-1/2k} . \tag{3.8}$$

Thus

$$|u(t)| \leqq (2kL)^{-1/2k}t^{-1/2k} . \tag{3.9}$$

On the other hand, $f(u) \leqq -Lu^{2(k+h)+1}$ for $u \geqq 0$ and $f(u) \geqq -Lu^{2(k+h)+1}$ for $u \leqq 0$, so in the same way as above we get $|u(t)| \leqq (2L(k+h))^{-1/2(k+h)}t^{-1/2(k+h)}$. Therefore,

$$|u(t)| \leqq C \min (t^{-1/2k}, t^{-1/2(k+h)}) , \tag{3.10}$$

where $C = \max((2kL)^{-1/2k}, (2L(k+h))^{-1/2(k+h)})$.

We can show (3.3) (ii) analogously.

Next we assume that $f(u)$ $(\neq 0)$ in (3.1) satisfies condition (M_+). By Lemma 2.2 we have

$$\begin{cases} |f(u)| \leqq K|u|^k(1+|u|^h) & \text{for all} \quad u \in \boldsymbol{C}\,, \\ f(u) \leqq -Lu^{k+1}(1+u^h) & \text{for all} \quad u \geqq 0\,, \end{cases} \tag{3.11}$$

where k, h are nonnegative integers and K, $L>0$.

PROPOSITION 3.2. *Suppose that $f(u)$ satisfies condition (M_+) and is not a linear function. If the initial value a is nonnegative, then for the solution $u(t)$ of (3.1) the following estimate holds:*

(i) *if* $f'(0)=0$ $(k \geqq 1)$,

$$0 \leqq u(t) \leqq C \min(t^{-1/k}, t^{-1/(k+h)})\,, \tag{3.12}$$

(ii) *if* $f'(0) \neq 0$ $(k=0, h \geqq 1)$

$$0 \leqq u(t) \leqq \min(e^{-Lt}|a|, Ct^{-1/h})\,,$$

C being independent of a.

PROOF. First we assume $f'(0)=0$ $(k \geqq 1)$. Since $a \geqq 0$ and $f(0)=0$, $u(t) \geqq 0$. From (3.11)

$$f(u) \leqq -Lu^{k+1}(1+u^h) \leqq -Lu^{k+1}\,, \tag{3.13}$$

and

$$f(u) \leqq -Lu^{k+h+1} \quad \text{for} \quad u \geqq 0\,. \tag{3.14}$$

Let us consider auxiliary equations as in the proof of Proposition 3.1;

$$\frac{dv}{dt} = -Lv^{k+1} \qquad v(0)=a\,, \tag{3.15}$$

and

$$\frac{dv}{dt} = -Lv^{k+h+1} \qquad v(0)=a\,. \tag{3.16}$$

The solution of (3.15) ((3.16)) is $v(t)=a(1+a^k kLt)^{-1/k}$ (resp. $a(1+a^{k+h+1}(k+h)Lt)^{-1/(k+h)}$). Making use of the comparison theorem for ordinary differential equations, we have

$$0 \leqq u(t) \leqq a(1+a^k kLt)^{-1/k} < Ct^{-1/k}\,, \tag{3.17}$$

$$0 \leqq u(t) \leqq a(1+a^{(k+h)}(k+h)Lt)^{-1/(k+h)} < Ct^{-1/(k+h)} \tag{3.18}$$

where $C = \max((kL)^{-1/k}, ((k+h)L)^{-1(k+h)})$.

Thus from (3.17) and (3.18) we have

$$0 \leqq u(t) \leqq C \min (t^{-1/k}, t^{-1/(k+h)}) . \tag{3.19}$$

The proof of (3.12) (ii) is the same as above.

§ 4. Construction of a local (in time) solution.

In this section we shall construct a holomorphic solution locally in time by the iteration method.

Now we consider the initial boundary value problem with the initial time $t=\tau$ $(\tau \geqq 0)$

$$(\mathrm{IBVP})_\tau \qquad \begin{cases} \dfrac{\partial u}{\partial t} = Au + f(u) , \\ Bu = 0 , \\ u(\tau, x) = a(x) . \end{cases}$$

PROPOSITION 4.1. *In $(IBVP)_\tau$ assume that*

(i) $a(x)$ is a boundedly continuous function,

(ii) $f(u)$ is a polynomial such that

$$|f(u)| \leqq K|u|^{p+1}(1+|u|^q) \quad (p \geqq 0, q \geqq 0) ,$$

(iii) the Green's function $U(z, x, y)$ of the linear equation has property (U.B.) (cf. Definition 1.4), that is, for every boundedly continuous function $g(x)$ there are constants M, θ $\left(0<\theta<\frac{\pi}{2}\right)$ such that

$$\sup_{z \in \Sigma_\theta, x \in \Omega} \left| \int_\Omega U(z, x, y) g(y) dy \right| \leqq M \sup_{x \in \Omega} |g(x)| = M \|g\|_\infty , \tag{4.1}$$

where $\Sigma_\theta = \{z = t+it' ; |\arg z| < \theta\}$.

Then there exists a solution $u(z, x)$ of $(IBVP)_\tau$ which is holomorphic in time in $\Delta_\tau = \left\{z = t+it' ; |\arg (z-\tau)| < \theta, |z-\tau| \leqq \dfrac{1}{A\|a\|_\infty^p (1+B\|a\|_\infty^q)}\right\}$ where $A = K(2M)^{p+1}$ and $B = (2M)^q$.

PROOF. $(\mathrm{IBVP})_\tau$ is transformed to the integral equation:

$$u(z, x) = \int_\Omega U(z-\tau, x, y) a(y) dy + \int_\tau^z ds \int_\Omega U(z-s, x, y) f(u(s, y)) dy . \tag{4.2}$$

We shall solve this integral equation (4.2) by successive approximation:

$$\begin{cases} u_0(z, x) = \displaystyle\int_\Omega U(z-\tau, x, y) a(y) dy , \\ u_{n+1}(z, x) = u_0(z, x) + \displaystyle\int_\tau^z ds \int_\Omega U(z-s, x, y) f(u_n(s, y)) dy \quad (n=0, 1, 2, \cdots) . \end{cases} \tag{4.3}$$

Let us show that $u_n(z, x)$ is holomorphic in z and

$$\|u_n(z, \cdot)\|_\infty \leqq 2M\|a\|_\infty , \tag{4.4}$$

for $z \in \Lambda_\tau$ by induction on n. Obviously $u_0(z, x)$ is a holomorphic function of z and by the assumption (iii)

$$\|u_0(z, \cdot)\|_\infty \leqq M\|a\|_\infty \quad (z \in \Lambda_\tau) . \tag{4.5}$$

Assume that $u_n(z, x)$ is holomorphic in z $(z \in \Lambda_\tau)$ and

$$\|u_n(z, \cdot)\| \leqq 2M\|a\|_\infty \quad (z \in \Lambda_\tau) .$$

Clearly $u_{n+1}(z, x)$ is holomorphic in z for $z \in \Lambda_\tau$. From (4.3) we have

$$\|u_{n+1}(z, \cdot)\|_\infty \leqq M\|a\|_\infty + |z-\tau| KM(2M\|a\|_\infty)^{p+1}(1+(2M\|a\|_\infty)^q) . \tag{4.6}$$

Since

$$|z-\tau| \leqq \frac{1}{A\|a\|_\infty^p(1+B\|a\|_\infty^q)} ,$$

where

$$A = K(2M)^{p+1} , \qquad B = (2M)^q ,$$

we have

$$\|u_{n+1}(z, \cdot)\|_\infty \leqq 2M\|a\|_\infty \quad (z \in \Lambda_\tau) . \tag{4.7}$$

We now consider the equality

$$u_{n+1}(z, x) - u_n(z, x) = \int_\tau^z ds \int_\Omega U(z-s, x, y)\{f(u_n(s, y)) - f(u_{n-1}(s, y))\} dy . \tag{4.8}$$

Since $\|u_n(z, \cdot)\|_\infty \leqq 2M\|a\|_\infty$ $(z \in \Lambda_\tau)$, there is a constant H depending on $\|a\|_\infty$ such that

$$\|f(u_n(s, \cdot)) - f(u_{n-1}(s, \cdot))\|_\infty \leqq H\|u_n(s, \cdot) - u_{n-1}(s, \cdot)\|_\infty , \tag{4.9}$$

for $s \in \Lambda_\tau$ and $n = 1, 2, 3, \cdots$. We can derive from (4.8)

$$\|u_{n+1}(z, \cdot) - u_n(z, \cdot)\|_\infty \leqq \int_\tau^z MH\|u_n(s, \cdot) - u_{n-1}(s, \cdot)\|_\infty |ds| . \tag{4.10}$$

We shall prove

$$\|u_{n+1}(z, \cdot) - u_n(z, \cdot)\|_\infty \leqq \frac{|z-\tau|^n}{n!}(MH)^n \gamma , \tag{4.11}$$

where $\gamma = \sup_{z \in \Lambda_\tau} \|u_1(z, \cdot) - u_0(z, \cdot)\|_\infty$. In the case $n=1$ (4.11) obviously holds. If (4.11) is true for $n = m-1$, then from (4.10)

$$\|u_{m+1}(z,\cdot)-u_m(z,\cdot)\| \leqq \int_\tau^z (MH)^m \frac{|s-\tau|^{m-1}}{(m-1)!}\gamma|ds| \tag{4.12}$$
$$\leqq (MH)^m \frac{|z-\tau|^m}{m!}\gamma .$$

In view of (4.11) it is clear that $\sum_{n=0}^{\infty}\|u_{n+1}(z,\cdot)-u_n(z,\cdot)\|_\infty$ converges. This implies that $u_n(z, x)$ converges $u(z, x)$ uniformly in Λ_τ. By making $n\to\infty$ in (4.3), $u(z, x)$ satisfies (4.2).

§ 5. Global solutions of (IBVP).

In this section we shall prove Theorem 1.6 and Theorem 1.7. First let us assume that $f(u)$ in (IBVP) satisfies condition (M). First of all we shall establish the uniqueness and global existence of the solution of (IBVP).

PROPOSITION 5.1. *Assume that the initial value $a(x)$ is boundedly continuous. Then the solution of (IBVP) is unique. Furthermore, if $a(x)$ is real-valued, then the solution exists globally in time.*

PROOF. Since $f(u)$ is a locally Lipschitz continuous function, it is obvious by the standard argument that the solution of IBVP is unique (cf. Segal [18]). Existence of the local solution is guaranteed by Proposition 4.1. In Proposition 4.1 the interval in which the solution exists depends only on the norm $\|a\|_\infty$. Hence in order to extend the solution $u(t, x)$ to the interval $[0, \infty)$ it is sufficient to obtain an a priori estimate, that is, to estimate $\|u(t,\cdot)\|_\infty$ in terms of $\|a\|_\infty$ (cf. Segal [18]). Now $f(u)$ satisfies condition (M), so we can put

$$f(u)=-uh(u) . \tag{5.1}$$

Here $h(u)\geqq 0$. Therefore the equation (1.1) has the form

$$\frac{\partial u}{\partial t}=Au-q(t, x)u , \tag{5.2}$$

where $q(t, x)=h(u(t, x))\geqq 0$. Making use of the comparison theorem we have $|u(t, x)| \leqq v(t, x)$, where $v(t, x)$ is the solution of the linear equation

$$\begin{cases} \dfrac{\partial v}{\partial t}=Av , \\ Bv=0 , \\ v(0, x)=|a(x)| . \end{cases} \tag{5.3}$$

For $v(t, x)$ the estimate

$$0\leqq v(t, x)\leqq \exp(ct)\|a\|_\infty \tag{5.4}$$

holds, where $c=\sup_{x\in\Omega} c(x)$ (cf. Ito [10]).

Thus we have the desired a priori estimate

$$\|u(t,\cdot)\|_\infty \leqq \exp(ct)\|a\|_\infty . \tag{5.5}$$

REMARK 5.2. For a treatment for (IBVP) in the space of continuous functions with the aid of the theory of nonlinear semi-groups we refer to Konishi [15].

We shall give an estimate for the solution $u(t,x)$ which is important for proofs of main theorems. In the following k, h, C, and L are those in Lemma 2.1 and Proposition 3.1.

THEOREM 5.3. *In (IBVP), assume that $a(x)$ is real-valued and boundedly continuous and $f(u)$ is not a linear function. Then we have*

(i) *if $f'(0)=0$, i.e. $k\geqq 1$, then*

$$|u(t,x)|\leqq C\min(t^{-1/2k}, t^{-1/2(k+h)}) , \tag{5.6}$$

and

(ii) *if $f'(0)\neq 0$, i.e. $k=0$, $h\geqq 1$, then*

$$|u(t,x)|\leqq \min(Ct^{-1/2h}, e^{-Lt}\|a\|_\infty) . \tag{5.7}$$

PROOF. Consider the ordinary differential equation for $w(t)$

$$\frac{dw}{dt}=f(w) \qquad w(0)=\|a\|_\infty . \tag{5.8}$$

Since $c(x)\leqq 0$ and $\sigma(x)\geqq 0$, $w(t)$ satisfies the differential inequality

$$\begin{cases} \dfrac{\partial w}{\partial t}\geqq Aw+f(w) , \\ Bw\geqq 0 . \end{cases} \tag{5.9}$$

Therefore, by applying the comparison theorem for parabolic differential inequalities we have

$$|u(t,x)|\leqq w(t) \tag{5.10}$$

because of $|u(0,x)|=|a(x)|\leqq\|a\|_\infty$. Consequently, by Proposition 3.1, this implies (i) and (ii).

Now we are going to prove Theorem 1.6.

PROOF OF THEOREM 1.6. We notice that by Proposition 5.1 there exists the unique solution $u(t,x)$ on the nonnegative interval $R^+=[0,\infty)$. When $f(u)$ is a linear function, the statement of Theorem 1.6 is clear. So we may assume that $f(u)$ is not a linear function.

Consider the initial boundary value problem $(\mathrm{IBVP})_\tau$ with the initial value $u(\tau, x)$ at $t=\tau$. According to Proposition 4.1 it follows from Lemma 2.1 (i) that there is the unique solution $u(z, x)$ which is holomorphic in z in

$$(5.11) \qquad \Lambda_\tau=\left\{z\,;\ |\arg(z-\tau)|<\theta,\ |z-\tau|\leqq \frac{1}{A\|u(\tau,\cdot)\|_\infty^{2k}(1+B\|u(\tau,\cdot)\|_\infty^{2h})}\right\}.$$

Of course $u(z, x)$ is coincident with $u(t, x)$ on $[0, \infty)$ because of the uniqueness.

In the following we shall give the proof, dividing two cases.

(i) The case $f'(0)=0$ $(k\geqq 1)$. According to Theorem 5.3

$$|u(\tau, x)|\leqq C\tau^{-1/2k}.$$

Therefore the domain

$$(5.12) \qquad \tilde{\Lambda}_\tau=\left\{z\,;\ |\arg(z-\tau)|<\theta,\ |z-\tau|\leqq \frac{1}{A_1\tau^{-1}(1+B_1\tau^{-h/k})}\right\},$$

where $A_1=AC^{2k}$ and $B_1=BC^{2h}$, is contained in Λ_τ. This implies that there is a constant δ_1 such that when $\tau\geqq 1$, the domain

$$(5.13) \qquad \{z\,;\ |\arg(z-\tau)|<\theta,\ |z-\tau|\leqq\delta_1\tau\}$$

is contained in Λ_τ. On the other hand by Theorem 5.3

$$|u(\tau, x)|\leqq C\tau^{-1/2(k+h)}.$$

So $u(z, x)$ exists in

$$(5.14) \qquad \left\{z\,;\ |\arg(z-\tau)|<\theta,\ |z-\tau|\leqq \frac{1}{A_1\tau^{-k/(h+k)}(1+B_1\tau^{-h/(k+h)})}\right\}.$$

Hence $u(z, x)$ exists, if $\tau\leqq 1$, in the domain

$$(5.15) \qquad \{z\,;\ |\arg(z-\tau)|<\theta,\ |z-\tau|\leqq\delta_2\tau\}.$$

Thus $u(t, x)$ is extensible holomorphically in t to the domain

$$(5.16) \qquad \{z\,;\ |\arg(z-\tau)|<\theta,\ |z-\tau|\leqq\delta\tau\},$$

δ being $\min(\delta_1, \delta_2)$. It is easy to see that

$$(5.17) \qquad \bigcup_{\tau\geqq 0}\{z\,;\ |\arg(z-\tau)|<\theta,\ |z-\tau|\leqq\delta\tau\},$$

contains a sector

$$(5.18) \qquad \Sigma_{\theta_0}=\{z=t+it'\,;\ |\arg z|<\theta_0,\ 0<\theta_0<\theta\}.$$

(ii) The case $f'(0)\neq 0$. As a preparation for the proof for this case, we state a simple lemma.

LEMMA 5.4. *Let $U(t, x, y)$ be the Green's function of the linear system of equations (1.7), (1.8), (1.9) and q be a positive number. Suppose that $U(t, x, y)$ has property (U.B.). Then*

$$\tilde{U}(t, x, y) = U(t, x, y)e^{-qt} , \tag{5.19}$$

is the Green's function of the linear system of equations

$$\begin{cases} \dfrac{\partial u}{\partial t} = Au - qu = \tilde{A}u , \\ Bu = 0 , \\ u(t, x) = \delta(x-y) , \end{cases} \tag{5.20}$$

and for every bounded continuous function $g(x)$

$$\left|\int_0^z ds \int_\Omega \tilde{U}(s, x, y)g(y)dy\right| \leqq N\|g\|_\infty \qquad (|\arg z| < \theta) , \tag{5.21}$$

N being a positive constant which is independent of z and $g(x)$.

PROOF. The first statement is clear. Since $U(t, x, y)$ has property (U.B.) (cf. Definition 1.4),

$$\left|\int_0^z ds \int_\Omega \tilde{U}(s, x, y)g(y)dy\right| \leqq \int_0^z Me^{-q(\mathrm{Re}\, s)}\|g\|_\infty |ds| \leqq N\|g\|_\infty .$$

Now we shall proceed to the proof of Theorem 1.6 for the case $f'(0) \neq 0$. Put $h(u) = qu + f(u)$. Here $q = -f'(0)$ $(q > 0)$. So $h(0) = h'(0) = 0$ and

$$|h(u)| \leqq \tilde{K}(|u|^r + |u|^{2h+1}) \quad (r \geqq 2,\ h \geqq 1) , \tag{5.22}$$

$(\mathrm{IBVP})_\tau$ with the initial value $u(\tau, x)$ is rewritten into the form:

$$\begin{cases} \dfrac{\partial u}{\partial t} = Au - qu + (qu + f(u)) = \tilde{A}u + h(u) , \\ Bu = 0 . \end{cases} \tag{5.23}$$

(5.23) is transformed to the integral equation:

$$u(z, x) = \int_\Omega \tilde{U}(z-\tau, x, y)u(\tau, y)dy + \int_\tau^z ds \int_\Omega \tilde{U}(s-\tau, x, y)h(u(s, y))dy . \tag{5.24}$$

We shall solve (5.24) by the iteration. The iteration procedure is the same as in the proof of Proposition 4.1:

$$u_0(z, x) = \int_\Omega \tilde{U}(z-\tau, x, y)u(\tau, y)dy , \tag{5.25}$$

$$u_{n+1}(z, x) = u_0(z, x) + \int_\tau^z ds \int_\Omega \tilde{U}(s-\tau, x, y)h(u_n(s, y))dy . \tag{5.26}$$

We now verify that when $|\arg(z-\tau)|<\theta$,

$$|u_n(z, x)|\leqq 2MC\tau^{-1/2h} \tag{5.27}$$

for sufficiently large τ. Obviously by Theorem 5.3 $|u_0(z, x)|\leqq MC\tau^{-1/2h}$. Assuming (5.27) for n, we have from (5.26)

$$\begin{aligned}|u_{n+1}(z, x)|&\leqq MC\tau^{-1/2h}+\left|\int_\tau^z ds\int_\Omega \tilde{U}(s-\tau, x, y)h(u_n(s, y))dy\right| \\ &\leqq MC\tau^{-1/2h}+N\tilde{K}((2MC)^r\tau^{(2-r)/2h}+(2MC)^{2h+1}\tau^{(1-2h)/2h})\tau^{-1/h}\ ,\end{aligned} \tag{5.28}$$

making use of (5.22) and Lemma 5.4. For sufficiently large τ $(\tau\geqq\tau_0)$

$$N\tilde{K}((2MC)^r\tau^{(2-r)/2h}+(2MC)^{2h+1}\tau^{(1-2h)/2h})\tau^{-1/h}\leqq MC\tau^{-1/2h}\ . \tag{5.29}$$

Thus we have (5.27) for $z\in\Sigma_{\theta,\tau}$,

$$\Sigma_{\theta,\tau}=\{z,\ |\arg(z-\tau)|<\theta\}\quad(\tau\geqq\tau_0)\ . \tag{5.30}$$

By means of (5.27) and by an argument similar to the proof of Proposition 4.1, we can show that $(IBVP)_{\tau_0}$ with the initial value $u(\tau_0, x)$ exist in Σ_{θ,τ_0}.

For $\tau\leqq\tau_0$, using the estimate of $u(\tau, x)$ we can prove that $u(t, x)$ is extensible holomorphically in t to the domain

$$\{z;\ |\arg(z-\tau)|<\theta,\ |z-\tau|\leqq\tilde{\delta}\tau\}\quad(\tau\leqq\tau_0) \tag{5.31}$$

in the same way as for the case $f'(0)=0$. Therefore the solution of (IBVP) is holomorphic in t in a sector. Theorem 1.6 is proved.

Next let us assume that $f(u)$ in (IBVP) satisfies condition (M_+). Corresponding to Proposition 5.1, we have

PROPOSITION 5.5. *In addition to the assumption as above, suppose that the initial value $a(x)$ is boundedly continuous and nonnegative. Then the unique solution of $(IBVP)$ exists globally in time.*

PROOF. Since $f(0)=0$ and $f'(u)\leqq 0$ for $u\geqq 0$,

$$0\leqq u(t, x)\leqq\|a\|_\infty\ . \tag{5.32}$$

Noting (5.32), we can show the proposition by an argument similar to the proof of Proposition 5.1.

THEOREM 5.6. *In addition to the assumptions of Proposition 5.5, suppose that $f(u)$ is not a linear function. Then we have*

(i) *if $f'(0)=0$, i.e. $k\geqq 1$, then*

$$0\leqq u(t, x)\leqq C\min(t^{-1/k}, t^{-1/(k+h)})\ , \tag{5.33}$$

and

(ii) *if* $f'(0)\neq 0$, *i.e.* $k=0$, $h\geqq 1$, *then*

$$0\leqq u(t,x)\leqq \min\left(Ct^{-1/h},\ e^{-Lt}\|a\|_\infty\right). \tag{5.34}$$

Here k, h, C *and* L *are those in Proposition* 3.2.

The inequalities (5.33) and (5.34) can be obtained in the same way as for Theorem 5.3, so we omit the proof.

We can prove Theorem 1.7 in the same way as for Theorem 1.6 by making use of Propositions 4.1, 5.5 and Theorem 5.6.

§ 6. Examples.

In this section we shall give sufficient conditions for the elliptic operator A and boundary condition B under which Green's function of the linear system of equations (1.7), (1.8), (1.9) has property (U.B.).

Now let us consider the following boundary value problem for $u(x)$:

$$\text{(BVP)}\qquad \begin{cases} (\lambda-A)u=v(x)\ , \\ Bu=0\ . \end{cases}$$

The following facts concerning (BVP) are well-known: The operator A with the domain $D(A)$,

$$D(A)=\{u\ ;\ u\in W^{2,2}(\Omega),\ Bu=0\}\ ,$$

$W^{2,2}(\Omega)$ being the Sobolev space of order 2, is a closed operator in $L^2(\Omega)$. The resolvent kernel of (BVP) $G_\lambda(x,y)$ exists for λ outside the spectrum $\sigma(A)$, and

$$\sigma(A)\subset\{\lambda\ ;\ \mathrm{Re}\,\lambda\leqq-(\mathrm{Im}\,\lambda-a)^2+b\}\quad \text{for some real } a,\ b. \tag{6.1}$$

PROPOSITION 6.1. *Suppose that one of conditions (i), (ii), (iii) below, concerning the elliptic operator A or the boundary condition B holds*:

(i) $c(x)\leqq -c_0<0$,

(ii) $Bu=u$ *(the Dirichlet condition)*,

(iii) $Bu=\dfrac{\partial}{\partial\nu}u+\sigma(x)u\ ,\qquad \sigma(x)\geqq\sigma_0>0\ .$

Then the Green's function $U(z,x,y)$ *of the linear equations (1.7), (1.8), (1.9) has property (U.B.), i.e. is uniformly bounded.*

In order to prove Proposition 6.1 we give lemmas. Set for a boundedly continuous function $g(x)$

$$T_z g=\int_\Omega U(z,x,y)g(y)dy\quad (z=t+it')\ . \tag{6.2}$$

LEMMA 6.2. *Assume that one of the conditions (i), (ii), (iii) in Proposition*

6.1 holds. Then there are positive constants M and ω such that

(6.3) $$\|T_t g\|_\infty \leqq M e^{-\omega t}\|g\|_\infty .$$

For proof we refer to ch. 6 in Friedman [8].

LEMMA 6.3. *Under the same assumptions as in Lemma 6.2 there exist $a<0$ and $0<\alpha<\frac{\pi}{2}$ such that*

$$\sigma(A)\subset\left\{\lambda\,;\ \pi\geqq|\arg(\lambda-a)|\geqq\alpha+\frac{\pi}{2}\right\}.$$

PROOF. From (6.3) for $\operatorname{Re}\lambda>-\omega$ we have

(6.4) $$\begin{aligned}\int_\Omega G_\lambda(x, y)g(y)dy&=\int_0^\infty e^{-\lambda t}T_t g dt\\&=\int_0^\infty e^{-\lambda t}dt\int_\Omega U(t, x, y)g(y)dy .\end{aligned}$$

Hence, the resolvent kernel $G_\lambda(x, y)$ exists for $\operatorname{Re}\lambda>-\omega$. On the other hand $\sigma(A)$ is discrete. Therefore, from (6.1) the statement of Lemma 6.3 easily follows.

PROOF OF PROPOSITION 6.1. The Green's function $U(z, x, y)$ is represented by the resolvent kernel $G_\lambda(x, y)$ of $(\lambda-A)^{-1}$ of (VBP) as

(6.5) $$\begin{aligned}T_z g&=\frac{1}{2\pi i}\int_\Gamma e^{\lambda z}(\lambda-A)^{-1}g d\lambda\\&=\frac{1}{2\pi i}\int_\Gamma e^{\lambda z}d\lambda\int_\Omega G_\lambda(x, y)g(y)dy .\end{aligned}$$

From Lemma 6.3 we can take the path $\Gamma=\Gamma^+\cup\Gamma^-$,

$$\Gamma^+=\left\{\frac{a}{2}+re^{i\left(\alpha+\frac{\pi}{2}\right)},\ 0\leqq r<\infty\right\},\quad \Gamma^-=\left\{\frac{a}{2}+re^{-i\left(\alpha+\frac{\pi}{2}\right)},\ 0\leqq r<\infty\right\}.$$

Hence, if $|\arg z|<\alpha$, the Green's function $U(z, x, y)$ is uniformly bounded, because $a<0$ (cf. Remark 6.5 stated below and Tanabe [19]).

Combining Proposition 6.1 with Theorem 1.6 or 1.7, we have

THEOREM 6.4. *Assume that one of the conditions (i), (ii), (iii) in Proposition 6.1 holds. If the nonlinear term $f(u)$ in (IBVP) satisfies condition (M) ((M_+)), then every solution whose initial value is real-valued (resp. nonnegative) is holomorphically extensible in t to a sector $\Sigma_{\theta_0}=\{z\,;\ |\arg z|<\theta_0\}$ which is independent of initial values.*

REMARK 6.5. The operator T_z defined by (6.2) has semi-group property for $z\in\Sigma_{\pi/2}$

(i) $T_z T_w=T_{z+w}\quad (z, w\in\Sigma_{\pi/2})$,

(ii) $T_0=I$,

(iii) $\lim_{z\to w} T_z g = T_w g$ $(w\neq 0)$ in X,

X being $L^p(\Omega)$ $(1<p<\infty)$ or $C(\Omega)$ which is the space of boundedly continuous functions $g(x)$ with the norm $\|g\|_\infty=\sup_{x\in\Omega}|g(x)|$. The continuity at the origin

$$\lim_{z\to 0}(T_z g)(x)=g(x) \quad \left(|\arg z|<\theta<\frac{\pi}{2}\right), \tag{6.6}$$

is uniform on any compact set in Ω and (6.6) holds also in $L^p(\Omega)$-topology. As for boundedness of T_z,

$$\|T_z g\|_\infty \leqq M_{\theta,R}\exp(\omega_{\theta,R}|z|)\|g\|_\infty, \tag{6.7}$$

for $z\in\{z;\ |\arg z|\leqq\theta,\ |z|\geqq R\}$. This follows from (i) and (iii).

On the other hand for $z\in\{z;\ |\arg z|\leqq\theta,\ |z|\leqq r\}$,

$$|U(z,x,y)|\leqq C_{\theta,r}|z|^{-n/2}\exp(-c_{\theta,r}|x-y|^2/|z|^{1/2}) \tag{6.8}$$

holds (cf. Arima [1], Eidelman [7], Friedman [8], Ilin, Kalashnikov and Oleinik [9], Ito [10], Tanabe [19]). After simple calculations we have

$$\|T_z g\|_\infty\leqq\tilde{M}_{\theta,r}\|g\|_\infty. \tag{6.9}$$

The relation (6.4) and (6.5) follow from the theory of semi-groups. If we can take $\omega_{\theta,R}\leqq 0$ for some θ in (6.7), then $U(z,x,y)$ has property (U.B.).

We shall give another sufficient condition under which the Green's function of the linear equations (1.7), (1.8), (1.9) has property (U.B.).

PROPOSITION 6.6. *Suppose that the elliptic operator A has the form*

$$Au=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}u\right)+c(x)u \quad (c(x)\leqq 0)$$

and the boundary condition B is

$$Bu=\frac{\partial}{\partial\nu}u+\sigma(x)u, \quad (\sigma(x)\geqq 0).$$

Then the Green's function $U(z,x,y)$ of the linear equations (1.7), (1.8), (1.9) is uniformly bounded.

Before proving the proposition, we note that the operator A with the boundary condition B is non-positive self-adjoint. Then $T_z g$ defined by (6.2) is represented by the spectral measure $E(\tau)$ associated with A:

$$T_z g=\int_{-\infty}^{0}\exp(\tau z)dE(\tau)g. \tag{6.10}$$

The following lemma is essentially due to Beals [2].

LEMMA 6.7 *Let l be an integer such that $l>\frac{n}{4}$. Then for $g\in D(A^l)$ (the domain of A^l) the estimate*

$$\|g\|_\infty \leqq C(\|A^l g\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}) \,, \tag{6.11}$$

holds.

PROOF. Set $A_1=1-A$. Then A_1^{-1} exists. From the assumption we can choose $2=q_1<q_2<\cdots<q_s(l\geqq s)$ such that $1/q_{i+1}>1/q_i-2/n$, $0>1/q_s-2/n$. A_1^{-1} maps L^{q_i} to the Sobolev space $W^{2,q_i}\subset L^{q_{i+1}}$. So A_1^{-s} maps L^2 into $W^{2,q_s}\subset L^\infty$. Since $A_1^{-l}=A_1^{-s}A_1^{s-l}$, we have

$$\|g\|_\infty \leqq \tilde{C}\|A_1^l g\|_{L^2(\Omega)} \,. \tag{6.12}$$

Making use of spectral representation of A, we have (6.11) from (6.12).

LEMMA 6.8. *The inequality*

$$\|A^l T_z g\|_{L^2(\Omega)} \leqq \frac{M_l}{(\mathrm{Re}\, z)^l}\|g\|_{L^2(\Omega)} \,. \tag{6.13}$$

holds for $\mathrm{Re}\, z>0$.

PROOF. (6.13) follows from the equality

$$A^l T_z g = \frac{d^l}{dz^l} T_z g = \int_{-\infty}^0 \tau^l \exp(\tau z) dE(\tau) g \,.$$

PROOF OF PROPOSITION 6.6. Combining Lemma 6.7 with Lemma 6.8 we have for a boundedly continuous function $g(x)$

$$\begin{aligned}\|T_z g\|_\infty &\leqq C(\|A^l T_z g\|_{L^2(\Omega)} + \|T_z g\|_{L^2(\Omega)}) \\ &\leqq C\left(\frac{M_l}{(\mathrm{Re}\, z)^l}+1\right)\|g\|_{L^2(\Omega)} \,.\end{aligned} \tag{6.14}$$

Since Ω is bounded, for $\mathrm{Re}\, z\geqq 1$ and $|\arg z|<\theta<\frac{\pi}{2}$

$$\|(T_z g)(x)\|_\infty = \left\|\int_\Omega U(z,x,y)g(y)dy\right\|_\infty \leqq M\|g\|_\infty \,. \tag{6.15}$$

For $\mathrm{Re}\, z\leqq 1$ and $|\arg z|<\theta$, there is a constant $\tilde{M}$ such that

$$\|T_z g\|_\infty \leqq \tilde{M}\|g\|_\infty \,, \tag{6.16}$$

by (6.9), (6.15) and (6.16) imply that Green's function $U(z,x,y)$ is uniformly bounded.

REMARK 6.9. By arguments similar to that in the proof of Proposition 6.6 we can show the same result as stated in Proposition 6.6, for elliptic operators with coercive boundary conditions which are non-positive self-adjoint.

THEOREM 6.10. *Let the same assumptions as in Proposition 6.6 hold. Then the same result as stated in Theorem 6.4 is true.*

§7. Concluding remarks.

(i) *L^p-theory.* In preceding sections we discussed under the condition that initial values are boundedly continuous. When initial values are real-valued and belong to $L^p(\Omega)$ $(1\leqq p<\infty)$, the fact described below is known under some regularity conditions by applying the theory of nonlinear semi-groups:

There is a unique generalized solution $u(t, x)$ of (IBVP) and the mapping $S_t a=u(t, \cdot)$ has the semi-group property,

$$S_t S_s a=S_{t+s}a \quad (t, s\geqq 0)\ , \tag{7.1}$$

$$S_0 a=a\ , \tag{7.2}$$

$$\lim_{t\to S} S_t a=S_s a \quad \text{in} \quad L^p(\Omega)\ , \tag{7.3}$$

$$\|S_t a-S_t b\|_{L^p(\Omega)}\leqq e^{\omega t}\|a-b\|_{L^p(\Omega)}\ . \tag{7.4}$$

(See Brezis [3], Brezis, Crandall and Pazy [4], Brezis and Strauss [5], Konishi [14], [15], [16].) In view of these results, we shall give remarks on solutions with initial values in $L^p(\Omega)$.

Let Σ_{θ_0} be the sector in Theorem 1.6 and K be an arbitrary compact set in Σ_{θ_0}. According to the proof of Proposition 4.1, there is a constant C_K which is independent of initial values such that

$$|u(z, x)|\leqq C_K \quad (z\in K)\ . \tag{7.5}$$

Since $f(u)$ is a polynomial

$$|f(u)-f(v)|\leqq A_M|u-v| \quad \text{for } |u|,\ |v|\leqq M\ . \tag{7.6}$$

Combining (7.5) with (7.6)

$$\|u(z, \cdot)-v(z, \cdot)\|_{L^p(\Omega)}\leqq \tilde{C}_K\|a-b\|_{L^p(\Omega)}\ , \tag{7.7}$$

where $u(z, x)$ ($v(z, x)$) is a solution of IBVP with the initial value $a(x)$ (resp. $b(x)$). The inequality (7.7) implies

$$\|S_z a-S_z b\|_{L^p(\Omega)}\leqq \tilde{C}_K\|a-b\|_{L^p(\Omega)} \quad (z\in K)\ , \tag{7.8}$$

for boundedly continuous functions $a(x)$ and $b(x)$. As continuous functions are dense in $L^p(\Omega)$, (7.8) holds for any $a(x)$, $b(x)\in L^p(\Omega)$. Thus solutions obtained by the method of nonlinear semi-groups are holomorphically in time extensible to the sector Σ_{θ_0}. Analogously Theorem 1.7 can be extended for solutions with nonnegative initial values in $L^p(\Omega)$.

(ii) *Unbounded domains.* The boundedness of Ω required above is not essential. Our arguments are based on the following:

(α) uniform estimates of bounds of solutions of

$$\frac{du}{dt}=f(u)\ ,$$

(β) comparison theorems for parabolic differential inequalities,

(γ) uniform boundedness of Green's function of parabolic equations with boundary conditions.

For example, consider the initial value problem

$$(IVP)\qquad \begin{cases} \dfrac{\partial u}{\partial t}=\Delta u+f(u)\ , \\ u(0,x)=a(x)\ , \end{cases}$$

in $[0,\infty)\times R^n$. The Green's function of heat equation $U(z,x,y)$ is given by

$$U(z,x,y)=\frac{1}{(4\pi z)^{\frac{n}{2}}}\exp(-|x-y|^2/4z)\ ,$$

and satisfies

$$\left|\int_{R^n}U(z,x,y)g(y)dy\right|\leqq M_\theta\|g\|_\infty\ .\quad (|\arg z|<\theta)\ . \tag{7.9}$$

The inequality (7.9) implies the uniform boundedness of the Green's function. Therefore, from arguments similar to the proof of Theorem 1.6 or 1.7, we have

THEOREM 7.1. *Assume that $f(u)$ in (IVP) satisfies condition (M) $((M_+))$ and the initial value $a(x)$ is real-valued (resp. nonnegative) and boundedly continuous. Then every bounded solution is holomorphic in time in a sector $\Sigma_{\theta_0}=\{z\,;\ |\arg z|<\theta_0\}$ which does not depend on initial values.*

(iii) *Abstract theory.* For abstract evolution equations we can obtain results similar to those in preceding sections. Let X be a complex Banach space with the norm $\|\cdot\|_X$. Let Y be a real subspace of X which is a Banach lattice with unit 1. Consider the abstract evolution equation

$$(AEQ)\qquad \begin{cases} \dfrac{du}{dt}=Au+f(u)\ , \\ u(0)=a\ . \end{cases}$$

We assume that A is a generator of a holomorphic semi-group T_t, which satisfies

(7.10) $$\|T_z a\|_X \leqq M\|a\|_X \quad \text{for } z \in \{z\,;\ |\arg z| < \theta\}\ ,$$

(7.11) $$\|T_t a\|_X \leqq \|a\|_X \quad a \in Y\ ,$$

and $T_t Y_+ \subset Y_+$, Y_+ being the totality of nonnegative elements of Y, and

(7.12) $$f(u) = \sum_{k=1}^{l} B_k(u, u, \cdots, u)\ ,$$

where $B_k(u_1, u_2, u_3, \cdots, u_k)$ is a multiple linear mapping, and $f(u)$ is monotone on $Y(Y_+)$, that is,

$$f(u) \geqq f(v) \quad \text{for} \quad u \leqq v \text{ (resp. } 0 \leqq u \leqq v)\ .$$

Under certain conditions on $f(u)$ which assure uniform estimates of solutions of ordinary differential equation,

$$\frac{du}{dt} = f(u)\ ,$$

$$u(0) = a\ , \qquad a \in Y\ ,$$

we are able to show results analogous to main theorems.

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