Singularities in contact geometry and degenerate pseudo-differential equations

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To study differential equations, one of the most powerful approach is to transform them into simple forms. For example, in the theory of non-linear partial differential equations of the first order with one unknown function, established by Lagrange, Hamilton, Jacobi [6], Lie, Carathéodory [1] and others, it is well-known that the equations without singularities can be transformed into equilibrium systems by contact transformations. Recently, also in the theory of linear partial differential equations, this approach has been proved to be greatly useful. In fact, we can employ a wide class of transformations when we consider the equations in the category of pseudo-differential operators. The transformations are initiated by Maslov [9] and Egorov [2] and called Fourier integral operators by Hörmander [4] or quantized contact transformations by Sato, Kawai and Kashiwara [14] (which we abbreviate to S-K-K hereafter in this paper). In S-K-K, it is shown that any system of pseudo-differential equations with one unknown function and simple characteristics can be transformed micro-locally into a partial de Rham system by a suitable quantized contact transformation in the complex domain and the micro-local structure of the system becomes clear. The pseudo-differential operators are defined on the cotangential projective bundle P^*X of a complex manifold X and "micro-local" means "local on P^*X". Note that P^*X has a natural contact structure and that the support of any system of pseudo-differential equations with simple characteristics is regular as an involutory analytic set in P^*X.

In this paper, we investigate the structure of the systems of pseudo-differential equations whose supports are involutory submanifolds with singularities arising from the contact structure. For this purpose, we must investigate the structure of singularities of involutory submanifolds in a contact manifold, while, of course, it is a very interesting problem in itself to study singularities in contact geometry.

In §1.1 we quote a result from Oshima [10] about the maps defined by linear differential operators of the first order with degenerate symbols. In §1.2 we examine the local structure of vector fields degenerate at the origin. The results in §1.1 and §1.2 are lemmas for theorems in the following sections.

In §2 we study the structure of singularities of involutory submanifolds in a contact manifold. In §2.1 we define a very simple class of such singularities and in §2.2 we give a sufficient condition for a singularity to be of this class. This condition shows that the singularities generically belong to this class. Then, we give an application to non-linear partial differential equations of the first order with singularities.

In §3.1 we decide the condition for two pseudo-differential operators to be conjugate when the operators have not simple characteristics and when the gradients of their principal symbols do not vanish. In §3.2 we transform the systems of pseudo-differential equations whose supports are not regular into simple systems.

Throughout this paper, we treat these problems (micro-)locally in the complex analytic category or real analytic category.

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1. First order linear differential operators.

Let

$$P=\sum_{i=1}^{n} a_i(x)\frac{\partial}{\partial x_i}+b(x) \tag{1.1}$$

be a first order linear differential operator with analytic coefficients at the origin of $\boldsymbol{C}^n$. We denote by $\mathcal{O}$ the stalk at the origin of the sheaf of the ring of holomorphic functions over $\boldsymbol{C}^n$. For the operator (1.1), $\mathfrak{a}$ and $\mathfrak{b}$ denote the ideals of $\mathcal{O}$ generated by $a_1(x), \cdots, a_n(x)$ and $a_1(x), \cdots, a_n(x), b(x)$ respectively and $D(P)$ denotes the analytic set corresponding to $\mathfrak{a}$. Throughout this section we assume that $\mathfrak{a}$ does not equal $\mathcal{O}$.

The first problem is to decide the kernel and the image of the operator P: $\mathcal{O}\to\mathcal{O}$. In §1.1 we quote a result about this problem from Oshima [10], which is needed in §3.

In §1.2 we assume $b(x)\equiv 0$, namely, P is a degenerate vector field. It is the second problem to decide the equivalence classes of such vector fields under the coordinate transformations at the origin. The case when $\mathfrak{a}$ is the maximal ideal of $\mathcal{O}$ is treated in Poincaré [13], Siegel [15] etc. We generalize them in the case when $\mathfrak{a}$ is a simple ideal. The result of this section is a main tool to solve the problem in §2.

§1.1. The theorem of Cauchy-Kowalevsky.

Let

$$M(x)=\frac{\partial(a_1,\cdots,a_n)}{\partial(x_1,\cdots,x_n)}(x)\,,\quad x\in D(P) \tag{1.2}$$

be the Jacobian matrix of $a_1,\cdots,a_n$ on $D(P)$. Taking a different coordinate system, $M(0)$ is transformed into $G^{-1}M(0)G$, where G is the Jacobian matrix of the coordinate transformation at the origin. This shows that the set of simple elementary divisors of $\lambda I-M(0)$, which we denote by

$$E(P)=\{(\lambda-\mu_i)^{m_i};\ 1\leqq i\leqq k\}\,, \tag{1.3}$$

is independent of the choice of coordinate systems.

Now we give the following conditions to formulate a theorem.

A.1.1. *$\mathfrak{a}$ is a proper and simple ideal of $\mathcal{O}$, i.e. $\mathcal{O}/\mathfrak{a}$ is a non-zero regular local ring.*

A.1.2. *If $\mu_i=0$, then $m_i=1$.*

A.1.3. *There exists a real number θ such that $\theta<\arg\mu_i<\theta+\pi$ for $\mu_i\neq 0$, where $\arg\mu_i$ denotes the argument of complex number μ_i.*

A.1.4. *If $\alpha\in N^k$ and $\sum_{i=1}^{k}|\alpha_i\mu_i|\neq 0$, then $b(0)+\sum_{i=1}^{k}\alpha_i\mu_i\neq 0$. Here $N=\{0,1,2,\cdots\}$.*

THEOREM 1.1. *Assuming conditions A.1.1, A.1.2, A.1.3 and A.1.4, we have the following conclusion:*

$$\operatorname{Ker} P\cong\begin{cases}\mathcal{O}/\mathfrak{b} & \text{if}\ \ \mathfrak{a}=\mathfrak{b}\,,\\ 0 & \text{if}\ \ \mathfrak{a}\neq\mathfrak{b}\,,\end{cases}$$

$$\operatorname{Im} P=\mathfrak{b}\,.$$

That is, an analytic solution u of the equation

$$Pu=f \tag{1.4}$$

exists locally if and only if $f\in\mathfrak{b}$. If $\mathfrak{a}\neq\mathfrak{b}$, u is uniquely determined by f, and if $\mathfrak{a}=\mathfrak{b}$, there is a one-one correspondence between the solution u and the Cauchy data $u|_{D(P)}$.

For the proof of the above theorem and for examples which do not satisfy the conditions, we refer to Oshima [10], [11]. §1.2 tells us that A.1.3 can be weakened in Theorem 1.1 (see Remark 1.10).

When the coefficients $a_i(x)$, $b(x)$ of (1.1) are elements of the ring of formal power series $\hat{\mathcal{O}}$, we consider the same problem for the operator $\hat{P}$: $\hat{\mathcal{O}}\to\hat{\mathcal{O}}$, where $\hat{P}$ expresses the operator P in the ring $\hat{\mathcal{O}}$. $\hat{\mathfrak{a}}$ and $\hat{\mathfrak{b}}$ denote the ideals of $\hat{\mathcal{O}}$ generated by $a_1(x), \cdots, a_n(x)$ and $a_1(x), \cdots, a_n(x)$, $b(x)$ respectively. Then the following is an easy corollary of the proof of Theorem 1.1.

THEOREM 1.2. *Assuming conditions A.1.1′, A.1.2 and A.1.4 or assuming that* $\hat{\mathfrak{a}} \neq \hat{\mathcal{O}}$ *and A.1.4′, then we have*

$$\operatorname{Ker} \hat{P} \cong \begin{cases} \hat{\mathcal{O}}/\hat{\mathfrak{b}} & \text{if} \quad \hat{\mathfrak{a}} = \hat{\mathfrak{b}} , \\ 0 & \text{if} \quad \hat{\mathfrak{a}} \neq \hat{\mathfrak{b}} , \end{cases}$$

$$\operatorname{Im} \hat{P} = \hat{\mathfrak{b}} ,$$

where

A.1.1′. $\hat{\mathfrak{a}}$ *is a proper and simple ideal of* $\hat{\mathcal{O}}$.

A.1.4′. $b(0) + \sum_{i=1}^{k} \alpha_i \mu_i \neq 0$ *for* $\alpha \in N^k$.

§ 1.2. Degenerate vector fields.

Let

$$P = \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i} \tag{1.5}$$

and

$$P' = \sum_{i=1}^{n} a'_i(x) \frac{\partial}{\partial x_i} \tag{1.6}$$

be analytic vector fields at the origin of C^n. Then we define the following equivalence relation.

DEFINITION 1.3. $P \sim P' \Leftrightarrow P$ is transformed into P' by a suitable coordinate transformation at the origin.

Throughout this section we assume A.1.1. Considering what we mentioned at the first part of § 1.1, we see that the following condition is necessary for $P \sim P'$.

A.1.5. *There exist an analytic diffeomorphism* $F: D(P) \to D(P')$ *and a matrix* $G(x) \in GL(n, \mathcal{O}(D(P)))$ *such that*

$$G(x)M(x)G(x)^{-1} = M'(F(x)) \quad \textit{for} \quad x \in D(P) ,$$

where $\mathcal{O}(D(P)) = \mathcal{O}/\mathfrak{a}$.

THEOREM 1.4. *Assume the following* i) *or* ii) *for P. Then A.1.5 is a necessary and sufficient condition for* $P \sim P'$.

i) *A.1.1, A.1.2 and A.1.3 hold and moreover*

A.1.6. *if* $\alpha \in N^l$ *and* $|\alpha| \equiv \sum_{i=1}^{l} \alpha_i \geqq 2$, *then* $\sum_{i=1}^{l} \alpha_i \rho_i - \alpha_j \neq 0$ *for* $1 \leqq j \leqq l$, *where*

$\{\rho_j;\ 1\leqq j\leqq l\}$ *is the set of all the non-zero eigenvalues of* $M(0)$.

ii) *A.1.1 holds and there exists* $G(x)\in GL(n, \mathcal{O}(D(P)))$ *so that* $G(x)M(x)G(x)^{-1}$ *is a diagonal matrix and moreover*

A.1.7. *there exist positive numbers* ε, ν *such that if* $x\in D(P)$, $|x|\equiv\sum_{i=1}^{n}|x_i|<\varepsilon$, $\alpha\in N^l$ *and* $|\alpha|\geqq 2$, *then* $|\sum_{i=1}^{l}\alpha_i\rho_i(x)-\rho_j(x)|>|\alpha|^{-\nu}$ *for* $1\leqq j\leqq l$, *where* $\{\rho_j(x);\ 1\leqq j\leqq l\}$ *is the set of the non-zero diagonal elements of* $G(x)M(x)G(x)^{-1}$.

COROLLARY 1.5. *Suppose that* P *satisfies* i). *Then under a suitable coordinate system* $(x_1, \cdots, x_l, y_1, \cdots, y_{l'})$, *we have*

$$P=\sum_{i,j=1}^{l}\rho_j^i(y)x_i\frac{\partial}{\partial x_j}\ . \tag{1.7}$$

Suppose that P *satisfies* ii). *Then under a suitable coordinate system,*

$$P=\sum_{i=1}^{l}\rho_i(y)x_i\frac{\partial}{\partial x_i}\ . \tag{1.8}$$

The case when $\mathfrak{a}$ is the maximal ideal of $\mathcal{O}$ is proved on the assumption ii) by Siegel [15] and Sternberg [16].

PROOF OF THEOREM 1.4. Considering coordinate transformations of the following type:

$$\begin{cases} x_i\to\sum_{k=1}^{l}f_{ik}(y)x_i\ , & 1\leqq i\leqq l\ , \\ y_j\to g_j(y)\ , & 1\leqq j\leqq l'\ , \end{cases}$$

we get easily Theorem 1.4 from Corollary 1.5, so we have only to prove Corollary 1.5. (Note that $D(P)=\{x_1=\cdots=x_l=0\}$.)

We remark that it is equivalent to think coordinate transformations of (1.5) and to think those of the autonomous system of the ordinary differential equations

$$\dot{x}_j=a_j(x)\ ,\qquad 1\leqq j\leqq n\ . \tag{1.9}$$

It follows from A.1.1 that $D(P)=\{x_1=\cdots=x_l=0\}$ under a suitable coordinate system $(x_1, \cdots, x_l, y_1, \cdots, y_{l'})$. Then (1.9) is of the following form:

$$\begin{cases} \dot{x}_i=\sum_{k=1}^{l}c_{ik}(y)x_k+\sum_{|\alpha|\geqq 2}c_{i\alpha}(y)x^\alpha\ , \\ \dot{y}_j=\sum_{k=1}^{l}d_{jk}(y)x_k+\sum_{|\alpha|\geqq 2}d_{j\alpha}(y)x^\alpha\ , \end{cases} \tag{1.10}$$

and $(c_{ik}(y))\in GL(l, \mathcal{O}(D(P)))$. By a coordinate transformation

$$\begin{cases} x_i \to x_i \\ y_j \to y_j + \sum_{k=1}^{l} h_{jk}(y) x_k \ , \end{cases}$$

we can make $d_{jk}(y) \equiv 0$. Moreover considering a transformation

$$\begin{cases} x_i \to \sum_{k=1}^{l} h'_{ik}(y) x_i \ , \\ y_j \to y_j \ . \end{cases}$$

we can assume from the beginning that

$$\begin{cases} \dot{x}_i = \sum_{k=1}^{l} c_{ik}(y) x_k + H_i(x, y) \ , \\ \dot{y}_j = H_{j+l}(x, y) \ , \end{cases} \tag{1.11}$$

where $\mathrm{grad}_x H_j|_{x=0} \equiv 0$ and $H_j(0, y) \equiv 0$ for $1 \leqq j \leqq n$ and in the case i)

$$\begin{cases} c_{ik}(0) = \rho_i & \text{if} \quad i = k \ , \\ c_{ik}(0) = 0 & \text{if} \quad i > k \ , \end{cases}$$

in the case ii)

$$\begin{cases} c_{ik}(y) = \rho_i(y) & \text{if} \quad i = k \ , \\ c_{ik}(y) = 0 & \text{if} \quad i \neq k \ . \end{cases}$$

Consider the following coordinate system:

$$\begin{cases} x_i = z_i + \varphi_i(z, w) \ , & 1 \leqq i \leqq l \ , \\ y_j = w_j + \varphi_{j+l}(z, w) \ , & 1 \leqq j \leqq l' \ , \end{cases} \tag{1.12}$$

where $\varphi_j(0, w) \equiv 0$ and $\mathrm{grad}_z \varphi_j|_{z=0} \equiv 0$. Then

$$\begin{aligned} \dot{x}_i - \sum_{k=1}^{l} c_{ik}(y) x_k &= \sum_{p=1}^{l} \dot{z}_p \frac{\partial x_i}{\partial z_p} + \sum_{q=1}^{l'} \dot{w}_q \frac{\partial x_i}{\partial w_q} - \sum_{k=1}^{l} c_{ik}(y)(z_k + \varphi_k) \\ &= \sum_{p=1}^{l} \Big(\dot{z}_p - \sum_{k=1}^{l} c_{pk}(w) z_k\Big) \frac{\partial x_i}{\partial z_p} + \sum_{q=1}^{l'} \dot{w}_q \frac{\partial x_i}{\partial w_q} \\ &\quad + \sum_{p,k=1}^{l} c_{pk}(w) z_k \frac{\partial \varphi_i}{\partial z_p} - \sum_{k=1}^{l} c_{ik}(w) \varphi_k \\ &\quad + \sum_{k=1}^{l} (c_{ik}(w) - c_{ik}(y))(z_k + \varphi_k) \ , \\ \dot{y}_j &= \sum_{p=1}^{l} \Big(\dot{z}_p - \sum_{k=1}^{l} c_{pk}(w) z_k\Big) \frac{\partial y_j}{\partial z_p} + \sum_{q=1}^{l'} \dot{w}_q \frac{\partial y_j}{\partial w_q} \\ &\quad + \sum_{p,k=1}^{l} c_{pk}(w) z_k \frac{\partial \varphi_{j+l}}{\partial z_p} \ . \end{aligned}$$

In view of $\begin{pmatrix} \partial x_i/\partial z_p, & \partial x_i/\partial w_q \\ \partial y_j/\partial z_p, & \partial y_j/\partial w_q \end{pmatrix} \in GL(n, C)$, if φ_i satisfy

$$
(1.13)\qquad \begin{cases} \displaystyle\sum_{p,k=1}^{l} c_{pk}(w) z_k \frac{\partial \varphi_i}{\partial z_p} - \sum_{k=1}^{l} c_{ik}(w)\varphi_k \\ \displaystyle\qquad = \sum_{k=1}^{l} (c_{ik}(y) - c_{ik}(w))(z_k + \varphi_k) + H_i(x, y) , \\ \displaystyle\sum_{p,k=1}^{l} c_{pk}(w) z_k \frac{\partial \varphi_{j+l}}{\partial z_p} = H_{j+l}(x, y) , \end{cases}
$$

(1.11) changes into the following system:

$$
(1.14)\qquad \begin{cases} \dot{z}_i = \displaystyle\sum_{k=1}^{l} c_{ik}(w) z_k , \\ \dot{w}_j = 0 . \end{cases}
$$

Therefore, it is sufficient to show the existence of a solution of (1.13).

Since the assumption says that

$$
\sum_{p=1}^{l} \alpha_p c_{pp}(0) - c_{jj}(0) \neq 0 , \qquad \sum_{p=1}^{l} \alpha_p c_{pp}(0) \neq 0 , \qquad \text{for} \quad |\alpha| \geqq 2,\ 1 \leqq j \leqq l ,
$$

we see that (1.13) has a unique solution of a formal power series

$$
(1.15)\qquad \varphi_j = \sum_{|\alpha| \geqq 2} \varphi_{j\alpha}(w) x^\alpha = \sum_{|\alpha| \geqq 2, |\beta| \geqq 0} \varphi_{j\alpha\beta} w^\beta x^\alpha , \qquad 1 \leqq j \leqq n .
$$

In fact, $\varphi_{j\alpha\beta} \in C$ are determined inductively by (1.13) in the lexicographic order of $(|\alpha|, |\beta|, \sum_{p=1}^{l} p\alpha_p, n-j)$ (see (1.11) and (1.12)). The rest part of the proof is to show that (1.15) is analytic at the origin.

In the case i). The assumption shows the existence of a positive number C such that

$$
(1.16)\qquad \Big|\sum_{i=1}^{l} \alpha_i \rho_i - \rho_j\Big| \geqq C|\alpha| \quad \text{for} \quad |\alpha| \geqq 2,\ 1 \leqq j \leqq l.
$$

Considering a coordinate transformation

$$
\begin{cases} x_i \to e_i x_i \\ y_j \to e_{j+l} y_j \end{cases} \qquad e_j \in C ,
$$

we may assume the existence of the majorant series

$$
\Big\{ H_j(x, y) \ll \frac{M(x_1 + \cdots + x_l)^2}{1 - (x_1 + \cdots + x_l + y_1 + \cdots + y_{l'})} ,
$$

$$
(1.17)\qquad \left\{ c_{ik}(y) \ll \frac{M(y_1+\cdots+y_{l'})}{1-(y_1+\cdots+y_{l'})} + \begin{cases} 0 \text{ if } i>k,\ \dfrac{C}{2n^2} \text{ if } i<k, \\ |\rho_i| \text{ if } i=k\ , \end{cases} \right.
$$

where M is a positive number.

We set $s=z_1+\cdots+z_l$ and $t=w_1+\cdots+w_{l'}$. Suppose the formal power series $u=\sum_{|\alpha|\geqq 2, |\beta|\geqq 0} u_{\alpha\beta}w^\beta z^\alpha$ satisfies $\varphi_j(w, z)\ll u(w, z)$. Then we have

$$
\begin{aligned}
&-\sum_{p,k=1}^{l} c_{pk}(w)z_k\frac{\partial\varphi_i}{\partial z_p}+\sum_{p=1}^{l} c_{pp}(0)z_p\frac{\partial\varphi_i}{\partial z_p}+\sum_{k=1}^{l} c_{ik}(w)\varphi_k-c_{ii}(0)\varphi_i \\
&+\sum_{k=1}^{l}(c_{ik}(y)-c_{ik}(w))(z_k+\varphi_k)+H_i(x, y) \\
&\ll\left(\frac{Mt}{1-t}+\frac{C}{2n^2}\right)\left(s\sum_{p=1}^{l}\frac{\partial u}{\partial z_p}+lu\right) \\
&+\left(\frac{M}{1-t-l'u}-\frac{M}{1-t}\right)(s+lu)+\frac{M(s+lu)^2}{1-s-t-nu}\ .
\end{aligned}
$$

$\left(\text{Note that } u\ll\frac{1}{2}s\sum_{p=1}^{l}\frac{\partial u}{\partial z_p}.\right)$

$$
\ll s\left\{\left(\frac{3C}{4n}+f_1(s, t, u)\right)\sum_{p=1}^{l}\frac{\partial u}{\partial z_p}+f_2(s, t, u)s\right\},
$$

where $f_1(s, t, u)$ and $f_2(s, t, u)$ are analytic functions with three variables and the coefficients of their Taylor expansions are nonnegative real numbers and $f_1(0, 0, 0)=0$. For simplicity we set

$$
F\left(s, t, u, \frac{\partial u}{\partial z}\right)\equiv s\left\{\left(\frac{3C}{4n}+f_1(s, t, u)\right)\sum_{p=1}^{l}\frac{\partial u}{\partial z_p}+f_2(s, t, u)s\right\},
$$

then it is clear that

$$
\begin{aligned}
-\sum_{p,k=1}^{l} c_{pk}(w)z_k\frac{\partial\varphi_{j+k}}{\partial z_p}+\sum_{p=1}^{l} c_{pp}(0)z_p\frac{\partial\varphi_{j+k}}{\partial z_p}+H_{j+l}(x, y) \\
\ll F\left(s, t, u, \frac{\partial u}{\partial z}\right)
\end{aligned}
$$

holds also. Combining the above estimates with (1.16), we have $\varphi_j\ll u$ if a formal power series u satisfies

$$
(1.18)\qquad C\sum_{p=1}^{l} z_p\frac{\partial u}{\partial z_p}\ll F\left(s, t, u, \frac{\partial u}{\partial z}\right).
$$

On the other hand the solution $u(s, t)$ of the equation

$$(1.19)\qquad \begin{cases} \left\{\dfrac{C}{4}-lf_1(s,t,u)\right\}\dfrac{\partial u}{\partial s}=f_2(s,t,u)s\ , \\ u|_{s=0}=0 \end{cases}$$

is analytic at the origin and satisfies (1.18) and $\left.\dfrac{\partial u}{\partial s}\right|_{s=0}=0$. Thus we conclude that φ_j are analytic by the method of majorant.

In the case ii). As in the case i) we may assume

$$(1.17)'\qquad \begin{cases} H_j(x,y)\ll\dfrac{M(x_1+\cdots+x_l)^2}{2-(x_1+\cdots+x_l+y_1+\cdots+y_{l'})}\ , \\ \rho_j(y)=c_{jj}(y)\ll\dfrac{M}{2-(y_1+\cdots+y_{l'})}\ . \end{cases}$$

Let

$$\varepsilon_{j\alpha}(w)=\sum_{p=1}^{l}\alpha_p\rho_p(w)-\rho_j(w)\ ,$$

where we set $\rho_j(w)=0$ for $j>l$. Since $\varepsilon_{j\alpha}(0)\neq 0$ for $|\alpha|\geqq 2$, $\varphi_{j\alpha}(w)$ of (1.15) are analytic functions of w. Therefore it is sufficient to prove the following claim: "There exist positive numbers δ and N so that $\varphi_j(w^0,z)$ are holomorphic functions in the domain $\{z\in C^l;\ |z|<\delta\}$ and satisfy $|\varphi_j(w^0,z)|<N$ for any fixed $w^0\in C^{l'}$ satisfying $|w^0|<\delta''$.

Hence we fix w^0 satisfying $|w^0|<1$ and omitting w^0, we write simply $\varphi_{j\alpha}$, $\varepsilon_{j\alpha}$ etc. instead of $\varphi_{j\alpha}(w^0)$, $\varepsilon_{j\alpha}(w^0)$ etc.

Let

$$\begin{cases} \varphi_\alpha=\sum_{j=1}^{n}|\varphi_{j\alpha}| \quad \text{for} \quad |\alpha|\geqq 2\ , \\ \varphi_\alpha=1 \quad \text{for} \quad |\alpha|=1\ , \quad \varphi_0=0\ . \end{cases}$$

Then

$$\begin{cases} x_i\ll\sum_\alpha\varphi_\alpha z^\alpha\ , \quad y_j\ll 1+\sum_\alpha\varphi_\alpha z^\alpha\ , \\ x_1+\cdots+x_l+y_1+\cdots+y_{l'}\ll 1+\sum_\alpha\varphi_\alpha z^\alpha\ . \end{cases}$$

(1.13) and (1.17)′ imply that

$$\begin{aligned}\sum_\alpha\varepsilon_{j\alpha}\varphi_{j\alpha}z^\alpha &\ll\frac{M(\sum_\alpha\varphi_\alpha z^\alpha)^2}{1-\sum_\alpha\varphi_\alpha z^\alpha}+\left(\frac{M}{1-\sum_\alpha\varphi_\alpha z^\alpha}-M\right)(\sum_\alpha\varphi_\alpha z^\alpha)\\ &\ll 2M\sum_{r=2}^{\infty}(\sum_\alpha\varphi_\alpha z^\alpha)^r \quad \text{for} \quad 1\leqq j\leqq n\ .\end{aligned}$$

$$\therefore \quad |\varepsilon_{j\alpha}\varphi_{j\alpha}| \leqq 2M \sum_{\substack{r>1 \\ \alpha_1+\cdots+\alpha_r=\alpha}} \varphi_{\alpha_1}\cdots\varphi_{\alpha_r} \quad \text{for} \quad |\alpha|\geqq 2 .$$

$$\therefore \quad \varphi_\alpha \leqq 2M\mu_\alpha \sum_{\substack{r>1 \\ \alpha_1+\cdots+\alpha_r=\alpha}} \varphi_{\alpha_1}\cdots\varphi_{\alpha_r} ,$$

where $\mu_\alpha=\sum_{j=1}^{n}|\varepsilon_{j\alpha}|^{-1}$. It follows from the assumption that there exist positive real numbers δ and ν' so that

$$|\varepsilon_{j\alpha}|^{-1} > 2n|\alpha|^{-\nu'} \quad \text{for} \quad |w^0|<\delta \quad \text{and} \quad |\alpha|\geqq 2 .$$

Then the same argument as in Siegel [15] pp. 24-29 proves that

$$\varphi_j(w^0, z) \ll \frac{z_1+\cdots+z_l}{1-2^{8\nu'+4}M(z_1+\cdots+z_l)}-(z_1+\cdots+z_l) \quad \text{for} \quad |w^0|<\delta .$$

This completes the proof of Theorem 1.4.

The above proof shows the following:

COROLLARY 1.6. *Suppose that the assumption in Theorem 1.4 holds and that the two coordinate transformations of formal power series*

$$x_i \to \varphi_i(x) , \qquad 1\leqq i\leqq n ,$$

$$x_i \to \varphi'_i(x) , \qquad 1\leqq i\leqq n ,$$

do not change the expression (1.5). Then φ_i *and* φ'_i *are equal if* $\varphi_i|_{D(P)}\equiv\varphi'_i|_{D(P)}$ *and* $\operatorname{grad}\varphi_i|_{D(P)}\equiv \operatorname{grad}\varphi'_i|_{D(P)}$ *for* $1\leqq i\leqq n$. *And moreover if* $\varphi_i|_{D(P)}$ *and* $\operatorname{grad}\varphi_i|_{D(P)}$ *are analytic functions on* $D(P)$, *then the coordinate transformation is analytic at the origin. Especially when* $D(P)$ *is the origin, all the coordinate transformations of formal power series that do not change the expression (1.5) are analytic.*

REMARK 1.7. In the real analytic category (namely $a_i(x)$ are real-valued analytic functions and coordinate transformations are real analytic), we see that Theorem 1.4 holds also in view of the above proof. But the existence of the complex analytic coordinate transformation at the origin which transforms (1.5) into (1.6) does not assure the existence of real analytic one. We cite a counter-example.

EXAMPLE 1.8.
$$P=x\frac{\partial}{\partial x}+2y\frac{\partial}{\partial y}+(4z+y^2+x^4)\frac{\partial}{\partial z} ,$$

$$P'=x\frac{\partial}{\partial x}+2y\frac{\partial}{\partial y}+(4z-y^2+x^4)\frac{\partial}{\partial z} .$$

The equations

$$(1.20)\qquad \begin{cases} (P-1)u_1=0\ , \\ (P-2)u_2=0\ , \\ (P-4)u_3=u_1^4-u_2^2 \end{cases}$$

have no real-valued analytic solution satisfying $\left|\dfrac{\partial(u_1, u_2, u_3)}{\partial(x, y, z)}(0)\right|\neq 0$. On the other hand the equations (1.20)$'$ which we get from (1.20) by replacing P by P' have a solution $(u_1, u_2, u_3)=(x, y, z)$. This implies the non-existence of the real analytic coordinate transformation which changes P into P'. But the transformation $y\to\sqrt{-1}y$ changes P into P'.

REMARK 1.9. Assume P of (1.5) satisfies A.1.3 and that $\mathfrak{a}$ is the maximal ideal of $\mathcal{O}$. Then under a suitable coordinate system,

$$P=\sum_{i=1}^{n}\rho_i x_i\frac{\partial}{\partial x_i}+\sum_{i=1}^{n-1}\delta_i x_i\frac{\partial}{\partial x_{i+1}}+\sum_{\substack{1\leq i\leq n, |\alpha|\geq 2\\ \langle\rho,\alpha\rangle=\rho_i}} C_{i\alpha}x^\alpha\frac{\partial}{\partial x_i}\ .$$

Here the Jordan canonical matrix of $M(0)$ is $\begin{pmatrix} \rho_1 & \delta_1 & & \\ & \ddots & \ddots & \\ & & \ddots & \delta_{n-1} \\ & & & \rho_n \end{pmatrix}$ and $\langle\rho, \alpha\rangle=\sum_{i=1}^{n}\rho_i\alpha_i$ and $C_{i\alpha}$ are complex numbers.

The proof of the above statement is as follows. Let θ be a real number satisfying A.1.3 and assume

$$\operatorname{Re}\rho_1 e^{-\theta\sqrt{-1}}\leqq \operatorname{Re}\rho_2 e^{-\theta\sqrt{-1}}\leqq \cdots \leqq \operatorname{Re}\rho_n e^{-\theta\sqrt{-1}}\ .$$

We can choose inductively analytic functions u_i and complex numbers $C_{i\alpha}$ so that $(u_1, \cdots, n_n)$ is a coordinate system satisfying

$$(1.21)\qquad (P-\rho_i)u_i+\sum_{\substack{\langle\rho,\alpha\rangle=\rho_i\\ |\alpha|\geq 2}} C_{i\alpha}u^\alpha=\delta_{i-1}u_{i-1}\ .$$

If $\langle\rho, \alpha\rangle=\rho_i$, then $\alpha_j=0$ for $j\geqq i$. Therefore the existence of $C_{i\alpha}$ and formal power series u_i satisfying (1.21) are clear and their convergence can be proved similarly as in the proof of Theorem 1.1. (Cf. the method in Karlin and McGregor [7].)

REMARK 1.10. Suppose the first order part of

$$(1.1)\qquad P=\sum_{i=1}^{n}a_i(x)\frac{\partial}{\partial x_i}+b(x)$$

satisfies the assumption ii) in Theorem 1.4 and that there exist positive numbers ε and ν such that

$$\left|\sum_{i=1}^{l}\alpha_i\rho_i(x)+b(x)\right|>|\alpha|^{-\nu}$$

for

$$x \in D(P), \quad |x|<\varepsilon, \quad \alpha \in N^l \quad \text{and} \quad |\alpha| \geqq 1.$$

Then we have the same conclusion as in Theorem 1.1.

Using Corollary 1.5, we can assume the first order part of (1.1) is of the form (1.8). And the same argument as in the proof of Theorem 1.4 shows the above statement.

2. Contact geometry.

A *contact manifold* is an odd-dimensional manifold X with a line subbundle $\mathscr{L}$ of the cotangential vector bundle T^*X of X satisfying that for any point in X there exists a section ω of $\mathscr{L}$ in a neighbourhood of the point such that the $(2n+1)$-from $\omega \wedge (d\omega)^n$ nowhere vanishes, where $2n+1$ is the dimension of X. ω is called a *fundamental* 1-*form* on X. We treat mainly the case when X is a complex manifold, so a contact manifold means a complex contact manifold in this section.

We refer to Carathéodory [1] for the general facts that will be mentioned below.

A map $f: X \to Y$ between contact manifolds is called a *contact transformation* when $f^*\omega_Y$ is a fundamental 1-form on X for any fundamental 1-form ω_Y on Y. It follows in this case that the dimension of X and that of Y are equal and f is a local isomorphism. We treat a local theory in this section. Therefore, we fix a point $P \in X$, call it the origin and denote also by $\mathscr{O}$ and $\mathscr{L}$ the stalks of $\mathscr{O}$ and $\mathscr{L}$ at the origin respectively. $\mathscr{L}^{\otimes(r)}$ denotes the Whitney product of r copies of $\mathscr{L}$ and $\mathscr{L}^{\otimes(-r)}$ the dual line bundle of $\mathscr{L}^{\otimes(r)}$ and $\mathscr{O}$ is $\mathscr{L}^{\otimes(0)}$.

There exists a local coordinate system $(p_1, \cdots, p_n, x_1, \cdots, x_n, z)$ of X at the origin so that a fundamental 1-form is expressed as

$$\omega = dz - p_1 dx_1 - \cdots - p_n dx_n .$$

This coordinate system is called *canonical*. We define a map

$$\begin{array}{ccc} \mathscr{L}^{\otimes(-r)} \times \mathscr{L}^{\otimes(-s)} & \to & \mathscr{L}^{\otimes(1-r-s)} \\ \text{\rotatebox{90}{$\in$}} & & \text{\rotatebox{90}{$\in$}} \\ (f, g) & \mapsto & [f, g] \end{array}$$

by

$$[f, g] = \left\{ r\varphi \frac{\partial \psi}{\partial z} - s\psi \frac{\partial \varphi}{\partial z} + \sum_{j=1}^{n} \left(\frac{\partial \varphi}{\partial x_j} + p_j \frac{\partial \varphi}{\partial z} \right) \frac{\partial \psi}{\partial p_j} - \sum_{j=1}^{n} \left(\frac{\partial \psi}{\partial x_j} + p_j \frac{\partial \psi}{\partial z} \right) \frac{\partial \varphi}{\partial p_j} \right\} \otimes \omega^{\otimes(1-r-s)} ,$$

for $f=\varphi\otimes\omega^{\otimes(-r)}$, $g=\psi\otimes\omega^{\otimes(-s)}$. $[f, g]$ is called the *Lagrangean bracket* of f and g.

We also use the coordinate system $(x_1, \cdots, x_n, x_{n+1}, \xi_1, \cdots, \xi_n, \xi_{n+1})$ satisfying

$$\begin{cases} \xi_1 : \cdots : \xi_n : \xi_{n+1} = (-p_1) : \cdots : (-p_n) : 1 , \\ x_{n+1} = z , \end{cases}$$

where $(\xi_1, \cdots, \xi_{n+1})$ is a homogeneous coordinate system. Then.

$$\omega = \xi_1 dx_1 + \cdots + \xi_n dx_n + \xi_{n+1} dx_{n+1} ,$$

and a section of $\mathscr{L}^{\otimes(-r)}$ is a function of (x, ξ) and homogeneous in ξ of the homogeneous degree r. Under this coordinate system,

$$[f, g] = \sum_{j=1}^{n+1} \frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial x_j} - \sum_{j=1}^{n+1} \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial \xi_j} ,$$

which equals the *Poisson bracket* $\{f, g\}$.

An analytic set $V \subset X$ is called *involutory* if $f, g \in I(V)$ implies $[f, g] \in I(V)$, where

$$I(V) = \{f \in \mathscr{O} ; f|_V \equiv 0\} .$$

DEFINITION 2.1. A point P of an involutory analytic set V is called *degenerate* if P is not a singular point of V as an analytic set but $\omega|_V(P)=0$. Here $\omega|_V$ is the inverse image of ω with respect to the canonical injection $V \subsetneq X$. We denote by $D(V)$ the set of all the degenerate points of V. We say that V is *regular* if V has neither a singular point nor a degenerate point.

DEFINITION 2.2. Analytic sets (more precisely the germs of them) V, V' are said to be *isomorphic* if there exists a local contact transformation f on X so that $f(V)=V'$. We express this equivalence relation as $V \sim V'$.

When an involutory analytic set V is regular, there exists a canonical coordinate system (p, x, z) so that $V=\{p_1= \cdots =p_d=0\}$ (cf. Carathéodory [1]). Therefore, all the regular involutory analytic sets with the same dimension are isomorphic.

The generic non-regular points in involutory analytic sets are degenerate. The singularity of this type is peculiar to contact geometry and does not appear in symplectic geometry, which is defined in the case when the dimension of the manifold is even. But all the problems in symplectic geometry can be treated in contact geometry of the one higher dimension.

§ 2.1. Linearly degenerate involutory manifolds.

DEFINITION 2.3. We denote by $\mathscr{V}$ the set of all the involutory analytic sets in the $(2n+1)$-dimensional contact manifold X that are degenerate at the origin and especially by $\mathscr{V}_d$ the set of all the d-codimensional ones in $\mathscr{V}$. Under a local coordinate system $(t_1, \cdots, t_k)$ of $V \in \mathscr{V}_d$ $(k=2n+1-d)$,

$$\omega|_V = a_1(t)dt_1 + \cdots + a_k(t)dt_k , \tag{2.1}$$

then $a_1(0) = \cdots = a_k(0) = 0$. If there exists a coordinate system so that $a_1(t), \cdots, a_k(t)$ are linear functions of t, we say the origin is *linearly degenerate* and denote by $\mathscr{V}_d^L$ the set of all such V.

PROPOSITION 2.4. *For any $V \in \mathscr{V}_d$ we can choose a canonical coordinate system so that*

$$V = \{p_1 = \cdots = p_{d-1} = z + h(x', p') = 0\} . \tag{2.2}$$

Here $x' = (x_d, \cdots, x_n)$, $p' = (p_d, \cdots, p_n)$ and $h(x', p')$ is a function without constant and linear terms.

COROLLARY 2.5. *If $d > n+1$, then $\mathscr{V}_d = \emptyset$ and the elements of $\mathscr{V}_{n+1}$ are isomorphic to each other.*

We call the elements of $\mathscr{V}_{n+1}$ Lagrangean manifolds.

PROOF OF PROPOSITION 2.4. Suppose that $V = \{f_1 = \cdots = f_d = 0\}$, where $f_i \in \mathscr{O}$ and $df_1(0), \cdots, df_d(0)$ are linearly independent. We take a canonical coordinate system (p, x, z). $\omega(0) = dz$, $df_1(0), \cdots, df_d(0)$ are linearly dependent because the origin is degenerate. This implies the existence of indices $j_1, \cdots, j_m; j_{m+1}, \cdots, j_{d-1}$ such that

$$\left| \frac{\partial(f_1, \cdots, f_d)}{\partial(p_{j_1}, \cdots, p_{j_m}, x_{j_{m+1}}, \cdots, x_{j_{d-1}}, z)} \right| \neq 0 .$$

Therefore, the implicit function theorem shows that V is expressed as the set of the common zeros of the functions

$$\begin{cases} f'_k = p_{j_k} + g_k(p', x') , & 1 \leq k \leq m , \\ f'_k = x_{j_k} + g_k(p', x') , & m+1 \leq k \leq d-1 , \\ f'_d = z + g_d(p', x') , \end{cases} \tag{2.3}$$

where p', x' are the variables excluded $p_{j_1}, \cdots, p_{j_m}, x_{j_{m+1}}, \cdots, x_{j_{d-1}}, z$.

$[f'_k, f'_l]$ $(1 \leq k \leq d-1, 1 \leq l \leq d-1)$ depend only on the variables x', p'. On the other hand, $[f'_k, f'_l]$ belong to the ideal of $\mathscr{O}$ generated by the functions (2.3). Considering the expression of (2.3), we see that $[f'_k, f'_l] \equiv 0$. Since $df_1(0), \cdots, df_{d-1}(0), \omega(0)$ are linearly independent, there exists a canonical coordinate system $(\hat{p}, \hat{x}, \hat{z})$ such that $\hat{p}_j = f_j$ for $1 \leq j \leq d-1$ (cf. Carathéodory [1]). V can be expressed as follows because $\partial f_d / \partial \hat{z}_d \neq 0$. Hereafter we will omit $\hat{\ }$.

$$V = \{p_1 = \cdots = p_{d-1} = z + h(x, p_d, \cdots, p_n) = 0\} .$$

Now, $[z+h(x, p'), p_j]=p_j+\dfrac{\partial h}{\partial x_j}(x, p')$ vanishes on V for $1\leqq j\leqq d-1$, so $\dfrac{\partial h}{\partial x_j}\equiv 0$. Therefore h is independent not only of $p_1, \cdots, p_{d-1}$ but also of $x_1, \cdots, x_{d-1}$ and has no linear terms because $dp_1, \cdots, dp_{d-1}, dz+dh(0), \omega(0)=dz$ are linearly dependent.

q.e.d.

PROPOSITION 2.6. *Two involutory analytic sets $V, V'\in \mathscr{V}_d$ satisfy $V\sim V'$ when and only when $\omega|_V$ is isomorphic to $\omega|_{V'}$ as 1-forms on $(2n+1-d)$-dimensional manifolds.*

PROOF. The preceding proposition shows the existence of canonical coordinate systems (p, x, z) and $(\hat{p}, \hat{x}, \hat{z})$ such that

$$V=\{p_1=\cdots=p_{d-1}=z+h(x', p')=0\}\ ,$$

$$V'=\{\hat{p}_1=\cdots=\hat{p}_{d-1}=\hat{z}+\hat{h}(\hat{x}', \hat{p}')=0\}\ .$$

We use the coordinate system $(p_d, \cdots, p_n, x_1, \cdots, x_n)$ for V, then

$$\omega|_V=p_d dx_d+\cdots+p_n dx_n+dh(x', p')\ .$$

Suppose $\omega|_V$ is isomorphic to $\omega|_{V'}$, then there exist functions $q_j(x', p')$, $y_j(x', p')$ for $d\leqq j\leqq n$ such that

$$p_d dx_d+\cdots+p_n dx_n+dh(x', p')=q_d dy_d+\cdots+q_n dy_n+d\hat{h}(y, q)\ .$$

Then

$$\begin{aligned}&dz-(p_1dx_1+\cdots+p_{d-1}dx_{d-1})-(p_d dx_d+\cdots+p_n dx_n)\\&=d(z+h(x', p')-\hat{h}(y, q))-(p_1dx_1+\cdots+p_{d-1}dx_{d-1})\\&\quad-(q_d dy_d+\cdots+q_n dy_n)\ .\end{aligned}$$

Therefore under the canonical coordinate system $(p_1, \cdots, p_{d-1}, q_d, \cdots, q_n, x_1, \cdots x_{d-1}, y_d, \cdots, y_n, w)$ where $w=z+h(x', p')-\hat{h}(y, q)$, we have

$$V=\{p_1=\cdots=p_{d-1}=w+\hat{h}(y, q)=0\}\ .$$

This implies that $V\sim V'$.

The "only if" part of the theorem is clear. q.e.d.

Next we prepare the following lemma to determine the structure of $\mathscr{V}_d^L$. For the proof we refer to Mal'cef [8] or Gantmacher [3].

LEMMA 2.7. *Two skew-symmetric linear transformations on a $2n$-dimensional symplectic vector space $\mathscr{S}$ over $\boldsymbol{C}$ are isomorphic (namely similar by an isometric transformation on $\mathscr{S}$) if and only if the elementary divisors of them coincide. We denote by $\mathscr{E}_{2n}$ the sets of the simple elementary divisors of the skew-symmetric linear transformations on $\mathscr{S}$. Then $E\in\mathscr{E}_{2n}$ when and only when*

i) *for* $\rho\neq 0$ *and* $k\in N$, *the number of* $(\lambda-\rho)^k$ *contained in* E *and that of* $(\lambda+\rho)^k$ *are equal,*

ii) *the number of* λ^{2k-1} *contained in* E *is even,*

iii) *the product of the elements of* E *is a polynomial of* λ *of degree* $2n$.

Using the expression (2.1), we set

$$M(t)=\frac{\partial(a_1,\cdots,a_k)}{\partial(t_1,\cdots,t_k)}(t)\quad\text{for } t\in D(V) \tag{2.4}$$

and

$$M(0)=M_1+M_2\ ,\quad {}^tM_1=-M_1\quad\text{and}\quad {}^tM_2=M_2\ ,$$

where ${}^tM$ means the transposed matrix of a matrix M.

Taking a different coordinate system, $M(0)$ is transformed into $GM(0){}^tG=N(0)$ and

$$N(0)=N_1+N_2\ ,\quad {}^tN_1=-N_1\quad\text{and}\quad {}^tN_2=N_2\ .$$

Since $GM_1{}^tG=N_1$ and $GM_2{}^tG=N_2$, the simple elementary divisors of λM_1-M_2 are independent of the choice of coordinate system of $V\in\mathscr{V}_d$. We denote them by $T_d(V)$.

Suppose $d=1$, then we see that $M_1\in GL(2n, C)$ by Proposition 2.4. Hence for $a={}^t(a_1,\cdots,a_{2n})$ and $b={}^t(b_1,\cdots,b_{2n})\in C^{2n}$, we define the non-degenerate skew-symmetric bilinear form $(a, b)={}^taM_1b\in C$. Thus we get a $2n$-dimensional symplectic vector space $\mathscr{S}$. The equation $(a, M_1^{-1}M_2b)=-(M_1^{-1}M_2a, b)$ holds, which shows that $M_1^{-1}M_2$ defines a skew-symmetric linear transformation. Therefore, we see that $T_1(V)\in\mathscr{E}_{2n}$ by the previous lemma.

For $d>1$, Proposition 2.4 and the argument just above shows that $T_d(V)\in\mathscr{E}_{2(n+1-d)}$. Consequently, we can define the following map:

DEFINITION 2.8. $T_d:\mathscr{V}_d/\sim\ \to\mathscr{E}_{2(n+1-d)}$, where $\mathscr{V}_d/\sim$ denotes the equivalence classes of $\mathscr{V}_d$ defined by Definition 2.2.

Then T_d characterizes the structure of linearly degenerate involutory manifolds.

THEOREM 2.9. *The map*

$$T_d:\mathscr{V}_d^L/\sim\ \to\mathscr{E}_{2(n+1-d)} \tag{2.5}$$

is bijective, where we denote also by T_d *the restriction of the map in Definition 2.8 on* $\mathscr{V}_d^L/\sim$.

PROOF. We will suppose $d=1$ because Proposition 2.4 and Proposition 2.6 reduce general cases to the case when $d=1$.

By giving a concrete form of h in the expression (2.2), we first prove that the map is surjective. We set

$$h=\langle p, Ap\rangle+2\langle p, Cx\rangle+\langle x, Bx\rangle\ , \tag{2.6}$$

$$p={}^t(p_1, \cdots, p_n)\ ,\quad x={}^t(x_1, \cdots, x_n)\ ,\quad {}^tA=A\ ,\quad {}^tB=B\ ,$$

where A, B and C are matrices with elements in C and $\langle a, b\rangle$ is ${}^tab$ for column vectors a and b. Taking the coordinate system (p, x) of V,

$$-\omega|_V=2\{\langle p, Adp\rangle+\langle dp, Cx\rangle+\langle p, Cdx\rangle+\langle x, Bdx\rangle\}+\langle p, dx\rangle\ .$$

Then the Jacobian matrix (2.4) is

$$\begin{aligned}-M(0)&=\begin{pmatrix}2A & 2C\\ 2{}^tC+I_n & 2B\end{pmatrix}\\ &=\frac{1}{2}\begin{pmatrix}0 & -I_n\\ I_n & 0\end{pmatrix}+\frac{1}{2}\begin{pmatrix}4A & 4C+I_n\\ 4{}^tC+I_n & 4B\end{pmatrix}.\end{aligned}$$

Therefore

$$(M(0)-{}^tM(0))^{-1}(M(0)+{}^tM(0))=\begin{pmatrix}4{}^tC+I_n & 4B\\ -4A & -4C-I_n\end{pmatrix}.$$

Take an arbitrary $E\in\mathscr{X}_{2n}$. For a pair $(\lambda-\rho_i)^{k_i}$ and $(\lambda+\rho_i)^{k_i}$ in E, we define the $k_i\times k_i$-matrices

$$C_i=\begin{pmatrix}\frac{1}{4}(\rho_i-1) & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1\\ & & & \frac{1}{4}(\rho_i-1)\end{pmatrix},\quad A_i=B_i=(0)\ . \tag{2.7}$$

For λ^{2k_j} in E, we define the $k_j\times k_j$-matrices

$$C_j=\begin{pmatrix}-\frac{1}{4} & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1\\ & & & -\frac{1}{4}\end{pmatrix},\quad A_j=\begin{pmatrix}0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & 1\end{pmatrix},\quad B_j=(0)\ . \tag{2.8}$$

And we define A, B and C by the direct sums of these A_i, B_i and C_i respectively. Thus we get $V\in\mathscr{V}_1^{\,l}$ satisfying $T_1(V)=E$.

Next we show that the map is injective. Assume $T_1(V)=T_1(V')$ for V, $V'\in\mathscr{V}_1^{\,l}$. Then we rewrite (2.1) as follows:

$$\omega|_V=\langle(M_1+M_2)t, dt\rangle\ ,\quad {}^tM_1=-M_1\ ,\quad {}^tM_2=M_2\ ,$$

$$\omega|_{V'}=\langle(N_1+N_2)t', dt'\rangle\ ,\quad {}^tN_1=-N_1\ ,\quad {}^tN_2=N_2\ .$$

The symplectic vector spaces $\mathscr{S}$ and $\mathscr{S}'$ defined by the bilinear forms $(a, b)={}^taM_1b$

and $(a, b)'={}^t a N_1 b$ are isomorphic because they are of the same dimension. Hence it follows from the assumption, Lemma 2.7 and Proposition 2.8 that the pairs $(\mathscr{S}, M_1^{-1}M_2)$ and $(\mathscr{S}', N_1^{-1}N_2)$ are isomorphic. Set the isomorphism $G: a \to Ga$, then

$$\begin{cases} G(M_1^{-1}M_2)=(N_1^{-1}N_2)G\ , \\ M_1={}^tGN_1G\ . \end{cases}$$

Hence

$$M_2=M_1G^{-1}N_1^{-1}N_2G={}^tGN_2G\ .$$

Under the coordinate system $t''=Gt'$, we have

$$\omega|_{V'}=\langle (M_1+M_2)t'', dt''\rangle\ ,$$

which shows $V\sim V'$ (cf. Proposition 2.6). q.e.d.

In the real analytic category the map (2.5) is neither surjective nor injective because Lemma 2.7 is not valid. We must consider simple elementary divisors with signs. Let A be a skew-symmetric linear transformation on a $2n$-dimensional symplectic vector space $\mathscr{S}$ over $\boldsymbol{R}$. Then $\mathscr{S}$ is expressed as the direct sum of the following A-invariant linear subspaces (cf. Yaglom [17]):

$$\mathscr{S}=\sum_{j=1}^{m}\oplus\mathscr{S}_j\ ,$$

where $\mathscr{S}_j\perp\mathscr{S}_{j'}$ for $j\neq j'$. Moreover each $\mathscr{S}_j$ satisfies one of the following conditions:

$1)^+$ *There exists* $e\in\mathscr{S}_j$ *such that*

$$(e, A^ke)=\begin{cases} 0 & \textit{if } k\neq 2m_j-1\ , \\ 1 & \textit{if } k=2m_j-1\ , \end{cases}$$

or

$$1)^- \qquad =\begin{cases} 0 & \textit{if } k\neq 2m_j-1\ , \\ -1 & \textit{if } k=2m_j-1\ , \end{cases}$$

where $\dim\mathscr{S}_j=2m_j$.

2) *There exist* e_1, $e_2\in\mathscr{S}_j$ *such that*

$$(e_1, A^ke_1)=(e_2, A^ke_2)=0\ ,$$

$$(e_1, A^ke_2)=\begin{cases} 0 & \textit{if } k\neq m_j-1\ , \\ 1 & \textit{if } k=m_j-1\ , \end{cases}$$

where $\dim \mathscr{S}_j = 2m_j$ *and* m_j *is odd.*

3)$^+$ *There exist* $e_1, e_2 \in \mathscr{S}_j$ *and* $\nu_j \in \boldsymbol{R}$ *such that* $\nu_j \neq 0$ *and*

$$(e, (A-\sqrt{-1}\nu_j)^k e) = 0 ,$$

$$(e, (A+\sqrt{-1}\nu_j)^k \bar{e}) = \begin{cases} 0 & \textit{if } k \neq m_j - 1 , \\ (\sqrt{-1})^{m_j} & \textit{if } k = m_j - 1 , \end{cases}$$

or

3)$^-$
$$= \begin{cases} 0 & \textit{if } k \neq m_j - 1 , \\ -(\sqrt{-1})^{m_j} & \textit{if } k = m_j - 1 , \end{cases}$$

where $e = e_1 + \sqrt{-1} e_2$ *and* $\dim \mathscr{S}_j = 2m_j$.

4) *There exist* $e_1, e_2, e_3, e_4 \in \mathscr{S}_j$ *and* $\mu_j, \nu_j \in \boldsymbol{R}$ *such that* $\mu_j \neq 0$, $\nu_j \neq 0$ *and*

$$(e, (A-\mu_j-\sqrt{-1}\nu_j)^k e) = (e, (A-\mu_j+\sqrt{-1}\nu_j)^k \bar{e}) = 0 ,$$

$$(e', (A+\mu_j+\sqrt{-1}\nu_j)^k e') = (e', (A+\mu_j-\sqrt{-1}\nu_j)^k \bar{e}') = 0 ,$$

$$(e, (A+\mu_j+\sqrt{-1}\nu_j)^k e') = \begin{cases} 0 & \textit{if } k \neq m_j - 1 , \\ 1 & \textit{if } k = m_j - 1 , \end{cases}$$

where $e = e_1 + \sqrt{-1} e_2$, $e' = e_3 + \sqrt{-1} e_4$ *and* $\dim \mathscr{S}_j = 4m_j$.

5) *There exist* $e_1, e_2 \in \mathscr{S}_j$ *and* $\mu_j \in \boldsymbol{R}$ *such that* $\mu_j \neq 0$ *and*

$$(e_1, (A-\mu_j)^k e_1) = (e_2, (A+\mu_j)^k e_2) = 0 ,$$

$$(e_1, (A+\mu_j)^k e_2) = \begin{cases} 0 & \textit{if } k \neq m_j - 1 , \\ 1 & \textit{if } k = m_j - 1 . \end{cases}$$

where $\dim \mathscr{S}_j = 2m_j$.

In the above situation, for A we assign to each $\mathscr{S}_j$ the following polynomial called a simple elementary divisor with a sign:

1)$^+$ $+\lambda^{2m_j}$, 1)$^-$ $-\lambda^{2m_j}$,

2) λ^{m_j}, λ^{m_j},

3)$^+$ $+(\lambda^2+\nu_j^2)^{m_j}$, 3)$^-$ $-(\lambda^2+\nu_j^2)^{m_j}$,

4) $(\lambda^2-2\mu_j\lambda+\mu_j^2+\nu_j^2)^{m_j}$, $(\lambda^2+2\mu_j\lambda+\mu_j^2+\nu_j^2)^{m_j}$,

5) $(\lambda-\mu_j)^{m_j}$, $(\lambda+\mu_j)^{m_j}$.

Then two skew-symmetric linear transformations on $\mathscr{S}$ are isomorphic if and only if their simple elementary divisors with signs coincide. We denote by $\mathscr{Z}_{2n}^{R}$ the sets of these simple elementary divisors with signs corresponding to some skew-symmetric linear transformations on $\mathscr{S}$.

In the real analytic category, by (2.4) and by

$$M(0)=M_1+M_2\ ,\quad {}^tM_1=-M_1\ ,\quad {}^tM_2=M_2\ ,$$

we define the skew-symmetric bilinear form on $\boldsymbol{R}^{2n}$

$$(a,b)={}^taM_1b\in\boldsymbol{R}\ .$$

Then $\boldsymbol{R}^{2n}$ is expressed as the direct sum of the two M_2-invariant linear subspaces

$$\mathscr{S}\oplus\mathscr{S}_0=\boldsymbol{R}^{2n}\ ,$$

such that $\mathscr{S}_0\perp\boldsymbol{R}^{2n}$ and $M_1|_{\mathscr{S}}$ is regular. Set $A=(M_1|_{\mathscr{S}})^{-1}M_2|_{\mathscr{S}}$, then by the above correspondence between $(A,\mathscr{S})$ and simple elementary divisors with signs, we can define the map (cf. Definition 2.8)

$$T_d^R:\mathscr{V}_d/\sim\ \longrightarrow\mathscr{Q}^R_{2(n+1-d)}\ , \tag{2.9}$$

which characterizes the structure of linearly degenerate real analytic involutory manifolds. In fact, the same argument of the proof of Theorem 2.9 shows the following theorem.

THEOREM 2.10. *In the real analytic category, the map*

$$T_d^R:\mathscr{V}_d^L/\sim\ \longrightarrow\mathscr{Q}^R_{2(n+1-d)}$$

is bijective.

§2.2. The structure of degenerate involutory manifolds.

Our main purpose in this section is to show that almost all degenerate involutory manifolds are linearly degenerate.

DEFINITION 2.11. For an analytic set V in the $(2n+1)$-dimensional contact manifold X, $\Sigma(V)$ denotes the set of all the singular points of V. We call V is an *integral analytic set* when $\omega|_{V-\Sigma(V)}\equiv 0$. Moreover when $\Sigma(V)=\emptyset$, we call V is an *integral manifold.*

PROPOSITION 2.12. *For any integral manifold V, we can choose a canonical coordinate system so that*

$$V=\{p_1=\cdots=p_n=x_1=\cdots=x_d=z=0\}\ . \tag{2.10}$$

PROOF. We use the induction for the dimension of X. When $n=0$, the proposition is clear, so we assume $n\geqq 1$.

In the case when there exist $f,\ g\in\mathscr{O}$ so that $f|_V\equiv g|_V\equiv 0$ and $[f,g](0)\neq 0$, under a suitable canonical coordinate system, we have

$$Y=\{f=g=0\}=\{p_1=x_1=0\}\ .$$

Since $(Y,\omega|_Y)$ is a $(2n-1)$-dimensional contact manifold and V is an integral manifold in Y, the expression (2.10) follows from the assumption of the induction.

In the other case, choose a canonical coordinate system so that $\{p_1=0\}\supset V$ and set $Y=\{p_1=x_1=0\}$, then $V\cap Y$ is regular. It follows from the assumption of the induction that $V\cap Y=\{p_1=\cdots=p_n=x_1=z=0\}$ and the dimension of V is n. Hence

$$V=\{p_1=p_2+f_2x_1=\cdots=p_n+f_nx_1=z+gx_1=0\}\ ,$$

where the functions f_i and g depend only on the variables $x_1,\cdots,x_n$. We choose $(x_1,\cdots,x_n)$ as a local coordinate system of V. Since $\omega|_V\equiv 0$, its coefficient of dx_1 shows that $x_1\dfrac{\partial g}{\partial x_1}+g=0$, so $g\equiv 0$ (see Theorem 1.1). Next we have $f_i\equiv 0$ by the coefficient of dx_i for $2\leqq i\leqq n$. q.e.d.

COROLLARY 2.12. *The dimension of any integral analytic set is not larger than n and the same dimensional integral manifolds are isomorphic. Especially, every n-dimensional integral manifold is a Lagrangean manifold.*

PROPOSITION 2.13. *Assume that* $V=\{f(p,x,z)=0\}\in\mathscr{V}_1$ *and that* $\dfrac{\partial f}{\partial z}(0)=1$. *Then the map*

$$\begin{array}{ccc} [f\otimes\omega^{(-1)},\cdot]:\mathscr{L}^{\otimes(-s)} & \to & \mathscr{L}^{\otimes(-s)} \\ \cup\!\!\!\!\!\!\cup & & \cup\!\!\!\!\!\!\cup \\ \varphi\otimes\omega^{\otimes(-s)} & \mapsto & \psi\otimes\omega^{\otimes(-s)} \end{array}$$

defines the map $\varphi\mapsto P\varphi=\psi$. *Note that P is a derivation which can be restricted on V. Suppose that we have*

$$(2.11)\qquad \begin{cases} P=\displaystyle\sum_{i=1}^{2n+1} b_i'(t')\frac{\partial}{\partial t_i'}+c'(t')\ , \\[2ex] P|_V=\displaystyle\sum_{i=1}^{2n} b_i(t)\frac{\partial}{\partial t_i}+c(t)\ , \end{cases}$$

under local coordinate systems t′ on X and t on V. Then the following hold.

i) *Let* $(b_1',\cdots,b_{2n+1}',f)$ *be the ideal of* $\mathscr{O}$ *generated by* $b_1',\cdots,b_{2n+1}'$ *and f. Using such an expression, we have*

$$(b_1',\cdots,b_{2n+1}',f)/(f)=(b_1,\cdots,b_{2n})=(a_1,\cdots,a_{2n})\ .$$

Here $(a_1,\cdots,a_{2n})$ *is the ideal of* $\mathscr{O}(V)$ *generated by* $a_1,\cdots,a_{2n}$ *in (2.1). And f belongs to* $(b_1',\cdots,b_{2n+1}')$ *if* $r\neq 0$.

ii) $c(0)=c'(0)=-s$.

iii) *Suppose that* $T_1(V)=\{(\lambda-\rho_i)^{m_i};\ 1\leqq i\leqq k\}$, *then*

$$E(P)=\left\{\left(\lambda-\frac{1}{2}(1+\rho_i)\right)^{m_i},\lambda-r;\ 1\leqq i\leqq k\right\}\ ,$$

$$E(P|_V)=\left\{\left(\lambda-\frac{1}{2}(1+\rho_i)\right)^{m_i};\ 1\leqq i\leqq k\right\}.$$

(See Definition 2.8 and (1.3).)

PROOF. According to the definition of Lagrangean bracket,

$$P=\sum_{j=1}^{n}\left(\frac{\partial f}{\partial x_j}+p_j\frac{\partial f}{\partial z}\right)\frac{\partial}{\partial p_j}-\sum_{j=1}^{n}\frac{\partial f}{\partial p_j}\frac{\partial}{\partial x_j}+\left(rf-\sum_{j=1}^{n}p_j\frac{\partial f}{\partial p_j}\right)\frac{\partial}{\partial z}-s\frac{\partial f}{\partial z}. \tag{2.12}$$

Set $V=\{z+h(x,p)=0\}$, where $h(x,p)=\langle p,Ap\rangle+2\langle p,Cx\rangle+\langle x,Bx\rangle+H(x,p)$ and ${}^tA=A$, ${}^tB=B$ and H has no linear term. By the assumption, we can write $f=(z+h)\cdot g$ and $g(0)=1$. Hence under the coordinate system (p,x,z), $M(0)$ in (1.2) corresponding to P equals

$$\begin{pmatrix} 2{}^tC+I_n & 2B & 0\\ -2A & -2C & 0\\ 0 & 0 & r\end{pmatrix}.$$

On the other hand, $T_1(V)$ is the set of simple elementary divisors of

$$\begin{pmatrix} 4{}^tC+I_n & 4B\\ -4A & -4C\end{pmatrix}$$

as in the proof of Theorem 2.9. Therefore we can know $E(P)$ from $T_1(V)$. In the same way we can know $E(P|_V)$ from $T_1(V)$. The rest part of the proof is clear. q.e.d.

DEFINITION 2.14. We say that an involutory analytic set V is *maximally degenerate* when $\dim D(V)=n$.

Note that $\dim D(V)\leqq n$ since $D(V)$ is an integral analytic set for any $V\in\mathscr{V}_d$.

THEOREM 2.15. *All the maximally degenerate involutory manifolds of the same dimension are isomorphic.*

PROOF. Considering Proposition 2.4 and Proposition 2.6, we have only to prove the theorem in the case when the manifolds are hypersurfaces in X.

Assume $V\in\mathscr{V}_1$ and $\dim D(V)=n$. Using the same notations as in the preceding proposition, we have $D(V)=D(P|_V)$, which shows that

$$\operatorname{rank}\frac{(b_1,\cdots,b_{2n})}{(t_1,\cdots,t_{2n})}(0)\leqq n. \tag{2.13}$$

Combining this with Lemma 2.7 and Definition 2.8 and Proposition 2.13 iii), we see that

$$E(P|_V)=\left\{\left(\lambda-\frac{1}{2}(1-\rho_j)\right),\left(\lambda-\frac{1}{2}(1+\rho_j)\right);\ \rho_j=1,1\leqq j\leqq n\right\},$$

and the left part of (2.13) equals n. Hence $\Sigma(D(V))=\emptyset$ and $D(V)$ is a Lagrangean manifold (cf. Corollary 2.12).

Choose a canonical coordinate system so that $D(V)=\{x_1=\cdots=x_n=z=0\}$ and suppose $V=\{z+g(x,p)=0\}$. Set

$$h(x,p)=g(x,p)+\sum_{i=1}^{n}x_ip_i\,.$$

Then $\omega|_{D(V)}\equiv 0$ implies $\left.\frac{\partial h}{\partial p_i}\right|_{D(V)}\equiv\left.\frac{\partial h}{\partial x_i}\right|_{D(V)}\equiv 0$ for $1\leqq i\leqq n$. Hence h has no linear term, so we can write

$$h=\sum_{\substack{1\leqq i\leqq n\\1\leqq j\leqq n}}h_{ij}(x,p)x_ix_j\,.$$

Consider the equation

$$(2.14)\qquad\begin{cases}\Omega-\displaystyle\sum_{i=1}^{n}x_i\frac{\partial\Omega}{\partial x_i}=\sum_{\substack{1\leqq i\leqq n\\1\leqq j\leqq n}}h_{ij}\left(x_1,\cdots,x_n,-\frac{\partial\Omega}{\partial x_1},\cdots,-\frac{\partial\Omega}{\partial x_n}\right)x_ix_j\,,\\[2ex]\left.\dfrac{\partial\Omega}{\partial x_i}\right|_{x_1=\cdots=x_n=0}=y_i\quad\text{for }1\leqq i\leqq n\,.\end{cases}$$

Then we see that there exists the unique formal solution of the form $\Omega=\sum_{i=1}^{n}y_ix_i+\sum_{|\alpha|\geqq 2}\varphi_\alpha(y)x$, where $\varphi_\alpha(y)$ are analytic at the origin.

We can prove that Ω is analytic by the method of majorant (cf. Oshima [11]), but we employ here another method. Set

$$(2.15)\qquad p_j=-\frac{\partial\Omega}{\partial x_j}\,,\quad q_j=\frac{\partial\Omega}{\partial y_j}\quad\text{for}\quad 1\leqq j\leqq n\,.$$

Then y_j and q_j are expressed by (2.15) as formal power series of (x,p) and we have

$$(2.16)\qquad\begin{cases}y_j|_{x=0}=-p_j\,,\quad q_j|_{x=0}=0\,,\\[1ex]\left.\dfrac{\partial y_j}{\partial x_i}\right|_{x=0}=h_{ij}(0,\cdots,0,p_1,\cdots,p_n)+h_{ji}(0,\cdots,0,p_1,\cdots,p_n)\\[1ex]\qquad\qquad=2\,\dfrac{\partial^2h}{\partial x_i\partial x_j}(0,p)\,,\\[1ex]\left.\dfrac{\partial q_j}{\partial x_i}\right|_{x=0}=0\ \text{ if }i\neq j\text{ and }1\text{ if }i=j.\end{cases}$$

On the other hand, (2.14) and (2.15) show that

$$\sum_{i=1}^{n} p_i dx_i + d\left(h - \sum_{i=1}^{n} x_i p_i\right) = \sum_{i=1}^{n} p_i dx_i + d\Omega$$
$$= \sum_{i=1}^{n}\left(p_i + \frac{\partial\Omega}{\partial x_i}\right)dx_i + \sum_{i=1}^{n}\frac{\partial\Omega}{\partial y_i}dy_i$$
$$= \sum_{i=1}^{n} q_i dy_i\,.$$

Hence

$$dz - \sum_{i=1}^{n} p_i dx_i = d(z+g) - \sum_{i=1}^{n} q_i dy_i\,.$$

Therefore the map F in the ring of formal power series defined by $(p, x, z) \mapsto (p', x', z') = (q, y, z+g)$ is a contact transformation. Then if we show the map is analytic, the above implies the existence of the canonical coordinate system (p', x', z') satisfying $V=\{z'=0\}$ which completes the proof.

For V we can rewrite $P|_V$ in the case when $s=0$ (cf. Proposition 2.13):

$$P|_V = \sum_{j=1}^{n}\left(\frac{\partial g}{\partial x_j} + p_j\right)\frac{\partial}{\partial p_j} - \sum_{j=1}^{n}\frac{\partial g}{\partial p_j}\frac{\partial}{\partial x_j}\,, \tag{2.17}$$

$$= \sum_{j=1}^{n} p'_j \frac{\partial}{\partial p'_j}\,. \tag{2.18}$$

Now it follows from the first part of the proof of the theorem that $P|_V$ satisfies the assumption in Theorem 1.4, which assures the existence of an analytic coordinate transformation G changing (2.18) into (2.17). Then the formal transformation FG does not change the expression (2.18). In view of (2.16), we can apply Corollary 1.6 to FG, thus we see that FG is an analytic coordinate transformation. Hence the map F is also analytic. q.e.d.

THEOREM 2.16. *If $V \in \mathscr{V}_d$ satisfies the following assumption, then V is linearly degenerate.*

A.2.1. *Suppose $T_d(V) = \{(\lambda+1-\mu_i)^{m_i};\ 1 \leqq i \leqq k\}$, then one of the following holds:*

i) *There exists $\theta \in \boldsymbol{R}$ such that $\theta < \arg\mu_i < \theta+\pi$ for $1 \leqq i \leqq k$ and $\sum_{j=1}^{k} \alpha_j\mu_j - \mu_i \neq 0$ for $\alpha \in \boldsymbol{N}^k$, $|\alpha| \geqq 2$ and $1 \leqq i \leqq k$.*

ii) *$m_i = 1$ for $1 \leqq i \leqq k$ and there exists $\nu \in \boldsymbol{R}$ such that $\left|\sum_{j=1}^{k} \alpha_j\mu_j - \mu_i\right| \geqq |\alpha|^{-\nu}$ for $\alpha \in \boldsymbol{N}^k$, $|\alpha| \geqq 2$, $1 \leqq i \leqq k$.*

REMARK 2.17. For $V \in \mathscr{V}_d$ we can write the product of the elements of $T_d(V)$ as the form $\prod_{j=1}^{n+1-d} (\lambda-\rho_j)(\lambda+\rho_j)$. In this case, $(\rho_1, \cdots, \rho_{n+1-d})$ can be any point in

C^{n+1-d}. Now it is known that for any $\varepsilon>0$ there exists a subset S of C^{n+1-d} whose Lebesgue measure in C^{n+1-d} is zero and satisfying the following: If $(\rho_1, \cdots, \rho_{n+1-d}) \in S$, there exists a positive number C such that

$$|\beta_{n+2-d}+\sum_{j=1}^{n+1-d}\beta_j\rho_j| \geqq C|\beta|^{-(n+1-d+\varepsilon)}, \tag{2.19}$$

$$\text{for } \beta\in Z^{n+2-d} \text{ satisfying } |\beta|\equiv\sum_{j=1}^{n+2-d}|\beta_j|\neq 0.$$

Since (2.19) assures A.2.1 ii), we can say that the assumption in Theorem 2.16 generically holds. On the other hand, for example, when $|\mathrm{Re}\,\rho_j|<\frac{1}{3}$ for $1\leqq j\leqq n$, the condition A.2.1 i) holds even if (2.19) is not valid.

REMARK 2.18. Combining this theorem with Theorem 2.9, we see that the map $T_d: \mathscr{V}_d/\sim\ \to \mathscr{H}_{2(n+1-d)}$ is surjective and "generically" injective. Consequently we have the following: Assume $V\in\mathscr{V}_d$ satisfies A.2.1, then we can choose a canonical coordinate system under which V is expressed in a very simple form (cf. (2.7) and (2.8)). For example, if V satisfies A.2.1 ii), then we can write:

$$V=\{p_1=\cdots=p_{d-1}=z+\sum_{i=d}^{n}C_ip_ix_i=0\}, \tag{2.20}$$

where using ρ_j in Remark 2.17, $C_i=\frac{1}{2}(\rho_{i+1-d}-1)$ for $d\leqq i\leqq n$.

PROOF OF THEOREM 2.16. We employ a similar method as in the proof of Theorem 2.15. We may assume V is a hypersurface, then we can write $V=\{z+h(x, p)=0\}$. Set

$$h(x, p)=\langle p, Ap\rangle+2\langle p, Cx\rangle+\langle x, Bx\rangle+H(x, p),$$

as in the proof of Proposition 2.13. Define a quadratic polynomial g of $2n$ variables by

$$g(y, q)=\langle q, Bq\rangle+\langle q, (2{}^tC+I_n)y\rangle+\langle y, Ay\rangle,$$

and consider the equation

$$\begin{cases} \Omega=h\left(x, -\dfrac{\partial\Omega}{\partial x}\right)-g\left(y, \dfrac{\partial\Omega}{\partial y}\right), \\ \Omega=\langle x, y\rangle+(\text{higher order terms}). \end{cases} \tag{2.21}$$

Set $u=\Omega-\langle x, y\rangle$, then (2.21) is transformed into

$$\begin{cases} (P-1)u=-R, \\ u \text{ has no term of degree less than three,} \end{cases} \tag{2.22}$$

where

$$P=\left\langle -2Cx+2Ay, \frac{\partial}{\partial x}\right\rangle+\left\langle -2Bx+(2{}^tC+I_n)y, \frac{\partial}{\partial y}\right\rangle$$

and

$$R=\left\langle A\frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}\right\rangle+\left\langle B\frac{\partial u}{\partial y}, \frac{\partial u}{\partial y}\right\rangle+\text{(a power series of the variables } x,\ y \text{ and } \frac{\partial u}{\partial x} \text{ with no terms of degree less than three).}$$

The Jacobian matrix of the coefficients of P equals $\begin{pmatrix} -2C & 2A \\ -2B & 2{}^tC+I_n \end{pmatrix}$ at the origin, which implies that $E(P)=\left\{\left(\lambda-\frac{1}{2}\mu_i\right)^{m_i};\ 1\leqq i\leqq k\right\}$ (cf. the proof of Proposition 2.13). Hence A.2.1 assures that P satisfies the assumption in Theorem 1.4, so we can choose a coordinate system $(t_1, \cdots, t_{2n})$ so that

$$P=\sum_{j=1}^{2n}\lambda_j t_j\frac{\partial}{\partial t_j}+\sum_{j=1}^{2n-1}\delta_j t_{j+1}\frac{\partial}{\partial t_j},$$

where $\delta_j=0$ or 1 and $2\lambda_j\in\{\mu_1, \cdots, \mu_k\}$ for $1\leqq j\leqq 2n$. Then $R=\left(\text{a quadratic polynomial of } \frac{\partial u}{\partial t_1}, \cdots, \frac{\partial u}{\partial t_{2n}}\right)+\left(\text{a power series of the variables } t_1, \cdots, t_{2n}, \frac{\partial u}{\partial t_1}, \cdots, \frac{\partial u}{\partial t_{2n}} \text{ with no term of degree less than three}\right)$. Hence the equation (2.22) has a unique solution of a formal power series if

$$\sum_{j=1}^{2n}\alpha_j\lambda_j-1\neq 0 \quad \text{for } \alpha\in N^{2n} \text{ satisfying } |\alpha|\geqq 3\,. \tag{2.23}$$

Definition 2.8 shows that for any μ_i there exists μ_j satisfying $\mu_i+\mu_j=2$. Combining this with A.2.1, we see that (2.23) holds.

Setting $p_j=-\frac{\partial\Omega}{\partial x_j}$, $q_j=\frac{\partial\Omega}{\partial y_j}$ for a formal solution of (2.21) we can define a formal coordinate transformation by $(p, x)\mapsto(q, y)$. When we take the coordinate system (p, x) or (q, y) for V, we have

$$\begin{aligned} -\omega|_V &= dh+\sum_{j=1}^{n}p_j dx_j \\ &= d(\Omega+g)+\sum_{j=1}^{n}p_j dx_j \\ &= dg+\sum_{j=1}^{n}q_j dy_j\,. \end{aligned}$$

Since g is a quadratic polynomial of y and q, V is linearly degenerate if the transformation is analytic.

The analyticity of the transformation $(p, x) \mapsto (q, y)$ is similarly proved as in the proof of Theorem 2.15. In fact, its proof is easier than in Theorem 2.15 because $D(P)$ is a point (cf. Corollary 1.6). q.e.d.

REMARK 2.19. In the real analytic category, we see that Theorem 2.15 and Theorem 2.16 also hold in view of their proofs. In the case when $1 \leqq \dim D(V) \leqq n-1$, we must consider the matrix (2.4) with parameter $t \in D(V)$, but here we do not go farther (cf. Theorem 1.4).

We give an involutory manifold which is not linearly degenerate:

EXAMPLE 2.20. Assume $n=1$ and let

$$V=\left\{z-\frac{1}{3}px=0\right\} \in \mathscr{V}_1^L$$

and

$$V'=\left\{z-\frac{1}{3}px+x^3=0\right\} .$$

Then $T_1(V)=T_1(V')=\left\{\left(\lambda-\frac{1}{3}\right), \left(\lambda+\frac{1}{3}\right)\right\}$. We define two differential operators on V or V':

$$\begin{aligned} P|_V &\equiv \left[\left(z-\frac{1}{3}px\right)\otimes\omega^{(-1)}, \cdot\right]\Big|_V \\ &= \frac{2}{3}p\frac{\partial}{\partial p}+\frac{1}{3}x\frac{\partial}{\partial x} , \end{aligned}$$

and

$$\begin{aligned} P'|_{V'} &\equiv \left[\left(z-\frac{1}{3}px+x^3\right)\otimes\omega^{(-1)}, \cdot\right]\Big|_{V'} \\ &= \left(\frac{2}{3}p+3x^2\right)\frac{\partial}{\partial p}+\frac{1}{3}x\frac{\partial}{\partial x} . \end{aligned}$$

The equation $\left(P|_V-\frac{2}{3}\right)u=0$ has a solution p, but the equation $\left(P'|_{V'}-\frac{2}{3}\right)u=0$ has no non-trivial solution. This implies $P|_V \not\sim P'|_{V'}$. Hence we see that $V \not\sim V' \notin \mathscr{V}_1^L$.

In the rest of this section we will mention an application of the results to first order differential equations. We will treat their local theory in the analytic category. Let

$$\text{(2.24)} \qquad \begin{cases} f_i(p, x, z)=0 \quad \text{for} \quad 1 \leqq i \leqq k , \\ z(0)=p_1(0)=\cdots=p_n(0)=0 , \end{cases}$$

be a system of differential equations with one unknown function z of n variables $(x_1, \cdots, x_n)=x$, where p_j denote $\frac{\partial z}{\partial x_j}$ for $1 \leqq j \leqq n$. We regard the space of $2n+1$ variables (p, x, z) as a contact manifold X with the fundamental 1-form $\omega = dz - p_1 dx_1 - \cdots - p_n dx_n$ and denote by π the projection from X to the space Y of n variables $(x_1, \cdots, x_n)$. By the well-known result in the theory of first order differential equations (cf. Carathéodory [1]), we have a one-one correspondence between the solutions of (2.24) and the Lagrangean manifolds contained in $V=\{f_i(p, x, z)=0, 1 \leqq i \leqq k\} \subset X$ and satisfying that the restrictions of π to the manifolds are smooth. The Lagrangean manifolds contained in V are also contained in the maximal involutory analytic set contained in V, which is defined by the set of zeros of the closure of $I(V)$ with respect to Lagrangean bracket. Hence we may assume that V is involutory.

For the sake of simplicity we consider a single equation

$$(2.25) \qquad \begin{cases} f(p, x, z)=0\ , \\ z(0)=p_1(0)= \cdots =p_n(0)=0\ . \end{cases}$$

Then V is an involutory hypersurface. In the case when $\frac{\partial f}{\partial p_1}(0) \neq 0$, there is a one-one correspondence between the solutions of (2.25) and their Cauchy data u on $\{x_1=0\}$ satisfying $u(0)=0$ and $(\operatorname{grad} u)(0)=0$. When $(\operatorname{grad}_p f)(0)=0$ and $(\operatorname{grad}_x f)(0) \neq 0$, (2.25) has no solution. In this case, V is regular and there exist Lagrangean manifolds contained in V as in the case before, but the restriction of π to any one of the manifolds is not smooth. In the case when $(\operatorname{grad}_p f)(0)=(\operatorname{grad}_x f)(0)=0$ and $(\operatorname{grad}_z f)(0) \neq 0$, V is degenerate at the origin. Hence we can apply the results in this chapter to this case.

For example, consider the ordinary differential equation

$$(2.26) \qquad \begin{cases} \left(\frac{dz}{dx}\right)^2 - 2x\frac{dz}{dx} - 4z - 4x^2 = 0\ , \\ z(0) = \frac{dz}{dx}(0) = 0\ . \end{cases}$$

Then $V=\{4z-p^2+2px+4x^2=0\}$ and $T_1(V)=\{(\lambda-2\sqrt{2}), (\lambda+2\sqrt{2})\}$. By the theorem of Roth about the problem of Diophantine approximation for algebraic numbers, we see that V satisfies A.2.1. Hence by Remark 2.18 (or simply by Theorem 2.9), we have $V \sim V' \equiv \{2z'-(2\sqrt{2}+1)p'x'=0\}$. In fact, the contact transformation is given by

$$\begin{cases} p'=p-2(\sqrt{2}+1)x\ , \\ x'=\dfrac{\sqrt{2}}{8}p+\dfrac{2-\sqrt{2}}{4}x\ , \\ z'=z+\dfrac{\sqrt{2}}{16}p^2-\dfrac{\sqrt{2}}{4}x^2-\dfrac{2+\sqrt{2}}{4}px\ . \end{cases}$$

There exist two Lagrangean manifolds $\{p'=z'=0\}$ and $\{x'=z'=0\}$ in V'. The former corresponds to the solution $z=(1+\sqrt{2})x^2$ and the latter to $z=(1-\sqrt{2})x^2$.

In the same way we see that the equation

$$(2.26)' \qquad \begin{cases} \left(\dfrac{dz}{dx}\right)^3+\left(\dfrac{dz}{dx}\right)^2-2(x+z)\dfrac{dz}{dx}-4z-4x^2=0 \\ z(0)=\dfrac{dz}{dx}(0)=0 \end{cases}$$

has two solutions.

In general we have the following:

THEOREM 2.21. *Consider the equation (2.25) and assume that* $\dfrac{\partial f}{\partial z}(0)\neq 0$ *and that* $V=\{f=0\}$ *is linearly degenerate at the origin. Set*

$$f(p,x,0)=f_1(p,x)+f_2(p,x)\ ,$$

and

$$g(p,x,z)=\frac{\partial f}{\partial z}(0)\cdot z+f_1(p,x)\ ,$$

so that f_1 *is a quadratic polynomial and* f_2 *has no term of degree less than three as a power series of* (p,x). *Then the number of the solutions of (2.25) equals that of the equation*

$$(2.27) \qquad \begin{cases} g(p,x,z)=0\ , \\ z(0)=p_1(0)=\cdots=p_n(0)=0\ . \end{cases}$$

The number may be zero or may be infinite. But especially when V *satisfies A.2.1 ii) and the roots of the elements of* $T_1(V)$ *differ from each other, (2.25) has at least one solution and the number of the solution is not more than* 2^n.

EXAMPLE 2.22. Assume $n=1$. The equation

$$(2.28) \qquad x\frac{dz}{dx}-2z-x^2=0$$

has no solution, but the equation

$$x\frac{dz}{dx}-2z=0\ , \tag{2.29}$$

has infinitely many solutions $z=Cx^2$ for $C\in\boldsymbol{C}$. The equation

$$x\frac{dz}{dx}=f(x, z) \tag{2.30}$$

is called a Briot-Bouquet differential equation (cf. Hukuhara [5]). We can apply Theorem 2.21 to (2.30) if $\frac{\partial f}{\partial z}(0)\neq 0$.

To prove the theorem we prepare a lemma.

LEMMA 2.23. *Suppose that (p, x, z) and (p', x', z') are canonical coordinate systems of X. Then $\frac{\partial z'}{\partial z}(0)\neq 0$, $\frac{\partial z'}{\partial p_i}(0)=\frac{\partial z'}{\partial x_i}(0)=0$ for $1\leqq i\leqq n$ and there exists a subset I of $\{1, \cdots, n\}$ such that $\left|\frac{\partial(x_1', \cdots, x_n')}{\partial(t_1, \cdots, t_n)}\right|(0)\neq 0$, where we set $t_i=\begin{cases}x_i & \text{if } i\notin I\ ,\\ p_i & \text{if } i\in I\ .\end{cases}$*

PROOF. Since there exists a non-vanishing function c such that $dz-\sum_{i=1}^{n}p_i dx_i=c(dz'-\sum_{i=1}^{n}p_i'dx_i')$, we have $\frac{\partial z'}{\partial z}(0)=c(0)\neq 0$ and $\frac{\partial z'}{\partial p_i}(0)=\frac{\partial z'}{\partial x_i}(0)=0$.

Assume that the last statement in the lemma does not hold. Then rearranging the indices if necessary, we have an integer $m<n$ and a subset J of $\{1, \cdots, m\}$ such that $\left|\frac{\partial(x_1', \cdots, x_m')}{\partial(t_1, \cdots, t_m)}\right|(0)\neq 0$ and $\left|\frac{\partial(x_1', \cdots, x_m', x_{m+1}')}{\partial(t_1, \cdots, t_m, x_i)}\right|(0)=\left|\frac{\partial(x_1', \cdots, x_m', x_{m+1}')}{\partial(t_1, \cdots, t_m, p_i)}\right|(0)=0$ for $m+1\leqq i\leqq n$, where $t_j=\begin{cases}x_j & \text{if } j\notin J,\\ p_j & \text{if } j\in J.\end{cases}$ Set $s_j=\begin{cases}p_j & \text{if } j\notin J,\\ -x_j & \text{if } j\in J,\end{cases}$ then we can write:

$$\begin{aligned}&\frac{\partial(x_1', \cdots, x_m', x_{m+1}')}{\partial(t_1, \cdots, t_m, x_{m+1}, \cdots, x_n, s_1, \cdots, s_m, p_{m+1}, \cdots, p_n)}(0)\\ &\quad=\begin{pmatrix} A & AB & D & AC\\ a & aB & b & aC\end{pmatrix},\end{aligned} \tag{2.31}$$

where a and b are row vectors of length m and A, B, C and D are matrices of sizes $m\times m$, $m\times(n-m)$, $m\times(n-m)$ and $m\times m$ respectively. In view of $[x_i', x_j']\equiv 0$, we see that $\begin{pmatrix} A & AB\\ a & aB\end{pmatrix}{}^t\begin{pmatrix} D & AC\\ b & aC\end{pmatrix}$ is a symmetric matrix. Hence we have

$$A\,{}^tD+AB\,{}^tC\,{}^tA=D\,{}^tA+AC\,{}^tB\,{}^tA$$

and

$$a\,{}^tD+aB\,{}^tC\,{}^tA=b\,{}^tA+aC\,{}^tB\,{}^tA\ .$$

Hence

$$b=a({}^tD+B\,{}^tC\,{}^tA-C\,{}^tB\,{}^tA)\,{}^tA^{-1} \\ =aA^{-1}D\,. \tag{2.32}$$

This implies that the rank of the matrix (2.31) equals m. Hence the rank of $\dfrac{\partial(x'_1, \cdots, x'_{m+1}, z')}{\partial(p_1, \cdots, p_n, x_1, \cdots, x_n, z)}(0)$ equals $m+1$. This induces a contradiction. q.e.d.

PROOF OF THEOREM 2.21. In view of Theorem 2.9, we can define a canonical coordinate system (p', x', z') so that $V=\{f(p, x, z)=0\}=\{g(p', x', z')=0\}$ and $\dfrac{\partial(p', x', z')}{\partial(p, x, z)}(0)=I_{2n+1}$. We denote by π' the projection from X to the space of variables x' defined by $(p', x', z')\mapsto x'$. Then for a Lagrangean manifold W contained in V, $\pi|_W$ is smooth when and only when $\pi'|_W$ is smooth. The first statement of the theorem follows from this.

Assume V satisfies $A.2.1$ and that the roots of the elements of $T_1(V)$ differ from each other. Then we can choose a canonical coordinate system (p'', x'', z'') so that $V=\{z''+\sum_{i=1}^{n} C_i p''_i x''_i=0\}$ (see Remark 2.18).

First we will prove that every Lagrangean manifold W contained in V is of the following form:

$$W=\{z''=y_1=\cdots=y_n=0\}\,, \tag{2.33}$$

where $y_i=p''_i$ or x''_i for $1\leqq i\leqq n$. Take a canonical coordinate system $(\hat{p}, \hat{x}, \hat{z})$ such that $W=\{\hat{z}=\hat{x}_1=\cdots=\hat{x}_n=0\}$. Since Lemma 2.23 says the existence of a subset $I\subset\{1, \cdots, n\}$ satisfying $\left|\dfrac{\partial(\hat{x}_1, \cdots, \hat{x}_n)}{\partial(y_1, \cdots, y_n)}\right|(0)\neq 0$, we can write:

$$W=\{w+h(q)=y_1+h_1(q)=\cdots=y_n+h_n(q)=0\}\,,$$

where $y_i=\begin{cases} x''_i & \text{if } i\notin I, \\ p''_i & \text{if } i\in I \end{cases}$ and $q_i=\begin{cases} p''_i & \text{if } i\notin I, \\ -x''_i & \text{if } i\in I \end{cases}$ and $w=z''-\sum_{i\in I} p''_i x''_i$. Note that (q, y, w) is a canonical coordinate system. When we write $V=\{w+\sum_{i=1}^{n} C'_i q_i y_i=0\}$, we have

$$T_1(V)=\{(\lambda-2C'_i-1), (\lambda+2C'_i+1)\,;\ 1\leqq i\leqq n\} \tag{2.34}$$

and

$$[y_k+h_k(q), w+\sum_{i=1}^{n} C'_i q_i y_i]=C'_k y_k-\sum_{i=1}^{n}(C'_i+1)q_i\frac{\partial h_k}{\partial q_i}\,. \tag{2.35}$$

Since the function (2.35) vanishes on W, we have

$$\left\{\sum_{i=1}^{n}(C_i'+1)q_i\frac{\partial}{\partial q_i}+C_k\right\}h_k(q)=0\,. \tag{2.36}$$

Combining the assumption with (2.34), we have $\sum_{i=1}^{n}\alpha_i(C_i'+1)+C_k'\neq 0$ for $\alpha\in N^n$. Hence by Theorem 1.2, we see that $h_k\equiv 0$ for $1\leqq k\leqq n$. Moreover we have $h\equiv 0$ in view of $W\subset V$. Thus we have proved that W is of the form (2.33) and therefore the number of the Lagrangean manifolds contained in V equals 2^n.

To complete the proof we will show that there exists a Lagrangean manifold W satisfying that the map $\pi|_W$ is smooth. Applying Lemma 2.23 to (p'', x'', z'') and (p, x, z), we see the existence of a subset $I\subset\{1, \cdots, n\}$ satisfying $\left|\frac{\partial(x_1, \cdots, x_n)}{\partial(y_1, \cdots, y_n)}\right|(0)$ $\neq 0$, where we define (q, y, w) as before. This implies that the Lagrangean manifold $W=\{w=q_1=\cdots=q_n=0\}$ has the required property. q.e.d.

3. Pseudo-differential operators.

Let X be an $(n+1)$-dimensional complex manifold. The cotangential projective bundle of X, which we denote by P^*X, has a natural contact structure. Let $\mathscr{P}^f$ be the sheaf of ring of pseudo-differential operators of finite order on P^*X. For the definition and fundamental properties of $\mathscr{P}^f$ we refer to S-K-K.

In this section we fix a point $x_0^*\in P^*X$ and treat the local theory at that point. We employ local coordinate systems $(x_1, \cdots, x_{n+1}, \eta_1, \cdots, \eta_{n+1})$ and $(p_1, \cdots, p_n, x_1, \cdots, x_n, z)$, where the former corresponds to the point $(x_1, \cdots, x_{n+1}\,;\ \sum_{j=1}^{n+1}\eta_j dx_j)$ and x_0^* corresponds to $(0, \cdots, 0\,;\ dx_{n+1})$ ($\eta_1, \cdots, \eta_{n+1}$ are homogeneous coordinates). The relations $z=x_{n+1}$ and $p_j=-\eta_j/\eta_{n+1}$ $(1\leqq j\leqq n)$ hold between them. The fundamental 1-form is $\omega=\sum_{j=1}^{n+1}\eta_j dx_j=dz-\sum_{j=1}^{n}p_j dx_j$. (Cf. §2.)

Let $P(x, D_x)$ and $Q(x, D_x)$ be pseudo-differential operators of order m defined at x_0^*. Assume that their principal symbols $P_m(x, \eta)$ and $Q_m(x, \eta)$ coincide and that $\operatorname{grad}_{(x,\eta)}P_m(x_0^*)\neq 0$. Then if $V=\{P_m=0\}\neq\emptyset$ is regular as an involutory manifold in P^*X, we can find locally an invertible pseudo-differential operator $U(x, D_x)$ so that $P(x, D_x)U(x, D_x)=U(x, D_x)Q(x, D_x)$ (see S-K-K, Chapter II, Theorem 2.1.2). In §3.1 we consider the same problem when V is degenerate at x_0^*. In this case the next symbols play an important role.

Given a local contact transformation $F: P^*X\to P^*Y$, we can find a local isomorphism $\Phi: F^{-1}\mathscr{P}_X^f\to\mathscr{P}_Y^f$ so that the isomorphism between the principal symbols induced by Φ equals the natural one defined by F (see S-K-K, Chapter II, §3.3 and §4.3). We call Φ a *quantized contact transformation* corresponding to F.

A *pseudo-differential equation* means a coherent left $\mathscr{P}^f$-Module. We treat here the equation $\mathscr{M}$ with one unknown function, that is, there exists a left ideal $\mathscr{J}$ of $\mathscr{P}^f$ so that $\mathscr{M}=\mathscr{P}^f/\mathscr{J}$. The *symbol ideal* J of $\mathscr{P}^f/\mathscr{J}$ is defined by $J\equiv$ {the principal symbol of P; $P\in\mathscr{J}$ and ord $P=0$}, where ord P is the order of P. If the set of zeros of J, which is called the *support* of $\mathscr{M}$ and proved to be always involutory, has no degenerate point and if J is a simple ideal of $\mathscr{O}$, (in this case we say that $\mathscr{M}$ is an equation with *simple characteristics*), then $\mathscr{M}$ is locally transformed into a partial de Rham system by a quantized contact transformation (see S-K-K, Chapter II, Theorem 5.1.2). In §3.2 we investigate the case when the support of $\mathscr{M}$ is degenerate at x_0^*.

§3.1. The equivalence of pseudo-differential operators.

In this section we treat a pseudo-differential operator $P(x, D_x)$ satisfying the following:

A.3.1. *Let $P_m(x, \eta)$ be the principal symbol of $P(x, D_x)$, then* $\mathrm{grad}_{(x,\eta)} P_m(x_0^*)=0$ *and $V=\{P_m=0\}$ is degenerate at x_0^*, where $m=$* ord P.

$P(x, D_x)$ has the expansion $P(x, D_x)=\sum_{j\leqq m} P_j(x, D_x)$, where $P_j(x, \eta)\in\mathscr{L}^{\otimes(-j)}=\mathscr{O}\otimes\omega^{\otimes(-j)}$. In the sequel P_j is used as the above meaning.

Given an invertible pseudo-differential operator of the first order $U(x, D_x)$, we denote by $J_{m-1}(x, \eta)\in\mathscr{L}^{\otimes(-m+1)}$ the principal symbol of $UPU^{-1}-P$. Then we have the following theorem:

THEOREM 3.1. i) *$J_{m-1}(x, \eta)|_{D(V)}$ is determined only by $P_m(x, \eta)$, where $D(V)$ is the set of the degenerate points of V.*

ii) *Assume the principal symbol of $Q(x, D_x)\in\mathscr{P}^f$ equals that of $P(x, D_x)$. If there exist invertible operators $V(x, D_x)$, $W(x, D_x)\in\mathscr{P}^f$ so that*

$$P(x, D_x)V(x, D_x)=W(x, D_x)Q(x, D_x)\ , \tag{3.1}$$

then

$$(P_{m-1}(x, \eta)-Q_{m-1}(x, \eta))|_{D(V)}\in J_{m-1}(x, \eta)|_{D(V)}\boldsymbol{Z}\ . \tag{3.2}$$

PROOF. i) It is clear that J_{m-1} does not depend on P_j for $j\leqq m-1$. Hence we have only to prove that the symbol of order $m-1$ of $UPU^{-1}-U'PU'^{-1}$ vanishes on $D(V)$ for invertible operators of the first order U and U'. The symbols of order $m-1$ of $U'^{-1}(UPU^{-1}-U'PU'^{-1})U=(U'^{-1}U)P-P(U'^{-1}U)$ equals $[V_0(x, \eta), P_m(x, \eta)]$, where V_0 denotes the principal symbol of $U'^{-1}U$. Hence by Proposition 2.13 we see that its restriction to $D(V)$ is identically zero.

ii) (3.1) implies that the principal symbols of V and W coincide. Let the order of V be k and U be an invertible pseudo-differential operator of the first order. As in i) we see that the symbol of order $m-1$ of $(WU^{-k})^{-1}P(VU^{-k})-P = U^kQU^{-k}-P$ vanishes on $D(V)$, because WU^{-k} and VU^{-k} are pseudo-differential operators of order 0 with the same principal symbol. On the other hand, the restriction to $D(V)$ of the principal symbol of $U^kQU^{-k}-Q$ equals $kJ_{m-1}(x, \eta)|_{D(V)}$. Hence we have $(P_{m-1}-Q_{m-1})|_{D(V)}=kJ_{m-1}|_{D(V)}$. q.e.d.

Conversely we have the following:

THEOREM 3.2. *Assume that* $P(x, D_x)$ *and* $Q(x, D_x)\in\mathscr{P}^f$ *have the same principal symbol and satisfy A.3.1. Let* $T_1(V)=\{(\lambda-\rho_i)^{m_i};\ 1\leqq i\leqq K\}$ *(cf. Definition 2.8). Moreover assume that* ord $P>0$ *and that (3.2) and the following A.3.2, A.3.3 and A.3.4 hold. Then we can find an invertible operator* $U(x, D_x)\in\mathscr{P}^f$ *at* x_0^* *so that*

$$P(x, D_x)U(x, D_x)=U(x, D_x)Q(x, D_x)\ . \tag{3.3}$$

A.3.2. *The dimension of* $D(V)$ *equals the number of the elements of* $T_1(V)$ *satisfying* $\rho_i=-1$. *In this case we will say that* V *is simply degenerate.*

A.3.3. *If* $(\lambda+1)^{m_i}\in T_1(V)$, *then* $m_i=1$.

A.3.4. *There exists* $\theta\in\boldsymbol{R}$ *such that* $\rho_i=-1$ *or* $\theta<\arg(\rho_i+1)<\theta+\pi$ *for* $1\leqq i\leqq K$.

PROOF. (Cf. S-K-K, Chapter II, Theorem 2.1.2.) To simplify the indices we assume that Q is of the first order, but the proof in the general case (i.e. ord $Q>0$) is the same as in the case ord $Q=1$. We will use the notations such as

$$Q(x, D_x)=\sum_{j\geqq -1} Q_{-j}(x, D_x)$$

and

$$Q_{-j}(x, \eta)=q_{-j}(x', z, p)\otimes\omega^{-j}\ ,$$

where $x'=(x_1, \cdots, x_n)$.

Considering the transformation $\left(\dfrac{\partial}{\partial x_{n+1}}\right)^i Q(x, D_x)\left(\dfrac{\partial}{\partial x_{n+1}}\right)^{-i}$, we may assume in the proof that $P_0(x, \eta)|_{D(V)}\equiv Q_0(x, \eta)|_{D(V)}$ (cf. the proof of Theorem 3.1) and multiplying Q by a constant number we may also assume $\dfrac{\partial q_1}{\partial z}(0)=1$.

Set $R=P-Q$, then we have by (3.3)

$$U(x, D_x)Q(x, D_x)-Q(x, D_x)U(x, D_x)=R(x, D_x)U(x, D_x) \tag{3.4}$$

and by the above assumption

$$r_0(x', z, p)|_{D(V)}\equiv 0\ . \tag{3.5}$$

For $F\in\mathscr{L}^{\otimes k_1}$, $G\in\mathscr{L}^{\otimes k_2}$, $\alpha\in\boldsymbol{N}^n$ and $i\in\boldsymbol{N}$, we use the notation

$$(3.6) \qquad (F, G)^{\alpha, i} = \frac{1}{\alpha! i!} D_\eta^{(\alpha, i)} F(x, \eta) D_x^{(\alpha, i)} G(x, \eta) ,$$

where $D_x^{(\alpha,i)} = \dfrac{\partial^{|\alpha|+1}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n} \partial x_{n+1}^i}$ and $D^{(\alpha,i)} = \dfrac{\partial^{|\alpha|+1}}{\partial \eta_1^{\alpha_1} \cdots \partial \eta_n^{\alpha_n} \partial \eta_{n+1}^i}$. Then by comparing the symbols of order $-k$ of the both sides of (3.4) we have

$$(3.7) \qquad [U_{-k}, Q_1] - R_0 U_{-k} = \sum_{\substack{k=i+j+|\alpha|+l \\ j<k}} \{(R_{-i}, U_{-j})^{\alpha,l} + (Q_{-i}, U_{-j})^{\alpha,l} - (U_{-j}, Q_{-i})^{\alpha,l}\} .$$

Especially when $k=0$, we have

$$(3.7)' \qquad [U_0, Q_1] - R_0 U_0 = 0 .$$

Set $H_k = [\cdot, Q_1] - R_0 : \mathscr{L}^{\otimes k} \to \mathscr{L}^{\otimes k}$. Combining the assumptions with Definition 2.8 and Proposition 2.18, we see that H_k satisfies the assumption in Theorem 1.1. Thus we know that (3.7)′ has a unique solution with the Cauchy datam $U_0|_{D(V)} \equiv 1$ (see (3.5)) and that U_{-k} $(k \geqq 1)$ are inductively and uniquely determined by (3.7).

We want to prove that $\sum_{j=0}^{\infty} U_{-j}(x, D_x)$ defines a pseudo-differential operator, which is equivalent to say that there exists a neighbourhood V of x_0^* so that

$$(3.8) \qquad \overline{\lim_{j\to\infty}} \sqrt[j]{\sup_{x^* \in V} |u_{-j}(x^*)|/j!} < \infty ,$$

where $U_{-j}(x, \eta) = u_{-j}(x', z, p) \otimes \omega^{\otimes j}$. We will employ the method of majorant.

Using the coordinate system (p, x', z), for $\varphi \in \mathscr{O}$ we have

$$(3.9) \qquad \begin{cases} D_{x_i}(\varphi \otimes \omega^{\otimes k}) = \dfrac{\partial \varphi}{\partial x_i} \otimes \omega^{\otimes k} \quad \text{for } 1 \leqq i \leqq n , \\[2ex] D_{x_{n+1}}(\varphi \otimes \omega^{\otimes k}) = \dfrac{\partial \varphi}{\partial z} \otimes \omega^{\otimes k} , \\[2ex] D_{\eta_i}(\varphi \otimes \omega^{\otimes k}) = -\dfrac{\partial \varphi}{\partial p_i} \otimes \omega^{\otimes k+1} \quad \text{for } 1 \leqq i \leqq n , \\[2ex] D_{\eta_{n+1}}(\varphi \otimes \omega^{\otimes k}) = \left(-k - p_i \displaystyle\sum_{i=1}^{n} \frac{\partial \varphi}{\partial p_i}\right) \otimes \omega^{\otimes k+1} . \end{cases}$$

Let $\lambda_1, \cdots, \lambda_{K'}$ be all the non-zero elements of $\{1+\rho_i ; 1 \leqq i \leqq K\}$. Then A.3.4 assures the existence of a positive number C_1 satisfying

$$\left| \frac{1}{2} \sum_{j=1}^{K'} \beta_j \lambda_j + l \right| \geqq C_1(2|\beta| + l) \quad \text{for } \beta \in N^{K'} \text{ and } l \in N .$$

Hence in view of the proof of Theorem 1.1 (see Oshima [10]), we can choose a coordinate system $(y_1, \cdots, y_m, z_1, \cdots, z_{m'})$ (where $m = \operatorname{codim} D(V)$ and $m' = \dim D(V)$)

so that the following hold:

Set $s=y_1+\cdots+y_m$, $t=z_1+\cdots+z_{m'}$ and $u=s+t$. Then there exist positive numbers $r\leqq 1$ and C_2 such that when we define H_k^* by

$$(3.10)\qquad H_k^*=\left(C_1-\frac{C_2u}{r-u}\right)s\frac{\partial}{\partial s}-\frac{C_2u}{r-u}s\frac{\partial}{\partial t}+kC_1-k\frac{C_2u}{r-u}-\frac{C_2u}{r-u}\,,$$

then "$H_k^*\varphi(s,t)\gg H_k\psi(y,z)$" implies "$\varphi\gg\psi$" for $k\geqq 1$. Here and in the sequel, we use the notation $\gg$ in the ring of formal power series of the variables (y,z).

Now we introduce the following notations:

$$v=\frac{1}{\rho-u}=\frac{1}{\rho-(y_1+\cdots+y_m+z_1+\cdots+z_{m'})}\,,\quad\text{where } \rho>0\,,$$

$$\mathscr{F}=\{Cv^i;\ C>0 \text{ and } i\in N-\{0\}\}\,,$$

$$D=\frac{d}{du}:\begin{array}{ccc}\mathscr{F} & \to & \mathscr{F}\\ \cup & & \cup\\ Cv^i & \mapsto & Civ^{i+1}\,.\end{array}$$

And we define the notation "$F_1\gg F_2$" for two maps F_1 and F_2 from $\mathscr{F}$ into the ring of formal power series of (y,z), which means $F_1\varphi\gg F_2\varphi$ for any $\varphi\in\mathscr{F}$. Here we fix the positive number ρ so that the following hold:

$$(3.11)\qquad\begin{cases}\rho<\dfrac{C_1r}{C_1+2C_2}\,,\\ u_0\ll v,\ r_{-i}\ll(i+1)!\,v^{i+2} \text{ for } i\geqq 0,\ q_{-j}\ll(j+1)!\,v^{j+2} \text{ for } j\geqq -1,\\ \displaystyle\sum_{\tau=1}^{n}p_\tau\frac{\partial}{\partial p_\tau}\ll vD,\ \frac{\partial}{\partial z}\ll vD,\ \frac{\partial}{\partial p_\tau}\ll vD \text{ and } \frac{\partial}{\partial x_\tau}\ll vD \text{ for } 1\leqq\tau\leqq n.\end{cases}$$

In view of (3.7), (3.9), (3.10) and (3.11), we see that if $\varphi_k\in\mathscr{F}$ satisfy the following (3.12), then $\varphi_k\gg u_{-k}$.

$$(3.12)\qquad\begin{cases}\varphi_0\gg v\\ H_k^*\varphi_k\gg W_k \text{ for } k=1,2,3,\cdots\,,\end{cases}$$

where

$$W_k=\sum_{\substack{k=i+j+|\alpha|+l\\ i\geqq -1,\,j<k}}2\frac{(i+1)!}{\alpha!\,l!}\{(vD)^{|\alpha|}\prod_{\tau=1}^{l}(|i+\tau-1|+vD)v^{i+2}\}(vD)^{|\alpha|+l}\varphi_j$$
$$+\frac{(i+1)!}{\alpha!\,l!}\{(vD)^{|\alpha|}\prod_{\tau=1}^{l}(j+\tau-1+vD)\varphi_j\}(vD)^{|\alpha|+l}v^{i+2}\,.$$

First we will compute W_k.

Since $D^i\gg ivD^{i-1}$, we have by induction

(3.13)
$$\begin{aligned}(vD)^i &\ll (vD)(2^{i-1}v^{i-1}D^{i-1})\\ &=2^{i-1}v^i((i-1)vD^{i-1}+D^i)\\ &\ll 2^i v^i D^i \text{ for } i\geqq 1\,.\end{aligned}$$

If $i\geqq -1$, $j\geqq 1$ and $j\geqq i$, we have

$$\begin{aligned}\prod_{\tau=1}^{l}(|1+\tau-1|+vD)v^j &\ll \prod_{\tau=1}^{l}(i+\tau+1+vD)v^j\\ &\ll\left\{\prod_{\tau=1}^{n}\left(1+\frac{i+\tau+1}{j+2\tau-2}\right)\right\}(vD)^l v^j\,,\end{aligned}$$

because $(\tau+vD)v^j\ll(\tau+j)v^{j+2}=\left(1+\frac{\tau}{j}\right)(vD)v^j$. Now $\frac{i+\tau+1}{j+2\tau-2}\leqq 3$ in the above, so we have

(3.14)
$$\prod_{\tau=1}^{l}(|i+\tau-1|+vD)v^j\ll 4^l(vD)^l v^j\,.$$

Now we set

(3.15)
$$\varphi_k=k!\,M^k v^{Nk+1}\,,$$

where the positive integers M and N will be determined later. In view of $N\geqq 1$, we can apply (3.14) and have the following:

$$\begin{aligned}W_k &\ll \sum_{\substack{k=i+j+|\alpha|+l\\ i\geqq -1,\, j<k}}\frac{3(i+1)!}{\alpha!\,l!}4^l\{(vD)^{|\alpha|+l}v^{i+2}\}(vD)^{|\alpha|+l}\varphi_j\\ &\ll \sum_{\substack{k=i+j+\tau\\ i\geqq -1,\, j<k}}\frac{3(i+1)!}{\tau!}4^\tau(n+1)^\tau\{(vD)^\tau v^{i+2}\}(vD)^\tau\varphi_j\end{aligned}$$

(Apply (3.13) and set $C_3=2^4(n+1)$.)

$$\begin{aligned}&\ll \sum_{\substack{k=i+j+\tau\\ i\geqq -1,\, j<k}}\frac{3(i+1)!}{\tau!}C_3^\tau v^{2\tau}(D^\tau v^{i+2})(D^\tau\varphi_j)\\ &= \sum_{\substack{k=i+j+\tau\\ i\geqq -1,\, j<k}}3C_3^\tau(i+1)!\,v^{3\tau+i+2}\frac{(\tau+i+1)!}{\tau!\,(i+1)!}D^\tau\varphi_j\\ &\ll \sum_{\substack{k=i+j+\tau-1\\ i,j,\tau\geqq 0,\, j<k}}(2C_3)^{i+\tau+1}i!\,v^{3\tau+i+1}D^\tau\varphi_j\,.\end{aligned}$$

On the other hand,

$$H_k^* v^j=\left(C_1-\frac{2C_2u}{r-u}\right)sDv^j+k\left(C_1-\frac{2C_2u}{r-u}\right)v^j+(k-1)\frac{C_2u}{r-u}v^j$$

and

$$\left(C_1-\frac{2C_2u}{r-u}\right)v=\frac{C_1+2C_2}{r-u}\left(\frac{C_1r}{C_1+2C_2}-u\right)\frac{1}{\rho-u}\gg\frac{C_1+2C_2}{r-u}\gg\frac{C_1+2C_2}{r}\ .$$

Hence we have

$$H_k^*v^{Nk+1}\gg k\frac{C_1+2C_2}{r}v^{Nk}\ .$$

Set $N=10$. Then for non-negative integers i, j, k and τ satisfying $k=i+j+\tau-1$ and $k>j$, we have

$$\begin{aligned}&Nk-(3\tau+i+1)+\tau+(Nj+1)=10(k-j)-(4\tau+2+i)\\&\geqq 10(k-j)-4(\tau+i)-2=6(k-j)-6\geqq 0\ ,\end{aligned}$$

and

$$D^\tau\varphi_j=j!M^j\frac{(10j+\tau)!}{(10j)!}v^{\tau+10j+1}\ .$$

Therefore, combining the above facts, we see that if M satisfies

$$(3.16)\qquad (k!M^k)\left(k\frac{C_1+2C_2}{r}\right)\geqq\sum_{\substack{k=i+j+\tau-1\\ j<k}}(2C_3)^{i+\tau+1}i!j!M^j\frac{(10j+\tau)!}{(10j)!}$$

for $k\geqq 1$, then (3.15) satisfies (3.12) and in view of $\varphi_k\gg u_k$, we have (3.8) for $V=\{(y,z)\,;\ |y_1|+\cdots+|y_m|+|z_1|+\cdots+|z_{m'}|<\rho/2\}$. Consequently, to complete the proof, it is sufficient to prove the existence of M satisfying (3.16).

Consider the following inequalities:

$$\begin{aligned}(3.17)\qquad &\sum_{\substack{k=i+j+\tau-1\\ j<k}}\frac{1}{k!M^kk}\cdot(2C_3)^{i+\tau+1}i!j!M^j\frac{(10j+\tau)!}{(10j)!}\\
&=\sum_{\substack{k=i+j+\tau-1\\ j<k}}(2C_3)^2\frac{i!j!(10j+\tau)!}{k!k(10j)!}\left(\frac{2C_3}{M}\right)^{k-j}\\
&=4C_3^2\sum_{\substack{1\leqq l\leqq k\\ 0\leqq i\leqq l+1}}\frac{i!(k-l)!(10k-9l-i+1)!}{kk!(10k-10l)!}\left(\frac{2C_3}{M}\right)^{k-j}\\
&\leqq 12C_3^2\sum_{1\leqq l\leqq k}\frac{(k-l)!(10k-9l+1)!}{kk!(10k-10l)!}\left(\frac{2C_3}{M}\right)^l\\
&=12C_3^2\sum_{1\leqq l\leqq k}\frac{10k-9l+1}{k}\left(\prod_{\tau=0}^{l-1}\frac{10k-9l-\tau}{k-l}\right)\left(\frac{2C_3}{M}\right)^l\\
&\leqq 120C_3^2\sum_{1\leqq l\leqq k}\left(\frac{20C_3}{M}\right)^l\ .\end{aligned}$$

Take M so that $M\geqq\dfrac{4800C_3^3r}{C_1+2C_2}+40C_3$, then (3.17)$<\dfrac{C_1+2C_2}{r}$, which implies (3.16).

q.e.d.

The above proof shows the following:

COROLLARY 3.3. *Suppose $P(x, D_x) \in \mathscr{P}^f$ satisfies A.3.1, A.3.2, A.3.3, A.3.4 and* ord $P>0$, *then we can characterize the invertible operator $U(x, D_x) \in \mathscr{P}^f$ so that* $P(x, D_x)U(x, D_x)=U(x, D_x)P(x, D_x)$:

The operator U is of the first order and uniquely determined by the datum $U_0(x, \eta)|_{D(V)}$, which can be any analytic function satisfying $U_0(x_0^*)\neq 0$. Moreover, if we assume $D(V)$ is the point x_0^*, then the invertible operator which commutes with P is the operator of multiplication by a constant number.

REMARK 3.4. In Theorem 3.2, the assumption ord $P>0$ is necessary. In fact, suppose that the principal symbols of $P(x, D_x)$ and $Q(x, D_x)$ are x_{n+1}. Then, for the existence of an invertible operator $U \in \mathscr{P}^f$ satisfying (3.3), it is necessary and sufficient that $P_{-1}(x, \eta)|_{V'} \equiv Q_{-1}(x, \eta)|_{V'} \bmod \boldsymbol{Z}$ holds, where $V'=\{\eta_1=\cdots=\eta_n=0\}$. On the other hand, $D(V)=\{\eta_1=\cdots=\eta_n=x_{n+1}=0\}$, and if $P_{-1}(x, \eta)|_{D(V)} \equiv Q_{-1}(x, \eta)|_{D(V)}$ mod $\boldsymbol{Z}$, then there exist invertible operators V and W satisfying (3.1). This is proved by applying Theorem 3.2 after multiplying P by an invertible operator to make the order of P positive.

§ 3.2. The structure of pseudo-differential equations.

In this section we investigate the local structure of pseudo-differential equations. Let V be an involutory manifold containing x_0^*. Consider the pseudo-differential equations with support V and satisfying the following condition:

A.3.5. *The equations have one unknown function respectively and their symbol ideals are simple at x_0^*.*

Then the equations are transformed by a quantized contact transformation into the following equations $\mathscr{N}$ at x_0^*:

THEOREM 3.5. (S-K-K, Chapter II, Theorem 5.1.2.) *If V is regular involutory manifold of codimension d, then*

$$\mathscr{N}: \frac{\partial u}{\partial x_i}=0, \quad i=1, \cdots, d. \tag{3.18}$$

THEOREM 3.6. *If V is a maximally degenerate involutory manifold of codimension d, then*

$$\mathscr{N}_f \begin{cases} \dfrac{\partial u}{\partial x_i}=0, \quad i=1, \cdots, d-1, \\ x_{n+1}\dfrac{\partial u}{\partial x_{n+1}}+f(x_d, \cdots, x_n)u=0, \end{cases} \tag{3.19}$$

where f is an analytic function of variables $x_d, \cdots, x_n$. In this case, $\mathcal{N}_f$ is isomorphic to $\mathcal{N}_{f+l}$ for $l \in \boldsymbol{Z}$.

THEOREM 3.7. *If V is of codimension d and satisfies A.2.1 i) and A.2.1 ii), then*

$$\mathcal{N}_C \begin{cases} \dfrac{\partial u}{\partial x_i}=0, \quad i=1, \cdots, d-1, \\ x_{n+1}\dfrac{\partial u}{\partial x_{n+1}}+\sum_{j=d}^{n} C_j x_j \dfrac{\partial u}{\partial x_j}+Cu=0, \end{cases} \tag{3.20}$$

where C_i are constant numbers determined by V (see Remark 2.18) and C is a number depending on the corresponding equation. In this case, $\mathcal{N}_C$ is isomorphic to $\mathcal{N}_{C+l}$ for any integer l.

Using Theorem 2.15 (resp. Theorem 2.9 and Theorem 2.16) and Theorem 3.2, we can prove Theorem 3.6 (resp. Theorem 3.7) in the same way as in the proof of Theorem 3.5 in S-K-K. We leave the details to the reader.

REMARK 3.8. Omitting A.2.1 ii) from the assumption in Theorem 3.7, we have a similar result by applying Theorem 2.9. For example, assume $n=d=1$ and that $T_1(V)=\{\lambda^2\}$, then

$$\mathcal{N}_C: \frac{\partial^2 u}{\partial x_1^2}-\frac{1}{2}x_1\frac{\partial^2 u}{\partial x_1 \partial x_2}+x_2\frac{\partial^2 u}{\partial x_2^2}+C\frac{\partial u}{\partial x_2}=0.$$

Here $\mathcal{N}_C$ is defined at $x_0^*=(0; dx_2)$ and $\mathcal{N}_C$ is isomorphic to $\mathcal{N}_{C+l}$ for any integer l.

REMARK 3.9. When V is a Lagrangean manifold (i.e. $d=n+1$), the equations are maximally overdetermined systems. Then, Theorem 3.7 (or Theorem 3.6) says that they are characterized by the number C modulus $\boldsymbol{Z}$. (Cf. S-K-K, Chapter II, § 4).

REMARK 3.10. We have studied pseudo-differential equations in the complex domain. But also in the real domain, if their supports in the pure imaginary cosphere bundle are real (cf. S-K-K, Chapter II, § 2.1), we have a similar result by the same argument. In fact, both Theorem 3.5 and Theorem 3.6 hold and Theorem 3.7 changes a little in the real domain (cf. Theorem 2.10 and Remark 2.19).

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