On some finite subgroups of $GL(n, \boldsymbol{Q})$

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Introduction. In [6] Burnside determined all finite subgroups of $GL(n, \boldsymbol{Q})$ which contain the symmetric groups of degree n (=the group of permutation matrices) as a subgroup.

The purpose of this paper is to extend the determination of Burnside to the finite subgroups of $GL(n, \boldsymbol{Q})$ containing the alternating group of degree n. This extension is not trivial, although there arises no essentially new groups by this extended classification.

Our method of the proof is a little different from that of Burnside, and quite elementary, and this also gives an alternative proof of the result of Burnside. (And this also shows that the list of Burnside must be slightly corrected.)

Here we only treat the case $n \geqq 5$. This is not an essential restriction, because the complete classification of the finite subgroups of $GL(n, \boldsymbol{Q})$ for $n \leqq 4$ has been done by many people. (Cf. Dade [7], Bülow and Neubüser [4] and the several papers listed in its references.)

Before stating our result explicitly, we give some definitions and notation.

Definitions and Notation. For a field K, $GL(n, K)$ denotes the group of all the invertible matrices of degree n. $\boldsymbol{C}$, $\boldsymbol{R}$ and $\boldsymbol{Q}$ denote the complex, real and rational field respectively. $O(n)$ denotes the real orthogonal group defined by $O(n)=\{X \in GL(n, \boldsymbol{R}) \mid {}^tXX=I_n$, the identity matrix of degree $n\}$. $W(A_n)$ $(n \geqq 1)$, $W(B_n)$ $(n \geqq 2)$, $W(D_n)$ $(n \geqq 4)$ and $W(E_n)$ $(n=6, 7$ and $8)$ denote respectively the Weyl group of the respective type. For detailed exposition of these groups and related concept, see for example Bourbaki [3]. P_n denotes the group of the permutation matrices of degree n. P_n is isomorphic to S_n, the symmetric group of degree n, as an abstract group. For a group G, $[G, G]$ denotes the commutator subgroup of G. $[P_n, P_n]$ is isomorphic to A_n, the alternating group of degree n.

Let us define two subgroups of $GL(n, \boldsymbol{Q})$. $W(E_7)^{\sharp}$=the subgroup of $GL(8, \boldsymbol{Q})$ generated by P_8 and the reflection (in the ordinary sense) with respect to the vector $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$. ($W(E_7)^{\sharp} \cong W(E_7)$ as an abstract group.) $W(E_8)^{\sharp}$=the subgroup of $GL(9, \boldsymbol{Q})$ generated by P_9 and the reflec-

*) Supported in part by the Sakkokai Foundation.

tion (in the ordinary sense) with respect to the vector $\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$. ($W(E_8)^{\sharp} \cong W(E_8)$ as an abstract group.) Z_n denotes the matrix in $GL(n, \boldsymbol{Q})$ whose diagonal entries are $1-\frac{2}{n}$ and other entries are all $-\frac{2}{n}$.

Our main result is the following theorem:

THEOREM 1. *Let* $n \geqq 5$. *Let* G *be a finite subgroup of* $GL(n, \boldsymbol{Q})$ *containing the subgroup* $[P_n, P_n]$. *Then* G *is conjugate (in* $GL(n, \boldsymbol{Q})$*) to one of the following subgroups (also denoted by* G*)*:

1) $n \geqq 5$, $[P_n, P_n] \leqq G \leqq \{I_n, -I_n, Z_n, -Z_n\} \times P_n$,
2) $n \geqq 5$, $[W(A_n), W(A_n)] \leqq G \leqq \{I_n, -I_n\} \times W(A_n)$,
3) $n \geqq 5$, $[W(D_n), W(D_n)] \leqq G \leqq W(B_n)$,
4) $n=6$, $[W(E_6), W(E_6)] \leqq G \leqq \{I_n, -I_n\} \times W(E_6)$,
5) $n=7$, $[W(E_7), W(E_7)] \leqq G \leqq W(E_7)$,
6) $n=8$, $[W(E_8), W(E_8)] \leqq G \leqq W(E_8)$,
7) $n=8$, $[W(E_7)^{\sharp}, W(E_7)^{\sharp}] \leqq G \leqq \{I_n, -I_n\} \times W(E_7)^{\sharp}$,
8) $n=9$, $[W(E_8)^{\sharp}, W(E_8)^{\sharp}] \leqq G \leqq \{I_n, -I_n\} \times W(E_8)^{\sharp}$.

REMARK 1. (i) G is not irreducible (as a subgroup of $GL(n, \boldsymbol{Q})$, if G is in the cases 1) 7,) and 8). G is irreducible but imprimitive (in the sense of Blichfeldt [2]), if G is in the case 3). G is primitive (in the sense of Blichfeldt), if G is in the cases 2), 4), 5) and 8). (ii) In the statement of Theorem 1, the group in the left hand side is all normal in the group in the right hand side, and the indices are respectively 8 (in case 1)), 4 (in case 2)), 4 (case 3)), 4 (case 4)), 2 (case 5)), 2 (case 6)), 4 (case 7)) and 4 (case 8)). And it is an easy task to describe all the groups between the left and the right sides. Moreover, it is easily seen that the groups listed in Theorem 1 are not conjugate in $GL(n, \boldsymbol{Q})$ to each other.

REMARK 2. The groups $W(E_7)^{\sharp}$, $W(E_7)^{\sharp} \times \{I_n, -I_n\}$ are finite subgroups of $GL(8, \boldsymbol{Q})$ containing P_8. But these are not listed in Burnside's list in [6]. Also $W(E_8)^{\sharp}$ and $W(E_8)^{\sharp} \times \{I_n, -I_n\}$ are finite subgroups of $GL(9, \boldsymbol{Q})$ containing P_9, but not listed in Burnside [6]. (The existence of these groups are well known, and here novelty exists only in pointing out that they were escaped from the list of Burnside.)

§1. Outline of the proof of Theorem 1.

First, we will show that the proof of Theorem 1 is essentially reduced to the following Problem A.

PROBLEM A. Classify those finite subgroups G of $O(n)$ containing the group $[P_n, P_n]$ such that G is conjugate in $GL(n, \boldsymbol{C})$ to a subgroup of $GL(n, \boldsymbol{Q})$.

Because, if G is a finite subgroup of $GL(n, \boldsymbol{Q})$ containing the group $[P_n, P_n]$ (i.e., if G satisfies the assumption of Theorem 1), then there exists an element $c_1 \in GL(n, \boldsymbol{C})$ such that $c_1^{-1}Gc_1 \leqq O(n)$ and $c_1^{-1}[P_n, P_n]c_1 \leqq O(n)$. Since P_n and $c_1^{-1}P_nc_1$ are representations in $O(n)$, there exists an element $c_2 \in O(n)$ such that $c_2^{-1}c_1^{-1}[P_n, P_n]c_1c_2 =[P_n, P_n]$. Then, $c_1c_2Gc_2^{-1}c_1^{-1}$ satisfies the assumptions of Problem A. Noting the well known fact that two finite subgroups of $GL(n, K)$ which are conjugate in $GL(n, K')$ (with K' some extended field on K) are already conjugate in $GL(n, K)$, we have completed the proof of the reduction of Theorem 1 to Problem A.

From now on we consider about Problem A mainly. The outline of the rest of the proof is as follows.

Let G be a finite subgroup satisfying the assumptions of Problem A. Let X be the subset of G consisting of the elements T's such that the eigenvalues of T are 1 (with multiplicity $n-2$) and -1 (with multiplicity 2). Let $N=\langle X\rangle$ be the subgroup of G generated by the set X. Clearly N is a normal subgroup of G. Our first aim is the determination of the possible subgroup N. Once one knows the group N, then G (in Theorem 1) is contained in the normalizer of N' in $GL(n, \boldsymbol{Q})$ where N' is a conjugate of N with $N' \leqq GL(n, \boldsymbol{Q})$. But this is a relatively easy task to determine the normalizer, because for every possible N', we can calculate its automorphism group fairly easily. In order to determine the subgroup N (i.e., in order to determine the set X), we consider the configuration consists of -1 eigen spaces (2-dimensional subspaces of $\boldsymbol{R}^n$) of the elements in X. In practice, using some lemmas, we can determine the possible configurations consist of -1 eigenspaces of the elements of X. This is the most difficult part of the proof of Theorem 1.

§2. Proof of Theorem 1 (Determination of the possible configurations).

Let T be an element of G whose eigenvalues are 1 (with multiplicity $n-2$) and -1 (with multiplicity 2). Let $\boldsymbol{k}=(k_1, \cdots, k_n)$ and $\boldsymbol{l}=(l_1, \cdots, l_n)$ be two -1 eigenvectors such that $(\boldsymbol{k}, \boldsymbol{l})=0$, $(\boldsymbol{k}, \boldsymbol{k})=(\boldsymbol{l}, \boldsymbol{l})=2$, where $(\ ,\ )$ means the ordinary inner product of the vectors in $\boldsymbol{R}^n$. Then we have the following lemma:

LEMMA 1. *Let T, $\boldsymbol{k}$ and $\boldsymbol{l}$ be as above. Then the following two properties* (i) *and* (ii) *hold.*

(i) *The value of* $(k_{i_1}-k_{i_2})^2+(l_{i_1}-l_{i_2})^2+(k_{i_3}-k_{i_4})^2+(l_{i_3}-l_{i_4})^2$ *is one of the nine* *integers* 0, 1, 2, 3, 4, 5, 6, 7 *and* 8 *for every mutually distinct four indices* i_1, i_2, i_3 *and* i_4.

(ii) *The value of* $(k_{i_1}-k_{i_2})^2+(l_{i_1}-l_{i_2})^2+(k_{i_2}-k_{i_3})^2+(l_{i_2}-l_{i_3})^2+(k_{i_3}-k_{i_1})^2+(l_{i_3}-l_{i_1})^2$ *is one of the eight even integers* 0, 2, 4, 6, 8, 10, 12 *and* 14 *for every mutually distinct three indices* i_1, i_2, *and* i_3.

PROOF. Let $S_1=(i_1i_2)(i_3i_4)$, $S_2=(i_1i_2i_3)$ are permutations in G and $\boldsymbol{a}=(a_1,\cdots,a_n)$, $\boldsymbol{b}=(b_1,\cdots,b_n)$ and $\boldsymbol{c}=(c_1,\cdots,c_n)$ are vectors in $\boldsymbol{R}^n$, where $a_{i_1}=1$, $a_{i_2}=-1$, $a_i=0$ for $i\neq i_1$, i_2, $b_{i_3}=1$, $b_{i_4}=-1$, $b_i=0$ for $i\neq i_3$, i_4 and $c_{i_2}=1$, $c_{i_3}=-1$, $c_i=0$ for $i\neq i_2$, i_3. (Here the group consisting of all the even permutation matrices is identified with the alternating group for convenience.) Then we obtain the following equations:

$$T\cdot\boldsymbol{k}=-\boldsymbol{k}\,,\quad T\cdot\boldsymbol{l}=-\boldsymbol{l}\,,$$

$$T\cdot\boldsymbol{a}=-(k_{i_1}-k_{i_2})\boldsymbol{k}-(l_{i_1}-l_{i_2})\boldsymbol{l}+\boldsymbol{a}\,,$$

$$T\cdot\boldsymbol{b}=-(k_{i_3}-k_{i_4})\boldsymbol{k}-(l_{i_3}-l_{i_4})\boldsymbol{l}+\boldsymbol{b}\,,$$

$$T\cdot\boldsymbol{c}=-(k_{i_2}-k_{i_3})\boldsymbol{k}-(l_{i_2}-l_{i_3})\boldsymbol{l}+\boldsymbol{c}\,,$$

$$S_1\cdot\boldsymbol{k}=\boldsymbol{k}-(k_{i_1}-k_{i_2})\boldsymbol{a}-(k_{i_3}-k_{i_4})\boldsymbol{b}\,,$$

$$S_1\cdot\boldsymbol{l}=\boldsymbol{l}-(l_{i_1}-l_{i_2})\boldsymbol{a}-(l_{i_3}-l_{i_4})\boldsymbol{b}\,,$$

$$S_1\cdot\boldsymbol{a}=-\boldsymbol{a}\,,\quad S_1\cdot\boldsymbol{b}=-\boldsymbol{b}\,,$$

$$S_2\cdot\boldsymbol{k}=\boldsymbol{k}+(k_{i_3}-k_{i_1})\boldsymbol{a}-(k_{i_2}-k_{i_3})\boldsymbol{c}\,,$$

$$S_2\cdot\boldsymbol{l}=\boldsymbol{l}+(l_{i_3}-l_{i_1})\boldsymbol{a}-(l_{i_2}-l_{i_3})\boldsymbol{c}\,,$$

$$S_2\cdot\boldsymbol{a}=\boldsymbol{c},\quad S_2\cdot\boldsymbol{c}=-\boldsymbol{a}-\boldsymbol{c}\,,$$

here $X\cdot\boldsymbol{y}=(y_1,\cdots,y_n)(x_{ij})^t$ for $X=(x_{ij})\in GL(n,\boldsymbol{R})$ and $\boldsymbol{y}=(y_1,\cdots,y_n)\in\boldsymbol{R}^n$. Hence the subgroup $\langle T,S_1\rangle$ of G generated by T and S_1 maps the subspace spanned by $\boldsymbol{k}$, $\boldsymbol{l}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ into itself and fixes its orthogonal complement pointwisely. If $\boldsymbol{k}$, $\boldsymbol{l}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly independent, then we obtain $\mathrm{Trace}(TS_1)=(k_{i_1}-k_{i_2})^2+(l_{i_1}-l_{i_2})^2+(k_{i_3}-k_{i_4})^2+(l_{i_3}-l_{i_4})^2-4+n-4$. If $\boldsymbol{k}$, $\boldsymbol{l}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ span a three dimensional subspace of $\boldsymbol{R}^n$, then three vectors of them are linearly independent and we obtain from the relation among them that $\mathrm{Trace}(TS_1)=(k_{i_1}-k_{i_2})^2+(l_{i_1}-l_{i_2})^2+(k_{i_3}-k_{i_4})^2+(l_{i_3}-l_{i_4})^2-5+n-3$. On the other hand, since G is conjugate to a subgroup of $GL(n,\boldsymbol{Q})$, $\mathrm{Trace}(TS_1)$ must be an integer in the closed interval $[-n,n]$. Then we obtain (i) of Lemma 1 in these two cases. If $\boldsymbol{k}$, $\boldsymbol{l}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ span a two dimentional subspace of $\boldsymbol{R}^n$, then we obtain from the relations among them that $(k_{i_1}-k_{i_2})^2+(l_{i_1}-l_{i_2})^2+(k_{i_3}-k_{i_4})^2+(l_{i_3}-l_{i_4})^2=8$. Hence the proof of Lemma 1 (i) is completed. Next we will prove Lemma 1 (ii). The subgroup $\langle T,S_2\rangle$ of

G generated by T and S_2 maps the subspace spanned by $\boldsymbol{k}$, $\boldsymbol{l}$, $\boldsymbol{a}$ and $\boldsymbol{c}$ into itself and fixes its orthogonal complement pointwisely. If $\boldsymbol{k}$, $\boldsymbol{l}$, $\boldsymbol{a}$ and $\boldsymbol{c}$ are linearly independent, then we have $\mathrm{Trace}(TS_2)=\frac{1}{2}\{(k_{i_1}-k_{i_2})^2+(l_{i_1}-l_{i_2})^2+(k_{i_2}-k_{i_3})^2+(l_{i_2}-l_{i_3})^2+(k_{i_3}-k_{i_1})^2+(l_{i_3}-l_{i_1})^2\}-3+n-4$. Even if $\boldsymbol{k}$, $\boldsymbol{l}$, $\boldsymbol{a}$ and $\boldsymbol{c}$ are not linearly independent, we obtain $\mathrm{Trace}(TS_2)=\frac{1}{2}\{(k_{i_1}-k_{i_2})^2+(l_{i_1}-l_{i_2})^2+(k_{i_2}-k_{i_3})^2+(l_{i_2}-l_{i_3})^2+(k_{i_3}-k_{i_1})^2+(l_{i_3}-l_{i_1})^2+n-7$ from the relations among the four vectors $\boldsymbol{k}$, $\boldsymbol{l}$, $\boldsymbol{a}$ and $\boldsymbol{c}$. Thus the proof of Lemma 1 (ii) is completed.

LEMMA 2. *$(k_i-k_j)^2+(l_i-l_j)^2$ is one of the five integers* 0, 1, 2, 3 *and* 4 *for every indices i and j.*

PROOF. Let us set $m_{ij}=(k_i-k_j)^2+(l_i-l_j)^2$. Then we obtain by Lemma 1 the following equation for every mutually distinct four indices:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} m_{i_1i_2} \\ m_{i_1i_3} \\ m_{i_1i_4} \\ m_{i_2i_3} \\ m_{i_2i_4} \\ m_{i_3i_4} \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{pmatrix}$$

where α_1, α_2, $\alpha_3\in\boldsymbol{Z}$ and α_4, α_5, $\alpha_6\in 2\boldsymbol{Z}$. Let A be the matrix in the left side of the equation, and A_{i1} be the $(i,1)$-cofactor of the matrix A. Then $\det A=-4$, $A_{11}=0$, $A_{21}=-2$ and $A_{31}=2$, hence we obtain $m_{i_1i_2}=\sum_{i=1}^{6}\frac{(-1)^{i-1}\alpha_iA_{i1}}{\det A}=\sum_{i=1}^{6}\frac{(-1)^{i-1}\alpha_iA_{i1}}{-4}=-\frac{\alpha_2+\alpha_3}{2}-\frac{-\alpha_4A_{41}+\alpha_5A_{51}-\alpha_6A_{61}}{4}\in\boldsymbol{Z}/2$. Since we have chosen i_1, i_2, i_3 and i_4 arbitrarily, m_{ij} is in $\boldsymbol{Z}/2$ for all i, j. On the other hand, since $m_{i_1i_2}+m_{i_2i_3}+m_{i_3i_1}$ is an even integer, at least one of $m_{i_1i_2}$, $m_{i_2i_3}$ and $m_{i_3i_1}$ must be an integer. Thus we have shown that there exist i_0 and j_0 such that $m_{i_0j_0}$ is an integer. Hence every m_{ij}, i, $j\notin\{i_0,j_0\}$, is an integer, because $m_{i_0j_0}+m_{ij}$ is an integer by Lemma 1. Moreover, every m_{ij}, $i\in\{i_0,j_0\}$, $j\notin\{i_0,j_0\}$, is an integer, because $n\geqq 5$ there exist i', $j'\notin\{i_0,j_0,j\}$ and $m_{ij}+m_{i'j'}$ is an integer. Thus we have shown that every m_{ij} is an integer. Next we will show that every $m_{ij}\leqq 4$. Let us set $\boldsymbol{k}'=\alpha\boldsymbol{k}+\beta\boldsymbol{l}$ and $\boldsymbol{l}'=\gamma\boldsymbol{k}+\delta\boldsymbol{l}$ where $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in O(2)$. Then $\boldsymbol{k}'$ and $\boldsymbol{l}'$ are the base of the -1 eigen space of T and $(\boldsymbol{k}',\boldsymbol{l}')=0$ and $(\boldsymbol{k}',\boldsymbol{k}')=(\boldsymbol{l}',\boldsymbol{l}')=2$. Therefore $\boldsymbol{k}'$ and $\boldsymbol{l}'$ satisfy the assumptions of Lemma 2 and $(k_i'-k_j')^2+(l_i'-l_j')^2$ must be an integer for every i and j as we have already shown. Let us set $\alpha=(l_i-l_j)/\sqrt{m_{ij}}$, $\beta=-(k_i-k_j)/\sqrt{m_{ij}}$, $\gamma=(k_i-k_j)/\sqrt{m_{ij}}$, and $\delta=(l_i-l_j)/\sqrt{m_{ij}}$. Then $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in O(2)$ and we have $k_i'=k_j'$ and

$l'_i - l'_j = m_{ij}$. Since $2 = \sum_{t=1}^{n} (l'_t)^2 \geqq (l'_i)^2 + (l'_j)^2 = 2\left(l'_i - \frac{\sqrt{m_{ij}}}{2}\right)^2 + m_{ij}/2$, we have $m_{ij} \leqq 4$. This completes the proof of Lemma 2.

LEMMA 3. *Let* $\boldsymbol{k}$ *and* $\boldsymbol{l}$ *be as before. Then* $\boldsymbol{k}$ *and* $\boldsymbol{l}$ *are transformed by some element of* $O(2)$ *into the following vectors or there exists a permutation matrix* S *such that* $S\cdot\boldsymbol{k}$ *and* $S\cdot\boldsymbol{l}$ *are among the following vectors.*

$n \geqq 5$

① $\begin{cases} \boldsymbol{k} = (1, 1, 0, 0, 0, \cdots, 0) \\ \boldsymbol{l} = (0, 0, 1, 1, 0, \cdots, 0) \end{cases}$
② $\begin{cases} \boldsymbol{k} = (1, 1, 0, \cdots, 0) \\ \boldsymbol{l} = (1, -1, 0, \cdots, 0) \end{cases}$

③ $\begin{cases} \boldsymbol{k} = (1, 1, 0, 0, 0, \cdots, 0) \\ \boldsymbol{l} = (0, 0, 1, -1, 0, \cdots, 0) \end{cases}$
④ $\begin{cases} \boldsymbol{k} = (0, 0, 1, -1, 0, \cdots, 0) \\ \boldsymbol{l} = (1, -1, 0, 0, 0, \cdots, 0) \end{cases}$

⑤ $\begin{cases} \boldsymbol{k} = (\sqrt{2}, 0, 0, 0, \cdots, 0) \\ \boldsymbol{l} = (0, 1, -1, 0, \cdots, 0) \end{cases}$
⑥ $\begin{cases} \boldsymbol{k} = (\sqrt{2}, 0, 0, 0, \cdots, 0) \\ \boldsymbol{l} = (0, 1, 1, 0, \cdots, 0) \end{cases}$

⑦ $\begin{cases} \boldsymbol{k} = (\alpha, \alpha, \alpha, \cdots, \alpha, \alpha+\sqrt{2}) \qquad \alpha = -2\sqrt{2}/n \\ \boldsymbol{l} = (1, -1, 0, \cdots, 0, 0) \end{cases}$

⑧ $\begin{cases} \boldsymbol{k} = (\alpha, \alpha, \alpha, \cdots, \alpha) \qquad \alpha = \sqrt{2/n} \\ \boldsymbol{l} = (1, -1, 0, \cdots, 0) \end{cases}$

⑨ $\begin{cases} \boldsymbol{k} = (\alpha, \alpha, \alpha, \alpha, \alpha+\sqrt{2}/2, \cdots, \alpha+\sqrt{2}/2) \qquad \alpha = 4\sqrt{2}/n - \sqrt{2}/2 \\ \boldsymbol{l} = (\beta, \beta, \beta-\sqrt{2}, \beta-\sqrt{2}, \beta-\sqrt{2}/2, \cdots, \beta-\sqrt{2}/2) \qquad \beta = \sqrt{2}/2 \end{cases}$

⑩ $\begin{cases} \boldsymbol{k} = (\alpha, \alpha, \alpha+1, \cdots, \alpha+1) \qquad \alpha = 4/n - 1 \\ \boldsymbol{l} = (1, -1, 0, \cdots, 0) \end{cases}$

⑪ $\begin{cases} \boldsymbol{k} = (\alpha, \alpha, \alpha, \cdots, \alpha, \alpha+1, \alpha+1) \qquad \alpha = -4/n \\ \boldsymbol{l} = (1, -1, 0, \cdots, 0, 0) \end{cases}$

⑫ $\begin{cases} \boldsymbol{k} = (\alpha, \alpha, \alpha, \cdots, \alpha, \alpha+1) \qquad \alpha = (-1 \pm \sqrt{1+n})/n \\ \boldsymbol{l} = (1, -1, 0, \cdots, 0, 0) \end{cases}$

⑬ $\begin{cases} \boldsymbol{k} = (\alpha, \alpha, \alpha, \alpha+\sqrt{6}/3, \cdots, \alpha+\sqrt{6}/3) \qquad \alpha = (6-n)\sqrt{6}/3n \\ \boldsymbol{l} = (2\sqrt{3}/3, -\sqrt{3}/3, -\sqrt{3}/3, 0, \cdots, 0) \end{cases}$

$n=5$

⑭ $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \alpha, \alpha+\sqrt{3/2}) & \alpha=(-\sqrt{6}\pm 4)/10 \\ \boldsymbol{l}=(\beta, \beta, \beta-\sqrt{2}, \beta-\sqrt{2}, \beta-\sqrt{2}/2) & \beta=\sqrt{2}/2 \end{cases}$

⑮ $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \alpha+\sqrt{3/2}, \alpha+\sqrt{3/2}) & \alpha=(-\sqrt{6}+\varepsilon)/5 \\ \boldsymbol{l}=(\beta, \beta-\sqrt{2}, \beta-\sqrt{2}, \beta-\sqrt{2}/2, \beta-\sqrt{2}/2) & \beta=(3\sqrt{2}-\sqrt{3}\varepsilon)/5 \\ \varepsilon=\pm 1 \end{cases}$

⑯ $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \alpha+\sqrt{5/3}, \alpha+\sqrt{5/3}) & \alpha=-2/\sqrt{15} \\ \boldsymbol{l}=(\beta, \beta-\sqrt{3}, \beta-\sqrt{3}, \beta-2\sqrt{3}/3, \beta-2\sqrt{3}/3) & \beta=2\sqrt{3}/3 \end{cases}$

$n=6$

⑰ $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \alpha+1, \cdots, \alpha+1) & \alpha=(-3\pm\sqrt{3})/6 \\ \boldsymbol{l}=(1, -1, 0, 0, \cdots, 0) \end{cases}$

⑱ $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \alpha, \alpha+\sqrt{3/2}, \alpha+\sqrt{3/2}) & \alpha=-\sqrt{6}/6 \\ \boldsymbol{l}=(\beta, \beta, \beta-\sqrt{2}, \beta-\sqrt{2}, \beta-\sqrt{2}/2, \beta-\sqrt{2}/2) & \beta=\sqrt{2}/2 \end{cases}$

⑲ $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \cdots, \alpha, \alpha+\sqrt{3/2}) & \alpha=(-\sqrt{3}+3\varepsilon)/6\sqrt{2} \\ \boldsymbol{l}=(\beta, \beta-\sqrt{2}, \cdots, \beta-\sqrt{2}, \beta-\sqrt{2}/2) & \beta=(3\sqrt{3}-\varepsilon)/2\sqrt{6} \\ \varepsilon=\pm 1 \end{cases}$

⑳ $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha+\sqrt{3/2}, \cdots, \alpha+\sqrt{3/2}) & \alpha=-\sqrt{6}/3 \\ \boldsymbol{l}=(\beta, \beta-\sqrt{2}, \beta-\sqrt{2}/2, \cdots, \beta-\sqrt{2}/2) & \beta=(3\pm\sqrt{3})/3\sqrt{2} \end{cases}$

㉑ $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \alpha, \alpha+\sqrt{2}/2, \alpha-\sqrt{2}/2) & \alpha=\pm 1/\sqrt{6} \\ \boldsymbol{l}=(\beta, \beta, \beta-\sqrt{2}, \beta-\sqrt{2}, \beta-\sqrt{2}/2, \beta-\sqrt{2}/2) & \beta=\sqrt{2}/2 \end{cases}$

$n=7$

㉒ $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \alpha+1, \cdots, \alpha+1) & \alpha=(-4\pm\sqrt{2})/7 \\ \boldsymbol{l}=(1, -1, 0, 0, \cdots, 0) \end{cases}$

㉓ $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \alpha, \alpha+1, \alpha+1, \alpha+1) & \alpha=(-3\pm\sqrt{2})/7 \\ \boldsymbol{l}=(1, -1, 0, 0, 0, 0, 0) \end{cases}$

㉔ $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \cdots, \alpha, \alpha+\sqrt{2}/2, \alpha+\sqrt{2}/2) & \alpha=(-2+3\sqrt{2}\varepsilon)/7\sqrt{2} \\ \boldsymbol{l}=(\beta, \beta-\sqrt{2}, \cdots, \beta-\sqrt{2}, \beta-\sqrt{2}/2, \beta-\sqrt{2}/2) & \beta=(10-\sqrt{2}\varepsilon)/7\sqrt{2} \\ \varepsilon=\pm 1 \end{cases}$

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$$\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \alpha, \alpha+\sqrt{2}/2, \alpha+\sqrt{2}/2, \alpha-\sqrt{2}/2) & \alpha=(-\sqrt{2}\pm 4)/14 \\ \boldsymbol{l}=(\beta, \beta, \beta-\sqrt{2}, \beta-\sqrt{2}, \beta-\sqrt{2}/2, \beta-\sqrt{2}/2, \beta-\sqrt{2}/2) & \beta=\sqrt{2}/2 \end{cases}$$

$n=8$

㉖
$$\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \cdots, \alpha, \alpha+1, \alpha+1, \alpha+1) & \alpha=-1/2 \text{ or } -1/4 \\ \boldsymbol{l}=(1, -1, 0, \cdots, 0, 0, 0, 0) \end{cases}$$

㉗
$$\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \alpha, \alpha+1, \alpha+1, \alpha+1, \alpha+1) & \alpha=-1/2 \\ \boldsymbol{l}=(1, -1, 0, 0, 0, 0, 0, 0) \end{cases}$$

㉘
$$\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \alpha+1, \cdots, \alpha+1) & \alpha=-1/2 \text{ or } -3/4 \\ \boldsymbol{l}=(1, -1, 0, 0, \cdots, 0) \end{cases}$$

㉙
$$\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \alpha, \alpha+1, \cdots, \alpha+1) & \alpha=\sqrt{2}(\varepsilon-1)/4 \\ \boldsymbol{l}=(\beta, \beta-\sqrt{2}, \beta-\sqrt{2}, \beta-\sqrt{2}, \beta-\sqrt{2}/2, \cdots, \beta-\sqrt{2}/2) \\ \quad \beta=\sqrt{2}(-\varepsilon+5)/8, \quad \varepsilon=\pm 1 \end{cases}$$

㉚
$$\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \cdots, \alpha, \alpha+\sqrt{2}/2) & \alpha=\sqrt{2}(5\varepsilon-1)/16 \\ \boldsymbol{l}=(\beta, \beta-\sqrt{2}, \cdots, \beta-\sqrt{2}, \beta-\sqrt{2}/2) & \beta=\sqrt{2}(-\varepsilon+13)/16 \\ \quad \varepsilon=\pm 1 \end{cases}$$

㉛
$$\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \alpha, \alpha, \alpha, \alpha, \alpha) & \alpha=1/2 \\ \boldsymbol{l}=(\beta, \beta, \beta, \beta, \beta-1, \beta-1, \beta-1, \beta-1) & \beta=1/2 \end{cases}$$

㉜
$$\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \alpha, \alpha+\sqrt{2}/2, \alpha+\sqrt{2}/2, \alpha-\sqrt{2}/2, \alpha-\sqrt{2}/2) & \alpha=0 \\ \boldsymbol{l}=(\beta, \beta, \beta-\sqrt{2}, \beta-\sqrt{2}, \beta-\sqrt{2}/2, \cdots, \beta-\sqrt{2}/2) & \beta=\sqrt{2}/2 \end{cases}$$

㉝
$$\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \alpha, \alpha+\sqrt{2}/2, \cdots, \alpha+\sqrt{2}/2, \alpha-\sqrt{2}/2) & \alpha=0 \text{ or } -\sqrt{2}/4 \\ \boldsymbol{l}=(\beta, \beta, \beta-\sqrt{2}, \beta-\sqrt{2}, \beta-\sqrt{2}/2, \cdots, \beta-\sqrt{2}/2) & \beta=\sqrt{2}/2 \end{cases}$$

㉞
$$\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \alpha, \alpha+\sqrt{2}/2, \cdots, \alpha+\sqrt{2}/2, \alpha-\sqrt{2}/2) & \alpha=(\varepsilon-1)\sqrt{2}/8 \\ \boldsymbol{l}=(\beta, \beta-\sqrt{2}, \beta-\sqrt{2}, \beta-\sqrt{2}, \beta-\sqrt{2}/2, \cdots, \beta-\sqrt{2}/2) & \beta=(5-\varepsilon)\sqrt{2}/8 \\ \quad \varepsilon=\pm 1 \end{cases}$$

㉟
$$\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \alpha+\sqrt{2}/2, \cdots, \alpha+\sqrt{2}/2, \alpha-\sqrt{2}/2) & \alpha=(\varepsilon-3)/8\sqrt{2} \\ \boldsymbol{l}=(\beta, \beta-\sqrt{2}, \beta-\sqrt{2}, \beta-\sqrt{2}/2, \cdots, \beta-\sqrt{2}/2) & \beta=(9-3\varepsilon)/8\sqrt{2}, \ \varepsilon=\pm 1 \end{cases}$$

㊱ $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha+\sqrt{2}/2, \cdots, \alpha+\sqrt{2}/2, \alpha-\sqrt{2}/2) & \alpha=-\sqrt{2}/4 \\ \boldsymbol{l}=(\beta, \beta-\sqrt{2}, \beta-\sqrt{2}/2, \cdots, \beta-\sqrt{2}/2) & \beta=(2+\varepsilon)\sqrt{2}/4, \quad \varepsilon=\pm 1 \end{cases}$

$n=9$

㊲ $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \alpha+1, \cdots, \alpha+1) & \alpha=-2/3 \\ \boldsymbol{l}=(1, -1, 0, 0, \cdots, 0) \end{cases}$

㊳ $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \cdots, \alpha, \alpha+1, \alpha+1, \alpha+1) & \alpha=-1/3 \\ \boldsymbol{l}=(1, -1, 0, \cdots, 0, 0, 0, 0) \end{cases}$

㊴ $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \alpha, \alpha, \alpha+\sqrt{7/8}, \cdots, \alpha+\sqrt{7/8}) & \alpha=(-2\sqrt{7}+\varepsilon)/9\sqrt{2} \\ \boldsymbol{l}=(\beta, \beta-\sqrt{2}, \beta-\sqrt{2}, \beta-\sqrt{2}, \beta-\sqrt{2}, \beta-3\sqrt{2}/4, \cdots, \beta-3\sqrt{2}/4) \\ \quad \beta=(14\sqrt{7}\varepsilon)/9\sqrt{2}, \quad \varepsilon=\pm 1 \end{cases}$

㊵ $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \alpha, \alpha, \alpha+\sqrt{3}/2, \cdots, \alpha+\sqrt{3}/2) & \alpha=(\varepsilon-2)\sqrt{3}/9 \\ \boldsymbol{l}=(\beta, \beta-1, \beta-1, \beta-1, \beta-1, \beta-1/2, \cdots, \beta-1/2) & \beta=(2-\varepsilon)/3, \quad \varepsilon=\pm 1 \end{cases}$

㊶ $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \alpha, \cdots, \alpha) & \alpha=\sqrt{2}/3 \\ \boldsymbol{l}=(\beta, \beta, \beta, \beta-1, \cdots, \beta-1) & \beta=2/3 \end{cases}$

㊷ $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \alpha, \alpha, \alpha, \alpha, \alpha, \alpha+\sqrt{3}/2) & \alpha=(4\varepsilon-1)\sqrt{3}/18 \\ \boldsymbol{l}=(\beta, \beta, \beta, \beta, \beta-1, \beta-1, \beta-1, \beta-1, \beta-1/2) & \beta=1/2, \quad \varepsilon=\pm 1 \end{cases}$

㊸ $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \alpha, \alpha+\sqrt{2}/2, \cdots, \alpha+\sqrt{2}/2, \alpha-\sqrt{2}/2) & \alpha=-\sqrt{2}/6 \\ \boldsymbol{l}=(\beta, \beta, \beta-\sqrt{2}, \beta-\sqrt{2}, \beta-\sqrt{2}/2, \cdots, \beta-\sqrt{2}/2) & \beta=\sqrt{2}/2 \end{cases}$

$n=10$

㊹ $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha+\sqrt{5}/2, \cdots, \alpha+\sqrt{5}/2) & \alpha=-2/\sqrt{5}, \\ \boldsymbol{l}=(\beta, \beta-\sqrt{3}, \beta-\sqrt{3}/2, \cdots, \beta-\sqrt{3}/2) & \beta=\sqrt{3}/2+1/2\sqrt{5} \end{cases}$

㊺ $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha+1/2, \alpha+1/2, \alpha+1/2, \alpha+1/2, \alpha-1/2, \cdots, \alpha-1/2) & \alpha=0 \\ \boldsymbol{l}=(\beta, \beta-\sqrt{3}, \beta-\sqrt{3}/2, \beta-\sqrt{3}/2, \beta-\sqrt{3}/2, \beta-\sqrt{3}/2, \beta-\sqrt{3}/2, \cdots, \\ \quad \beta-\sqrt{3}/2) \quad \beta=\sqrt{3}/2+1/2\sqrt{5} \end{cases}$

㊻ $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \alpha, \alpha, \alpha, \alpha, \alpha, \alpha+\sqrt{3}/2, \alpha+\sqrt{3}/2) & \alpha=(-\sqrt{3}+2\sqrt{2}\varepsilon)/10 \\ \boldsymbol{l}=(\beta, \beta, \beta, \beta, \beta-1, \beta-1, \beta-1, \beta-1, \beta-1/2, \beta-1/2) & \beta=1/2, \quad \varepsilon=\pm 1 \end{cases}$

(47) $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \alpha, \alpha, \alpha, \alpha+\sqrt{3}/2, \cdots, \alpha+\sqrt{3}/2) & \alpha=(-2\sqrt{3}+\sqrt{2}\varepsilon)/10 \\ \boldsymbol{l}=(\beta, \beta, \beta, \beta, \beta-1, \beta-1, \beta-1/2, \cdots, \beta-1/2) & \beta=(4+\sqrt{6}\varepsilon)/10,\ \varepsilon=\pm 1 \end{cases}$

$n=11$

(48) $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \alpha+\sqrt{7/8}, \cdots, \alpha+\sqrt{7/8}) & \alpha=(\varepsilon-2\sqrt{14})/11 \\ \boldsymbol{l}=(\beta, \beta-\sqrt{2}, \beta-\sqrt{2}, \beta-3\sqrt{2}/4, \cdots, \beta-3\sqrt{2}/4) & \beta=(8\sqrt{2}+7\varepsilon)/11, \\ \quad \varepsilon=\pm 1 \end{cases}$

(49) $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \cdots, \alpha, \alpha+\sqrt{7/8}, \alpha+\sqrt{7/8}) & \alpha=(-\sqrt{14}+5\varepsilon)/22 \\ \boldsymbol{l}=(\beta, \beta-\sqrt{2}, \cdots, \beta-\sqrt{2}, \beta-3\sqrt{2}/4, \beta-3\sqrt{2}/4) \\ \quad \beta=(19\sqrt{2}-\sqrt{7}\varepsilon)/22,\ \varepsilon=\pm 1 \end{cases}$

(50) $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \alpha, \cdots, \alpha, \alpha+\sqrt{3}/2, \alpha+\sqrt{3}/2, \alpha+\sqrt{3}/2, \alpha+\sqrt{3}/2) \\ \quad \alpha=(-2\sqrt{3}+\varepsilon)/11 \\ \boldsymbol{l}=(\beta, \beta, \beta, \beta, \beta-1, \beta-1, \beta-1, \beta-1/2, \cdots, \beta-1/2) & \beta=(5+\sqrt{3}\varepsilon)/11, \\ \quad \varepsilon=\pm 1 \end{cases}$

(51) $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \alpha, \cdots, \alpha, \alpha+\sqrt{3}/2, \alpha+\sqrt{3}/2, \alpha+\sqrt{3}/2) & \alpha=(-3\sqrt{3}\pm 4)/22 \\ \boldsymbol{l}=(\beta, \beta, \beta, \beta, \beta-1, \cdots, \beta-1, \beta-1/2, \beta-1/2, \beta-1/2) & \beta=1/2 \end{cases}$

$n=12$

(52) $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \alpha, \alpha+\sqrt{3}/2, \cdots, \alpha+\sqrt{3}/2) & \alpha=-\sqrt{3}/3 \\ \boldsymbol{l}=(\beta, \beta, \beta-1, \beta-1, \beta-1/2, \cdots, \beta-1/2) & \beta=1/2+\sqrt{3}/6 \end{cases}$

(53) $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \cdots, \alpha, \alpha+\sqrt{3}/2, \alpha+\sqrt{3}/2) & \alpha=(-\sqrt{3}+3\varepsilon)/12 \\ \boldsymbol{l}=(\beta, \beta, \beta-1, \cdots, \beta-1, \beta-1/2, \beta-1/2) & \beta=(9-\sqrt{3}\varepsilon)/12,\ \varepsilon=\pm 1 \end{cases}$

(54) $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \alpha, \alpha, \alpha, \alpha, \alpha, \alpha+\sqrt{3}/2, \cdots, \alpha+\sqrt{3}/2) & \alpha=-\sqrt{3}/6 \\ \boldsymbol{l}=(\beta, \beta, \beta, \beta, \beta-1, \beta-1, \beta-1, \beta-1, \beta-1/2, \cdots, \beta-1/2) & \beta=1/2 \end{cases}$

$n=18$

(55) $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha+1/2, \alpha+1/2, \alpha-1/2, \cdots, \alpha-1/2) & \alpha=1/3 \\ \boldsymbol{l}=(\beta, \beta-\sqrt{3}, \beta-\sqrt{3}/2, \beta-\sqrt{3}/2, \beta-\sqrt{3}/2, \cdots, \beta-\sqrt{3}/2) \\ \quad \beta=\sqrt{3}/2+1/6 \end{cases}$

⑯ $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha+\sqrt{7/8}, \cdots, \alpha+\sqrt{7/8}) & \alpha=(-4\sqrt{7}+2\varepsilon)/9\sqrt{2} \\ \boldsymbol{l}=(\beta, \beta-\sqrt{2}, \beta-3\sqrt{2}/4, \cdots, \beta-3\sqrt{2}/4) & \beta=(13+\sqrt{7}\varepsilon)/9\sqrt{2},\ \varepsilon=\pm 1 \end{cases}$

⑰ $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \cdots, \alpha, \alpha+\sqrt{7/8}) & \alpha(-\sqrt{7}+13\varepsilon)/36\sqrt{2} \\ \boldsymbol{l}=(\beta, \beta-\sqrt{2}, \cdots, \beta-\sqrt{2}, \beta-3\sqrt{2}/4) & \beta=(67-\sqrt{7}\varepsilon)/36\sqrt{2},\ \varepsilon=\pm 1 \end{cases}$

$n=19$

⑱ $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \alpha+\sqrt{3}/2, \cdots, \alpha+\sqrt{3}/2) & \alpha=(-8\sqrt{3}+\sqrt{2}\varepsilon)/19 \\ \boldsymbol{l}=(\beta, \beta-1, \beta-1, \beta-1/2, \cdots, \beta-1/2) & \beta=(10-2\sqrt{6}\varepsilon)/19,\ \varepsilon=\pm 1 \end{cases}$

⑲ $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \cdots, \alpha, \alpha+\sqrt{3}/2, \alpha+\sqrt{3}/2) & \alpha=(-2\sqrt{3}+5\sqrt{2}\varepsilon)/38 \\ \boldsymbol{l}=(\beta, \beta-1, \cdots, \beta-1, \beta-1/2, \beta-1/2) & \beta=(17\sqrt{2}-3\sqrt{3}\varepsilon)/19\sqrt{2},\ \varepsilon=\pm 1 \end{cases}$

⑳ $\begin{cases} \boldsymbol{k}=(\alpha, \alpha, \alpha, \cdots, \alpha, \alpha+\sqrt{3}/2) & \alpha=(-\sqrt{3}+7\sqrt{2}\varepsilon)/38 \\ \boldsymbol{l}=(\beta, \beta, \beta-1, \cdots, \beta-1, \beta-1/2) & \beta=(33-\sqrt{6}\varepsilon)/38,\ \varepsilon=\pm 1. \end{cases}$

PROOF. Let S be an arbitrary element in P_n, and $\boldsymbol{k}'=S\cdot\boldsymbol{k}=(k_{s(1)}, \cdots, k_{s(n)})$, $\boldsymbol{l}'=S\cdot\boldsymbol{l}=(l_{s(1)}, \cdots, l_{s(n)})$ and $T'=STS^{-1}$. Then $\boldsymbol{k}'$ and $\boldsymbol{l}'$ span the -1 eigenspace of T' and $\langle T', [P_n, P_n]\rangle=S\langle T, [P_n, P_n]\rangle S^{-1}$. Therefore, we may consider $\boldsymbol{k}$ and $\boldsymbol{l}$ up to permutations.

(I) At first we assume that there exist i and j such that $(k_i-k_j)^2+(l_i-l_j)^2=4$. We may set $i=1$ and $j=2$. Let us set $\boldsymbol{k}'=\frac{1}{2}(l_1-l_2)\boldsymbol{k}-\frac{1}{2}(k_1-k_2)\boldsymbol{l}$ and $\boldsymbol{l}'=\frac{1}{2}(k_1-k_2)\boldsymbol{k}+\frac{1}{2}(l_1-l_2)\boldsymbol{l}$. Then we have $k_1'=k_2'$ and $l_1'=l_2'+2$. For brevity we will omit the primes in the following argument. Since $2=\sum_{i=1}^{n}(l_i)^2=(l_1)^2+(l_1-2)^2+\sum_{i\neq 1,2}^{n}(l_i)^2\geqq 2(l_1-1)^2+2$, we have $l_1=1$, $l_2=-1$ and $l_i=0$ for all $i\neq 1, 2$. Since $(k_1-k_i)^2+(l_1-l_i)^2\in\{0, 1, 2, 3, 4\}$, $(k_1-k_i)^2\in\{0, 1, 2, 3\}$ for all i. Therefore, $k_i=k_1\pm\sqrt{3}$ or $k_1\pm\sqrt{2}$ or $k_1\pm 1$ or k_1. Since $2(k_1)^2+(k_1\pm\sqrt{3})^2=3(k_1\pm\sqrt{3}/3)^2+2$ and $n\geqq 5$, we have $k_i\neq k_1\pm\sqrt{3}$ for any i, $1\leqq i\leqq n$. If there exists i_0 such that $k_{i_0}=k_1+\sqrt{2}\varepsilon$ ($\varepsilon=1$ or -1), then $k_i=k_1$ or $k_i=k_1+\sqrt{2}\varepsilon$ $(=k_{i_0})$ for all i, because $(k_i-k_{i_0})^2\in\{0, 1, 2, 3, 4\}$. Let there be s k_i's such that $k_i=k_1+\sqrt{2}\varepsilon$. Since $2=(n-s)(k_1)^2+s(k_1+\sqrt{2}\varepsilon)^2$ and k_1 is a real number, we have $(n-s)\geqq s\cdot\{(n-s)-1\}$. Since $n-s\geqq 2$ and $n\geqq 5$, we obtain $s=0$ or 1. Hence we have obtained the vectors ⑤, ⑦ and ⑧ in the table. Therefore, we may assume that $k_i=k_1$ or $k_1\pm 1$ for all i. Since $(k_1+1)^2+(k_1-1)^2=2(k_1)^2+2$, we obtain ④ or $\boldsymbol{k}=(k_1, \cdots, k_1, k_1+\varepsilon, \cdots, k_1+\varepsilon)$. If we set $k_1'=-k_1$, then we have $(k_1', \cdots, k_1', k_1'-1, \cdots, k_1'-1)=-(k_1, \cdots, k_1, k_1+1, \cdots,$

k_1+1), and therefore we may assume that $\boldsymbol{k}=(k_1, \cdots, k_1, k_1+1, \cdots, k_1+1)$. Since $\sum_{i=1}^{n}(k_i)^2=2$, either there exist at most four k_i's or there exist at most four (k_1+1)'s, and so we obtain the vectors ②, ③, ⑩, ⑪, ⑫, ⑰, ㉒, ㉓, ㉖, ㉗, ㉘, ㊲, ㊳.

(II) Secondly, we assume that there exist i_1 and i_2 such that $(k_{i_1}-k_{i_2})^2+(l_{i_1}-l_{i_2})^2=3$ and $(k_i-k_j)^2+(l_i-l_j)^2\neq 4$ for all i and j. We may set $i_1=1$ and $i_2=2$. Let us set $\boldsymbol{k}'=\frac{(l_1-l_2)}{\sqrt{3}}\boldsymbol{k}-\frac{(k_1-k_2)}{\sqrt{3}}\boldsymbol{l}$, $\boldsymbol{l}'=\frac{(k_1-k_2)}{\sqrt{3}}\boldsymbol{k}+\frac{(l_1-l_2)}{\sqrt{3}}\boldsymbol{l}$. Then we have $k_1'=k_2'$ and $l_2'=l_1'-\sqrt{3}$. For brevity we will also omit the primes in the following argument. Since $(k_i-k_1)^2+(l_i-l_1)^2\in\{0, 1, 2, 3\}$ and $(k_i-k_2)^2+(l_i-l_2)^2\in\{0, 1, 2, 3\}$ and $k_1=k_2$ and $l_2=l_1-\sqrt{3}$, l_i takes only one of the following seven values (listed in the left hand side in the table (1) below) and the possible values of k_i are determined as stated below;

$$\left.\begin{array}{lll} \text{when} & l_i=l_1, \quad \text{then} & k_i=k_1 \\ & l_i=l_1-\sqrt{3}, & k_i=k_1 \\ & l_i=l_1-\sqrt{3}/3, & k_i=k_1\pm\sqrt{6}/3 \text{ or } k_1\pm\sqrt{5/3} \\ & l_i=l_1-2\sqrt{3}/3, & k_i=k_1\pm\sqrt{6}/3 \text{ or } k_1\pm\sqrt{5/3} \\ & l_i=l_1-\sqrt{3}/6, & k_i=k_1\pm\sqrt{11/12} \\ & l_i=l_1-5\sqrt{3}/6, & k_i=k_1\pm\sqrt{11/12} \\ & l_i=l_1-\sqrt{3}/2, & k_i=k_1\pm 1/2 \text{ or } k_1\pm\sqrt{5}/2 \text{ or } k_1\pm 3/2. \end{array}\right\} \quad (1)$$

Since $(l_1)^2+2(l_1-\sqrt{3})^2=3(l_1-2\sqrt{3}/3)^2+2\geqq 2$, there exist at most two i's such that $l_i=l_1-\sqrt{3}$. Subcase (i) If there exist two $(l_1-\sqrt{3})$'s then $l_1=2\sqrt{3}/3$ and $\boldsymbol{l}=(l_1, l_1-\sqrt{3}, l_1-\sqrt{3}, l_1-2\sqrt{3}/3, \cdots, l_1-2\sqrt{3}/3)$. Hence from the table (1) we obtain $k_1=k_2=k_3$ and k_i is one of $k_1+\sqrt{5/3}$, $k_1-\sqrt{5/3}$, $k_1+\sqrt{6}/3$ and $k_1-\sqrt{6}/3$ for $i\neq 1$, 2 and 3. Since $\{(k_1+\sqrt{5/3})-(k_1-\sqrt{5/3})\}^2=20/3\notin \boldsymbol{Z}$ and $\{(k_1+\sqrt{6}/3)-(k_1-\sqrt{6}/3)\}^2=8/3\notin \boldsymbol{Z}$ and $\{(k_1+\varepsilon\sqrt{5/3})-(k_1+\varepsilon'\sqrt{6}/3)\}^2\notin \boldsymbol{Q}$ (where ε and $\varepsilon'=1$ or -1), we obtain $k_i=k_4$ for all $i\neq 1$, 2 and 3. If $k_i=k_1-\alpha$ for $i\neq 1$, 2 and 3, then $(k_1, k_1, k_1, k_1-\alpha, \cdots, k_1-\alpha)=-(k_1', k_1', k_1', k_1'+\alpha, \cdots, k_1'+\alpha)$ where $\alpha=\sqrt{5/3}$ or $\sqrt{6}/3$ and $k_1'=-k_1$. Hence we have only to consider the following two cases $\boldsymbol{k}=(k_1, k_1, k_1, k_1+\sqrt{5/3}, \cdots, k_1+\sqrt{5/3})$ and $\boldsymbol{k}=(k_1, k_1, k_1, k_1+\sqrt{6}/3, \cdots, k_1+\sqrt{6}/3)$. From the former case we obtain the vectors listed as ⑯ because $\sum_{i=1}^{n}(k_i)^2=2$. From the latter case we have $2=n(k_1)^2+2\sqrt{6}(n-3)k_1/3+2(n-3)/3$ and therefore $\boldsymbol{k}$ is equal to the vector listed as ⑬ or $\boldsymbol{k}=(-\sqrt{6}/3, -\sqrt{6}/3, -\sqrt{6}/3, 0, \cdots, 0)$. Since $-(\sqrt{3}/3)\boldsymbol{k}+(\sqrt{6}/3)\boldsymbol{l}=(\sqrt{2}, 0, \cdots, 0)$ and $-(\sqrt{6}/3)\boldsymbol{k}-(\sqrt{3}/3)\boldsymbol{l}=(0, 1, 1, 0, \cdots, 0)$, we obtain the vectors listed as ⑥. Hence we may assume that there is only one $l_1-\sqrt{3}$. Since $2(l_1)^2+(l_1-\sqrt{3})^2=3(l_1-\sqrt{3}/3)^2+2\geqq 2$, there are at most two l_1's. Subcase (ii) If there are two l_1's, then we obtain $\boldsymbol{l}=(l_1, l_1, l_1-\sqrt{3}, l_1-\sqrt{3}/3, \cdots, l_1-\sqrt{3}/3)$ where $l_1=\sqrt{3}/3$. Since $(l_1, l_1, l_1-\sqrt{3}, l_1-\sqrt{3}/3, \cdots, l_1-\sqrt{3}/3)=-(l_1'-\sqrt{3}, l_1'-\sqrt{3}, l_1', l_1'-2\sqrt{3}/3, \cdots, l_1'-2\sqrt{3}/3)$ where $l_1'=-l_1+\sqrt{3}$, we obtain the vectors listed as ⑯, ⑬ and ⑥ as before. Hence we may assume that $l_i\neq l_1$ and $l_i\neq l_1-\sqrt{3}$ for all $i\neq 1$ and 2. Therefore $k_i\neq k_1$ for all $i\neq 1$ and 2. Subcase (iii) If there exists i_0 such that $l_{i_0}=l_1-\sqrt{3}/2$, then k_{i_0} is one of $k_1\pm\sqrt{5}/2$, $k_1\pm 3/2$, $k_1\pm 1/2$. If $k_{i_0}=k_1+\varepsilon\sqrt{5}/2$ (where $\varepsilon=1$ or -1), then $k_i=k_{i_0}$ for all $i\neq 1$ and 2 because $(k_i-k_{i_0})^2+(l_i-l_{i_0})^2\in\{0, 1, 2, 3\}$ and $(l_i-l_{i_0})^2\in\{0, 1/12, 1/3\}$. As we have shown before, we have only to consider the case $\varepsilon=1$. From the table (1) we obtain $l_i=l_{i_0}$ for all $i\neq 1$, 2. Since $\sum_{i=1}^{n}(k_i)^2=2$ and $\sum_{i=1}^{n}(l_i)^2=2$, we obtain $k_1=\dfrac{-(n-2)\sqrt{5}+\varepsilon_1\sqrt{20-2n}}{2n}$ and $l_1=\sqrt{3}/2+\varepsilon_2\sqrt{2n}/2n$ where ε_1 and ε_2 are 1 or -1. Since $\sum_{i=1}^{n}k_il_i=0$, we obtain $n=10$. Since $(\sqrt{3}/2-\sqrt{2n}/2n, -\sqrt{3}/2-\sqrt{2n}/2n, -\sqrt{2n}/2n, \cdots, -\sqrt{2n}/2n)=-(-\sqrt{3}/2+\sqrt{2n}/2n, \sqrt{3}/2+\sqrt{2n}/2n, \sqrt{2n}/2n, \cdots, \sqrt{2n}/2n)$, we obtain the vectors listed as ㊹. If $k_{i_0}=k_1+(3/2)\varepsilon$ (where $\varepsilon=1$ or -1) then $k_i=k_1+(3/2)\varepsilon$ or $k_1+(1/2)\varepsilon$ for all $i\neq 1$, 2 because $(k_i-k_{i_0})^2+(l_i-l_{i_0})^2\in\{0, 1, 2, 3\}$ and $(l_i-l_{i_0})^2\in\{0, 1/12, 1/3\}$. Therefore, we obtain from the table (1) $l_i=l_{i_0}=l_1-\sqrt{3}/2$ for all $i\neq 1$, 2. But from the conditions $\sum_{i=1}^{n}(k_i)^2=\sum_{i=1}^{n}(l_i)^2=2$ and $\sum_{i=1}^{n}k_il_i=0$, we can show there are no such vectors by the similar calculation as before. Hence we may assume that $k_i=k_1+1/2$ or $k_1-1/2$ for all $i\neq 1$, 2. Then $l_i=l_1-\sqrt{3}/2$ for all $i\neq 1$, 2. Let s be the number of i's such that $k_i=k_1+1/2$. Then from $\sum_{i=1}^{n}(k_i)^2=\sum_{i=1}^{n}(l_i)^2=2$, we obtain $k_1=\dfrac{(n-2s-2)+\varepsilon_1\sqrt{2(3-2s)n+4(s+1)^2}}{2n}$ and $l_1=\sqrt{3}/2\pm\sqrt{2n}/2n$ where $\varepsilon_1=\pm 1$. Since $\sum_{i=1}^{n}k_il_i=0$, we obtain $k_1=-(2+2s-n)/2n$ and $2(3-2s)n+4(s+1)^2=0$. Hence $n=2(s+1)^2/(2s-3)=s+7/2+25/2(2s-3)$. Since n is an integer $\geqq 5$, we obtain $s=2$ $(n=18)$ or $s=4$ $(n=10)$ or $s=14$ $(n=18)$. From $s=2$ and $s=14$ for $n=18$ we obtain the vectors listed as ㊿ and from $s=4$ for $n=10$ we obtain the vectors listed as ㊺. Subcase (iv) If there exists i_0 such that $l_{i_0}=l_1-\sqrt{3}/6$, then $k_{i_0}=k_1+\sqrt{11/12}$ or $k_1-\sqrt{11/12}$. Since $(k_i-k_{i_0})^2+(l_i-l_{i_0})^2\in\{0, 1, 2, 3\}$ and $(l_i-l_{i_0})^2\in\{0, 1/12, 1/3, 3/4, 4/3\}$, $k_i=k_{i_0}$ for all $i\neq 1$, 2. Therefore, $l_i=l_{i_0}=l_1-\sqrt{3}/6$ for all $i\neq 1$, 2. On the other hand $\sum_{i=1}^{n}(k_i)^2=\sum_{i=1}^{n}(l_i)^2=2$ and $\sum_{i=1}^{n}k_il_i=0$, we can show there are no such vectors by the similar calculation. Subcase (v) If there exists i_0 such that $l_{i_0}=l_1-5\sqrt{3}/6$, then by setting $l_1=-l_1'+\sqrt{3}$, we have $l_{i_0}=-l'_1+\sqrt{3}/6$ and therefore this case is already done. Subcase (vi) If there exists i_0 such that $l_{i_0}=l_1-\sqrt{3}/3$, then k_{i_0} is one of $k_1\pm\sqrt{5/3}$ and $k_1\pm\sqrt{6}/3$. Therefore, we have $k_i=k_{i_0}$ for all $i\neq 1, 2$, because $(k_i-k_{i_0})^2+(l_i-l_{i_0})^2\in\{0, 1, 2, 3\}$ and $(l_i-l_{i_0})^2\in\{0, 1/12, 1/3, 3/4\}$ for all $i\neq 1, 2$. Hence $l_i=l_{i_0}$ for all $i\neq 1, 2$. On the other hand $\sum_{i=1}^{n}(k_i)^2=\sum_{i=1}^{n}(l_i)^2=2$ and $\sum_{i=1}^{n}k_il_i=0$, we can show that there are no such vectors. Subcase (vii) If there exists i_0 such that $l_{i_0}=l_1-2\sqrt{3}/3$, then by setting $l_1=-l'_1+\sqrt{3}$ we have $l_{i_0}=-l'_1+\sqrt{3}/3$ and therefore this case is already done.

(III) Thirdly we assume that there exist i_1 and i_2 such that $(k_{i_1}-k_{i_2})^2+(l_{i_1}-l_{i_2})^2=2$ and $(k_i-k_j)^2+(l_i-l_j)^2\in\{0, 1, 2\}$ for all i and j. We may set $i_1=1$ and $i_2=2$. Let us set $\boldsymbol{k}'=\frac{(l_1-l_2)}{\sqrt{2}}\boldsymbol{k}-\frac{(k_1-k_2)}{\sqrt{2}}\boldsymbol{l}$ and $\boldsymbol{l}'=\frac{(k_1-k_2)}{\sqrt{2}}\boldsymbol{k}+\frac{(l_1-l_2)}{\sqrt{2}}\boldsymbol{l}$. Then we obtain $k'_1=k'_2$ and $l'_2=l'_1-\sqrt{2}$. We will also omit the primes in the following argument for brevity. Since $(k_1-k_i)^2+(l_1-l_i)^2\in\{0, 1, 2\}$ and $(k_2-k_i)^2+(l_2-l_i)^2\in\{0, 1, 2\}$ and $k_1=k_2$ and $l_2=l_1-\sqrt{2}$, l_i takes only one of the following five values and the possible values of k_i are determined as stated below;

$$\text{when}\quad\left.\begin{array}{ll} l_i=l_1, \text{ then} & k_i=k_1 \\ l_i=l_1-\sqrt{2}, & k_i=k_1 \\ l_i=l_1-\sqrt{2}/4, & k_i=k_1\pm\sqrt{7/8} \\ l_i=l_1-3\sqrt{2}/4, & k_i=k_1\pm\sqrt{7/8} \\ l_i=l_1-\sqrt{2}/2, & k_i=k_1\pm\sqrt{3/2} \text{ or } k_1\pm\sqrt{2}/2. \end{array}\right\}\qquad(2)$$

Since $2(l_1)^2+2(l_1-\sqrt{2})^2=4(l_1-\sqrt{2}/2)^2+2\geqq 2$, we have $\boldsymbol{l}=(l_1, l_1, l_1-\sqrt{2}, l_1-\sqrt{2}, l_1-\sqrt{2}/2, \cdots, l_1-\sqrt{2}/2)$ and $l_1=\sqrt{2}/2$, or $l_i\neq l_1$ for all $i\neq 1$, or $l_i\neq l_1-\sqrt{2}$ for all $i\neq 2$. Subcase (i) If $\boldsymbol{l}=(l_1, l_1, l_1-\sqrt{2}, l_1-\sqrt{2}, l_1-\sqrt{2}/2, \cdots, l_1-\sqrt{2}/2)$ and $l_1=\sqrt{2}/2$, then $k_4=k_3=k_2=k_1$ and $k_i=k_1+\alpha$ or $k_1-\alpha$ for all $i\neq 1, 2, 3$ and 4 (where $\alpha=\sqrt{3/2}$ or $\sqrt{2}/2$). If $\alpha=\sqrt{3/2}$, then $n\leqq 6$ because $\sum_{i=1}^{n}(k_i)^2=2$, and therefore we obtain the vectors listed as ⑭ and ⑱. If $\alpha=\sqrt{2}/2$, then $4\geqq s(n-s-4)$, because $\sum_{i=1}^{n}(k_i)^2=2$, where s is the number of i's such that $k_i=k_1+\sqrt{2}/2$. If $s<n-s-4$, then by setting $\boldsymbol{k}'=-\boldsymbol{k}$, these vectors are transformed into the vectors with $s>n-s-4$. Therefore we may assume $s\geqq 1$ and $s\geqq n-s-4$. Thus we have $n=6$, $s=1$; $n=7$, $s=2$; $n=8$, $s=2$; $n=8$, $s=3$; $n=9$, $s=4$, and $s=n-4$ for every $n\geqq 5$. Thus we obtain the vectors ㉑, ㉖, ㉜, ㉝, ㊸, ⑨ and $\boldsymbol{k}=(-\sqrt{2}/2, -\sqrt{2}/2, -\sqrt{2}/2, -\sqrt{2}/2, 0, \cdots, 0)$, $\boldsymbol{l}=(\sqrt{2}/2, \sqrt{2}/2, -\sqrt{2}/2, -\sqrt{2}/2, 0, \cdots, 0)$ respectively. The last pair is transformed into the vectors ① by an element in $O(2)$. Subcase (ii) Next we assume that $l_i \neq l_1$ for all $i \neq 1$. Subcase (ii-1) If there exists i_0 such that $l_{i_0}=l_1-\sqrt{2}/4$, then $k_{i_0}=k_1+\sqrt{7/8}$ or $k_1-\sqrt{7/8}$. Since $(k_i-k_{i_0})^2+(l_i-l_{i_0})^2 \in \{0, 1, 2\}$ and $(l_i-l_{i_0})^2 \in \{0, 1/8, 1/2, 9/8\}$ for all i, we have $k_i=k_1$ or $k_i=k_{i_0}$. Hence from the table (2) we have $l_i=l_1-\sqrt{2}$ or $l_1-\sqrt{2}/4$ $(=l_{i_0})$ for $i \neq 1$. Let s be the number of i's such that $l_i=l_1-\sqrt{2}$. Since $\sum_{i=1}^{n}(k_i)^2=\sum_{i=1}^{n}(l_i)^2=2$ we have $k_1= \dfrac{-\sqrt{7}(n-s-1)+\varepsilon_1\sqrt{7(s+1)^2-(7s-9)n}}{2\sqrt{2}\,n}$ and $l_1=\dfrac{(n+3s-1)+\varepsilon_2\sqrt{(3s-1)^2+(15-9s)n}}{2\sqrt{2}\,n}$ where ε_1 and ε_2 are 1 or -1. Since $\sum_{i=1}^{n}k_i l_i=0$, we obtain $-\varepsilon_1\varepsilon_2\sqrt{7}(n-s-1)(3s-1) =\sqrt{\{(3s-1)^2+(15-9s)n\}\{7(s+1)^2-(7s-9)n\}}$. Therefore, we obtain $\varepsilon_1\varepsilon_2=-1$ and $n=\dfrac{9s^2+s+8}{9s-8}=s+1+\dfrac{16}{9s-8}$. Since n is an integer $\geqq 5$, we have $s=1$ and $n=18$. Thus we obtain the vectors $\boldsymbol{k}=(\alpha, \alpha, \alpha+\sqrt{7/8}, \cdots, \alpha+\sqrt{7/8})$ and $\boldsymbol{l}=(\beta, \beta-\sqrt{2}, \beta-\sqrt{2}/4, \cdots, \beta-\sqrt{2}/4)$ where $\alpha=(-4\sqrt{7}+2\varepsilon)/9\sqrt{2}$ and $\beta=(-\sqrt{7}\varepsilon+5)/9\sqrt{2}$ and $\varepsilon=1$ or -1. These vectors are transformed into the vectors ㊽ by an element of $O(2)$ and a permutation. Subcase (ii-2) If there exists i_0 such that $l_{i_0}=l_1-3\sqrt{2}/4$, then $k_{i_0}=k_1+\sqrt{7/8}$ or $k_1-\sqrt{7/8}$. Since $(k_i-k_{i_0})^2+(l_i-l_{i_0})^2 \in \{0, 1, 2\}$ and $(l_i-l_{i_0})^2 \in \{0, 1/8, 1/2, 9/8\}$, we obtain $k_i=k_{i_0}$ or $k_i=k_1$ for all i and $l_i=l_1-\sqrt{2}$ or $l_i=l_1-3\sqrt{2}/4$. Therefore, we have only to consider the case $k_{i_0}=k_1+\sqrt{7/8}$. Let s be the number of i's such that $l_i=l_1-\sqrt{2}$. Since $\sum_{i=1}^{n}(k_i)^2=\sum_{i=1}^{n}(l_i)^2=2$ and $\sum_{i=1}^{n}k_i l_i=0$, we obtain by the similar calculation as before that $k_1=\dfrac{-\sqrt{7}(n-s-1)+\varepsilon_1\sqrt{7(s+1)^2+(9-7s)n}}{2\sqrt{2}\,n}$ and $l_1=\dfrac{(3n+s-3)+\varepsilon_2\sqrt{(s-3)^2+(7-s)n}}{2\sqrt{2}\,n}$ where ε_1 and ε_2 are 1 or -1 and $n=\dfrac{s^2+s+16}{s}$ $=s+1+16/s$ and $\varepsilon_1\varepsilon_2(n-s-1)(s-3)\leqq 0$. Therefore, we obtain $s=1$, $n=18$, $\varepsilon_1\varepsilon_2=1$; $s=2$, $n=11$, $\varepsilon_1\varepsilon_2=1$: $s=4$, $n=9$, $\varepsilon_1\varepsilon_2=-1$; $s=8$, $n=11$, $\varepsilon_1\varepsilon_2=-1$ and $s=16$, $n=18$, $\varepsilon_1\varepsilon_2=-1$. Thus we obtain the vectors ㊻, ㊽, ㊴, ㊾ and ㊼ respectively. Subcase (ii-3) If there exists i_0 such that $l_{i_0}=l_1-\sqrt{2}/2$ then k_{i_0} is one of $k_1\pm\sqrt{2}/2$ and $k_1\pm\sqrt{3/2}$. We may assume $k_{i_0}=k_1+\sqrt{3}/2$ or $k_1+\sqrt{2}/2$. If $k_{i_0}=k_1+\sqrt{3}/2$, then we have $k_i=k_{i_0}$ or $k_i=k_1$, because $(k_{i_0}-k_i)^2+(l_{i_0}-l_i)^2 \in \{0, 1, 2\}$ and $(l_{i_0}-l_i)^2 \in \{0, 1/8, 1/2\}$. Hence we obtain $l_i=l_{i_0}=l_1-\sqrt{2}/2$ or $l_i=l_1-\sqrt{2}$ for every $i \neq 1$. Let s be the number of i's such that $l_i=l_1-\sqrt{2}$. Since $\sum_{i=1}^{n}(k_i)^2=\sum_{i=1}^{n}(l_i)^2=2$ and $\sum_{i=1}^{n}k_i l_i=0$, we obtain $k_1=\dfrac{-(n-s-1)\sqrt{3}+\varepsilon_1\sqrt{3(s+1)^2+(1-3s)n}}{\sqrt{2}\,n}$, $l_1=\dfrac{(n+s-1)+\varepsilon_2\sqrt{(s-1)^2+(3-s)n}}{\sqrt{2}\,n}$, $n=1+s+4/s$ and $\varepsilon_1\varepsilon_2(s-1)\leqq 0$. Therefore we obtain $n=5$, $s=2$, $\varepsilon_1\varepsilon_2=-1$; $n=6$,

$s=4$, $\varepsilon_1\varepsilon_2=-1$ and $n=6$, $s=1$. Hence we obtain the vectors ⑮, ⑲, ⑳ respectively. If $k_{i_0}=k_1+\sqrt{2}/2$, then we have $k_i=k_1+\sqrt{2}/2$ or $k_i=k_1-\sqrt{2}/2$ or $k_i=k_1$, because $(k_{i_0}-k_i)^2+(l_{i_0}-l_i)^2\in\{0, 1, 2\}$ and $(l_{i_0}-l_i)^2\in\{0, 1/8, 1/2\}$. Hence we obtain $l_i=l_{i_0}=l_1-\sqrt{2}/2$ or $l_i=l_1-\sqrt{2}$ for every $i\neq 1$. Let s and t be the numbers of i's such that $k_i=k_1+\sqrt{2}/2$ and $k_i=k_1-\sqrt{2}/2$ respectively. Since $\sum_{i=1}^{n}(k_i)^2=\sum_{i=1}^{n}(l_i)^2=2$ and $\sum_{i=1}^{n}k_il_i=0$, we obtain $k_1=\dfrac{-(s-t)+\varepsilon_1\sqrt{(s-t)^2-(s+t-4)n}}{\sqrt{2}n}$,

$l_1=\dfrac{(2n-s-t-2)+\varepsilon_2\sqrt{(s+t+2)^2-(s+t)n}}{\sqrt{2}n}$, $n(st-s-t)=(st-s-t)(t+s+3)+st-4$ and $\varepsilon_1\varepsilon_2(t-s)(n-s-t-2)\geqq 0$. We may assume $s\geqq 1$ and $s\geqq t$. Since k_1 is a real number, we have $s+t\geqq st$. Therefore, we obtain $s=t=2$ or $t\leqq 1$. If $s=t=2$ then the vectors are transformed into the vectors in III Subcase (i) by a permutation after setting $\boldsymbol{l}'=\boldsymbol{k}$ and $\boldsymbol{k}'=\boldsymbol{l}$. If $t=1$, then $n=8$ and $s=5$, 4, 3, 2 or 1. We can easily show that $s\geqq 6-s$ by setting $\boldsymbol{k}'=\boldsymbol{l}$ and $\boldsymbol{l}'=\boldsymbol{k}$. If $t=0$, then $n=s+3+4/s$. Hence we obtain $n=7$, $t=0$, $s=2$, $\varepsilon_1\varepsilon_2=-1$; $n=8$, $t=0$, $s=4$, $\varepsilon_1\varepsilon_2=-1$; $n=8$, $t=0$, $s=1$, $\varepsilon_1\varepsilon_2=-1$; $n=8$, $t=1$, $s=3$, $\varepsilon_1\varepsilon_2=-1$; $n=8$, $t=1$, $s=4$, $\varepsilon_1\varepsilon_2=-1$ and $n=8$, $t=1$, $s=5$. Thus we have the vectors in ㉔, ㉙, ㉚, ㉞, ㉟ and ㊱ respectively. Subcase (ii-4) If $l_i=l_1-\sqrt{2}$ for all $i\neq 1$, then $k_i=k_1$ for all i. Since $2=\sum_{i=1}^{n}(k_i)^2=\sum_{i=1}^{n}(l_i)^2$ and $\sum_{i=1}^{n}k_il_i=0$, we can show there are no such vectors. Subcase (iii) At last we assume that $l_i\neq l_1-\sqrt{2}$ for all $i\neq 2$. Then by setting $l_1'=-l_1+\sqrt{2}$, we have $l_1=-(l_1'-\sqrt{2})$, $l_1-\sqrt{2}=-l_1'$, $l_1-\sqrt{2}/4=-(l_1'-3\sqrt{2}/4)$, $l_1-3\sqrt{2}/4=-(l_1'-\sqrt{2}/4)$ and $l_1-\sqrt{2}/2=-(l_1'-\sqrt{2}/2)$. Therefore, this case is already done.

(IV) At last we assume that $(k_i-k_j)^2+(l_i-l_j)^2=0$ or 1 for all i and j. Since $\sum_{i=1}^{n}(k_i)^2=\sum_{i=1}^{n}(l_i)^2=2$ and $\sum_{i=1}^{n}k_il_i=0$, there are at least two i_1 and i_2 such that $(k_{i_1}-k_{i_2})^2+(l_{i_1}-l_{i_2})^2=1$. We may set $i_1=1$ and $i_2=2$. Let us set $\boldsymbol{k}'=(l_1-l_2)\boldsymbol{k}-(k_1-k_2)\boldsymbol{l}$ and $\boldsymbol{l}'=(k_1-k_2)\boldsymbol{k}+(l_1-l_2)\boldsymbol{l}$, then we obtain $k_1'=k_2'$ and $l_2'=l_1'-1$. We will also omit the primes in the following argument for brevity. Since $(k_i-k_1)^2+(l_i-l_1)^2=0$ or 1 and $(k_i-k_2)^2+(l_i-l_2)^2=0$ or 1 and $k_1=k_2$ and $l_2=l_1-1$, l_i takes only one of the following three values and the possible values of k_i are determined as stated below;

$$\text{when}\quad \left.\begin{array}{ll} l_i=l_1, \text{ then} & k_i=k_1 \\ l_i=l_1-1, & k_i=k_1 \\ l_i=l_1-1/2, & k_i=k_1\pm\sqrt{3}/2 \end{array}\right\} \quad (3)$$

Let t and s be the numbers of i's such that $l_i=l_1$ and $l_i=l_1-1$ respectively. Then we may assume $l_1=l_2=\cdots=l_t$ and $l_{t+1}=l_{t+2}=\cdots=l_{t+s}=l_1-1$. Since

$\{(k_1+\sqrt{3}/2)-(k_1-\sqrt{3}/2)\}^2=3$, either there is no i such that $k_i=k_1+\sqrt{3}/2$, or there is no i such that $k_i=k_1-\sqrt{3}/2$. However, since $(k_1, k_1, \cdots, k_1, k_1-\sqrt{3}/2, k_1-\sqrt{3}/2, \cdots, k_1-\sqrt{3}/2)=-(k_1', \cdots, k_1', k_1'+\sqrt{3}/2, \cdots, k_1'+\sqrt{3}/2)$ where $k_1'=-k_1$, we have only to consider the case where there is no i such that $k_i=k_1-\sqrt{3}/2$. Since $\sum_{i=1}^{n}(k_i)^2=\sum_{i=1}^{n}(l_i)^2=2$ and $\sum_{i=1}^{n}k_i l_i=0$, we obtain $(8t+8s-3st-16)n=(t+s+2)(8t+8s-3st-16)+32-2st$ and $k_1=\dfrac{-\sqrt{3}(n-t-s)+\varepsilon_1\sqrt{3(t+s)^2+(8-3t-3s)n}}{2n}$ and $l_1=\dfrac{(n-t+s)+\varepsilon_2\sqrt{(s-t)^2+(8-t-s)n}}{2n}$ and $\varepsilon_1\varepsilon_2(n-t-s)(s-t)\leqq 0$ where ε_1 and $\varepsilon_2=1$ or -1. Since k_1 is a real number, we obtain $t+s\leqq 5$ or $n-(t+s)\leqq 5$. Since l_1 is a real number we obtain $s=t=4$ or $s\leqq 3$ or $t\leqq 3$. Therefore we can show by an elementary calculation that only the following cases occur;

$n=8$	$t=s=4$,		$\varepsilon_1\varepsilon_2=1$ or -1 listed as	(31)
$n=9$	$t=1$,	$s=4$,	$\varepsilon_1\varepsilon_2=-1$	(40)
	$t=4$,	$s=1$,	$\varepsilon_1\varepsilon_2=1$	(40)
	$t=3$,	$s=6$,	$\varepsilon_1\varepsilon_2=1$ or -1	(41)
	$t=6$,	$s=3$,	$\varepsilon_1\varepsilon_2=1$ or -1	(41)
	$t=4$,	$s=4$,	$\varepsilon_1\varepsilon_2=1$ or -1	(42)
$n=10$	$t=4$,	$s=4$,	$\varepsilon_1\varepsilon_2=1$ or -1	(46)
	$t=4$,	$s=2$,	$\varepsilon_1\varepsilon_2=1$	(47)
	$t=2$,	$s=4$,	$\varepsilon_1\varepsilon_2=-1$	(47)
$n=11$	$t=4$,	$s=4$,	$\varepsilon_1\varepsilon_2=1$ or -1	(51)
	$t=3$,	$s=4$,	$\varepsilon_1\varepsilon_2=-1$	(50)
	$t=4$,	$s=3$,	$\varepsilon_1\varepsilon_2=1$	(50)
$n=12$	$t=4$,	$s=4$,	$\varepsilon_1\varepsilon_2=1$ or -1	(54)
	$t=2$,	$s=2$,	$\varepsilon_1\varepsilon_2=1$ or -1	(52)
	$t=2$,	$s=8$,	$\varepsilon_1\varepsilon_2=-1$	(53)
	$t=8$,	$s=2$,	$\varepsilon_1\varepsilon_2=1$	(53)
$n=19$	$t=1$,	$s=2$,	$\varepsilon_1\varepsilon_2=-1$	(58)
	$t=2$,	$s=1$,	$\varepsilon_1\varepsilon_2=1$	(58)
	$t=1$,	$s=16$,	$\varepsilon_1\varepsilon_2=-1$	(59)
	$t=16$,	$s=1$,	$\varepsilon_1\varepsilon_2=1$	(59)
	$t=2$,	$s=16$,	$\varepsilon_1\varepsilon_2=-1$	(60)
	$t=16$,	$s=2$,	$\varepsilon_1\varepsilon_2=1$	(60)

Thus the proof of Lemma 3 is completed.

§ 3. **Proof of the Theorem 1** (Completion of the proof).

In Lemma 3 we have listed all possible configrations (i.e. 2-dimensional -1 eigenspace of $T \in X$). But the subgroup $\langle [P_n, P_n], T \rangle$ is not always of finite order. In this section, we will first show when $\langle [P_n, P_n], T \rangle$ is of finite order. We use the following important and well known lemma due to Burnside and Minkowski.

LEMMA OF BURNSIDE AND MINKOWSKI (For a proof, see e.g. [5], Note G.)

Let G be a finite subgroup of $GL(n, \mathbf{Q})$. Then the order of G is no more than

$$M_n = \prod_p p^{\left[\frac{n}{p-1}\right] + \left[\frac{n}{p(p-1)}\right] + \left[\frac{n}{p^2(p-1)}\right] + \cdots},$$

where p ranges over all primes, and $[z]$ denotes the greatest integer which does not exceed z.

Using Lemma of Burnside and Minkowski we can prove the following lemma:

LEMMA 4. *Let T be an element of G satisfying the assumption of Problem A and let $\boldsymbol{k}$ and $\boldsymbol{l}$ be as given in Lemma 3. If $\boldsymbol{k}$ and $\boldsymbol{l}$ are among the following list, then $\langle T, [P_n, P_n] \rangle$ is of infinite order*: ⑯, ⑱, ㉔, ㉙, ㊴, ㊵, ㊷, ㊹, $\cdots$, ㊿⑩.

In order to prove Lemma 4, we have only to show that for the vector $\boldsymbol{v} = (1, \cdots, 1)$, the number of images under the action of $\langle T, [P_n, P_n] \rangle$ is greater than $M_n / |[P_n, P_n]|$ $\left(\text{where } |[P_n, P_n]| = \frac{1}{2} n!\right)$. We have been able to prove this assertion for any T of the type listed in the statement of Lemma 4. Here we used a considerable amount of calculations, but these calculations were quite elementary, and so we omit the details. But we will show this by an example. Let us assume that $\boldsymbol{k}$ and $\boldsymbol{l}$ are vectors of number ㊼. Then we can choose an element X in $GL(10, R)$ whose diagonal entries are $1+m$ and other entries are all m where $m = (\varepsilon - \sqrt{6})/10\sqrt{6}$. Let us set $T = XTX^{-1}$. Since $S = XSX^{-1}$ for every element $S \in P_{10}$, we have $X^{-1} \langle T', [P_{10}, P_{10}] \rangle X = \langle T, [P_{10}, P_{10}] \rangle$. Therefore, it is sufficient if we show that $\langle T', [P_{10}, P_{10}] \rangle$ is of infinite order. We have $(1, 1, \cdots, 1) T' = (0, 0, 0, 0, 1, 1, 0, 0, 0, 0)$, $(0, 0, 0, 0, 1, 0, 1, 0, 0, 0) T' = (1/2, 1/2, 1/2, 1/2, 1, 0, 1, 0, 0, 0)$, $(1/2, 1/2, 0, 0, 1, 1/2, 0, 0, 1, 1/2) T' = (3/4, 3/4, 1/4, 1/4, 1, 1/2, 0, 0, 1, 1/2)$. From the last vector we obtain $2^8 \cdot 3^4 \cdot 5^2 \cdot 7$ distinct vectors by the action of the elements in $[P_{10}, P_{10}]$. Therefore, the number of images of $(1, 1, \cdots, 1)$ under the action of $\langle [T', [P_{10}, P_{10}] \rangle$ is greater than $2^8 \cdot 3^4 \cdot 5^2 \cdot 7$ and is greater than $M_{10} / |[P_{10}, P_{10}]| = 2^{11} \cdot 3^2$. Therefore, $\langle T', [P_{10}, P_{10}] \rangle$ is of infinite order.

LEMMA 5. *Let T be an element of G satisfying the assumption of Problem*

A and let $\boldsymbol{k}$ *and* $\boldsymbol{l}$ *be as given in Lemma 3. Then we obtain the following assertions:*

(i) *If* $\boldsymbol{k}$ *and* $\boldsymbol{l}$ *are of type* ④, *then* $T \in [P_n, P_n]$.

(ii) *If* $\boldsymbol{k}$ *and* $\boldsymbol{l}$ *are* ①, ②, ③, ④, ⑤, ⑥, ⑦, ⑨, ⑩, ⑪, ⑬, ⑭, ⑮, *then* $\langle T, [P_n, P_n]\rangle$ *is isomorphic to a subgroup which lies between* $[W(B_n), W(B_n)]$ *and* $W(B_n)$.

(iii) *If* $\boldsymbol{k}$ *and* $\boldsymbol{l}$ *are in* ⑧, *then* $\langle T, [P_n, P_n]\rangle$ *is isomorphic to a subgroup of index* 2 *of the group* $\{I_n, -I_n\} \times P_n$.

(iv) *If* $\boldsymbol{k}$ *and* $\boldsymbol{l}$ *are in* ⑫, *then* $\langle T, [P_n, P_n]\rangle$ *is isomorphic to the group* $[W(A_n), W(A_n)]$.

PROOF. The assertion of (i) is obvious. If $\boldsymbol{k}$ and $\boldsymbol{l}$ are in ①, ②, ③, ⑤, ⑥, ⑦, ⑩ or ⑪, then the assertion is clear, because of the fact that, if $\boldsymbol{k}$ is the following vector and if U denotes the set consists of all the vectors obtained from $(1, -1, 0, \cdots, 0)$ by permuting the entries, then U and $\boldsymbol{k}$ (by suitably multiplying by a scalar) span a certain root system. That is, if $\boldsymbol{k}$ is a permutation of $\pm(\alpha+1, \alpha+1, \alpha, \cdots, \alpha)$ with $\alpha=0$ or $-4/n$, then U and $\boldsymbol{k}$ span a root system of type D_n; if $\boldsymbol{k}$ is a permutation of $\pm(\alpha+\sqrt{2}, \alpha, \cdots, \alpha)$ with $\alpha=0$ or $-2\sqrt{2}/n$, then U and $\sqrt{2}\boldsymbol{k}$ span a root system of type B_n. If $\boldsymbol{k}$ and $\boldsymbol{l}$ is in ⑨, then by taking a suitable element in $GL(n, \boldsymbol{R})$ which normalizes the subgroup $[P_n, P_n]$, the subspace spanned by $\boldsymbol{k}$ and $\boldsymbol{l}$ in ⑨ is transformed to that of $\boldsymbol{k}$ and $\boldsymbol{l}$ in ①. (Here, the element which induces the transformation is given by Z_n.) Similarly, if $\boldsymbol{k}$ and $\boldsymbol{l}$ are in ⑬, then by Z_n, it is transformed into that of $\boldsymbol{k}$ and $\boldsymbol{l}$ in ⑥. If $\boldsymbol{k}$ and $\boldsymbol{l}$ are in ⑭ or ⑮, then we can obtain the assertion by an easy consideration. (cf. [9].). The assertions of (iii) and (iv) are clear.

The remaining cases that we must consider are the cases: ⑰, ⑲, ⑳ (for $n=6$); ㉒, ㉓ (for $n=7$); ㉖, ㉗, ㉘, ㉚, ㉛ (for $n=8$); and ㊲, ㊳, ㊶ (for $n=9$).

LEMMA 6. (i) *If* $\boldsymbol{k}$ *and* $\boldsymbol{l}$ *are in* ⑰, ⑲ *or* ⑳, ㉑, *then* $\langle T, [P_6, P_6]\rangle$ *is isomorphic to* $[W(E_6), W(E_6)]$ *(as an abstract group).*

(ii) *If* $\boldsymbol{k}$ *and* $\boldsymbol{l}$ *are in* ㉒, ㉓ *or* ㉕, *then* $\langle T, [P_7, P_7]\rangle$ *is isomorphic to* $[W(E_7), W(E_7)]$ *(as an abstract group).*

(iii-1) *If* $\boldsymbol{k}$ *and* $\boldsymbol{l}$ *are in* ㉖, ㉘, ㉚, ㉝, ㉞, ㉟ *or* ㊱, *then* $\langle T, [P_8, P_8]\rangle$ *is isomorphic to* $[W(E_8), W(E_8)]$ *(as an abstract group).*

(iii-2) *If* $\boldsymbol{k}$ *and* $\boldsymbol{l}$ *are in* ㉗ or ㉛, *then* $\langle T, [P_8, P_8]\rangle$ *is isomorphic to a subgroup of* $\{I_8, -I_8\} \times W(E_7)^{\#}$ *of index at most* 4.

(iv) *If* $\boldsymbol{k}$ *and* $\boldsymbol{l}$ *are* ㊲, ㊳ ㊶ *or* ㊸, *then* $\langle T, [P_9, P_9]\rangle$ *is isomorphic to a subgroup of* $\{I_9, -I_9\} \times W(E_8)^{\#}$ *of index at most* 4.

PROOF. (i) If ⑰ holds, then the assertion is obvious from the consideration of the root system of type E_6. Even if ⑲, ⑳ or ㉑ holds, then we can easily identify $\langle T, [P_6, P_6]\rangle$ with $[W(E_6), W(E_6)]$ directly. (Or by using the recent classification theorem of primitive and unimodular finite groups of $GL(6, \boldsymbol{C})$ due to Lindsay II cf. Feit [9].) (ii), (iii) and (iv) The assertion is obvious except for type ㉚, by considering the root systems. In the case of ㉚, the identification of $\langle T, [P_8, P_8]\rangle$ with $[W(E_8), W(E_8)]$ is not direct and given as follows: a triality automorphism σ (cf. [8]) (of order 3) of $[W(E_8), W(E_8)]/\{I_8, -I_8\}$ ($\cong FH(8, 2)$) is lifted to an automorphism of $[W(E_8), W(E_8)]$ (cf. Griess [10], page vii, viii, Assumed result (13)), and we can see by Dye [8], that σ does not preserve the eigenvalues of an element whose eigenvalues are -1 with multiplicity 2 and 1 with multiplicity 6. Therefore, there exist at least two *inequivalent* representations of $[W(E_8), W(E_8)]$ (containing A_n) in the following sense: for subgroups H_1, $H_2 \subset [W(E_8), W(E_8)]$ with $H_i (i=1, 2) \cong A_n$ (as an abstract group), two representations of $[W(E_8), W(E_8)]$ (contain A_n) are *equivalent* if there exists an element $T \in GL(n, \boldsymbol{C})$ such that $T^{-1}[W(E_8), W(E_8)]T = [W(E_8), W(E_8)]$, and $T^{-1}H_1T = H_2$. The two groups of type ㉖ and ㉘ are equivalent in this sense. Therefore, since there are at least two inequivalent representations because of the existence of the automorphism σ of $[W(E_8), W(E_8)]$, we obtain the group of type ㉚ must be also isomorphic to $[W(E_8), W(E_8)]$. Thus, we have completed the proof of Lemma 6.

The Final step of Proof of Theorem 1.

From Lemmas 3, 4, 5 and 6, we obtain that $\langle T, [P_n, P_n]\rangle$ must be one of the groups which lie among those listed in the statement of Theorem 1 (under certain changes of bases). Moreover we can see (together with a certain amount of, but very elementary, considerations) that the situation is same for $\langle T_1, \cdots, T_t, [P_n, P_n]\rangle$ for some $T_1, \cdots, T_t$ (which satisfy the assumption of Problem A, as far as $\langle T_1, \cdots, T_t, [P_n, P_n]\rangle$ (hence the set X) is of finite order). This consideration is a little complicated and not so uniform. But we will show by an example how our method is. That is, let us assume that T_1 corresponds to the pair of vectors in ⑩ and T_2 corresponds to the pair of vectors in ⑫. Then T_1 is the product of two reflections with respect to the vectors $\boldsymbol{k}_1 = (\alpha_1, \alpha_1, \alpha_1+1, \cdots, \alpha_1+1)$ and $\boldsymbol{l}_1 = (1, -1, 0, \cdots, 0)$ and T_2 is the product of two reflections with respect to the vectors $\boldsymbol{k}_2 = (\alpha_2, \cdots, \alpha_2, \alpha_2+1)$ and $\boldsymbol{l}_2 = (1, -1, 0, \cdots, 0)$ where $\alpha_1 = 4/n - 1$ and $\alpha_2 = (-1 \pm \sqrt{1+n})/n$. Then T_1T_2 is the product of two reflections with respect to $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$. But $(\boldsymbol{k}_1, \boldsymbol{k}_2) = (1 \pm \sqrt{1+n})/n$ is not equal to 0, ± 1, $\pm 1/2$, $\pm\sqrt{2}/2$ and $\pm\sqrt{3}/2$ if $n \geqq 5$ and $n \neq 8$. This shows that $\langle T_1, T_2, [P_n, P_n]\rangle$ is not of finite order for $n \geqq 5$ and $n \neq 8$. If

$n=8$ and $\alpha_2=(-1-\sqrt{1+8})/8=-1/2$, then $(\boldsymbol{k}_1, \boldsymbol{k}_2)=-1/4$ and so this shows that $\langle T_1, T_2, [P_8, P_8]\rangle$ is not of finite order for the same reason as above. If $n=8$ and $\alpha_2=(-1+\sqrt{1+8})/8=1/4$, then we can show that $\langle T_1, T_2, [P_8, P_8]\rangle$ is isomorphic to $[W(E_8), W(E_8)]$. Moreover, in cases 1), 4), 5), 6), 7) and 8) in Theorem 1, we can see that the group in the left hand side is a characteristic subgroup of any subgroup listed in Theorem 1. In cases 2) and 3), the diagonal subgroup is a characteristic subgroup. Therefore, any subgroup which satisfies the assumption of Theorem 1 must lie in the normalizer (in $GL(n, \boldsymbol{Q})$) of the subgroup of the left hand side (of the list in Theorem 1). The determination of the normalizer is comparatively easy, because this is a subset of the determination of the automorphism group. (cf. [1]) (See also Burnside [6].) Thus we can see that the normalizer (in $GL(n, \boldsymbol{Q})$) of the left hand side subgroup is just the right hand side subgroup in every case from 1) to 8) in the list of Theorem 1.

Thus we have completed the proof of Theorem 1.

Acknowledgment. The authors thank Professor T. Yokonuma for his valuable remarks by reading the manuscript of the present paper.

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(Received May 24, 1972)
(Revised June 25, 1973)

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