The limiting absorption principle for uniformly propagative systems

By Kenji YAJIMA

(Communicated by H. Fujita)

§0. Introductions.

The present paper is concerned with some spectral properties, especially with the limiting absorption principle, of uniformly propagative systems and their perturbed operators. The investigations of such systems were initiated by C. H. Wilcox [11]. The spectral property and the scattering theory of them have been investigated by C. H. Wilcox, J. R. Schulenberger, T. Kato, and others [2] [7] [8] [9]. In these works however, rather stronger assumptions on the rate of the decrease of the perturbation were assumed, except for a remark in [2].

On the other hand, S. Agmon [1] recently proved the limiting absorption principle for general elliptic operators under weaker assumptions on the perturbations.

In this paper we use Agmon's method, especially a result which we call here Agmon's Lemma, and prove that the limiting absorption principle holds for uniformly propagative systems under weaker assumptions. The writer is informed by T. Ikebe that he proved the limiting absorption principle in a similar but somewhat different way.

An almost equivalent result is recently obtained by T. Suzuki [10] using rather abstract methods.

Our main results are stated in Theorem 1.1, Theorem 1.3, and Theorem 1.7.

§1. Notations and Theorems.

The following notations will be used throughout this paper.

R^n: the n-dimensional euclidean space with a generic point $x={}^t(x_1, \cdots, x_n)$;

Ξ^n: the dual space of R^n with a generic point $\xi={}^t(\xi_1, \cdots, \xi_n)$;

C^m: the m-dimensional unitary space with a generic point $\zeta={}^t(\zeta_1, \cdots, \zeta_m)$.

Inner products in R^n, Ξ^n and C^m are denoted by $(x, x')=\sum_{j=1}^{n} x_j x_j'$, $(\xi, \xi')=\sum_{j=1}^{n} \xi_j \xi_j'$ and $(\zeta, \zeta')=\sum_{j=1}^{m} \zeta_j \bar{\zeta}_j'$, respectively. We write $(x, x)=|x|^2$, $(\xi, \xi)=|\xi|^2$ and $(\zeta, \zeta)=|\zeta|^2$ for brevity.

$D_j = -i\dfrac{\partial}{\partial x_j}$, $\quad j=1, 2, \cdots, n$;

$\mathscr{S}(R^n)$, $\mathscr{S}'(R^n)$: the space of rapidly decreasing functions and its dual space;

$\mathscr{D}(R^n)$, $\mathscr{D}'(R^n)$: the space of test functions and the space of distributions.

$\mathscr{F}$: the Fourier transform $\mathscr{S}' \to \mathscr{S}'$; its restriction to a subspace of $\mathscr{S}'$ is also denoted by $\mathscr{F}$, for example $\mathscr{F}: L^2 \to L^2$.

For the pair of Banach spaces X and Y, $B(X, Y)$ and $B_\infty(X, Y)$ denote the spaces of all bounded linear operators and compact linear operators from X to Y, respectively. When $X=Y$ we write $B(X, X)=B(X)$ and $B_\infty(X, X)=B_\infty(X)$ for brevity. For a closed operator $T: X\to Y$, $D(T)$, $R(T)$, $N(T)$, $\sigma(T)$, $\sigma_p(T)$, and $\rho(T)$ stand for the domain, the range, the nullspace, the spectrum, the point spectrum and the resolvent set of T, respectively. For $\zeta\in\rho(T)$ we put $R_T(\zeta)=(T-\zeta)^{-1}$. For a densely defined operator $T: X\to Y$ its adjoint operator is denoted by $T^*: Y^*\to X^*$.

In the following part of this paper we shall deal with the spectral properties of selfadjoint operators associated with differential systems of the following form operating on C^m-valued functions defined on R^n

$$(L_0) \qquad L_0(D)=\sum_{j=1}^{n} A_j D_j$$

$$(L) \qquad L(D)=M(x)^{-1}\sum_{j=1}^{n} A_j D_j\,.$$

Our assumptions are as follows.

(A.1) L_0 is the uniformly propagative system in the sense of Wilcox, i.e.,

(a.1.1) the $A_1, \cdots, A_n$ are $m\times m$ (Hermitian) symmetric matrices;

(a.1.2) the roots $\lambda_j(\xi)$ of the characteristic equation $p(\lambda, \xi)=\det(\lambda I-\sum_{j=1}^{n} A_j\xi_j)=0$ have constant multiplicity for all $\xi\in\Xi^n\setminus\{0\}$, and $\lambda_j(\xi)\neq 0$ for any $\xi\in\Xi^n\setminus\{0\}$ unless $\lambda_j(\xi)\equiv 0$ for all $\xi\in\Xi^n$. Here I is the $m\times m$ unit matrix.

(A.2) $M(x)=(m_{ij}(x))$ is an $m\times m$ matrix depending on $x\in R^n$ satisfying the following conditions;

(a.2.1) there exists a constant $C_1>0$ such that

$$(1.1) \qquad C_1|\xi|^2\leqq(\xi, M(x)\xi)\leqq C_1^{-1}|\xi|^2 \text{ for all } \xi\in\Xi^n \text{ and all } x\in R^n$$

(a.2.2) there exist constants $\delta>1$ and $C_2>0$ such that

$$(1.2) \qquad \sup|m_{ij}(x)-\delta_{ij}|\leqq C_2(1+|x|^2)^{-\delta/2} \text{ for all } x\in R, \text{ where } \delta_{ij}=1 \text{ if } i=j \text{ and } \delta_{ij}=0 \text{ if } i\neq j\,.$$

In order to state the theorems we have to define the operators L_0 and L

precisely and to introduce some auxiliary spaces. For $\sigma\in R^1$ and a non-negative integer s we put

$$H^s_{0,\sigma}=\left\{u\in\mathscr{S}'(R^n;C^m);\ \|u\|^2_{H^s_{0,\sigma}}\equiv\sum_{|\alpha|\leqq s}\int|D^\alpha u(x)|^2(1+|x|^2)^\sigma dx<\infty\right\}$$

$$H^s_{1,\sigma}=\left\{u\in\mathscr{S}'(R^n;C^m);\ \|u\|^2_{H^s_{1,\sigma}}\equiv\sum_{|\alpha|\leqq s}\int(D^\alpha u(x)\cdot M(x)D^\alpha u(x))(1+|x|^2)^\sigma dx<\infty\right\}.$$

We write for brevity $H^0_{0,0}, H^0_{0,\sigma}, H^s_{0,0}, H^0_{1,0}, H^0_{1,\sigma}$ and $H^s_{1,0}$ as $H_0, H_{0,\sigma}, H^s, H_1, H_{1,\sigma}$ and H^s_1, respectively.

Since the formal differential operator $L_0(D)$ has constant coefficients, it has a natural selfadjoint realization L_0 in the Hilbert space H_0. Namely, $L_0=\mathscr{F}^{-1}L_0(\xi)\mathscr{F}$, where the operator $L_0(\xi)$ is the maximal operator determined by the multiplication by the symbol $L_0(\xi)=\sum A_j\xi_j$. Let J be the identification operator from H_1 to H_0 defined as $(Ju)(x)=u(x)$ for all $u\in H_1$. Then J^* is an operator determined by the multiplication by $M(x)^{-1}$. Define the operator L by

$$Lu=J^*L_0Ju\ ,\qquad u\in D(L)=J^{-1}D(L_0)\ . \tag{1.3}$$

Obviously L is a selfadjoint operator in H_1.

Main results in this paper are summarized in the following theorems and corollaries.

THEOREM 1.1. *Let $I_0=R^1\setminus\{0\}$ and let $\Pi^\pm=\{\zeta\in C^1;\ \mathrm{Im}\,\zeta\gtrless0\}$ be the upper or the lower half plane of C^1. Let ε be any positive number. Under the assumption (A.1) the following statements hold.*

(1) *Let P_0 be the orthogonal projection onto $N(L_0)$ in H_0. Then $R_{L_0}(\zeta)(1-P_0)$ $(\mathrm{Im}\,\zeta\neq0)$ can be extended to $\Pi^\pm\cup I_0$ as a $B(H_{0,(1+\varepsilon)/2}, H^1_{0,-(1+\varepsilon)/2})$-valued locally Hölder continuous function. Furthermore $R_{L_0}(\zeta)$ $(\mathrm{Im}\,\zeta\neq0)$ itself can be extended as a $B(H_{0,(1+\varepsilon)/2}, H_{0,-(1+\varepsilon)/2})$-valued locally Hölder continuous function. We put $R_{L_0}(\lambda\pm i0)=\lim_{\varepsilon\downarrow0}R_{L_0}(\lambda\pm i\varepsilon)$ and $R'_{L_0}(\lambda\pm i0)=\lim_{\varepsilon\downarrow0}R_{L_0}(\lambda\pm i\varepsilon)(1-P_0)$, where the limit is considered in $B(H_{0,(1+\varepsilon)/2}, H_{0,-(1+\varepsilon)/2})$ and $B(H_{0,(1+\varepsilon)/2}, H^1_{0,-(1+\varepsilon)/2})$, respectively.*

(2) *For any $u\in H_{0,(1+\varepsilon)/2}$ and $\lambda\in I$, $(L_0-\lambda)R_{L_0}(\lambda\pm i0)u=u$ holds in the sense of $\mathscr{D}'$.*

COROLLARY 1.2. *Let $I_0=R^1\setminus\{0\}$ and let K be any compact subset of C^1 satisfying $K\cap R^1\subset I_0$. Then there exists a constant $C_{K,\varepsilon}$ depending only on K and $\varepsilon>0$ such that*

$$\|(1-P_0)u\|_{H^1_{0,-(1+\varepsilon)/2}}\leqq C_{K,\varepsilon}\|(L_0-\zeta)(1-P_0)u\|_{H_{0,(1+\varepsilon)/2}}\ , \tag{1.4}$$

$$\|u\|_{H_{0,-(1+\varepsilon)/2}} \leqq C_{K,\varepsilon} \|(L_0-\zeta)u\|_{H_{0,(1+\varepsilon)/2}} \tag{1.5}$$

for all $u \in H_{0,(1+\varepsilon)/2} \cap D(L_0)$ *and all* $\zeta \in K$.

THEOREM 1.3. *Let assumptions* (*A.1*) and (*A.2*) *be satisfied, and let* I *be any interval not intersecting with* $\sigma_p(L) \cup \{0\}$. *Then the following statements hold.*

(1) $R_L(\zeta)$ ($\mathrm{Im}\,\zeta \neq 0$) *can be extended to* $\Pi^{\pm} \cup I$ as a $B(H_{1,\delta/2}, H_{1,-\delta/2})$*-valued locally Hölder continuous function. We put* $R_L(\lambda \pm i0) = \lim_{\varepsilon \downarrow 0} R_L(\lambda \pm i\varepsilon)$.

(2) *For any* $u \in H_{1,\delta/2}$ *and* $\lambda \in I$, $(L-\lambda)R_L(\lambda \pm i0)u = u$ *holds in the sense of* $\mathscr{D}'$.

COROLLARY 1.4. *Let* I *be as in Theorem* 1.3 *and let* K *be a compact subset of* C^1 *with* $K \cap R^1 \subset I$. *Then there exists a constant* $C_K > 0$ *depending only on* K *such that*

$$\|u\|_{H_{1,-\delta/2}} \leqq C_K \|(L-\zeta)u\|_{H_{1,\delta/2}} \quad \text{for all } u \in D(L) \cap H_{1,\delta/2} \text{ and all } \zeta \in K. \tag{1.6}$$

COROLLARY 1.5. $\sigma(L) \setminus \sigma_p(L)$ *is absolutely continuous.*

Concerning the point spectrum $\sigma_p(L)$ and the corresponding eigenfunctions we obtain the following theorems.

THEOREM 1.6. *Let assumptions* (*A.1*) *and* (*A.2*) *be satisfied. Then each eigenfunction* u_λ of L *corresponding to eigenvalue* $\lambda \in \sigma_p(L) \setminus \{0\}$ *belongs to* $H_{1,\sigma}$ *for any* $\sigma > 0$. *Moreover there exists a constant* $C_{\lambda,\sigma}$ *such that* $\|u_\lambda\|_{H_{1,\sigma}} \leqq C_{\lambda,\sigma} \|u\|_{H_1}$ *holds for all eigenfunction* u_λ *corresponding to* $\lambda \in \sigma_p(L) \setminus \{0\}$, *where for fixed* $\sigma > 0$ *the constant* $C_{\lambda,\sigma}$ *can be taken independent of* λ *if* λ *is in a compact subset of* $\sigma_p(L) \setminus \{0\}$.

THEOREM 1.7. *Let assumptions* (*A.1*) *and* (*A.2*) *be satisfied. Then* $\sigma_p(L) \setminus \{0\}$ *is discrete and the only possible accumulation point is the origin; that is, the eigenspace corresponding to each eigenvalue* $\lambda \in \sigma_p(L) \setminus \{0\}$ *is finite dimensional and* $\sigma_p(L) \setminus \{0\}$ *does not have any limit point in* $R^1 \setminus \{0\}$.

§2. Proof of theorems.

In this section we will prove theorems and corollaries stated in §1. First we will introduce some notations which are necessary for the proofs. As the consequence of the assumption (A.1) the distinct roots of the characteristic equation $p(\lambda, \xi) = \det(\lambda I - \sum A_j \xi_j) = 0$ can be enumerated as $\lambda_\mu(\xi) > \lambda_{\mu-1}(\xi) > \cdots > \lambda_1(\xi) > \lambda_0(\xi) > \lambda_{-1}(\xi) > \cdots > \lambda_{-\mu+1}(\xi) > \lambda_{-\mu}(\xi)$, where $\lambda_0(\xi)$ is the constantly vanishing root when it exists, and will be omitted otherwise. The roots $\lambda_j(\xi)$ are positively homogeneous of degree 1, and they satisfy $-\lambda_j(\xi)=\lambda_{-j}(-\xi)$. We put $S_j=\{\xi\in\Xi^n, \lambda_j(\xi)=\text{sign } j\}$ $(j\neq 0)$, then S_j are C^∞ compact hypersurfaces without boundary and they satisfy $S_j=-S_{-j}$ (See [8]).

For each j, define

$$\hat{P}_j(\xi)=-\frac{1}{2\pi i}\int_{\Gamma_j(\xi)}(\sum A_j\xi_j-\lambda I)^{-1}d\lambda\,, \qquad \Gamma_j(\xi)=\{\lambda\in C^1;\ |\lambda-\lambda_j(\xi)|=\delta_j(\xi)>0\}\,,$$

where $\delta_j(\xi)$ is chosen so small that $\Gamma_j(\xi)$ does not enclose any root of $p(\lambda,\xi)=0$ except $\lambda_j(\xi)$. Let P_j be $\mathscr{F}^{-1}\hat{P}_j(\xi)\mathscr{F}$: $H_0\to H_0$, where $\hat{P}_j(\xi)$ is the maximal operator determined by the multiplication by $\hat{P}_j(\xi)$. Then the following statements hold. (See, for example, [8].)

(2.1) $\{P_j\}_{j=-\mu,\dots,\mu}$ is a complete system of projectors in H_0 reducing the operator L_0.

(2.2) $L_0P_ju=\mathscr{F}^{-1}(\lambda_j(\xi))\mathscr{F}P_ju$ for all $u\in H_0$.

In what follows, using the above results, we shall construct a spectral representation of L_0. We write $R_\pm=\{\lambda\gtrless 0; \lambda\in R^1\}$. The symbol sign j is used to denote $+$ or $-$ as well as $+1$ or -1 according to $j>0$ or $j<0$.

For each $j\neq 0$ we define the non-degenerate C^∞-mapping F_j: $\Xi^n\setminus\{0\}\to R_{\text{sign } j}\times S_j$ by $F_j(\xi)=(\lambda_j(\xi), (\text{sign } j)\xi/\lambda_j(\xi))$ and the operator $\hat{\Gamma}_j$: $\mathscr{D}(\Xi^n; C^m)\to\mathscr{D}(R_{\text{sign } j}; L^2(S_j;C^m))$ by $(\hat{\Gamma}_jf)(\mu,\omega)=f(F_j^{-1}(\mu,\omega))$ for $f\in\mathscr{D}(\Xi^n; C^m)$. Then the following fact is obvious.

(2.3) There exist a positive C^∞-measure $d\sigma_j(\omega_j)$ on S_j, and a measure $d\rho_\pm(\mu)$ on $R_\pm$ such that $\hat{\Gamma}_j$ can be extended to a unitary operator $\hat{\Gamma}_j$: $L^2(\Xi^n; C^m)\to L^2(d\rho_{\text{sign } j}; L^2(S_j, d\sigma_j; C^m))$.

Define Γ_j by $\Gamma_j=\hat{\Gamma}_j\mathscr{F}P_j=\hat{\Gamma}_j\hat{P}_j\mathscr{F}$, then the following relations (2.4), (2.5), (2.6) and (2.7) can be verified easily.

(2.4) $(\Gamma_jL_0u)(\mu)=\mu(\Gamma_ju)(\mu)$ a.e. $\mu\in R_{\text{sign } j}$, for all $u\in H_0$.

(2.5) $\Gamma_jP_i=0$ when $i\neq j$.

(2.6) If $u\in H_{0,\sigma}\left(\sigma>\frac{1}{2}\right)$ then $(\Gamma_ju)(\cdot)$: $R_{\text{sign } j}\to L^2(S_j, d\sigma_j; C^m)$ is locally Hölder continuous.

(2.7) If we define $\Gamma_j(\mu)$: $H_{0,\sigma}\to L^2(S_j, d\sigma_j; C^m)\left(\sigma>\frac{1}{2}\right)$ by $\Gamma_j(\mu)u=(\Gamma_ju)(\mu)$ for $\mu\in R_{\text{sign } j}$, then the mapping $\Gamma_j(\cdot)$: $R_{\text{sign } j}\to B(H_{0,\sigma}; L^2(S_j, d\sigma_j; C^m))$ is locally Hölder continuous.

PROOF OF THEOREM 1.1. For non-real $\zeta\in C^1$, we have $R_{L_0}(\zeta)=\sum_{j\neq 0}R_{L_0}(\zeta)P_j-\zeta^{-1}P_0$.

Therefore it is sufficient for the proof of the first part of the theorem to show that $R_{L_0}(\zeta)P_j$ $(j\neq 0)$ can be extended to $\Pi^{\pm}\cup I_0$ as a $B(H_{0,(1+\varepsilon)/2}, H^1_{0,-(1+\varepsilon)/2})$-valued locally Hölder continuous function. To this end we let M be the operator determined by the multiplication of the function $(1+|x|^2)^{-(1+\varepsilon)/4}$ and prove that $MD_kR_{L_0}(\zeta)P_jM$ $(\mathrm{Im}\,\zeta\neq 0)$ has an extension to $\Pi^{\pm}\cup I'$ as a $B(H_0)$-valued Hölder continuous function, where I' is an arbitrary closed interval contained in I_0. It is clear that the above mentioned result follows from this. We give the proof for the case $I'\subset(0,\infty)$. Let $\lambda\in I'$ and put $I''=R^1\setminus I'$. Let $E_0(d\lambda)$ be the spectral measure for L_0. Then the equation

$$MD_kR_{L_0}(\zeta)P_jM=MD_kR_{L_0}(\zeta)P_jE_0(I')M+MD_kR_{L_0}(\zeta)P_jE_0(I'')M \quad (\mathrm{Im}\,\zeta\neq 0) \tag{2.8}$$

holds. Since $\lambda\notin I''$ the limit value as $\Pi^{\pm}\ni\zeta\to\lambda$ of the second term on the right hand side of equation (2.8) obviously exists locally uniformly for $\lambda\in I'$. Since $I'\subset(0,\infty)$ the same is true for the first term if j is negative. Thus we assume $j>0$ in the following.

For $u, v\in H_0$ it is clear that Mu and $Mv\in H_{0,(1+\varepsilon)/2}$. Therefore by (2.3), (2.4), (2.6) and (2.7) we obtain the equation

$$\begin{aligned}(MD_kR_{L_0}(\zeta)P_jE_0(I')Mu, v)_{H_0}&=(P_jE_0(I')Mu, D_kP_jR_{L_0}(\bar\zeta)Mv)_{H_0}\\&=(\Gamma_j(E_0(I')Mu), \Gamma_j(D_kP_jR_{L_0}(\bar\zeta)Mv))_{L^2(d\rho_+, L^2(S_j, d\sigma_j; C^m))}\\&=\int_{R_+}(\Gamma_j(E_0(I')Mu)(\mu), \Gamma_j(D_kP_jR_{L_0}(\bar\zeta)Mv(\mu))_{L^2(S_j, d\sigma_j; C^m)}d\rho_+(\mu)\\&=\int_{I'}(\Gamma_j(Mu)(\mu), \Gamma_j(D_kP_jR_{L_0}(\bar\zeta)Mv)(\mu))_{L^2(S_j, d\sigma_j; C^m)}d\rho_+(\mu)\,.\end{aligned} \tag{2.9}$$

Since $D_k=\mathscr{F}^{-1}\xi_k\mathscr{F}$, where ξ_k is the maximal operator determined by the multiplication by ξ_k, we can express $\Gamma_j(D_kP_jR_{L_0}(\bar\zeta)Mv)(\mu)$ as $\Phi_k(\mu)(\Gamma_jMv)(\mu)/(\mu-\bar\zeta)=\Phi_k(\mu)\Gamma_j(\mu)Mv/(\mu-\bar\zeta)$. Here $\Phi_k(\mu)$ is a $B(L^2(S_j, d\sigma_j; C^m))$-valued C^∞-function on R_+. Using this expression, we see after simple calculations that the right hand side of (2.9) is equal to

$$\int_{I'}\left(\frac{1}{\mu-\zeta}M\Gamma_j(\mu)^*\Phi_k(\mu)^*\Gamma_j(\mu)Mu,\ v\right)_{H_0}d\rho_+(\mu)\,.$$

Therefore as an equation for $B(H_0)$-valued functions the following equation holds:

$$MD_kR_{L_0}(\zeta)P_jE_0(I')M=\int_{I'}\frac{1}{\mu-\zeta}M\Gamma_j(\mu)^*\Phi_k(\mu)^*\Gamma_j(\mu)Md\rho_+(\mu)\,. \tag{2.10}$$

Here $M\Gamma_j(\mu)^*\Phi_k(\mu)^*\Gamma_j(\mu)M$: $I'\to B(H_0)$ is a $B(H_0)$-valued locally Hölder continuous function. Therefore the application of the famous Privaloff's theorem shows that the left hand side of (2.10) can be extended to $\Pi^{\pm}\cup I'$ as a $B(H_0)$-valued locally

Hölder continuous function and the first part of the theorem is proved.

Statement (2) of the theorem can be proved easily using the theory of distributions. (Q.E.D.)

Here we record the following two lemmas for later use. The first one is proved implicitly in the proof of the theorem and the second is a direct consequence of the first.

COROLLARY 2.1. *Let $\lambda\in R^1\setminus\{0\}$ and $I\subset R^1\setminus\{0\}$ be a closed interval containing λ in its interior. Then in $B(H_{0,(1+\varepsilon)/2}, H_{0,-(1+\varepsilon)/2})$ the following equation holds:*

$$(2.11)\qquad R_{L_0}(\lambda\pm i0)=\text{p.v.}\int_{R_+\cap I}\frac{1}{\mu-\lambda}\sum_{j>0}\Gamma_j(\mu)^*\Gamma_j(\mu)d\rho_+(\mu)$$
$$+\text{p.v.}\int_{R_-\cap I}\frac{1}{\mu-\lambda}\sum_{j<0}\Gamma_j(\mu)^*\Gamma_j(\mu)d\rho_+(\mu)\mp i\pi\sum_{\text{sign}\,j=\text{sign}\,\lambda}\Gamma_j(\lambda)^*\Gamma_j(\lambda)\frac{d\rho_{\text{sign}\,\lambda}}{d\lambda}$$
$$+\sum_{j\neq 0}\int_{I^c}\frac{P_jE_0(d\lambda)}{\mu-\lambda}-\frac{1}{\lambda}P_0\,.$$

Here the symbol p.v. *stands for Cauchy's principal value.*

COROLLARY 2.2. *Let $\lambda\in R^1\setminus\{0\}$ and let $u\in H_{0,\sigma}(\sigma>1/2)$. Then if $\Gamma_j(\lambda)u=0$ for all j satisfying* $\text{sign}\,j=\text{sign}\,\lambda$, *the following equation holds*:

$$(2.12)\qquad \mathscr{F}(R_{L_0}(\lambda\pm i0)u)(\xi)=\sum_{j=-\mu}^{\mu}\frac{\hat P_j(\xi)(\mathscr{F}u)(\xi)}{\lambda_j(\xi)-\lambda}\,.$$

PROOF OF THEOREM 1.3. For $\zeta\in C^1\setminus R^1$ we put $G(\zeta)=(L_0-\zeta)JR_L(\zeta)$ and $G_0(\zeta)=(L-\zeta)J^{-1}R_{L_0}(\zeta)$. Then the following equations hold:

$$(2.13)\qquad R_{L_0}(\zeta)G(\zeta)=JR_L(\zeta)\,,\qquad G(\zeta)G_0(\zeta)=I_{H_0}\quad\text{and}\quad G_0(\zeta)G(\zeta)=I_{H_1}\,.$$

Hence, since it is known that $R_{L_0}(\zeta)$ $(\text{Im}\,\zeta\neq 0)$ can be extended as described in Theorem 1.1, it is sufficient to show that $G(\zeta)=G_0(\zeta)^{-1}$ $(\text{Im}\,\zeta\neq 0)$ can be extended to $\Pi^\pm\cup I$ as a $B(H_{1,\delta/2}, J_{0,\delta/2})$-valued locally Hölder continuous function. By assumption (A.2) the following relation at infinity can be verified easily.

$$(2.14)\qquad M(x)^{-1}-I=0(|x|^{-\delta})\quad\text{as}\quad |x|\to\infty\,.$$

Therefore by the result of Theorem 1.1 we find that

$$G_0(\zeta)=J^{-1}+(J^*-J^{-1})L_0R_{L_0}(\zeta)=J^{-1}+(J^*-J^{-1})L_0R_{L_0}(\zeta)(1-P_0)$$

$(\text{Im}\,\zeta\neq 0)$ can be extended to $\Pi^\pm\cup I$ as a $B(H_{0,\delta/2}, H_{1,\delta/2})$-valued locally Hölder continuous function. We denote the boundary value thus obtained by $G_0(\lambda\pm i0)$. Using the resolvent equation for $R_{L_0}(\zeta)$ and taking the limit as $\zeta\to\lambda\pm i0$ we get the following equation for $G_0(\lambda\pm i0)$ in $B(H_{0,\delta/2}, H_{1,\delta/2})$:

$$(2.15)\quad \begin{aligned} G_0(\lambda\pm i0) &= J^{-1}+[(J^*-J^{-1})L_0R_{L_0}(i)+(J^*-J^{-1})(\lambda-i)L_0R_{L_0}(i)R'_{L_0}(\lambda\pm i0)] \\ &= G_0(i)[I+(\lambda-i)G(i)(J^*-J^{-1})L_0R_{L_0}(i)R'_{L_0}(\lambda\pm i0)]\,. \end{aligned}$$

In taking the limit we used the following two facts: (A) $L_0R_{L_0}(i)$ can be considered as a bounded operator $H^s_{0,\sigma}\to H^s_{0,\sigma}$ for any $\sigma\in R^1$ and any integer s; (This fact can be proved easily by the use of the Fourier transform.) (B) Since $G(i)=J+(I-JJ^*)L_0JR_L(i)\in B(H_{1,\delta/2}, H_{0,\delta/2})$, $G(i)$ is the inverse of $G_0(i)\in B(H_{0,\delta/2}, H_{1,\delta/2})$.

Now we prove that $G_0(\lambda\pm i0)\in B(H_{0,\delta/2}, H_{1,\delta/2})$ is an invertible operator, that is, we prove the existence of $G_0(\lambda\pm i0)^{-1}\in B(H_{1,\delta/2}, H_{0,\delta/2})$. To this end we first show that

$$(2.16)\qquad K_\pm(\lambda)\equiv G(i)(J^*-J^{-1})L_0R_{L_0}(i)R'_{L_0}(\lambda\pm i0)\in B_\infty(H_{0,\delta/2})\,.$$

By Theorem 1.1 and fact (A) given above, we have $L_0R_{L_0}(i)R'_{L_0}(\lambda\pm i0)\in B(H_{0,(1+\varepsilon)/2}, H^1_{0,-1+\varepsilon/2})$ for arbitrary $\varepsilon>0$. Take $\varepsilon>0$ so small that $1+\varepsilon<\delta$. Then using Rellich theorem and the diagonal argument we can easily see that

$$(2.17)\qquad (J^*-J^{-1})L_0R_{L_0}(i)R'_{L_0}(\lambda\pm i0)\in B_\infty(H_{0,\delta/2}, H_{1,\delta/2})\,.$$

(2.17) and the fact (*B*) show that (2.16) holds. Now (2.15) can be written as

$$(2.15')\qquad G_0(\lambda\pm i0)=G_0(i)(I+(\lambda-i)K_\pm(\lambda))\,.$$

Hence if we are able to prove $-1\notin\sigma_p((\lambda-i)K_\pm(\lambda))$, then the application of Riesz-Schauder's theorem shows the existence of $G_0(\lambda\pm i0)^{-1}$. In what follows we prove that the supposition $-1\in\sigma_p((\lambda-i)K_\pm(\lambda))$ results in a contradiction. Let $-1\in\sigma_p((\lambda-i)K_\pm(\lambda))$ and $\varphi_\pm\in H_{0,\delta/2}$ be a corresponding eigenfunction. For any $\varepsilon>0$, we put $v_{\pm,\varepsilon}=R_{L_0}(\lambda\pm i\varepsilon)\varphi_\pm$ and $v_\pm=R_{L_0}(\lambda\pm i0)\varphi_\pm\in H_{0,-\delta/2}$. Then by Theorem 1.1 we have $J^{-1}v_\pm\neq 0$ obviously. Since $G_0(\lambda\pm i0)\varphi_\pm=0$ and $J^{-1}R_{L_0}(\lambda\pm i0)\varphi_\pm=J^{-1}v_\pm$, $\lim_{\varepsilon\downarrow 0}(L-\lambda)J^{-1}v_{\pm,\varepsilon}=0$ holds in $H_{1,-\delta/2}$, that is, $\lim_{\varepsilon\downarrow 0}LJ^{-1}v_{\pm,\varepsilon}=\lambda J^{-1}v_\pm$ holds in $H_{1,-\delta/2}$. This implies $\lim_{\varepsilon\downarrow 0}L_0v_{\pm,\varepsilon}=\lim_{\varepsilon\downarrow 0}J^*LJ^{-1}v_{\pm,\varepsilon}=\lambda J^*J^{-1}v_\pm$, and hence $L_0v_\pm-\lambda J^*J^{-1}v_\pm=0$ in the sense of distribution. Hence, if we can obtain $J^{-1}v_\pm\in H_1$, we get $(L-\lambda)J^{-1}v_\pm=0$ and the contradiction occurs, since we assumed $\lambda\in I$.

Let us prove $J^{-1}v_\pm\in H_1$ or the equivalent fact $v_\pm\in H_0$. Since other cases can be similarly proved we restrict ourselves to the case $I\subset R_+$. Since the above argument shows that $\varphi_\pm=(L_0-\lambda)v_\pm=\lambda(J^{*-1}J^{-1}-I)v_\pm$ we obtain

$$(2.18)\qquad \langle R_{L_0}(\lambda\pm i0)\varphi_\pm, \varphi_\pm\rangle=\langle v_\pm, \lambda(J^{*-1}J^{-1}-I)v_\pm\rangle$$

where $\langle\,,\rangle$ denotes the natural coupling between $H_{0,-\delta/2}$ and $H_{0,\delta/2}$. Since the definition of J^* means $(J^{*-1}J^{-1}v_\pm)(x)=M(x)v_\pm(x)$ and $M(x)$ is a Hermitian symmetric matrix by the assumption, the last member of (2.18) is a real number. Hence the left hand side $\langle R_{L_0}(\lambda\pm i0)\varphi_\pm,\varphi_\pm\rangle$ must be a real number also. Since the other case can be similarly proved we give the proof for the + case only, and the sign + is omitted in the following. Now applying Corollary 2.1 we obtain

$$\begin{aligned}(2.19)\qquad \langle R_{L_0}(\lambda+i0)\varphi,\varphi\rangle &= \text{p.v.}\int_I \frac{1}{\mu-\lambda}\sum_{j>0}(\Gamma_j(\mu)\varphi,\Gamma_j(\mu)\varphi)_{L^2(S_j,d\sigma_j;C^m)}d\rho_+(\mu)\\ &\quad -i\pi\sum_{j>0}\|\Gamma_j(\lambda)\varphi\|^2\frac{d\rho_+}{d\lambda}+\sum_{j\neq0}\int_{R\setminus I}\frac{1}{\mu-\lambda}(P_j\varphi,E_0(d\mu)P_j\varphi)_{H_0}\\ &\quad -\frac{1}{\lambda}(P_0\varphi,\varphi)_{H_0}\,.\end{aligned}$$

In the equation (2.19) all terms except for $-i\pi\sum_{j>0}\|\Gamma_j(\lambda)\varphi\|^2_{L^2(S_j,d\sigma_j;C^m)}\frac{d\rho_+}{d\lambda}$ are real numbers. Therefore $\Gamma_j(\lambda)\varphi=0$ for $j>0$. Then the application of Corollary 2.2 shows

$$(2.20)\qquad \mathscr{F}(R_{L_0}(\lambda+i0)\varphi)(\xi)=\sum_{j=-\mu}^{\mu}\frac{\hat{P}_j(\xi)(\mathscr{F}\varphi)(\xi)}{\lambda_j(\xi)-\lambda}\,.$$

And $\Gamma_j(\lambda)\varphi=0$ $(j>0)$ implies

(2.21) the trace of $\hat{P}_j(\xi)(\mathscr{F}\varphi)(\xi)$ on the hypersurface $\sum_\lambda^j$ vanishes, where $\sum_\lambda^j=\{\xi\in\Xi^n;\,\lambda_j(\xi)=\lambda\}$.

Let $\eta>0$ be so small that the ball $\{|\xi|<2\eta\}$ does not intersect with $\sum_\lambda^j$ for any $j\neq0$ and let $\phi(\xi)\in\mathscr{D}(\Xi^n;R^1)$ be a function satisfying

$$(2.22)\qquad \phi(\xi)=0 \text{ for } |\xi|>\frac{3}{2}|\eta|,\ \phi(\xi)=1 \text{ for } |\xi|<|\eta|, \text{ and } 0\leqq\phi(\xi)\leqq1 \text{ for all } \xi\in\Xi^n\,.$$

Then by (2.20) we get obviously

$$(2.20)'\qquad \mathscr{F}(R_{L_0}(\lambda+i0)\varphi)(\xi)=\sum_{j=-\mu}^{\mu}\phi(\xi)\frac{\hat{P}_j(\xi)(\mathscr{F}\varphi)(\xi)}{\lambda_j(\xi)-\lambda}+\sum_{j=-\mu}^{\mu}(1-\phi(\xi))\frac{\hat{P}_j(\xi)(\mathscr{F}\varphi)(\xi)}{\lambda_j(\xi)-\lambda}\,.$$

By the choice of $\phi(\xi)$, $\lambda_j(\xi)-\lambda\neq0$ on the support of $\phi(\xi)$. Hence the first term on the right hand side of (2.20)′ con be rewritten as $\phi(\xi)(\sum_{j=1}^n A_j\xi_j-\lambda)^{-1}(\mathscr{F}\varphi)(\xi)$ and has the same regularity as $(\mathscr{F}\varphi)(\xi)$. As for the second term, $(1-\phi(\xi))\hat{P}_j(\xi)$ are C^∞-class bounded functions. Therefore $(I-\phi(\xi))\hat{P}_j(\xi)(\mathscr{F}\varphi)(\xi)\in H^{\delta/2}$. Then using (2.21), by the repeated application of Agmon's Lemma (given in a lecture at Oberwolfach, 1971, see [12]) we get $\mathscr{F}(R_{L_0}(\lambda+i0)\varphi)\in H_0^{\min((\delta/2)-1+n\varepsilon,0)}$ for any positive integer n. Therefore $R_{L_0}(\lambda+i0)\varphi\in H_0$ and this means $v\in H_0$. Thus we have proved that $G_0(\lambda\pm i0)^{-1}\in B(H_{0,\delta/2},H_{1,\delta/2})$ exists. Local Hölder continuity of $G_0(\lambda\pm i0)$ and statement (2) of the theorem are proved easily. (Q.E.D.)

Corollary 1.4 follows from Theorem 1.3 obviously and Corollary 1.5 is proved easily by the application of Corollary 1.4.

For the preparation of the proof of Theorem 1.6 and Theorem 1.7 we prove the following lemma. In the remaining part of the paper we identify $u\in H_{1,\sigma}$ and $Ju\in H_{0,\sigma}$ for any $\sigma\in R^1$ and omit the symbols J or J^{-1} and the indices 0 or 1. We use the notation H^σ for any $\sigma\in R^1$ to denote the usual Sobolev space of order σ.

LEMMA 2.3. *Let assumptions (A.1) and (A.2) be satisfied. Let u_λ be an eigenfunction of L corresponding to the eigenvalue $\lambda\in\sigma_p(L)\setminus\{0\}$. Put $(L_0-\lambda)u_\lambda=f_\lambda$. Then the following statements hold*:

(2.23) $\qquad f_\lambda=(M(x)-I)u_\lambda \quad (\in H_\delta)$;

(2.24) $\qquad \Gamma_j(\lambda)f_\lambda=0$ *for all j satisfying* sign $j=$sign λ .

PROOF. $Lu_\lambda=\lambda u_\lambda$ means $L_0u_\lambda=\lambda M(x)u_\lambda$. Therefore $f_\lambda=(L_0-\lambda)u_\lambda=\lambda(M(x)-I)u_\lambda\in H_\delta$ which proves (2.23). Apply Γ_j to the both sides of $(L_0-\lambda)u_\lambda=f_\lambda$, then

(2.25) $$(\Gamma_jf_\lambda)(\mu)=(\mu-\lambda)(\Gamma_ju_\lambda)(\mu) \quad \text{for a.e. } \mu\in R_{\mathrm{sign}\,j},\ j\neq 0 .$$

Here in the right hand side of (2.25) $(\Gamma_ju_\lambda)(\mu)$ belongs to $L^2(d\rho_{\mathrm{sign}\,j};\, L^2(S_j, d\sigma_j; C^m))$ and in the left hand side $(\Gamma_ju_\lambda)(\mu)$ is continuous because $f_\lambda\in H_\delta$ $(\delta>1)$. Therefore $(\Gamma_jf_\lambda)(\lambda)=\Gamma_j(\lambda)f_\lambda=0$ for all j satisfying sign $j=$sign λ. (Q.E.D.)

We remark here again that (2.21) means that the trace of $\hat{P}_j(\xi)(\mathscr{F}f_\lambda)(\xi)$ on the hypersurface Σ_λ^j vanishes. This fact will be used frequently in the sequel.

PROOF OF THEOREM 1.6. Let u_λ be the eigenfunction of L corresponding to $\lambda\in\sigma_p(L)\setminus\{0\}$, and put $f_\lambda=(L_0-\lambda)u_\lambda$. Then by (2.23) of Lemma 2.3 $f_\lambda\in H_\delta$ and hence $(\mathscr{F}f_\lambda)\in H^\delta$. Applying the Fourier transform to $(L_0-\lambda)u_\lambda=f_\lambda$, we obtain

(2.26) $$\Big(\sum_{j=1}^n A_j\xi_j-\lambda\Big)(\mathscr{F}u_\lambda)(\xi)=(\mathscr{F}f_\lambda)(\xi) .$$

Let $\eta>0$ and $\phi(\xi)$ be a function determined as (2.22). Then $\big(\sum_{j=1}^n A_j\xi_j-\lambda\big)$ is non-singular on the support of $\phi(\xi)$. Therefore using Lemma 2.2 and Lemma 2.3, we get from (2.26)

(2.27) $$(\mathscr{F}u_\lambda)(\xi)=\phi(\xi)\Big(\sum_{j=1}^n A_j\xi_j-\lambda\Big)^{-1}(\mathscr{F}f_\lambda)(\xi)+\sum_{j=-\mu}^{\mu}(1-\phi(\xi))\frac{\hat{P}_j(\xi)(\mathscr{F}f_\lambda)(\xi)}{\lambda_j(\xi)-\lambda} ,$$

(2.21′) the trace of $\hat{P}_j(\xi)(\mathscr{F}f_\lambda)(\xi)$ on the hypersurface $\Sigma_{\lambda_k}^j$ vanishes for all j satisfying sign $j=$sign λ_k .

Then the application of Agmon's Lemma which was mentioned above implies

$$(\mathscr{F}u_\lambda)(\xi)\in H^{\delta-1} \quad \text{and} \quad \|u\|_{H^{\delta-1}}\leqq C_\lambda\|f\|_{H^\delta}\,, \tag{2.28}$$

where C_λ is the constant depending only on λ and δ, and which can be taken independent of λ when λ is in a compact subset of $R^1\setminus\{0\}$. Therefore remembering (2.23), we obtain

$$u_\lambda\in H_{\delta-1} \quad \text{and} \quad \|u_\lambda\|_{H_{\delta-1}}\leqq C_\lambda\|u_\lambda\|_H\,. \tag{2.29}$$

Now (2.23) and (2.29) imply $f_\lambda\in H_{2\delta-1}$. Then repeating the above process, we obtain with another constant C_λ which has the same property as above

$$u\in H_{2(\delta-1)} \quad \text{and} \quad \|u_\lambda\|_{H_{2(\delta-1)}}\leqq C_\lambda\|u_\lambda\|_H\,. \tag{2.30}$$

We can repeat the above process arbitrarily many times. Hence the statement of Theorem 1.6 holds. (Q.E.D.)

PROOF OF THEOREM 1.7. Let λ_k belong to $\sigma_p(L)\setminus\{0\}$, and u_k be corresponding eigenfunctions for $k=1,2,\cdots,n,\cdots$ which are orthonormalized. It is sufficient to prove that $\lambda_k\to\lambda\in\sigma_p(L)\setminus\{0\}$ never occurs. Let $\lambda_k\to\lambda\in\sigma_p(L)\setminus\{0\}$. Theorem 1.6 implies that $\{u_k\}$ forms a bounded subset of H_σ for any $\sigma>0$. Put $f_k=(L_0-\lambda_k)u_k$, then by Lemma 2.3 we have that $\{f_k\}$ forms a bounded subset of H_σ for any $\sigma>1$. Applying $G_0(\lambda_k+i0)$ to $f_k=(L_0-\lambda_k)u_k$, we obtain

$$\begin{aligned} G_0(\lambda_k+i0)f_k&=[\lim_{\varepsilon\downarrow 0}(L-(\lambda_k+i\varepsilon))R_{L_0}(\lambda_k+i\varepsilon)](L_0-\lambda_k)u_k \\ &=\lim_{\varepsilon\downarrow 0}[L-(\lambda_k+i\varepsilon)][u_k+i\varepsilon R_{L_0}(\lambda_k+i\varepsilon)u_k] \\ &=\lim_{\varepsilon\downarrow 0}[-i\varepsilon+i\varepsilon G_0(\lambda_k+i\varepsilon)]u_k=0\,. \end{aligned} \tag{2.31}$$

Therefore using (2.15) and (2.16) we obtain

$$f_k=-(\lambda_k-i)K(\lambda_k)f_k\,. \tag{2.32}$$

By (2.16), $K(\lambda)$ is $B_\infty(H_{\delta/2})$-valued locally Hölder continuous function on $R^1\setminus\{0\}$. Hence by (2.32), we can choose a subsequence $\{f_{k_l}\}$, of $\{f_k\}$ which converges to $f_\infty\in H_{\delta/2}$ in $H_{\delta/2}$. We denote this subsequence by f_k again. Choose $\phi(\xi)\in C^\infty(\Xi^n)$ as (2.22). Then by Lemma 2.3 and Corollary 2.2 we get

(2.33) the trace of $\hat{P}_j(\xi)(\mathscr{F}f_k)(\xi)$ on the hypersurface Σ_λ^j vanishes for all j satisfying $\operatorname{sign} j=\operatorname{sign}\lambda$;

$$\begin{aligned} \mathscr{F}(f_k)(\xi)&=\phi(\xi)\sum_{j\neq 0}\frac{\hat{P}_j(\xi)(\mathscr{F}f_k)(\xi)}{\lambda_j(\xi)-\lambda_k}+(1-\phi(\xi))\sum_{j\neq 0}\frac{\hat{P}_j(\xi)(\mathscr{F}f_k)(\xi)}{\lambda_j(\xi)-\lambda_k} \\ &\quad -\frac{\hat{P}_0(\xi)(\mathscr{F}f_k)(\xi)}{\lambda_k} \\ &=\mathscr{F}(R'_{L_0}(\lambda_k+i0)f_k)(\xi)-\frac{\hat{P}_0(\xi)(\mathscr{F}f_k)(\xi)}{\lambda_k}\,. \end{aligned} \tag{2.34}$$

The first and the third terms of the middle member are convergent in H. Put $h_k(\xi)=(1-\phi(\xi))\sum_{j\neq 0}\frac{\hat{P}_j(\xi)(\mathscr{F}f_k)(\xi)}{\lambda_j(\xi)-\lambda_k}$. Then as in the proof of Theorem 1.6 $\{h_k\}$ forms a bounded subset of H^σ for any $\sigma>0$. On the other hand Theorem 1.1 shows that $\mathscr{F}(R'_{L_0}(\lambda_k+i0)f_k)$ forms a bounded subset of $H_1^{-\delta/2}$ and so does $\{h_k\}$. Thus $\{\mathscr{F}^{-1}h_k\}$ is bounded simultaneously in H_σ and $H^1_{-\delta/2}$. Therefore Rellich's theorem and the diagonal argument show that there exists a subsequence $\{h_{k_\nu}\}$ which is convergent in H. Hence $\{\mathscr{F}(u_{k_\nu})\}$ is convergent in H, which means $\{u_{k_\nu}\}$ is convergent in H. This contradicts the the fact that $\{u_k\}$ is orthonormal. (Q.E.D.)

Concluding remarks.

REMARK 1. We record here the immediate corollary of Theorem 1.3 and the abstract stationary method of scattering theory developed by Kato-Kuroda [4] and Dieč [2].

THEOREM 2.4. *Let assumptions (A.1) and (A.2) be satisfied. Let P_{ac} and $P_{0,ac}$ be the projection operators in H_1 and H_0 onto the absolutely continuous subspaces of H_1 and H_0 with respect to L and L_0, respectively. Then the wave operators*

$$W_\pm(L, L_0; J^*)=\operatorname*{s-lim}_{t\to\pm\infty} e^{itL}J^*e^{-itL_0}P_{0,ac} \tag{2.35}$$

$$W_\pm(L_0, L; J)=\operatorname*{s-lim}_{t\to\pm\infty} e^{itL_0}Je^{-itL}P_{ac} \tag{2.36}$$

exist. Therefore the absolutely continuous parts of L and L_0 are unitarily equivalent.

REMARK 2. As another application of Theorem 1.3 we can obtain the eigenfunction expansions for the system in a generalized sense with their applications to scattering theory. We discuss the subjects elsewhere.

Acknowledgement.

The writer expresses his sincere gratitude to Professor S. T. Kuroda for his unceasing encouragement, valuable critism and advice.

References

[1] Agmon, S., Spectral properties of Schrödinger operators, Actes Congrès intern. Math., 1970, Tome 2, 679–683.

[2] Deič, V. G., A local stationary method in the theory of scattering for a pair of spaces, Soviet Math. Dokl. **12** (1971), 648-652. Dokl. Acad. Nauk., SSSR **197** (1971), 1247–1250.

[3] Ikebe, T., Eigenfunction expansions associated with the Schroedinger operators and

their application to scattering theory, Arch. Rat. Mech. Anal. **5** (1960), 1–34.
[4] Kato, T. and S. T. Kuroda, Theory of Simple Scattering and Eigenfunction Expansions, Functional Anal. and Related Fields., Springer Verlag, Berlin, 1970, 99–131.
[5] Kuroda, S. T., Absolutely continuous spectrum of selfadjoint elliptic differential operators., Seminar Notes in Applied Analysis, Univ. of Tokyo, 1973 (in Japanese).
[6] Mochizuki, K., Spectral and scattering theory for symmetric hyperbolic system in an exterior domain., Publ. RIMS, Kyoto Univ. **5** (1969), 219–258.
[7] Schulenberger, J. R. and C. H. Wilcox, A Rellich uniqueness theorem for steady state wave propagation in inhomogeneous isotropic media, Arch. Rat. Mech. Anal. **41** (1972), 13–45.
[8] Schulenberger, J. R. and C. H. Wilcox, The limiting absorption principle and spectral theory for steady state wave propagation in inhomogeneous inisotropic media, ibid, 46–65.
[9] Schulenberger, J. R. and C. H. Wilcox, Eigenfunction expansions and scattering theory for wave propagation problems of classical physics, ibid. **46** (1972), 280–319.
[10] Suzuki, T., The limiting absorption principle for a certain non-selfadjoint operator and its application., to appear.
[11] Wilcox, C. H., Wave operators and asymptotic solutions of wave propagation problems of classical physics., Arch. Rat. Mech. Anal. **22** (1966), 37–78.
[12] Yamada, N., On the principle of limiting absorption for the Dirac operator, Publ. RIMS, Kyoto Univ. **8** (1972–3), 576–606.

(Received December 20, 1973)

Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Tokyo
113 Japan