Schwarzian equations

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Introduction

The purpose of this paper is to provide an algebraic basis for the theory of Schwarzian differential equations defining uniformizations of algebraic curves.

In §1, we define the Schwarzian derivatives and the "S-operators" under an abstract setting. The well-known basic properties of Schwarzian derivatives are stated and verified under this abstract setting. Some examples in characteristic $p>0$ are given.

In §2, the equations defining uniformizations, or *the canonical S-operators*, are defined, and the main known results are reviewed.

After these expository parts, in §§3, 4, we state and prove the "k-rationality theorems" (Th. A, Th. B), whose corollary (Th. C) asserts that if $\mathscr{C}/k$ is the Shimura's canonical model of the quotient of the complex upper half plane by an arithmetic fuchsian group (which is defined over a certain classfield k), then the equation defining the uniformization of $\mathscr{C}$ is also "defined over k".

Most part of this paper came out of Ch. 2 of Vol. 2 of my previous lecture note [2.1].

In a forthcoming paper, we shall study more arithmetic properties of the equation; in particular discuss the "p-integrality" of the equation and algebraic solutions of its reduction mod p. Some of these results appeared in my previous notes [2.2], [2.3], [2.4] (available at Univ. of Tokyo).

§1. The S-operators.

§1-1. Differentials. Let K be a field. We shall fix a *one* dimensional vector space $D(K)$ over K and a *differentiation*

$$d: K \mapsto D(K)\ ,$$

which, by definition, satisfies

$$d(x+y)=dx+dy,\quad d(xy)=xdy+ydx \quad (x, y \in K)\ .$$

For each $h \geqq 1$, let $D^h(K)$ denote the tensor product of h copies of $D(K)$ over K, and put $D^0(K)=K$. Call an element of $D(K)=D^1(K)$ *a differential of* K, and an element of $D^h(K)$ *a differential of K of degree h*. Put $D(K)^\times=D(K)-\{0\}$, and let $\xi\in D(K)^{\times}$. Then, each element η of $D^{h}(K)$ $(h\geqq 1)$ is expressed uniquely in the form $\eta=a\cdot\xi^{h}$ $(a\in K)$, where $\xi^{h}=\xi\otimes\cdots\otimes\xi$ (h copies). We shall then write $a=\eta/\xi^{h}$. The kernel of d is a subfield of K, called the *constant field*, and is denoted by k;

$$k=\{x\in K\,|\,dx=0\}\ .$$

In the following, we shall assume that $d\neq 0$; or equivalently, $k\neq K$.

§1-2. **The symbol** $\langle\eta,\xi\rangle$. For each ξ, $\eta\in D(K)^{\times}$, an element $\langle\eta,\xi\rangle$ of $D^{2}(K)$ will be defined by

$$\langle\eta,\xi\rangle=\frac{2w_1w_3-3w_2^2}{w_1^2}\cdot\xi^2\ ,$$

where

$$w_1=\eta/\xi,\quad w_{i+1}=dw_i/\xi\quad (i\geqq 1)\ .$$

This is an abstract definition of the Schwarzian derivative. The following Propositions are well-known for the classical (analytic) Schwarzian derivatives:

PROPOSITION 1. $\langle\eta,\zeta\rangle-\langle\xi,\zeta\rangle=\langle\eta,\xi\rangle$.

COROLLARY. $\langle\xi,\eta\rangle=-\langle\eta,\xi\rangle$; $\langle\xi,\xi\rangle=0$.

PROPOSITION 2. *Let* ξ, $\eta\in D(K)^{\times}$. *Then the necessary and sufficient condition for* $\langle\eta,\xi\rangle=0$ *is the following.*

(i) *If* ξ *is exact, i.e.,* $\xi=dx$ $(x\in K)$, *then the condition is that* $\eta=dy$, *where* y *is some linear fractional transform of* x *over* k.

(ii) *If* ξ *is non-exact, then the condition is that* $\eta=a\cdot\xi$ *with some* $a\in k^{\times}$.

PROOF OF PROPOSITION 1. Put

$$w_1=\eta/\xi\ ,\qquad w_{i+1}=dw_i/\xi\ ,$$
$$x_1=\xi/\zeta\ ,\qquad x_{i+1}=dx_i/\zeta\ ,$$
$$y_1=\eta/\zeta\ ,\qquad y_{i+1}=dy_i/\zeta\ ,$$

$(i=1,2,3)$. Then $w_1=y_1/x_1$, $w_2=(dw_1/\zeta)/(\xi/\zeta)=x_1^{-3}(x_1y_2-x_2y_1)$, $w_3=(dw_2/\zeta)/(\xi/\zeta)$ $=x_1^{-5}\{x_1(x_1y_3-x_3y_1)-3x_2(x_1y_2-x_2y_1)\}$. It is enough to prove

$$\frac{2y_1y_3-3y_2^2}{y_1^2}-\frac{2x_1x_3-3x_2^2}{x_1^2}=\frac{2w_1w_3-3w_2^2}{w_1^2}x_1^2\ . \tag{1}$$

But (1) becomes an identity between the two rational functions of x_i and y_i $(1\leqq i\leqq 3)$, if we substitute our formula for w_i (by x_j and y_j) on the right side.

Q.E.D.

PROOF OF PROPOSITION 2. First, assume that the characteristic of k is different from 2. If $\xi=dx$, *then* $\langle\eta, \xi\rangle=0$ is equivalent to $2w_1w_1''=3(w_1')^2$, and hence to $(z')^2=2zz''$, where $z=w_1^{-1}$ and $'$ denotes the differentiation by x. Differentiating the last equation by x, we obtain $z'''=0$; hence $z=Ax^2+Bx+C$ with $A, B, C\in k$. But in order that z satisfy $(z')^2=2zz''$, it is necessary and sufficient that z is of the form $z=a(bx+c)^2$ ($a, b, c\in k$). But this is also equivalent to $\eta=dy$ with some linear fractional transform y of x over k. Now, let ξ be non-exact. It is clear that the condition is sufficient. Conversely, suppose that we had $\langle\eta, \xi\rangle=0$ but $w_1\notin k$. Then $w_1w_3=(3/2)w_2^2$; hence $d(w_1w_2^{-1})=w_2^{-2}(w_2^2-w_1w_3)\xi=(-1/2)\xi$; hence ξ is exact, which is a contradiction. Hence $w_1\in k$. In the case of characteristic 2, $\langle\eta, \xi\rangle=0$ is always equivalent to $w_1\in k$, and when $\xi=dx$, this is also equivalent to $\eta=dy$ with a linear fractional transform y of x over k.

(Note that $K^p\subset k$ if p is the characteristic.) Q.E.D.

§ **1-3. *S*-operators.** By Proposition 1, $\langle\eta, \xi\rangle$ behaves like the difference $\eta-\xi$. So, we shall introduce the following notion of the S-operator.

DEFINITION. *A map*

$$S: D(K)^\times\to D^2(K)$$

will be called an S-operator (of K) if

$$S\langle\eta\rangle-S\langle\xi\rangle=\langle\eta, \xi\rangle$$

holds for all $\xi, \eta\in D(K)^\times$.

Let ζ be any fixed element of $D(K)^\times$. Then the map S_ζ defined by $S_\zeta\langle\xi\rangle=\langle\xi, \zeta\rangle$ gives an S-operator (Proposition 1). By the definition of S-operators, the difference of two S-operators is a constant; hence all other S-operators are given by $S\langle\xi\rangle=S_\zeta\langle\xi\rangle+C$, where C is an arbitrary constant in $D^2(K)$. An S-operator of the form S_ζ is called an *inner* S-operator (w.r.t. ζ). An S-operator S is inner if and only if

$$S\langle\zeta\rangle=0$$

holds for some $\zeta\in D(K)^\times$. (Then S is inner w.r.t. ζ.) In general, not all S-operators are inner. This is equivalent to saying that not all elements of $D^2(K)$ are of the form $\langle\zeta, \zeta'\rangle$ ($\zeta, \zeta'\in D(K)^\times$).

Let σ be an automorphism of K over k, commuting with the differentiation d; i.e., $(dy/dx)^\sigma=d(y^\sigma)/d(x^\sigma)$ ($x, y\in K, x\notin k$). Then σ acts on $D^h(K)$ by $\xi=y(dx)^h\to\xi^\sigma=y^\sigma(d(x^\sigma))^h$. Since σ commutes with d, $\langle\eta, \xi\rangle^\sigma=\langle\eta^\sigma, \xi^\sigma\rangle$ holds for any $\xi, \eta\in D(K)^\times$. Let S be an S-operator of K. Then the map $S^\sigma: D(K)^\times\to D^2(K)$

defined by $S^\sigma\langle\xi\rangle=S\langle\xi^\sigma\rangle^{\sigma^{-1}}$ is again an S-operator of K. We say that S *is σ-invariant* if $S^\sigma=S$.

§1-4. Effect of separable extensions. Let K, $D(K)$, d be as in §1-1, and let L be a separably algebraic extension of K. Put $D(L)=D(K)\otimes_K L$. Then the differentiation $d: K\to D(K)$ can be extended uniquely to a differentiation $d_L: L\to D(L)$. This corresponds to (and follows immediately from) the well-known fact that any derivation of a field can be extended uniquely to that of any separably algebraic extension. If $f(x)=x^n+a_1x^{n-1}+\cdots+a_n=0$ is the monic irreducible equation for $x\in L$ over K, then dx is given by $-f'(x)^{-1}\{x^{n-1}da_1+\cdots+da_n\}$, where $f'(x)=nx^{n-1}+(n-1)a_1x^{n-2}+\cdots$ $(\neq 0)$. Since $x^{n-1},\cdots,1$ are linearly independent over K, we have $dx=0$ if and only if $da_1=\cdots=da_n=0$; i.e., if and only if x is separably algebraic over k. Therefore, the constant field of L coincides with the separable closure of k in L.

Let S be an S-operator of K. Then it can be extended uniquely to an S-operator S_L of L. Indeed, $S_L\langle\xi\rangle=\langle\xi,\zeta\rangle+S\langle\zeta\rangle$ gives the desired extension S_L, where ξ, ζ are any elements of $D(L)^\times$, $D(K)^\times$ respectively.

§1-5. Connection with linear differential equations of degree two. Here, we assume that the characteristic of K is different from 2. For each $\xi\in D(K)^\times$, $\boldsymbol{D}_\xi$ will denote the derivation of K over k defined by $x\to dx/\xi$. For each $\xi\in D(K)^\times$ and A, $B\in K$, $[\xi;\ A, B]$ will denote the differential equation

$$(2)\qquad (\boldsymbol{D}_\xi^2+A\cdot\boldsymbol{D}_\xi+B)\cdot u=0\ .$$

The differential equations of the type $[\xi;\ A, B]$ will simply be called *equations.* Let $\xi,\eta\in D(K)^\times$, and put $w_1=\eta/\xi$, $w_{i+1}=dw_i/\xi$ $(i\geqq 1)$. Then $\boldsymbol{D}_\xi=w_1\boldsymbol{D}_\eta$ and $\boldsymbol{D}_\xi^2=w_1^2\boldsymbol{D}_\eta^2+w_2\boldsymbol{D}_\eta$; hence the equation $[\xi;\ A, B]$ may be rewritten as $[\eta;\ A_1, B_1]$, where

$$(3)\qquad A_1=Aw_1^{-1}+w_2w_1^{-2}\ ,\quad B_1=Bw_1^{-2}\ .$$

We shall always identify two such equations (and consider them as different expressions of the same equation).

Let $C\in K^\times$. Then the equation obtained by substituting u by $\sqrt{C}^{-1}u$ in (2) will be denoted by $\sqrt{C}[\xi;\ A, B]$. More explicitly,

$$\sqrt{C}[\xi;\ A, B]=[\xi;\ A', B']\ ,$$

with

$$(4)\qquad \begin{cases} A'=A-\dfrac{D_\xi(C)}{C}\,, \\ B'=B-\dfrac{D_\xi(C)}{2C}A-\dfrac{2CD_\xi^2(C)-3\{D_\xi(C)\}^2}{4C^2}\,. \end{cases}$$

This definition of $\sqrt{C}$-multiple of an equation is independent of the way of expressing the equation. The two equations are called *equivalent* if one is a $\sqrt{C}$-multiple of the other for some $C\in K^\times$. This is an equivalence relation. Each equivalence class will be called *a class of equations.*

An equation will be called of *Schwarzian type* if it has an expression of the form $[\xi;\,0,B]$ (for some choice of ξ). In this case, such a differential ξ is determined uniquely up to $k^\times$-multiples. Indeed, by (3), $[\xi;\,0,B]=[\eta;\,0,B_1]$ implies $w_2=0$; hence $\eta/\xi\in k^\times$. By a simple calculation, we obtain

$$(5)\qquad \sqrt{(\eta/\xi)}[\xi;\,0,B]=[\eta;\,0,B_1']\,,$$

for any $\xi,\eta\in D(K)^\times$ and $B\in K$, where B_1' is given by

$$(6)\qquad 4B_1'\eta^2=\langle\xi,\eta\rangle+4B\xi^2\,.$$

Therefore, $\sqrt{C}$-multiples of Schwarzian type equations (for any $C\in K^\times$) are again of Schwarzian type. Hence we may speak of *Schwarzian type classes.* Equations in a fixed Schwarzian type class are in one-to-one correspondence with $D(K)^\times/k^\times$, by $[\xi;\,0,B]\to\xi \pmod{k^\times}$.

PROPOSITION 3. (i) *Let S be an S-operator of K, and for each $\xi\in D(K)^\times$, put $S\langle\xi\rangle=-4B_\xi\cdot\xi^2$ $(B_\xi\in K)$. Then the class $\mathfrak{K}$ of the equation $[\xi;\,0,B_\xi]$ is independent of ξ, and $S\to\mathfrak{K}$ gives a one-to-one correspondence between the set of all S-operators of K and that of all Schwarzian type classes.*

PROOF. By (5), (6), and by $S\langle\eta\rangle-S\langle\xi\rangle=\langle\eta,\xi\rangle$, we have

$$\sqrt{(\eta/\xi)}[\xi;\,0,B_\xi]=[\eta;\,0,B_\eta]\,;$$

hence the class $\mathfrak{K}$ of $[\xi;\,0,B_\xi]$ is independent of ξ. That $S\mapsto\mathfrak{K}$ is one-to-one follows immediately. Q.E.D.

We shall call the class $\mathfrak{K}$ corresponding to S *the class of type S.*

Now let K_{sep} be the separable closure of K. Then the derivations and the S-operators of K are uniquely extended to those of K_{sep}.

PROPOSITION 4. *Let S be an S-operator of K, and fix $\xi\in D(K)^\times$. For each $\eta\in D(K)^\times$, put $\eta=v^{-2}\cdot\xi$ $(v\in K_{sep})$. Then η is a solution of $S\langle\eta\rangle=0$ if and only if v is a solution of $[\xi;\,0,B_\xi]$.*

PROOF. Put $w_1=\eta/\xi=v^{-2}$, $w_{i+1}=dw_i/\xi$ $(i\geqq 1)$. Then $w_2=-2v^{-3}\cdot D_\xi v$, and

$w_3=6v^{-4}(D_\xi v)^2-2v^{-3}(D_\xi^2 v)$. Hence $\langle\eta,\xi\rangle=-4\xi^2\cdot D_\xi^2 v/v$. But $S\langle\eta\rangle=\langle\eta,\xi\rangle+S\langle\xi\rangle=\langle\eta,\xi\rangle-4B_\xi\cdot\xi^2$; hence $S\langle\eta\rangle=0$ if and only if $D_\xi^2 v+B_\xi\cdot v=0$. Q.E.D.

COROLLARY. *The notations being as in Proposition 4, suppose that K satisfies $K^\times=K^{\times 2}\cdot k^\times$. Let $\mathscr{V}$ be the vector space over k of all solutions $v\in K$ of the equation $[\xi;\ 0,B_\xi]$. Then*

$$\dim\mathscr{V}=0 \Longleftrightarrow S\langle\eta\rangle=0 \text{ has no solutions } \eta\in D(K)^\times,$$

$$\dim\mathscr{V}=1 \Longleftrightarrow S\langle\eta\rangle=0 \text{ has a non-exact solution } \eta,$$

and

$$\dim\mathscr{V}=2 \Longleftrightarrow S\langle\eta\rangle=0 \text{ has an exact solution } \eta.$$

In the last case, the general solutions η of $S\langle\eta\rangle=0$ are also given by $d(v_1/v_2)$, where v_1, v_2 are any k-basis of $\mathscr{V}$.

This follows immediately from Propositions 2, 4.

§1-6. **Examples.** Let k_0 be an algebraically closed field of characteristic $p>0$, $\neq 2$. Let $K=k_0(t)$ be the rational function field. Let $D(K)$, $d: K\to D(K)$ be the space of differentials and the differentiation of K, in the usual sense. Then $k=\mathrm{Ker}(d)=K^p$. The S-operators S of K and the elements F of K are in one-to-one correspondence, by $S\langle dt\rangle=F\cdot(dt)^2$. Let $v\in K^\times$ and put $\eta=v^{-2}\cdot dt$. Then the two equations

$$(\#)\qquad S\langle\eta\rangle=0\ ,$$

$$(\mathrm{b})\qquad \frac{d^2v}{dt^2}-\frac{F}{4}v=0\ ,$$

are equivalent, and since $K^\times=K^{\times 2}\cdot k^\times$, the Corollary of Proposition 4 can be applied. We shall be concerned with the space $\mathscr{V}$ of all solutions $v\in K$ of (b).

Note here that separable extensions of K will not increase the dimension of the solution space of (b). In fact, let L be a separable extension of K, and $l=L^p$ be its constant field. Then L is a p-dimensional vector space over l, spannned by $[1,t,\cdots,t^{p-1}]$; hence $L=K\otimes_k l$. Let Δv denote the left side of (b). Then the map $v\mapsto\Delta v$ is an l-linear endomorphism of L, and it maps K into itself. Hence $\mathrm{Ker}(\Delta)$, i.e., the solution space of (b) in L, is spanned by $\mathrm{Ker}(\Delta)\cap K=\mathscr{V}$*).

Hypergeometric cases. This is the case where F is of the form

$$F=\frac{\alpha t^2+\beta t+\gamma}{t^2(1-t)^2}\qquad(\alpha,\beta,\gamma\in k_0)\ .$$

*) The same holds when L is the completion K_P of K at a prime divisor P of K/k_0, or more generally, when L is any separable extension of K_P. Thus, for $p>0$, there is no essential distinction between local and global solutions.

First, we check that dim $\mathscr{V}\geqq 1$ *if and only if* $\alpha+1$, $\alpha+\beta+\gamma+1$ and $\gamma+1$ *are square elements of the prime field* $\boldsymbol{F}_p$. In fact, put

$$4t^2(1-t)^2\cdot\Delta[1, t, \cdots, t^{p-1}]=[1, t, \cdots, t^{p-1}]\Lambda,$$

$\Lambda=(\lambda_{ij})$ being a square matrix of degree p over k. Then λ_{ij} are given by

$$\begin{aligned}
\lambda_{jj}&=4j(j-1)-\gamma &&\cdots\cdot\ 0\leqq j\leqq p-1,\\
\lambda_{j+1,j}&=-8j(j-1)-\beta &&\cdots\cdot\ 0\leqq j\leqq p-2,\\
&=(-8j(j-1)-\beta)t^p &&\cdots\cdot\ j=p-1,\\
\lambda_{j+2,j}&=4j(j-1)-\alpha &&\cdots\cdot\ 0\leqq j\leqq p-3,\\
&=(4j(j-1)-\alpha)t^p &&\cdots\cdot\ j=p-2,\ p-1,\\
\lambda_{ij}&=0 &&\cdots\cdot\ i\neq j,\ j+1,\ j+2,
\end{aligned}$$

the subscripts i, j being counted mod p. Therefore, $|\Lambda|$ is of the form $|\Lambda|=Pt^{2p}+Qt^p+R$, with P, Q, $R\in k_0$. The explicit values of P, $P+Q+R$, R are given respectively by

$$\prod_{i=0}^{p-1}\{(2i-1)^2-1-\alpha\},\qquad \prod_{i=0}^{p-1}\{(2i-1)^2-1-\alpha-\beta-\gamma\},$$

$$\prod_{i=0}^{p-1}\{(2i-1)^2-1-\gamma\}.$$

Therefore, $|\Lambda|=0$ if and only if $\alpha+1$, $\alpha+\beta+\gamma+1$ and $\gamma+1$ are square elements of $\boldsymbol{F}_p$. Thus our assertion is proved.

Now put $\alpha+1=\rho_\infty^2$, $\alpha+\beta+\gamma+1=\rho_1^2$, $\gamma+1=\rho_0^2$, the signs of ρ_i being arbitrarily fixed. Assume $\rho_i\in\boldsymbol{F}_p$ $(i=0, 1, \infty)$, so that dim $\mathscr{V}\geqq 1$. Put

$$A=\frac{1}{2}(1-\rho_0-\rho_1-\rho_\infty),\qquad B=\frac{1}{2}(1-\rho_0-\rho_1+\rho_\infty),\qquad C=1-\rho_0,$$

and

$$v=t^{-\frac{1}{2}(1-\rho_0)}\cdot(1-t)^{-\frac{1}{2}(1-\rho_1)}\cdot u.$$

Thus, u is an element of K defined modulo $k^\times$-multiples. It is easy to check that v is a solution of (b) if and only if u is a solution of the equation

$$(\natural)\qquad t(1-t)\frac{d^2u}{dt^2}+\{C-(A+B+1)t\}\frac{du}{dt}-ABu=0.$$

Let Δ^*u denote the left side of $(\natural)$, and put

$$\Delta^*[1, t, \cdots, t^{p-1}]=[1, t, \cdots, t^{p-1}]\Lambda^*$$

with $\Lambda^*=(\lambda^*_{ij})$, $\lambda^*_{ij}\in k$. Then, λ^*_{ij} are given by

$$\lambda^*_{jj}=-(j+A)(j+B)\ , \qquad \lambda^*_{j-1,j}=j(j-1+C)\ ,$$

and $\lambda^*_{ij}=0$ for $i\neq j$, $j-1$. In particular, Λ^* is upper triangular, and the diagonal element λ^*_{jj} vanishes if and only if $j=-A$ or $-B$. Therefore, if $A=B$, rank (Λ^*) $=p-1$. On the other hand, if $A\neq B$, say $-A<-B$, then rank $(\Lambda^*)=p-2$ if and only if $\lambda^*_{j-1,j}=0$ for some j with $-A<j\leqq -B$. Here, where inequalities are concerned, the numbers of $\boldsymbol{F}_p$ should be replaced by the corresponding integers in the interval $[0, p)$. It is more convenient, however, to choose the representatives from $(0, p]$; we shall denote the representative of $X\in \boldsymbol{F}_p$ in $(0, p]$ by $X^\cdot$. Since dim $\mathscr{V}=p-$ rank (Λ^*), we conclude that:

dim $\mathscr{V}=1 \Longleftrightarrow$ one of the two intervals $[1, C^\cdot)$, $[C^\cdot, p]$ contains both $A^\cdot$ and $B^\cdot$.

dim $\mathscr{V}=2 \Longleftrightarrow A^\cdot, B^\cdot$ are separated by these two intervals.

The solutions u of ($\natural$) are explicitly given by the following table.

Cases	Dimension of the solutions u	Basis of solutions
$A^\cdot,\ B^\cdot\in[C^\cdot, p]$	1	$f(A, B; C; t)$
$A^\cdot,\ B^\cdot\in[1, C^\cdot)$	1	$t^{1-C}f(A-C+1, B-C+1; 2-C; t)$
otherwise	2	$f(A, B; C; t)$ $t^{1-C}f(A-C+1, B-C+1; 2-C; t)$

Here, in general, $f(A, B; C; t)$ is a polynomial of t defined by the hypergeometric series

$$1+\frac{A\cdot B}{1\cdot C}t+\frac{A(A+1)B(B+1)}{1\cdot 2C(C+1)}t^2+\cdots ,$$

where we stop the series as soon as the numerator vanishes. As can be checked easily, in each of the above cases, the denominator does not vanish before the numerator does. When they vanish at the same time, we stop the series right before that term.

For a numerical example, let $\alpha=\gamma=-1$, $\beta=1$, so that $\rho_0=\rho_1=\rho_\infty=0$. Then $A=B=1/2$, $C=1$, so that the interval $[1, C^\cdot)$ is empty. Therefore, dim $\mathscr{V}=1$. The solutions u, v, η of ($\natural$), (b), ($\sharp$), which are determined up to $k^\times$-multiples, are given by $u=f(1/2, 1/2; 1; t)=\sum_{i=0}^{r}\binom{r}{i}^2 t^i$ (where $r=(p-1)/2$), $v=\{t(1-t)\}^{-r}u$, and $\eta=\{t(1-t)\}^{-1}u^{-2}dt$. Since dim $\mathscr{V}=1$, the differential η is *non-exact* in K, and also in the separable closure L of K. Actually, there is a special solution η_0 of ($\sharp$) in $D(L)^\times$, characterized up to $\boldsymbol{F}_p^\times$-multiples by the condition that it is *logarithmically exact* (i.e., $\eta_0=z^{-1}dz$ with some $z\in L$). It is given by

$$\eta_0=\{t(1-t)\}^{-1}u^{\frac{2}{p-1}}dt \qquad (\text{up to } \boldsymbol{F}_p^\times\text{-multiples}) .$$

The logarithmic exactness of η_0 is proved later in a succeeding paper, and the significance of η_0 will become clear there.

§2. The canonical S-operator.

§2-1. The definition. Let $G_R=PSL_2(\boldsymbol{R})$ act on the complex upper half plane $\boldsymbol{H}=\{\tau\in\boldsymbol{C}|\operatorname{Im}\tau>0\}$ by

$$\tau\to g\tau=\frac{a\tau+b}{c\tau+d}; \qquad g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in G_R .$$

Let Δ be a fuchsian group of the first kind, i.e., a discrete subgroup of G_R with finite volume quotient. Then by the compactification of $\Delta\backslash\boldsymbol{H}$, we obtain a compact Riemann surface $\mathscr{C}$.

The canonical S-operator with respect to Δ is a certain S-operator on the field of automorphic functions with respect to Δ. Let K be the field of automorphic functions with respect to Δ, identified with the field of meromorphic functions on $\mathscr{C}$, and let $D(K)$ be the space of all differentials of K, identified with that of meromorphic differentials on $\mathscr{C}$. Let $d: K\to D(K)$ be the (usual) differentiation. To define the canonical S-operator, we need to consider the field $\tilde{K}$ of all meromorphic functions on $\boldsymbol{H}$, and the space $D(\tilde{K})$ of all meromorphic differentials on $\boldsymbol{H}$. The action of G_R on $\boldsymbol{H}$ induces that on $D^h(\tilde{K})$, denoted by $\xi\to\xi^g$ ($g\in G_R$). Let $h\geqq 0$. Then, by the covering map $\boldsymbol{H}\to\Delta\backslash\boldsymbol{H}\subsetneq\mathscr{C}$, we may identify $D^h(K)$ as a subspace of $D^h(\tilde{K})$ consisting of all $\xi\in D^h(\tilde{K})$ that are Δ-invariant and that have at most poles at each cusp of Δ. Let τ^* be any linear fractional function of τ, considered as an element of $\tilde{K}$. Let $\tilde{S}$ be the inner S-operator of $\tilde{K}$ defined by

$$\tilde{S}\langle\xi\rangle=\langle\xi, d\tau^*\rangle .$$

Then by Propositions 1, 2 (§1-2), S is independent of the choice of τ^*; hence we may replace τ^* by the identity function τ. Putting $\xi=\xi(\tau)d\tau$ ($\xi(\tau)\in\tilde{K}^\times$), we have

$$(7) \qquad \tilde{S}\langle\xi\rangle=\frac{2\xi(\tau)\xi''(\tau)-3\xi'(\tau)^2}{\xi(\tau)^2}(d\tau)^2 ,$$

where $'$ is the differentiation with respect to τ.

It is classically known (modulo difference of formulations) that $\tilde{S}$ *induces an S-operator of K.* This can be checked immediately as follows. Let $\delta\in\Delta$. Then $\langle d\tau, d\tau^\delta\rangle=0$, since τ^δ is a linear fractional transform of τ. Therefore, $\tilde{S}\langle\xi\rangle^\delta=\langle\xi^\delta, d(\tau^\delta)\rangle=\langle\xi^\delta, d\tau\rangle=\tilde{S}\langle\xi^\delta\rangle$ for any $\xi\in D(\tilde{K})^\times$. Hence Δ-invariance of ξ implies that of $\tilde{S}\langle\xi\rangle$. On the other hand, it can be checked immediately by using (7) that if ξ has at most poles at cusps, then $\tilde{S}\langle\xi\rangle$ also has the same property. Therefore, $\tilde{S}$ maps $D(K)^{\times}$ into $D^2(K)$. The equality $\tilde{S}\langle\eta\rangle-\tilde{S}\langle\xi\rangle=\langle\eta,\xi\rangle$ is satisfied for all ξ, $\eta\in D(K)^{\times}$, since it is satisfied for all ξ, $\eta\in D(\tilde{K})^{\times}$. Therefore, $\tilde{S}$ induces an S-operator S of K. We shall call this special S-operator S of K *the canonical S-operator with respect to Δ.*

REMARK. For each $h\geqq 0$, the differentials $\xi\in D^h(K)$ are in one-to-one correspondence with meromorphic automorphic forms $\xi(\tau)$ with weight $-2h$ with respect to Δ, by $\xi=\xi(\tau)(d\tau)^h$. Hence the above fact (that $\tilde{S}\langle D(K)^{\times}\rangle\subset D^2(K)$) is equivalent to the fact that if $\xi(\tau)$ is an automorphic form of weight -2, then $2\xi(\tau)\xi''(\tau)-3\xi'(\tau)^2$ is also an automorphic form of weight -8.

The following proposition follows immediately from the definition of the canonical S-operator and Proposition 2 (ii).

PROPOSITION 5. *Let Δ and Δ' be fuchsian groups of the first kind, with $\Delta'\subset\Delta$. Let K, K' be the corresponding fields of automorphic functions, with the natural inclusion $K\subset K'$. Let S, S' be the canonical S-operators with respect to Δ, Δ'. Then* (i) *S' is the unique S-operator of K' that extends S;* (ii) *if Δ normalizes Δ', then S' is invariant (in the sense of § 1-3) by the Galois group of K'/K (isomorphic to Δ/Δ').*

§ **2-2. Some other formulations** (A). Consider the pair $\{\mathscr{C}, e\}$ of a compact Riemann surface $\mathscr{C}$ and a $\boldsymbol{Z}^+\cup(\infty)$-valued function e on $\mathscr{C}$ satisfying

$$(e1)\qquad e(P)=1 \quad \text{for almost all } P\in\mathscr{C},$$

$$(e2)\qquad 2g-2+\sum_P (1-1/e(P))>0\,,$$

where g is the genus of $\mathscr{C}$. The two pairs $\{\mathscr{C}, e\}$, $\{\mathscr{C}', e'\}$ are called isomorphic if there is an isomorphism ι of $\mathscr{C}$ onto $\mathscr{C}'$ satisfying $e'\circ\iota=e$. Let Δ be a fuchsian group of the first kind, and let $\mathscr{C}$ be, as before, the compatification of $\Delta\backslash\boldsymbol{H}$. For each $P\in\mathscr{C}$, let $e(P)$ be the order of ramification of the covering map $\boldsymbol{H}\to\Delta\backslash\boldsymbol{H}\subsetneq\mathscr{C}$ at P. Thus, $e(P)=\infty$ if P is a cusp, $1<e(P)<1$ if P is an elliptic point, and $e(P)=1$ for all other P. Then by classical results on fuchsian groups, e satisfies the conditions (e1) (e2), and $\Delta\to\{\mathscr{C}, e\}$ gives *a one-to-one correspondence* between the set of all fuchsian groups of the first kind Δ (counted up to conjugacy in G_R) and that of all pairs $\{\mathscr{C}, e\}$ of compact Riemann surfaces $\mathscr{C}$ and $\boldsymbol{Z}^+\cup(\infty)$-valued functions e on $\mathscr{C}$ satisfying (e1), (e2) (counted up to isomorphisms). If Δ corresponds to $\{\mathscr{C}, e\}$ and γ is an element of G_R such that $\gamma^{-1}\Delta\gamma=\Delta$, then γ induces an automorphism of $\{\mathscr{C}, e\}$. Let $N(\Delta)$ be the group of all such γ (i.e., the normalizer of Δ in G_R). Then $(N(\Delta):\Delta)$ is finite, and the quotient $N(\Delta)/\Delta$ is thus identified with the group of *all* automorphisms of $\{\mathscr{C}, e\}$.

Now, take any $\{\mathscr{C}, e\}$, and let K be the field of meromorphic functions on $\mathscr{C}$. Then *the canonical S-operators S with respect to* $\{\mathscr{C}, e\}$ is an S-operator of K defined as follows. Let Δ be the corresponding fuchsian group and identify $\mathscr{C}$ and e with the compactification of $\Delta\backslash \boldsymbol{H}$ and the ramification index of the covering $\boldsymbol{H}\mapsto\mathscr{C}$, respectively. Then K is identified with the field of automorphic functions with respect to Δ, and *S is by definition the canonical S-operator with respect to* Δ. This definition is independent of the above identifications, since S is invariant by the automorphisms of $\{\mathscr{C}, e\}$ (above remark on $N(\Delta)/\Delta$, and Proposition 5 (ii)).

Take two $\{\mathscr{C}, e\}$, $\{\mathscr{C}', e'\}$. An *admissible covering* $f:\{\mathscr{C}', e'\}\mapsto\{\mathscr{C}, e\}$ is by definition a finite covering $\mathscr{C}'\mapsto\mathscr{C}$, satisfying $e(P)=e'(P')\rho(P'/P)$ for any $P'\in\mathscr{C}'$, where $P=f(P')$ and $\rho(P'/P)$ is the ramification index. Fix $\{\mathscr{C}, e\}$ and let Δ be the corresponding fuchsian group. Then it is clear that the Galois theory holds between admissible coverings of $\{\mathscr{C}, e\}$ and the subgroups of Δ of finite indices. Therefore, a translation of Proposition 5 reads as follows:

PROPOSITION 5′. *Let S be the canonical S-operator of* $\{\mathscr{C}, e\}$, *and let* $\{\mathscr{C}', e'\}\to\{\mathscr{C}, e\}$ *be an admissible covering. Then* (i) *the* (*unique*) *extension of S to* $\{\mathscr{C}', e'\}$ *is the canonical S-operator of* $\{\mathscr{C}', e'\}$; (ii) *S is invariant by the automorphisms of* $\{\mathscr{C}, e\}$.

We note finally that $\mathscr{C}$ may be replaced by equivalent algebraic objects; algebraic curves over $\boldsymbol{C}$, or algebraic function fields of one variable over $\boldsymbol{C}$. Thus, we use the notations and the terminologies: $D^h(\mathscr{C})$ and the canonical S-operator *of* $\{\mathscr{C}, e\}$, or $\{K, e\}$ and the canonical S-operator *of* $\{K, e\}$ etc., in an obvious sense.

§2-3. (I). **Generalized algebraic function fields of one variable.** Before giving another formulation of the canonical S-operator, we need some definitions.

Let k be any field of characteristic 0. By a generalized algebraic function field of one variable (abbrev. g.a.f.f.) L/k, we mean a one-dimensional extension L of k, not assumed to be finitely generated, satisfying the following conditions $(L0)$, $(L1)$, $(L2)$:

$(L0)$ k is algebraically closed in L.

$(L1)$ Let $\mathfrak{F}(L)$ be the set of all intermediate fields K of L/k such that K/k is finitely generated and L/K is normally algebraic. Then $\mathfrak{F}(L)$ is non-empty.

$(L2)$ Let K be any element of $\mathfrak{F}(L)$, and for each prime divisor P of the algebraic function field K/k, let $e(P)$ denote the ramification index of P in L/K. Then $e(P)=1$ for almost all P, and the quantity

$$\mu(K)=2g-2+\sum_P\left(1-\frac{1}{e(P)}\right)\deg P\,,\quad (g:\text{ the genus of } K/k)$$

is positive.

This concept of generalized algebraic function field of one variable is a formalization of the field of automorphic functions, in the following sense. Take any fuchsian group of the first kind Δ, and let $\{\Delta_\lambda\}$ be any lattice of normal subgroups of finite indices of Δ. Then the composite L of the fields of automorphic functions K_λ with respect to Δ_λ is a g.a.f.f. over $\boldsymbol{C}$, and conversely, all g.a.f.f. over $\boldsymbol{C}$ can be obtained in this manner. Since we are only considering the fields of characteristic 0, these exhaust essentially all examples.

The following elementary properties of L/k are well-known for the case $k=\boldsymbol{C}$.

(i) For K, $K'\in\mathfrak{F}(L)$ with $K\subset K'$, we have $\mu(K')=[K':K]\mu(K)$; by the Hurwitz formula. (Hence the conditions in $(L2)$ are satisfied for all $K\in\mathfrak{F}(L)$ if so for *one* $K\in\mathfrak{F}(L)$.)

(ii) $\mu(K)\geqq 1/42$ for any $K\in\mathfrak{F}(L)$ (Siegel); in particular, $\mathfrak{F}(L)$ satisfies the minimal condition.

Let G be the group of all automorphisms of L over k. Then G carries a unique topology with which $\mathrm{Gal}\,(L/K)$ $(K\in\mathfrak{F}(L))$ are open subgroups. With this topology, G is locally compact, and the elements of $\mathfrak{F}(L)$ correspond in one-to-one manner with open compact subgroups of G.

Now let $V=\mathrm{Gal}\,(L/K)$ $(K\in\mathfrak{F}(L))$ be any open compact subgroup of G.

(iii) Let $N(V)$ be the normalizer of V in G. Then $(N(V):V)<\infty$.

In fact, the group of automorphisms of K/k leaving $e(P)$ invariant is finite; since by $(L2)$, the sum of $\deg P$ for all such P that $e(P)\neq 1$ is at least three for $g=0$ and at least one for $g=1$.

(iv) V is topologically finitely generated.

This follows immediately from the well-known theorem for the case of $k=\boldsymbol{C}$.

(v) V is contained in at most finitely many compact subgroups of G.

This follows directly from (i) (ii) (iii) and (iv).

(vi) G/V is at most a countable set.

In fact, the number of gV with given group index $(V:V\cap gVg^{-1})$ is finite, by (iii) (iv) and (v).

(vii) There exists a G-invariant subfield L' of L such that $L'\cdot k=L$ and that $L'\cap k$ is of at most countable transcendence degree over the rationals.

This follows directly from (iv) and (vi).

Now, by (i) (ii), $\mathfrak{F}(L)$ contains at least one minimal element.

DEFINITION. We call L *simple* if $\mathfrak{F}(L)$ contains only one minimal element, and *ample* (or *arithmetic*) if otherwise.

If L/k is finitely generated, then $L\in\mathfrak{F}(L)$ and $\mu(L)>0$, so that the genus of L is at least two; hence G is finite and its fixed field is the unique minimal element of $\mathfrak{F}(L)$. Thus, such an L is simple.

In general, if L is simple and K_0 is the minimal element of $\mathfrak{F}(L)$, then $\mathfrak{F}(L)$ consists of all finite extensions of K_0 in L. In this case, G coincides with Gal (L/K_0) and hence G is compact. (In fact, G leaves the unique minimal field K_0 invariant, and hence $(G:\text{Gal}\,(L/K_0))<\infty$ by (iii); hence $G=\text{Gal}\,(L/K_0)$ by the minimality of K_0.) On the other hand, if L is ample and K_0, K_0' are two distinct minimal elements of $\mathfrak{F}(L)$, then $K_0\cap K_0'=k$. In this case, the subgroup of G generated by Gal (L/K_0) and Gal (L/K_0'), and hence also G itself, are non-compact. Therefore, L is simple (resp. ample) if and only if G is compact (resp. non-compact). In the case where $k=\boldsymbol{C}$ and L is obtained from the system $\{\Delta_\lambda\}$ of *all* normal subgroups of Δ of finite indices, L is ample if and only if the "commensurabilizer"

$$\{g\in PSL_2(R)\,;\ g^{-1}\Delta g\sim\Delta\}\quad(\sim:\text{ the commensurability relation})$$

of Δ in $PSL_2(\boldsymbol{R})$ is dense in $PSL_2(\boldsymbol{R})$. As is well-known, this condition is satisfied by all arithmetically defined fuchsian groups. (This is why it may also be proposed to call arithmetic instead of ample.) Some of the related literatures are [8], [2-1] I, Ch. II, and [7-2].

(II). **Some other formulations** (B). Let L be a generalized algebraic function field of one variable over $\boldsymbol{C}$. Take any $K\in\mathfrak{F}(L)$, and for each prime divisor P of K, let $e(P)$ be the ramification index of P in L/K. Then e satisfies (e1) (e2); hence the canonical S-operator S^K of $\{K,e\}$ is defined. The (unique) extension S of S^K to L is independent of K, as is clear by the definition of e and Proposition 5′ (i). We shall call S *the canonical S-operator of L*. Invariance of S by automorphisms of $L/\boldsymbol{C}$ is proved in §4-3 (the Corollary of Lemma B).

§**2-4. Local properties.** Let $\{\mathscr{C},e\}$ be as in §2-2, and let S be the canonical S-operator. Then we have the following:

PROPOSITION 6. *Let* $P\in\mathscr{C}$ *and* $\xi\in D(\mathscr{C})^\times$. *Then*

(i) $\text{ord}_P\,S\langle\xi\rangle\geqq-2$

(ii) *Put* $e=e(P)$, $n=\text{ord}_P\,\xi$; *take* $t\in K$ *such that* $\text{ord}_P\,t=1$, *and put*

$$\xi=ct^n(1+c_1t+\cdots)dt\,,\qquad c\neq0,\ c_1,\ c_2\cdots\in\boldsymbol{C}\,.$$

Then

$$S\langle\xi\rangle=\left\{\frac{a_0}{t^2}+\frac{a_1}{t}+\cdots\right\}(dt)^2\,,$$

with
$$\begin{cases} a_0 = \dfrac{1}{e^2} - (n+1)^2 , & \dots \textit{for any } P, \\ a_1 = -2nc_1 & \dots \textit{if } e(P)=1. \end{cases}$$

PROOF. Let ν be any rational number, and consider a Puiseux series:

$$\eta/dt = Ct^{\nu}(1+C_1t+\cdots)\,;\ C\neq 0,\, C_1 \cdots \in \boldsymbol{C}\,.$$

Then a straightforward computation shows

$$\langle \eta, dt\rangle = -\{(\nu^2+2\nu)+2C_1\nu t+\cdots\}(dt/t)^2\,.$$

But if $\eta=\xi$, then $\nu=n$ and $C_1=c_1$; and if $\eta=d\tau$, then $\nu=(1/e)-1$. Since $S\langle\xi\rangle = \langle\xi, dt\rangle - \langle d\tau, dt\rangle$, our Proposition follows immediately. Q.E.D.

REMARK. Put $S\langle\xi\rangle = -4B_\xi\cdot\xi^2$. Then, Proposition 6 is equivalent to the following assertions on the properties of the equation $[\xi;\,0,\,B_\xi]$ of §1-5.

(i) The equation $[\xi;\,0,\,B_\xi]$ is fuchsian, i.e., regular at each $P\in\mathscr{C}$.

(ii) At each P, the exponents are given by

$$\frac{1}{2}\left(1+n+\frac{1}{e}\right),\quad \frac{1}{2}\left(1+n-\frac{1}{e}\right).$$

(iii) Unless $e(P)=\infty$, the local solutions of the equation do not contain logarithms.

Since $[\eta;\,0,\,B_\eta]=\sqrt{\eta/\xi}[\xi;\,0,\,B_\xi]$ for any $\eta\in D(\mathscr{C})^\times$, the exponents of $[\eta;\,0,\,B_\eta]$ are obtained by adding $\dfrac{1}{2}\mathrm{ord}_P\,\eta$ to the exponents of $[\xi;\,0,\,B_\xi]$. Hence $[\eta;\,0,\,B_\eta]$ satisfies (i)~(iii) if and only if $[\xi;\,0,\,B_\xi]$ does. This shows that the assertions of Proposition 6 are *independent of* ξ; i.e., if *any* S-operator S of $\mathscr{C}$ satisfies the assertions for *one* ξ, then it satisfies the assertions for *all* ξ.

Unfortunately, Proposition 6 does not characterize S. In fact, the S-operator $S'\langle\xi\rangle = S\langle\xi\rangle + C$ $(C\in D^2(\mathscr{C}))$ also satisfies the assertions if and only if $(C) > \prod_{e(P)>1} P^{-1}$, where (C) is the divisor of C. By the Riemann-Roch theorem, the dimension of the space of such differentials C is given by

$$\mu = 3g-3+\#\{P\,|\,e(P)>1\}\,.$$

Therefore, the only case where Proposition 6 characterizes S is that of $\mu=0$. This case is called the *triangular case*. It is the case of $g=0$ and $\#\{P\,|\,e(P)>1\}=3$. (Note that if $g=1$ and $e(P)=1$ for all P, then e does not satisfy (e2).) If $\{\mathscr{C}, e\}$ is triangular, then we may assume that $\mathscr{C}$ is the rational x-curve, and the three points P of $e(P)>1$ are those with $x(P)=0,\,1,\,\infty$. Call those points P_0, P_1, P_∞ respectively, and put $e_i=e(P_i)$ $(i=0\ 1,\,\infty)$. Then

COROLLARY. *The notations being as above, the canonical S-operator for the triangular* $\{\mathscr{C}, e\}$ *is given by*

$$S\langle\xi\rangle=\langle\xi, dx\rangle+\frac{ax^2+bx+c}{x^2(1-x)^2}(dx)^2, \qquad (\xi\in D(\mathscr{C})^\times),$$

where

$$a=\frac{1}{e_\infty^2}-1, \quad a+b+c=\frac{1}{e_1^2}-1, \quad c=\frac{1}{e_0^2}-1.$$

As far as the author knows, the combinations of Proposition 5′ and this corollary are the only known methods for the explicit determination of S.

§2-5. **Example** (R. A. Rankin, M. Eichler). Let $\mathscr{C}$ be the algebraic curve over C defined by the equation

$$y^m=x^n(x^p-1)^q,$$

where m, n, p, q are integers with $m, p>0$, $n, q\geqq 0$, $(m, n, q)=1$. Then,

$$g=\frac{1}{2}\{(m-(m, q))p-(m, n)-(m, pq+n)+2\}.$$

Let $g\geqq 2$ and put $e(P)=1$ for all $P\in\mathscr{C}$. Let $\mathscr{C}_0$ be the rational x^p-curve, and let P_0, P_1, P_∞ be the points on $\mathscr{C}_0$ at which $x^p=0, 1, \infty$ respectively. Put $e_0(P)=1$ for $P\in\mathscr{C}_0$ with $P\neq P_0, P_1, P_\infty$, and put

$$e_0(P_0)=\frac{mp}{(m, n)}, \quad e_0(P_1)=\frac{m}{(m, q)}, \quad e_0(P_\infty)=\frac{mp}{(m, pq+n)}.$$

Then $\{\mathscr{C}_0, e_0\}$ is triangular, and $(x, y)\mapsto x^p$ gives an admissible covering $\{\mathscr{C}, e\}\mapsto\{\mathscr{C}_0, e_0\}$. Therefore, by Proposition 5′ and the corollary of Proposition 6, we immediately obtain an explicit formula for the canonical S-operator of $\{\mathscr{C}, e\}$. It is given by

$$S\langle\xi\rangle=\langle\xi, dx\rangle+\frac{Ax^{2p}+Bx^p+C}{m^2x^2(x^p-1)^2}(dx)^2, \quad (\xi\in D(\mathscr{C})^\times),$$

where

$$A=(m, pq+n)^2-m^2, \qquad A+B+C=((m, q)^2-m^2)p^2,$$
$$C=(m, n)^2-m^2.$$

§3. The k-rationality theorems, and an application

§3-1. **The first formulation.** (I) Let Δ be a fuchsian group of the first kind, and let $\mathscr{C}$ be a compactified quotient of $\boldsymbol{H}$ modulo Δ, considered as an *algebraic* *curve* over $\boldsymbol{C}$. Let k $(\subset \boldsymbol{C})$ be a field of definition of $\mathscr{C}$, and let S be an S-operator of $\mathscr{C}$. We shall say that S is *k-rational* if $S\langle\xi\rangle$ is k-rational for any k-rational differential $\xi \neq 0$. Let ζ $(\neq 0)$ be a fixed k-rational differential of $\mathscr{C}$, so that $S\langle\xi\rangle = \langle\xi, \zeta\rangle + C$ with some $C \in D^2(\mathscr{C})$. Then, S is k-rational if and only if C is so. One might *conjecture* that *the canonical S-operator is k-rational* if ($\mathscr{C}$ is defined over k and) moreover all points $P \in \mathscr{C}$ with $e(P) > 1$ are k-rational. But we find no reasons to expect this to be true.

We shall prove the k-rationality of the canonical S-operator in the following cases. Put $\Delta = \Delta_1$, $\mathscr{C} = \mathscr{C}_1$, and $S = S_1$. Assume that there is another fuchsian group Δ_2, such that *the two groups* Δ_1 *and* Δ_2 *are commensurable and generate a dense subgroup of* $G_R = PSL_2(\boldsymbol{R})$. Put $\Delta_0 = \Delta_1 \cap \Delta_2$, and for each $i = 0, 2$, let $\mathscr{C}_i$ be an algebraic curve corresponding to Δ_i. Then we have canonical projections $\mathrm{pr}_i : \mathscr{C}_0 \mapsto \mathscr{C}_i$ $(i = 1, 2)$.

$$\begin{array}{ccc} & \mathscr{C}_0 & \\ {}^{\mathrm{pr}_1}\swarrow & & \searrow^{\mathrm{pr}_2} \\ \mathscr{C}_1 & & \mathscr{C}_2 \end{array} \qquad\qquad \begin{array}{ccc} & \Delta_0 = \Delta_1 \cap \Delta_2 & \\ \subset & & \supset \\ \Delta_1 & & \Delta_2 \end{array}$$

Let S_i $(i = 0, 2)$ be the corresponding canonical S-operators. We know that S_0 is the unique extension of S_i to $\mathscr{C}_0$ for $i = 1, 2$ (Proposition 5 (i) of § 2-1).

THEOREM A. *The notations and assumptions being as above, let k be a common field of definition of* $\mathscr{C}_i$ $(i = 0, 1, 2)$ *and of* pr_i $(i = 1, 2)$. *Then* S_i $(i = 0, 1, 2)$ *are k-rational.*

(II) A slightly weaker version, which is more convenient for applications, is the following. Let Δ_i, $\mathscr{C}_i$, pr_i etc. be as above, and assume moreover that Δ_2 is of the form $\Delta_2 = \varepsilon^{-1}\Delta_1\varepsilon$ $(\varepsilon \in G_R)$. Then the automorphism $\tau \to \varepsilon\tau$ of $\boldsymbol{H}$ induces an isomorphism $\sigma : \mathscr{C}_2 \to \mathscr{C}_1$. Let φ be the rational map of $\mathscr{C}_0$ into $\mathscr{C} \times \mathscr{C}$ defined by

$$\varphi(P) = \mathrm{pr}_1(P) \times \sigma \circ \mathrm{pr}_2(P)\ , \quad (P \in \mathscr{C}_0)\ .$$

Then φ induces a birational morphism of $\mathscr{C}_0$ onto its image $\mathfrak{X} = \varphi(\mathscr{C}_0)$. $\mathfrak{X}$ is an irreducible algebraic curve on $\mathscr{C} \times \mathscr{C}$, and can be considered as an algebraic correspondece of $\mathscr{C}$.

COROLLARY. *Let k be a common field of definition of* $\mathscr{C}$ *and* $\mathfrak{X}$. *Then S is k-rational.*

Indeed, if $\mathscr{C}$ and $\mathfrak{X}$ are defined over k, then one may re-select k-rational models for $\mathscr{C}_i$ $(i = 0, 2)$ and pr_i $(i = 1, 2)$. This corollary has an application to all arithmetic type fuchsian groups (§ 3-3).

REMARK. Actually, the assumption "Δ_1 and $\varepsilon^{-1}\Delta_1\varepsilon$ generate a dense subgroup of G_R" can be weakened to "Δ_1 and ε generate a dense subgroup", by some minor modifications of the proof (see § 4-2).

§ 3-2. **The second formulation.** The readers are advised to recall the definitions of ample fields and the properties of invariant subfields of the ample fields (§ 2-3, I).

THEOREM B. *Let L/k be a generalized algebraic function field of one variable. Suppose that L is ample, and let Φ be any open non-compact subgroup of* $\mathrm{Aut}_k L$. *Then* (i) *there exists a unique Φ-invariant S-operator S of L*; (ii) *if $k=C$, then S is the canonical S-operator*; (iii) *if L' is any Φ-invariant subfield of L not contained in k (so that L'/k' is again an ample field, where $k'=L'\cap k$), then S is k'-rational in the sense that $S\langle\xi\rangle\in D^2(L')$ for any $\xi\in D(L')^\times$.*

This can be applied, for example, to the $G_\mathfrak{p}$-fields of characteristic 0 ([2-1], Vol. 1, Ch. 2).

§ 3-3. **An application to Shimura curves.** As an application of the corollary of Theorem A (§3-1), we shall prove that the canonical S-operators of arithmetic fuchsian groups are k-rational with respect to Shimura's models $\mathscr{C}/k$. Let F be a totally real algebraic number field of finite degree, and let $\mathfrak{o}$ be the ring of integers of F. Let $F_{>0}$ denote the multiplicative group of *totally positive* elements of F, and let $\mathfrak{o}_{>0}^\times$ be the group of totally positive units of F. For each integral $\mathfrak{o}$-ideal $\mathfrak{c}$, $C(F,\mathfrak{c})$ will denote the classfield over F corresponding to the ideal group

$$\{x\mathfrak{o}\,|\,x\in F_{>0},\ x\equiv 1 \bmod^* \mathfrak{c}\}\ ,$$

where mod* denotes the multiplicative congruence. Let B be a quaternion algebra over F, and let $\mathfrak{O}$ be a maximal $\mathfrak{o}$-order of B. For each prime divisor $\mathfrak{p}$ of F which is unramified in B, we fix an $F_\mathfrak{p}$-isomorphism $B\otimes_F F_\mathfrak{p}\cong M_2(F_\mathfrak{p})$. Here, $F_\mathfrak{p}$ denotes the completion of F with respect to $\mathfrak{p}$. If $\mathfrak{p}$ is finite, this isomorphism can be so chosen that $\mathfrak{O}\otimes_\mathfrak{o}\mathfrak{o}_\mathfrak{p}$ corresponds to $M_2(\mathfrak{o}_\mathfrak{p})$, $\mathfrak{o}_\mathfrak{p}$ being the $\mathfrak{p}$-adic completion of $\mathfrak{o}$. This isomorphism induces an injective isomorphism of $B^\times/F^\times$ into $PL_2(F_\mathfrak{p})$, which will be denoted by $\iota_\mathfrak{p}$. Now, put $B_{>0}=\{x\in B\,|\,N_{B/F}(x)\in F_{>0}\}$, and let φ be an infinite prime divisor of F unramified in B. Then ι_φ induces an isomorphism of $B_{>0}/F^\times$ into the subgroup $G_R=PSL_2(\boldsymbol{R})$ of $PL_2(\boldsymbol{R})$. Let e be an integral two-sided $\mathfrak{O}$-ideal, and put $\mathfrak{c}=e\cap\mathfrak{o}$. Let $\Delta=\Delta(\mathfrak{O},e)$ be the quotient of the group

$$\{\delta\in\mathfrak{O}\,|\,N_{B/F}(\delta)\in\mathfrak{o}_{>0}^\times,\ \delta\equiv 1 \bmod^* e\}\ ,$$

modulo its center

$$\{\delta \in \mathfrak{o}^{\times} \mid \delta \equiv 1 \bmod^{*} \mathfrak{c}\} .$$

Assume now that B is unramified at *exactly one* infinite prime divisor φ of F. Then by ι_{φ}, Δ can be regarded as a subgroup of G_R, which is a fuchsian group of the first kind. For such a fuchsian group Δ, Shimura proved the existence of a very nice complete non-singular model V of $\Delta\backslash \boldsymbol{H}$, which is defined over the classfield $C(F, \mathfrak{c})$ ([7.1] Main Theorem I; p. 73). Here, we call it $\mathscr{C}$ instead of V, to correspond with our previous notations. Our theorem is as follows:

THEOREM C. *Let S be the canonical S-operator with respect to $\Delta=\Delta(\mathfrak{O}, e)$, and let $\mathscr{C}$ be the Shimura model of $\Delta\backslash \boldsymbol{H}$ defined over the classfield $k=C(F, \mathfrak{c})$. Consider S as an S-operator on $\mathscr{C}$. Then S is k-rational.*

PROOF. In view of the Corollary of Theorem A, it suffices to find an element $\varepsilon \in G_R$ such that the two groups Δ, $\varepsilon^{-1}\Delta\varepsilon$ are commensurable and generate a dense subgroup of G_R, and that the correspondence $\mathfrak{X}$ (§3-1) is also defined over k. First, take any $\varepsilon \in B_{>0}/F^{\times}$, and identify it with an element of G_R by ι_{φ}. Then Δ and $\varepsilon^{-1}\Delta\varepsilon$ are commensurable. Let $D(B/F)$ denote the discriminant of B over F, and let $\mathfrak{p}$ be any prime ideal of $\mathfrak{o}$ not dividing $\mathfrak{c}\cdot D(B/F)$. Put

$$\mathfrak{o}^{(\mathfrak{p})} = \bigcup_{n=0}^{\infty} \mathfrak{p}^{-n}\mathfrak{o} , \qquad \mathfrak{O}^{(\mathfrak{p})} = \mathfrak{O} \otimes_{\mathfrak{o}} \mathfrak{o}^{(\mathfrak{p})} ,$$

and let $\Delta^{(\mathfrak{p})}$ be the quotient of the group

$$\{\delta \in \mathfrak{O}^{(\mathfrak{p})} \mid N_{B/F}(\delta) \in \mathfrak{o}^{(\mathfrak{p})\times}_{>0},\ \delta \equiv 1 \bmod^{*} e\} ,$$

modulo its center

$$\{\delta \in \mathfrak{o}^{(\mathfrak{p})\times} \mid \delta \equiv 1 \bmod^{*} \mathfrak{c}\} ,$$

where $\mathfrak{o}^{(\mathfrak{p})\times}_{>0} = \mathfrak{o}^{(\mathfrak{p})\times} \cap F_{>0}$. By the isomorphism $\iota_{\mathfrak{p}}$, $\Delta^{(\mathfrak{p})}$ is embedded into $PL_2(F_{\mathfrak{p}})$. The $\mathfrak{p}$-adic closures of $\iota_{\mathfrak{p}}(\Delta^{(\mathfrak{p})})$ (resp. $\iota_{\mathfrak{p}}(\Delta)$) contain $PSL_2(F_{\mathfrak{p}})$ (resp. $PSL_2(\mathfrak{o}_{\mathfrak{p}})$), by the Eichler-Kneser approximation theorem. Take any $\varepsilon \in \Delta^{(\mathfrak{p})}$ such that $\iota_{\mathfrak{p}}(\varepsilon) \notin PL_2(\mathfrak{o}_{\mathfrak{p}})$. Let Γ be the subgroup of $\Delta^{(\mathfrak{p})}$ generated by Δ and $\varepsilon^{-1}\Delta\varepsilon$. Then the closure of $\iota_{\mathfrak{p}}(\Gamma)$ contains $U=PSL_2(\mathfrak{o}_{\mathfrak{p}})$ and $U'=\iota_{\mathfrak{p}}(\varepsilon)^{-1}U\iota_{\mathfrak{p}}(\varepsilon)$. Since $\iota_{\mathfrak{p}}(\varepsilon) \notin PL_2(\mathfrak{o}_{\mathfrak{p}})$, the two subgroups U, U' of $PSL_2(F_{\mathfrak{p}})$ do not contain each other. But U is a maximal subgroup of $PSL_2(F_{\mathfrak{p}})$; hence U and U' generate $PSL_2(F_{\mathfrak{p}})$. Therefore, $(\Gamma:\Delta)=\infty$; hence, as a subgroup of G_R (by the embedding ι_{φ}), Γ is dense. Now, let $\mathfrak{X}$ be the correspondence of $\mathscr{C}$ defined in §3-1 with respect to such an element ε. Then ε is coprime to $\mathfrak{c}D(B/F)\mathfrak{O}$ (in the sense of [7-1] 2-14, p. 71); hence by 11.4 of [7-1], $\mathfrak{X}$ is defined over $k=C(F, \mathfrak{c})$. *Q.E.D.*

§ 4. Proof of the theorems

§ 4-1. **The key lemma (A).** Let $\mathrm{pr}_i : \mathscr{C}_0 \to \mathscr{C}_i$ $(i=1, 2)$ be as in § 3-1 (I). Then, pr_i^* gives a natural inclusion $D^h(\mathscr{C}_i) \subsetneq D^h(\mathscr{C}_0)$ for $h \geqq 0$, $i=1, 2$.

LEMMA A. $D^h(\mathscr{C}_1) \cap D^h(\mathscr{C}_2) = \{0\}$ *for* $h>0$.

As in § 3-1 (I), let S_i be the canonical S-operator with respect to Δ_i $(i=0, 1, 2)$. Then we obtain the following *algebraic characterization of* S_0:

COROLLARY. *The canonical S-operator* S_0 *is the unique S-operator of* $\mathscr{C}_0$ *satisfying*

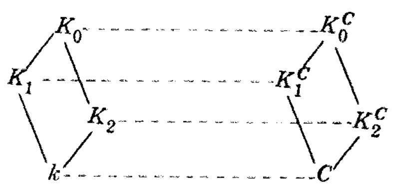
$$S_0\langle D(\mathscr{C}_i)^\times \rangle \subset D^2(\mathscr{C}_i) \ ,$$

for both $i=1, 2$.

PROOF OF THE COROLLARY. Since S_0 is the (unique) extension of S_i to $\mathscr{C}_0$ for $i=1, 2$, we have $S_0\langle D(\mathscr{C}_i)^\times \rangle \subset D^2(\mathscr{C}_i)$ $(i=1, 2)$. On the other hand, suppose that S_0+C $(C \in D^2(\mathscr{C}_0))$ also satisfies this condition. Then $C \in D^2(\mathscr{C}_1) \cap D^2(\mathscr{C}_2)$; hence $C=0$ by Lemma A. Q.E.D.

PROOF OF LEMMA A. Let $C \in D^h(\mathscr{C}_1) \cap D^h(\mathscr{C}_2)$, and put $C=C(\tau)(d\tau)^h$. Then $C(\tau)$ is a meromorphic automorphic form of weight $-2h$ with respect to *both* Δ_1 *and* Δ_2. But since Δ_1 and Δ_2 generate a dense subgroup of $G_{\boldsymbol{R}}$ by assumption, $C(\tau)$ must satisfy the functional equation $C(g\tau)=(c\tau+d)^{-2h}C(\tau)$ for all $g \in G_{\boldsymbol{R}}$. In particular, $C(\tau)$ must be invariant by all translations $\tau \to \tau+b$ $(b \in \boldsymbol{R})$. But since $C(\tau)$ is meromorphic, $C(\tau)$ must be a constant. But $C(-\tau^{-1})=\tau^{-2h}C(\tau)$, and $h>0$. Therefore $C=0$. Q.E.D.

§ 4-2. **Proof of Theorem A.** Let K_i (resp. $K_i^{\boldsymbol{C}}$) be the fields of k-rational (resp. $\boldsymbol{C}$-rational) functions on $\mathscr{C}_i$ $(i=0, 1, 2)$.



For each automorphism ρ of $\boldsymbol{C}$ over k, let $\tilde{\rho}$ be the unique automorphism of $K_0^{\boldsymbol{C}}$ that coincides with ρ on $\boldsymbol{C}$ and with the identity on K_0. Clearly, $\widetilde{\rho_1\rho_2}=\tilde{\rho}_1\tilde{\rho}_2$, $\widetilde{\rho^{-1}}=\tilde{\rho}^{-1}$; hence the set of all $\tilde{\rho}$ forms a group. Let K_0' be its fixed field. Then $K_0' \cap \boldsymbol{C}=k$; hence K_0' and $\boldsymbol{C}$ are linearly disjoint over k. But then, K_0' and $K_0^{\boldsymbol{C}}= K_0 \cdot \boldsymbol{C}$ must be linearly disjoint over K_0; hence $K_0'=K_0$. On the other hand, we have $d(y^{\tilde{\rho}})/d(x^{\tilde{\rho}})=(dy/dx)^{\tilde{\rho}}$ for each x, $y\in K_0^C$ with $x\in C$. This can be checked immediately if we write down the algebraic relation $F(x,y)=0$ between x and y over C and perform the two operations on $F(x,y)=0$, the differentiation and the isomorphism $\tilde{\rho}$ in two different orders. Thus, $\tilde{\rho}$ acts on the space of all differentials (of degree h) on K_0^C, by $\{Z(dx)^h\}^{\tilde{\rho}}=Z^{\tilde{\rho}}\{d(x^{\tilde{\rho}})\}^h$. Since $\tilde{\rho}$ commutes with the differentiation, $\langle\eta^{\tilde{\rho}},\xi^{\tilde{\rho}}\rangle=\langle\eta,\xi\rangle^{\tilde{\rho}}$ holds for all differentials ξ, $\eta\neq 0$ of K_0^C.

Now for each ρ, define an S-operator S_0^ρ of K_0^C by $S_0^\rho\langle\xi\rangle=S_0\langle\xi^{\tilde{\rho}}\rangle^{\tilde{\rho}^{-1}}$. Since $\tilde{\rho}$ leaves K_i^C invariant for $i=1,2$, S_0^ρ maps $D(K_i^C)^\times$ into $D^2(K_i^C)$ for both $i=1,2$. Therefore, by the corollary of Lemma A, we conclude $S_0^\rho=S_0$; hence $S_0\langle\xi\rangle^{\tilde{\rho}}=S_0\langle\xi^{\tilde{\rho}}\rangle$. Now let $\xi\in D(K_0)^\times$. Then $\xi^{\tilde{\rho}}=\xi$; hence $S_0\langle\xi\rangle$ is $\tilde{\rho}$-invariant. Hence if we put $S_0\langle\xi\rangle=A\xi^2$ $(A\in K_0^C)$, then A is $\tilde{\rho}$-invariant for all ρ; hence $A\in K_0$; hence $S_0\langle\xi\rangle\in D^2(K_0)$. On the other hand, if $\xi\in D(K_i)^\times$ $(i=1,2)$, then $S_i\langle\xi\rangle=S_0\langle\xi\rangle\in D^2(K_i^C)$; hence $S_i\langle\xi\rangle\ni D^2(K_i^C)\cap D^2(K_0)=D^2(K_i)$. This implies the k-rationality of S_i $(i=0,1,2)$. Q.E.D.

REMARK. We can also prove the corollary of Theorem A with the *weakened* assumption given in the remark of §3-1, in a similar way. Namely, for this proof, it suffices to use the following modification of the corollary of Lemma A:

S_0 is the unique S-operator of $\mathscr{C}_0$ satisfying

$$S_0(D(\mathscr{C}_1)^\times)\subset D^2(\mathscr{C}_1)$$

and

$$S_0\langle\xi^{\sigma^*}\rangle=S_0\langle\xi\rangle^{\sigma^*},\quad (\xi\in D(\mathscr{C}_1)^\times)\,.$$

Here, σ^* is the map

$$\bigcup_{h\geqq 0}D^h(\mathscr{C}_1)\to\bigcup_{h\geqq 0}D^h(\mathscr{C}_2)$$

induced from σ. The σ^*-invariance of S_0 is obvious, since σ^* is in essence a linear fractional transform. The uniqueness proof goes exactly as that of Lemma A.

§4-3. The key lemma (B).

LEMMA B. *Let L/k be ample, and let Φ be any open non-compact subgroup of* $\mathrm{Aut}_k L$. *Let $h>0$. Then the only Φ-invariant element of $D^h(L)$ is* 0.

PROOF. Let $C\in D^h(L)$ be Φ-invariant. Take an open compact subgroup V_1 of $\mathrm{Aut}_k L$ such that $C\in D^h(K_1)$, where K_1 is the fixed field of V_1. From the properties (ii), (iii), (v) of L/k given in §2-3 I, it follows immediately that there exists some $\varepsilon\in\Phi$ such that V_1 and $\varepsilon^{-1}V_1\varepsilon$ generate a non-compact subgroup of $\mathrm{Aut}_k L$. Put $V_2=\varepsilon^{-1}V_1\varepsilon$, $V_0=V_1\cap V_2$, and let K_i be the fixed field of V_i $(i=0,1,$

2). For each $i=0, 1, 2$, take a complete non-singular algebraic curve $\mathscr{C}_i$ over k representing K_i, and for each $P \in \mathscr{C}_i$, let $e_i(P)$ be the ramification index of P in L/K_i $(i=0, 1, 2)$. Let pr_i be the canonical projection $\mathscr{C}_0 \to \mathscr{C}_i$ $(i=1, 2)$. Let k_0 $(\subset k)$ be a common field of definition for $\mathscr{C}_i$ $(i=0, 1, 2)$, pr_i $(i=1, 2)$, and C. We may assume that k_0 is finitely generated over $\boldsymbol{Q}$. Hence by any embedding of k_0 into $\boldsymbol{C}$, we may regard $\mathscr{C}_i$ and pr_i as complex algebraic curves. By the definition of e_i, the projections pr_i are admissible coverings $\{\mathscr{C}_0, e_0\} \to \{\mathscr{C}_i, e_i\}$. Let Δ_0 be the fuchsian group corresponding to $\{\mathscr{C}_0, e_0\}$, let ι_0 be the corresponding covering $\boldsymbol{H} \to \mathscr{C}_0$, and let Δ_i be the covering group of $\mathrm{pr}_i \circ \iota_0$ $(i=1, 2)$. Then since V_1 and V_2 generate a non-compact group, Δ_1 and Δ_2 generate a dense subgroup of G_R. Since $C^\iota = C$ and $K_2 = K_1^\iota$, C belongs to $D^h(\mathscr{C}_i)$ for $i=1, 2$; hence $C=0$ by Lemma A.

COROLLARY. *Let L/C be a generalized algebraic function field of one variable, and let S be the canonical S-operator of L. Then,* (i) *S is* $\mathrm{Aut}_C L$-*invariant*; (ii) *let L/C be ample and Φ be any open non-compact subgroup of* $\mathrm{Aut}_C L$. *Then S is the uniqe Φ-invariant S-operator of L.*

PROOF. Let $K \in \mathfrak{F}(L)$ and put $V = \mathrm{Aut}_K L$. By its definition, S is an extension of an S-operator of K; hence it is V-invariant. If L is simple, take K to be the unique minimal element of $\mathfrak{F}(L)$. Then $\mathrm{Aut}_C L = V$; hence S is $\mathrm{Aut}_C L$-invariant. Now let L be ample and let G_0 be the subgroup of $\mathrm{Aut}_C L$ generated by all open compact subgroups of $\mathrm{Aut}_C L$. Then S is G_0-invariant, and moreover G_0 is open and non-compact (see §2-3 I). Hence by Lemma B, the only G_0-invariant element of $D^2(L)$ is 0. This implies that S is the only G_0-invariant S-operator of L. On the other hand, G_0 is a normal (and indeed, a characteristic) subgroup of the topological group $\mathrm{Aut}_C L$. Therefore, for each $\sigma \in \mathrm{Aut}_C L$, S^σ is again G_0-invariant; hence $S^\sigma = S$. This proves that S is $\mathrm{Aut}_C L$-invariant. Now, (ii) follows immediately from Lemma B. Q.E.D.

§4-4. **Proof of Theorem B.** In view of the above corollary, it suffices to prove (i) (iii). However, (iii) is an immediate consequence of (i). Indeed, by (i) applied to L'/k', there is a unique Φ-invariant S-operator S' of L'. But S' can be uniquely extended to an S-operator $S^\cdot$ of L (by $S^\cdot\langle\xi\rangle = S'\langle\zeta\rangle + \langle\xi, \zeta\rangle$; $\xi \in D(L)^\times$, $\zeta \in D(L')^\times$). Since $S^\cdot$ is Φ-invariant, we have $S^\cdot = S$ by (i). Therefore, S' is the restriction of S to L'.

So, it is enough to prove (i). But the uniqueness is immediate by Lemma B. Hence it is enough to prove the existence part of (i), assuming $\Phi = \mathrm{Aut}_k L$. In view of the property (vii) of L/k given in §2-3 I, we may assume that k is at most of countable transcendence degree over the rationals. Take any embedding of k into C. Let L_C be the quotient field of $L \underset{k}{\otimes} C$. Then L_C/C is a generalized algebraic function field of one variable and $\mathrm{Aut}_C L_C$ contains Φ as an open subgroup. Hence L_C is ample. Let S_C be the canonical S-operator of L_C. Then by the corollary of Lemma B, S_C is Φ-invariant. It is enough to prove $S_C\langle D(L)^\times\rangle \subset D^2(L)$. But this can be proved exactly in the same manner as in Theorem A (use Lemma B instead of Lemma A). This completes the proof of Theorem B.

References

[1] Deligne, P., Équations différentielles à points singuliers réguliers, Springer lecture note, 163, 1970.

[2.1] Ihara, Y., On congruence monodromy problems, Vols. 1, 2, Lecture notes, Univ. Tokyo, 1968-9.

[2.2] Ihara, Y., Modular transforms of p-power degrees and induced multiple differentials of characteristic p, mimeographed note, 1970.

[2.3] Ihara, Y., Non-abelian invariant differentials, mimeographed note, 1971.

[2.4] Ihara, Y., Non-abelian invariant differentials and Schwarzian equations in the p-adic theory of automorphic functions, Proc. U.S. Japan Seminar on "Modern methods in Number Theory", 1971.

[3] Lehner, J., On Schwarz's triangle function, Pacific J. Math. **4** (1954), 243-249.

[4] Poincaré, H., Oeuvres, Tome II, Sur les groupes des équations linéaires, Paris, 1928.

[5.1] Rankin, R. A., Sir Edmund Whittaker's work on automorphic functions, Proc. Edinburgh, Math. Soc. **11** (1958-59), 25-30.

[5.2] Rankin, R. A., The differential equations associated with the uniformisation of certain algebraic curves, Proc. Royal Soc. Edinburgh, Ser. A. **65** (1958), 35-62.

[5.3] Rankin, R. A., The Schwarzian derivative and uniformisation, J. d'Analyse Math. **6** (1958), 149-167.

[6] Schwarz, H. A., Gesammelte mathematische Abhandlungen, Berlin, 1890.

[7.1] Shimura, G., Construction of class fields and zeta functions of algebraic curves, Ann. of Math. **85** (1967), 58-159.

[7.2] Shimura, G., Introduction to the arithmetic theory of automorphic functions, Publ. Math. Soc. Japan, **11**, 1971.

[8] Pjateckii-Shapiro and Shafarevič, Galois theory of transcendental extensions and uniformization, (in Russian), Izv. Akad. Nauk SSSR, ser. Mat **30** (1966), 671-704. (Amer. Math. Transl. **69** (1968), 111-145.)

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