Finite groups with Sylow 2-subgroups of type $PS_p(4, q)$, q odd

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Introduction. In this paper we continue our investigations of finite fusion simple and, in particular, simple groups of 2-rank 3 or 4, [7], [8], [9], [10], [11]. Here we study the case of finite groups with Sylow 2-subgroups S of type $PS_p(4, q)$, q odd, (that is, S is isomorphic to a Sylow 2-subgroup of $PS_p(4, q)$ for some odd q).

Such a 2-group S can be described as follows: S contains a normal subgroup of index 2 which is the central product of two generalized quaternion 2-groups Q_1, Q_2 with Q_1, Q_2 being interchanged under conjugation by an involution of S.

We have already studied the case $|S|=64$ (and Q_1, Q_2 quaternion) in some detail in [8]. In particular, we note that, in addition to the groups $PS_p(4, q)$ themselves with $q \equiv 3, 5 \pmod 8$, the fusion-simple groups A_8, A_9, $A_5 \cdot E_{16}^{(1)}$, and $GL(3, 2) \cdot E_8^{(1)}$ have Sylow 2-groups of this type. Here $A_5 \cdot E_{16}^{(1)}$ and $GL(3, 2) \cdot E_8^{(1)}$ denote, as usual, respectively the unique nontrivial split extension of an elementary abelian group E of order 16 by A_5 acting nontransitively on the involutions of E and of an elementary abelian group of order 8 by $GL(3, 2)$.

Our main result is the following:

THEOREM A. *If G is a perfect fusion-simple group with Sylow 2-subgroups of type $PS_p(4, q)$, q odd, then G is isomorphic to either A_8, A_9, $A_5 \cdot E_{16}^{(1)}$, $GL(3, 2) \cdot E_8^{(1)}$, or $PS_p(4, q)$ for some odd q.*

In particular, if G is a simple group with such Sylow 2-groups, then G must be isomorphic to A_8, A_9, or $SP_p(4, q)$, q odd. On the other hand, if G is an arbitrary such fusion-simple group (and hence not necessarily perfect), then essentially the same result holds except that the last alternative now reads that G is isomorphic to a subgroup of $P\Gamma S_p(4, q)$ containing $PS_p(4, q)$ for some odd q.

Harris [14] has recently obtained a characterization of the groups $PS_p(4, q)$, q odd, which extends an earlier characterization of these groups established by Wong [17]. Using Harris' result, we obtain Theorem A as a corollary of the following theorem:

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THEOREM B. *If G is a fusion-simple group with Sylow 2-subgroups of type $PS_p(4, q)$, q odd, then one of the following holds:*

(i) $G \cong A_8$, A_9, $A_5 \cdot E_{16}^{(1)}$, *or* $GL(3, 2) \cdot E_8^{(1)}$; *or*

(ii) *If z is an involution in the center of a Sylow 2-subgroup of G and $M = C_G(z)$, then*

(1) $O(M) = 1$;

(2) *M contains a normal subgroup which is the central product of two subgroups L_1, L_2 isomorphic to $SL(2, q)$ for some odd q;*

(3) *M/L_1L_2 has Sylow 2-subgroups of order 2 and M contains an involution which interchanges L_1 and L_2 under conjugation.*

Using [8, Theorem A*] together with the classification of groups with Sylow 2-subgroups of type $D_{2^n} \times D_{2^n}$ obtained in [10], we shall show that either part (i) of Theorem B holds or else that $\bar{M} = M/O(M)$ satisfies conditions (1), (2), and (3) of part (ii) of Theorem B. This will have the effect of reducing Theorem B to the proof of the single assertion that $O(M) = 1$.

We use essentially the same procedures as in our previous papers to obtain the desired conclusion. However, in the present case, as in [10], the concept of ordinary balance is inadequate. Indeed, the groups $PS_p(4, q)$, q odd, themselves are, in general, not balanced. This comes about because $PS_p(4, q)$, q odd, possesses two conjugacy classes of involutions and except when q is a Fermat or Mersenne prime or 9, the centralizer of an involution in the second class has a nontrivial core. It turns out that the notion of *2-balance*, introduced in [6], is the proper concept to work with in the present situation.

We conclude this discussion by introducing some terminology which will be important in the ensuing analysis. First we recall from [8], that a group G is said to have the *involution fusion pattern* of the group X if there exists an isomorphism Ψ of a Sylow 2-subgroup S of G onto a Sylow 2-subgroup of X such that two involutions a, b of S are conjugate in G if and only if the involutions $a\Psi$, $b\Psi$ of $S\Psi$ are conjugate in X.

In the present context, the class of groups which will act as "prototypes" for our fusion-simple group G with Sylow 2-subgroups of type $PS_p(4, q)$, q odd, are the groups X which contain a normal subgroup Y of odd index isomorphic to $PS_p(4, q)$ for some odd q with $C_X(Y) = 1$. Any such group X is isomorphic to a subgroup of $P\Gamma S_p(4, q)$ containing $PS_p(4, q)$. In addition, X has the involution fusion pattern of $PS_p(4, q)$. In particular, X has exactly two conjugacy classes of involutions. Furthermore, if z is a central involution of X, then $C_X(z)$ satisfies the conclusions of part (ii) of Theorem B. On the other hand, if z is a noncentral involution, then $C_X(z)/O(C_X(z))$ possesses a normal subgroup of index twice an odd number isomorphic to $PSL(2, q)\times D_{2^n}$, where n is determined by the condition that a Sylow 2-subgroup of X has order 2^{2n+2}. (In addition, $O(C_X(z))$ is cyclic of order $(q+\delta/2^n$, where $\delta=\pm 1$ and $q+\delta\equiv 0 \pmod 4$)).

Now let G be a fusion-simple group with Sylow 2-subgroup S of type $PS_p(4, q)$, q odd. In view of the preceding remarks, we shall say that the centralizer of an involution y of G is of *type* $PS_p(4, q)$ if y is a central involution and $C_G(y)/O(C_G(y))$ has the structure given in part (ii) of Theorem B for the specified value of q or if y is a noncentral involution and $C_G(y)/O(C_G(y))$ contains a normal subgroup of index twice an odd number isomorphic to $PSL(2, q)\times D_{2^n}$, where $|S|=2^{2n+2}$.

Finally we shall say that G has the *centralizer of involution pattern* of $PS_p(4, q)$ for some odd q provided the following conditions hold:

(1) G has the involution fusion pattern of $PS_p(4, q)$; and

(2) The centralizer of every involution of G is of type $PS_p(4, q)$.

In such a case, the integer q will be called the *characteristic power* of G.

An essential step in the proof of Theorem B is the assertion that either part (i) of Theorem B holds for G or else G has the centralizer of involution pattern of $PS_p(4, q)$ for some odd q. This will be established in Section 3.

Finally we remark that the reason we are able to establish 2-balance in G is that we can reduce the problem to the verification of a specific property of certain proper subgroups K of G (the usual *covering* local subgroups). Because G will be a minimal counterexample to Theorem B, $X=K/O(K)$ will have the structure of one of our "prototypes". To establish 2-balance, it turns out to suffice to prove that the following condition holds for any four subgroup T of X:

$$\Delta_X(T)=\bigcap_{t\in T^\#} O(C_X(t))=1 .$$

To see that this is indeed the case when X contains a normal subgroup Y of odd index isomorphic to $PS_p(4, q)$, q odd, with $C_X(Y)=1$, we clearly need only treat the case that $O(C_X(t))\neq 1$ for each t in $T^\#$, otherwise the assertion is obvious. In particular, each involution t of T is then noncentral and q is not a Fermat or Mersenne prime or 9. However, as $O(C_X(t))$ is cyclic $\Delta_X(T)$ is characteristic in $O(C_X(t))$ and so is normal in $C_X(t)$ for each t in $T^\#$. Thus

$$\Delta_X(T)\triangleleft\langle C_X(t)|t\in T^\#\rangle .$$

On the other hand, as $Y\cong PS_p(4, q)$, it is easily verified that $Y=\langle C_Y(t)|t\in T^\#\rangle$. Likewise we have that $X=YC_X(T)$ and consequently $\langle C_X(t)|t\in T^\#\rangle=X$. Therefore $\Delta_X(T)$ is, in fact, normal in X. Since $O(X)=1$ and $|\Delta_X(T)|$ is odd, the desired conclusion $\Delta_X(T)=1$ now follows.

In general, our notation is standard and includes the use of the bar convention for homomorphic images.

2. Assumed results. We assume the reader is familiar with the notions of *balance, connectedness, 2-generation* and *p-stability with respect to a p-subgroup*. However, we note that the original definition of p-stability with respect to a p-group, given in [2], has been emended in [18]. Proofs of the following two theorems can be found in [6] or [13] and in [18] respectively.

THEOREM 2.1. *If G is a balanced, connected group of 2-rank at least 3 with $O(G)=1$ in which the centralizer of every involution is 2-generated, then $O(C_G(x))=1$ for every involution x of G.*

Next we state the extended form of Glauberman's ZJ-theorem.

THEOREM 2.2. *If H is a group with $O_p(H)\neq 1$, p odd, which is p-constrained and is p-stable with respect to the p-subgroup P of H, then*

$$H=O_{p'}(H)N_H(Z(J(P))) .$$

We recall some terminology from [6]. We let $\mathscr{E}_k(G)$ denote the set of elementary abelian 2-subgroups of rank k of the group G. Moreover, if $T\in\mathscr{E}_k(G)$, we set

$$\Delta_G(T)=\bigcap_{t\in T^\#} O(C_G(t)) .$$

With this notation, we say that G is *k-balanced* if for each T in $\mathscr{E}_k(G)$ and each involution b of G which centralizes T, we have

$$\Delta_G(T)\cap C_G(b)\subseteq O(C_G(b)) .$$

Our concern here will be with the case $k=2$. If G is a 2-balanced group and A is an elementary abelian 2-subgroup of G of rank at least 4, it is shown in [6, Section 5.1] that if we define for each a in $A^\#$

$$\theta(C_G(a))=\langle C_G(a)\cap\Delta_G(T)\mid T\in\mathscr{E}_2(A)\rangle ,$$

then θ is a *solvable A-signalizer functor* on G (that is, $\theta(C_G(a))$ is an A-invariant solvable subgroup of $C_G(a)$ of odd order such that $\theta(C_G(a))\cap C_G(b)\subseteq\theta(C_G(b))$ for each a, b in $A^\#$). Hence using Goldschmidt's version of the signalizer functor theorem [4], we have

THEOREM 2.3. *If G is a 2-balanced group and A is an elementary abelian 2-subgroup of G of rank at least 4, then the subgroup*

$$W_A=\langle\Delta_G(T)\mid T\in\mathscr{E}_2(A)\rangle$$

of G is of odd order.

In addition, the argument of [6, Section 4.1] yields

THEOREM 2.4. *If G, A, and W_A are as in the preceding theorem, then for each B in $\mathscr{E}_3(A)$, we have*

$$N_G(B)\subseteq N_G(W_A)\ .$$

Finally we state Harris' theorem [14].

THEOREM 2.5. *Let G be a fusion-simple group with Sylow 2-subgroups of type $PS_p(4, q)$, q odd. If z is an involution in the center of a Sylow 2-subgroup of G and if $C_G(z)$ is of type $PS_p(4, q)$ with $O(C_G(z))=1$, then G possesses a normal subgroup of odd index isomorphic to $PS_p(4, q)$ for some odd q.*

In particular, if G is perfect, then $G\cong PS_p(4, q)$.

3. The involutions of G. Let G be a fusion-simple group with Sylow 2-subgroup S of type $PS_p(4, q)$, q odd.

We shall use the following precise description of S. S is generated by six elements a_1, a_2, b_1, b_2, t, u with a_1, a_2 of order 2^{n-1} for some integer $n\geqq 2$ and b_1, b_2, t, u of order 2 which satisfy the following relations:

$$(1)\quad \begin{cases} a_1^u=a_2,\quad b_1^u=b_2,\quad a_i^{b_i}=a_i^{-1},\quad b_i^t=a_ib_i,\quad i=1, 2,\quad \text{and} \\ [\langle a_1, b_1\rangle, \langle a_2, b_2\rangle]=[t, u]=1\ . \end{cases}$$

(The action of t on a_i is given implicitly since we have $b_i^tb_i=a_i$, whence $a_i^t=b_ib_i^t=a_i^{-1}$, $i=1, 2$.)

We note that in the case of $n=2$, our notation here differs from the description of S given in [8]. We have the following correspondence of generators: a_1, b_1, a_2, b_2, t, u correspond respectively to c, da, ca, dab, f, e.

We have that S has order 2^{2n+2} and S is of 2-rank 4. Moreover, $SCN_3(S)$ is empty if $n\geqq 3$, while $SCN_4(S)$ is nonempty if $n=2$. However, S is connected for all values of n. The maximal subgroup $R=\langle a_1, a_2, b_1b_2, t, u\rangle$ is the central product of its subgroups $Q_1=\langle tb_1b_2, ub_1b_2z_1\rangle$ and $Q_2=\langle ta_1b_1b_2, uz_1\rangle$, which are generalized quaternion and are interchanged by the involution b_1. Hence S has the structure stated in the introduction. The integer n will be called the *height* of S.

In this section, we shall study the fusion pattern of involutions of G, the approximate structure of the centralizers of the involutions of each class, and some related local structure. Our primary aim will be to show that either Theorem B holds or G has the centralizer of involution pattern of $PS_p(4, q)$ for some odd q.

The center of S is of order 2 and we let z be the involution of $Z(S)$. We set $M=C_G(z)$ and fix this notation.

We first restate some of the results of [8] for the case that S is of height 2, in which case S is also of type A_8.

PROPOSITION 3.1. *If S has height 2, then one of the following holds:*

(i) $G\cong A_8$, A_9, $A_5\cdot E_{16}^{(1)}$, *or* $GL(3,2)\cdot E_8^{(1)}$; *or*

(ii) *G has the involution fusion pattern of $PS_p(4, q)$ with $q\equiv 3$, 5 (mod 8).*

This is [8, Theorem A*]. In addition, we have proved [8, Propositions 3.1, 3.2 and Corollary A*]:

PROPOSITION 3.2. *If S has height 2 and G has the involution fusion pattern of $PS_p(4, q)$ with $q\equiv 3$, 5 (mod 8), then*

(i) *M is of type $PS_p(4, q)$ for some $q\equiv 3$, 5 (mod 8);*

(ii) *If $q=3$, then $G\cong PS_p(4, 3)$.*

In view of Propositions 3.1 and 3.2, we see that either Theorem B holds or else G has the involution fusion pattern of $PS_p(4, q)$, $q\equiv 3$, 5 (mod 8), M is of type $PS_p(4, q)$ for some $q\equiv 3$, 5 (mod 8) and $q\geqq 5$. As a consequence, we shall make the following assumption throughout the balance of the paper:

If S is of height 2, then G has the involution fusion pattern of $PS_p(4, q)$, $q\equiv 3$, 5 (mod 8), M is of type $PS_p(4, q)$ for some $q\equiv 3$, 5 (mod 8), and $q\geqq 5$.

We note finally an additional result established in the course of the proof of [8, Propositions 3.1 and 3.2], which we shall need.

LEMMA 3.3. *If S is of height 2 and A is an elementary abelian subgroup of S of order 16, then $N_G(A)/O(N_G(A))\cong A_5\cdot E_{16}^{(1)}$.*

REMARK. If in Lemma 3.3, G is not fusion-simple, but satisfies the weaker condition that $Z^*(G)=1$, then [8, Proposition 3.9] and the remark following it yield the weaker result that $N_G(A)-C_G(A)$ contains a 3-element which acts regularly on A.

Our principal objective in this section is to establish the following result:

PROPOSITION 3.4. *G satisfies the following conditions:*

(i) *G has the centralizer of involution pattern of $PS_p(4, q)$ for some odd $q\geqq 5$;*

(ii) *If A is an arbitrary elementary abelian subgroup of G of order 16, then $N_G(A)/C_G(A)$ is isomorphic to A_5 or S_5 according as $n=2$ or $n\geqq 3$.*

The proof of the proposition is long and will be divided into a number of lemmas. Because of our assumption in the case $n=2$, part (i) of the proposition will hold once we show that the centralizer of every noncentral involution of G is of type $PS_p(4, q)$. Moreover, part (ii) of the proposition holds by Lemma 3.3. Hence the bulk of the proof will deal with the case that $n\geqq 3$.

Thus in Lemmas 3.5-3.18, we assume that $n\geqq 3$.

Furthermore, for our subsequent analysis of the subgroup structure of G, we shall need some information about arbitrary groups with Sylow 2-subgroups of type $PS_p(4, q)$, q odd, and hence not necessarily fusion-simple.

Thus in Lemmas 3.5-3.12, *we drop the assumption that G is fusion-simple.*

We introduce some additional notation (in all cases). First of all, we recall from [11] that a 2-group T is called a *crown product* if T is of the form $(D_1\times D_2)\langle y\rangle$, where D_i is dihedral, y is an involution which normalizes D_i, and $F_i=D_i\langle y\rangle$ is dihedral, $i=1, 2$. We write $T=F_1\wedge F_2$. We note that if D_1, D_2 are each four groups, then $T\cong Z_2\times Z_2\int Z_2$.

We set $z_1=a_1^{2^{n-2}}$, $z_2=a_2^{2^{n-2}}$, $z=z_1z_2$, $Z=\langle z_1, z_2\rangle$, $D=\langle a_1, b_1\rangle\times\langle a_2, b_2\rangle$, $T=\langle D, t\rangle$, $R=\langle a_1, a_2, b_1b_2, t, u\rangle$, $A_1=\langle b_1b_2, z_1, z_2\rangle$, $A_2=\langle b_1, a_2b_2, z_1, z_2\rangle$, $Q_1=\langle tb_1b_2, ub_1b_2z_1\rangle$, and $Q_2=\langle ta_1b_1b_2, uz_1\rangle$. The following properties of S and its subgroups are easy consequences of the definition of S. We leave their verification to the reader.

LEMMA 3.5. *The following conditions hold:*

(i) $Z(S)=\langle z\rangle\cong Z_2$.

(ii) $\mho^{n-1}(S)=\langle z_1, z_2\rangle$ *and* $S'=\langle a_1, a_2, b_1b_2\rangle$.

(iii) *S has nine conjugacy classes of involutions represented by* z, z_1, b_1, b_1z, b_1b_2, $a_1b_1b_2$, t, u, tu.

(iv) *If r is an involution of S distinct from z which is not conjugate to b_1 or b_1z, then r is conjugate to rz in S.*

(v) (1) $C_S(z_1)=\langle a_1, b_1, a_2, b_2, t\rangle=\langle D, t\rangle=T$;

(2) $C_S(b_1)=C_S(b_1z)=\langle b_1, z_1\rangle\times\langle a_2, b_2\rangle\cong Z_2\times Z_2\times D_{2^n}$;

(3) $C_S(b_1b_2)=\langle b_1, z_1, b_2, z_2, u\rangle\cong Z_2\times Z_2\int Z_2$

(4) $C_S(a_1b_1b_2)=\langle a_1b_1, z_1, b_2, z_2, tu\rangle\cong Z_2\times Z_2\int Z_2$;

(5) $C_S(t)=\langle t, u, z_1, z_2\rangle\cong Z_2\times D_8$;

(6) $C_S(u)=\langle t, u, b_1b_2, a_1a_2\rangle\cong Z_2\times D_{2^{n+1}}$;

(7) $C_S(tu)=\langle t, u, a_1a_2^{-1}, b_1a_2b_2\rangle\cong Z_2\times D_{2^{n+1}}$.

(vi) *The centralizer of every involution of S has 2-rank three or four.*

(vii) *S is connected of 2-rank* 4.

(viii) $\mho^{n-1}(C_S(u))=\mho^{n-1}(C_S(tu))=\langle z\rangle$ *and* $\mho^{n-1}(T)=Z(T)=Z$.

(ix) *Every elementary abelian subgroup of S of order 16 is contained in T and is conjugate in S to A_1 or A_2.*

(x) $N_S(A_i)\cong D_8\int Z_2$, $C_S(A_i)=A_i$, *and* $N_S(A_i)/A_i\cong D_8$, $i=1, 2$.

(xi) $R=Q_1Q_2=Q_1*Q_2\cong Q_{2^{n+1}}*Q_{2^{n+1}}$, *and b_1 interchanges Q_1, Q_2 under conjugation. Furthermore,* $Q=\langle t, u\rangle^S$.

(xii) $T\cong D_{2^{n+1}}\wedge D_{2^{n+1}}$ *and is the unique maximal subgroup of S with a noncyclic center.*

(xiii) Aut(T) *and* Aut(S) *are 2-groups.*

(xiv) Aut(S) *contains an element which interchanges A_1 and A_2 and an* *element which interchanges* b_1 *and* b_1z.

(xv) $Z/\langle z\rangle$ *is the center of* $S/\langle z\rangle$.

(xvi) $|N_S(C_T(b_1)) : C_T(b_1)|=2$.

(xvii) $\langle D, u\rangle\cong\langle D, tu\rangle\cong D_{2^n}\wr Z_2$.

REMARK. We note that (x), (xiii), and (xv) as well as the assertion $\mho^{n-1}(S)=\langle z_1, z_2\rangle$ do not hold if $n=2$.

We first prove.

LEMMA 3.6. *$N_G(Z)$ has a normal 2-complement.*

PROOF: Clearly T is a Sylow 2-subgroup of $H=C_G(Z)$. But $T=D\langle t\rangle\cong D_{2^{n+1}}\curlywedge D_{2^{n+1}}$ by Lemma 3.5 (xii). It follows therefore from [11, Lemma 8.5] that H has a normal subgroup K of index 2 with Sylow 2-subgroup D. But $D\cong D_{2^n}\times D_{2^n}$, $n\geqq 3$, and $Z=Z(D)\subseteq Z(K)$. We conclude therefore from [11, Lemma 8.4] that K has a normal 2-complement. Hence H also has a normal 2-complement. But Aut(T) is a 2-group by Lemma 3.5 (xiii) and so $N_G(Z)/H$ is a 2-group by the Frattini argument. Thus also $N_G(Z)$ has a normal 2-complement, as asserted.

Now we begin our analysis of the fusion of involutions.

LEMMA 3.7. *z is not conjugate to z_1, u, or tu in G.*

PROOF: Suppose $z_1^g=z$ for some g in G. We can choose g so that $C_S(z_1)^g=T^g\subseteq S$. Since T is the unique maximal subgroup of S with a noncyclic center, $T^g=T$, whence $g\in N_G(T)\subseteq N_G(Z)$. Since $N_G(Z)$ has a normal 2-complement by the preceding lemma, we clearly have a contradiction. Thus z is not conjugate to z_1 in G.

Suppose next that $u^g=z$ for some g in G. We can assume $C_S(u)^g\subseteq S$. By Lemma 3.5 (ii) and (viii), we have $\mho^{n-1}(C_S(u))=\langle z\rangle$ and $\mho^{n-1}(S)=Z$, so $z^g\in Z=\langle z_1, z_2\rangle$. Hence by the preceding paragraph, we must have $z^g=z$, contrary to the fact that $u^g=z$. Thus z is not conjugate to u in G. Similarly using the fact that also $\mho^{n-1}(C_S(tu))=\langle z\rangle$, we conclude that z is not conjugate to tu in G.

LEMMA 3.8. *Any two elementary abelian subgroups of S of order* 16 *that are conjugate in G are conjugate in S.*

PROOF: Let A, B be two such subgroups of S that are conjugate in G. By Lemma 3.5 (ix), A and B are conjugate in S to A_1 or A_2. Hence to establish the lemma, it will suffice to prove that A_1 and A_2 themselves are not conjugate in G; so assume the contrary.

Using Alperin's fusion Theorem, [1 or 5, Theorem 7.2.6], it follows that S contains a subgroup S_1, satisfying

(a) $S_1\supseteq A_1$;

(b) $S_1^*=N_S(S_1)$ is a Sylow 2-subgroup of $N_G(S_1)$;

(c) A_1^x is conjugate in S to A_2 for some x in $N_G(S_1)$.

By Lemma 3.5 (xiii), $S_1 \subset S$ and consequently $S_1^* \supset S_1$.

Let T_1 be the subgroup of S_1 generated by its elementary subgroups of order 16, so that T_1 contains A_1 and $A_3 = A_1^x$. Since $T_1 \subseteq T$, $\Omega_1(Z(T_1)) \supseteq Z$. Suppose that $\Omega_1(Z(T_1)) \supset Z$. Since T_1 is not elementary of order 16, it follows from Lemma 3.5 (v) that $\Omega_1(Z(T_1))$ contains an involution b which is conjugate in S to b_1. Clearly we can assume without loss that $b = b_1$, in which case $T_1 \subseteq C_T(b_1) = \langle b_1, z_1 \rangle \times \langle a_2, b_2 \rangle$. Since $A_3 \subseteq T_1$, and A_3 is not conjugate to A_1 in S, we must have that $\langle A_1, A_3 \rangle = C_T(b_1)$ and hence that $T_1 = C_T(b_1)$.

By Lemma 3.5 (xvi), $|N_S(T_1) : T_1| = 2$. Since T_1 is characteristic in S_1 and $S_1^* \supset S_1$, we conclude now that $S_1 = T_1$. Thus $S_1 \cong Z_2 \times Z_2 \times D_{2^n}$. It is easily seen from the structure of $\mathrm{Aut}(S_1)$ that any element of $N_G(S_1)$ of odd order necessarily leaves invariant every elementary subgroup of S_1 of order 16. Hence $A_3 = A_1^x = A_1^y$ for some y in $S_1^* \subseteq S$, which is not the case. We conclude therefore that $\Omega_1(Z(T_1)) = Z$.

Hence $N_G(S_1) \subseteq N_G(T_1) \subseteq N_G(Z)$. However, by Lemma 3.6, $N_G(Z)$ has a normal 2-complement and we reach the same contradiction as in the preceding case.

LEMMA 3.9. *If A is an elementary abelian subgroup of S or order 16 and an involution a of A is conjugate to z in G, then a is conjugate to z in $N_G(A)$.*

PROOF: As usual, $a^g = z$ for some g in G such that $C_S(a)^g \subseteq S$. Hence $A^g \subseteq S$. By the preceding lemma, $A^{gs} = A$ for some s in S. Since $a^{gs} = z^s = z$, our assertion is proved.

LEMMA 3.10. *If $N_G(A_i) \not\subseteq C_G(z)$ for $i=1$ or 2, then we have*

(i) *$N_G(A_i) \not\subseteq C_G(z)$ for both $i=1$ and 2;*

(ii) *$N_G(A_i)/C_G(A_i)$ is isomorphic to either S_5 or to a Sylow 3-normalizer of A_8, $i=1$ and 2;*

(iii) *With a suitable choice of notation, we have*

(1) *$z \sim b_1 z$ and $z_1 \sim b_1$;*

(2) *$b_1 b_2 \sim z$ or z_1 and $a_1 b_1 b_2 \sim z$ or z_1.*

PROOF: By Lemma 3.5 (xiv), $\mathrm{Aut}(S)$ contains an element which interchanges A_1 and A_2. Hence by symmetry, we can assume without loss that $N_G(A_1) \not\subseteq C_G(z)$. Set $H_1 = N_G(A_1)$ and $C_1 = C_G(A_1)$. By Lemma 3.5 (ix) and (x), $N_S(A_1)$ is a Sylow 2-subgroup of H_1 and a Sylow 2-subgroup of H_1/C_1 is isomorphic to D_8. Moreover, z is not conjugate to z_1 in G by Lemma 3.7, so H_1 does not act transitively on $A_1^\#$. Now $N_S(A_1)$ acts on $A_1^\#$ with orbit lengths 1, 2, 4, 4, 4. Since z is not conjugate to z_1 and since $13 = 1+4+4+4$ does not divide the order of $\bar{H}_1 = H_1/C_1$, which is isomorphic to a subgroup of $GL(4, 2) \cong A_8$, we see that z has precisely 5 or 9 conjugates in $A_1^\#$. Correspondingly, 5 or 9 divides $|\bar{H}_1|$.

We note next if $\bar{H}_1$ contained a subgroup isomorphic to A_6 or $Z_3\times A_5$, then H_1 would act transitively on $A_1^{\#}$, which is not the case. Examining the subgroups of A_8 having a dihedral Sylow 2-subgroup of order 8, we conclude easily that $\bar{H}_1$ is isomorphic to S_6 or to a Sylow 3-normalizer in A_8.

Since a Sylow 2-subgroup of $N_G(A_1)$ is a 2-group of type A_{10} by Lemma 3.5 (x), we can apply a result by Kondo [15, Lemma 3.2] to obtain that $z_1\sim b_1$ if and only if $z\sim b_1z$. On the other hand, by the structure of $\bar{H}_1$, H_1 has exactly two orbits on $A_1^{\#}$, one containing z and the other containing z_1. Hence $b_1\sim z$ or z_1 in H_1. But $b_1\sim b_1z$ in Aut(S) by Lemma 3.5 (xiv). We conclude therefore that for a suitable choice of notation, we have $z_1\sim b_1$ and $z\sim b_1z$ in H_1. Clearly $b_1b_2\sim z$ or z_1.

Since $z\sim b_1z\in A_2$, it follows from the preceding lemma that $N_G(A_2)\not\subseteq C_G(z)$. Hence repeating the analysis for A_2, we obtain that $N_G(A_2)/C_G(A_2)$ is also isomorphic to S_6 or to a Sylow 3-normalizer in A_8. Likewise $N_G(A_2)$ has two orbits in its action on $A_2^{\#}$, again represented by z and z_1, so $a_1b_1b_2\sim z$ or z_1. All parts of the lemma now follow.

REMARK. We note that at this point we cannot yet assert that $N_G(A_1)/C_G(A_1)\cong N_G(A_2)/C_G(A_2)$. This we shall prove later.

LEMMA 3.11. *If G has no isolated involution, then $N_G(A)\not\subseteq C_G(z)$ for any elementary abelian subgroup A of S of order* 16.

PROOF: In view of the preceding lemma and Lemma 3.5 (ix), it will be enough to prove that $N_G(A_i)\not\subseteq C_G(z)$, $i=1$ or 2, under the given assumption. Since z is not conjugate to z_1, u or tu by Lemma 3.7, z must be conjugate in G to b_1, b_1z, b_1b_2, $a_1b_1b_2$, or t by Lemma 3.5 (iii) as z is not isolated in S with respect to G. Each of the first four elements lies in A_1 or A_2. Hence if z is conjugate to one of them, then the desired conclusion will follow from Lemma 3.9. Thus to establish the lemma, it will suffice to show that if $z\sim t$ in G, then z is also conjugate to one of the first four elements listed.

As usual, there then exists g in G such that $t^g=z$ and $C_S(t)^g\subseteq S$. Then $(C_S(t)')^g\subseteq S'$. But $z\in C_S(t)'$ and $S'=\langle a_1, a_2, b_1b_2\rangle$ by Lemma 3.5 (ii) and (v). We conclude at once from this and the fact that z is not conjugate to z_1 that z must be conjugate to b_1b_2 or $a_1b_1b_2$.

Combined with Lemma 3.10, Lemma 3.11 has the following corollary:

LEMMA 3.12. *If G has no isolated involution, then for a suitable choice of notation, $z\sim b_1z$ and $z_1\sim b_1$.*

Henceforth, *we assume that G is fusion-simple.* In addition, we assume the notation so chosen that $z\sim b_1z$ and $z_1\sim b_1$.

LEMMA 3.13. *The following conditions hold:*

(i) *G has exactly two conjugacy classes of involutions represented by* z *and* z_1;

(ii) $z_1 \sim u \sim tu$.

PROOF: The conjugacy classes of involutions of S intersecting D nontrivially are represented by z, z_1, b_1, b_1z, b_1b_2 and $a_1b_1b_2$. Lemmas 3.10 and 3.12 show that each of these six elements is conjugate in G to z or z_1. Hence every involution of D is conjugate in G to z or z_1. We shall argue now that also one of t, u, or tu is conjugate to z or z_1 in G. Assume false. Since $\langle D, u\rangle$ is maximal in S and G is fusion-simple, Thompson's fusion lemma implies that t must be conjugate to u in G. Since u is not conjugate to z or z_1, we conclude easily from Lemma 3.5 (v) that u is extremal in S; that is, $C_S(u)$ is a Sylow 2-subgroup of $C_G(u)$. Hence there exists g in G such that $t^g = u$ and $C_S(t)^g \subseteq C_S(u)$. But $C_S(t) = \langle t, u, z_1, z_2\rangle \cong Z_2 \times D_8$, and $C_S(u) = \langle t, u, b_1b_2, a_1a_2\rangle \cong Z_2 \times D_{2^{n+1}}$ by Lemma 3.5 (v). Hence $\langle t, u, z\rangle^g$ and $\langle t, z_1, z_2\rangle^g$ are elementary subgroups of $C_S(u)$ of order 8. Moreover, as $n \geqq 3$, they are, in fact, conjugate in $C_S(u)$. We conclude therefore that $\langle t, u, z\rangle$ is conjugate to $\langle t, z_1, z_2\rangle$ in G, which clearly implies that t, u, or tu is conjugate to z or z_1. Thus we have shown that t, u, or tu is conjugate to z or z_1 in G.

Hence all the involutions of $\langle D, t\rangle$, $\langle D, u\rangle$, or $\langle D, tu\rangle$ are conjugate to z or z_1 in G. However, each of these groups is maximal in S and we conclude now by another application of Thompson's fusion lemma that t, u, *and* tu are conjugate to z or z_1 in G. Lemma 3.5 (iii) together with Lemma 3.7 now yields that G has exactly two conjugacy classes of involutions, represented by z and z_1. Furthermore, by the same lemma, z is not conjugate to u or tu in G, so we also have $z_1 \sim t \sim tu$.

LEMMA 3.14. *We have* $z_1 \sim u \sim tu$ *in* M.

PROOF: We have already shown that these elements are conjugate in G, so there exists g in G such that $u^g = z_1$ and $C_S(u)^g \subseteq C_S(z_1) = T$. (Here we have used the fact that T is a Sylow 2-subgroup of $C_G(z_1)$ as z_1 is not conjugate to z in G). By Lemma 3.5 (viii) we now have $\langle z\rangle^g = (\mho^{n-1}(C_S(u)))^g \subseteq \mho^{n-1}(T) = Z$. Since z is not conjugate to z_1 or to z_2, if follows that $z^g = z$, so $g \in M$. Similarly z_1 is conjugate to tu in M.

LEMMA 3.15. *R is a Sylow 2-subgroup of $O^2(M)$. In particular,* $|M : O^2(M)| = 2$.

PROOF: Set $K = O^2(M)$. In view of the preceding lemma, M does not have a normal 2-complement and hence neither does $\bar{M} = M/\langle z\rangle$. Thus $\bar{K}$ is of even order. Since $\bar{K} \lhd \bar{M}$, it follows that $Z(\bar{S}) \cap \bar{K} \neq 1$. But $\bar{Z} = Z(\bar{S})$ by Lemma 3.5 (xv) and so $\bar{Z} \subseteq \bar{K}$. Similarly $\langle z\rangle = Z(S) \subseteq K$ and hence $Z \subseteq K$. But now u and tu are also in K by the preceding lemma. However, $R = \langle t, u\rangle^S$ by Lemma 3.5 (xi). Since

$S\cap K\lhd S$, we conclude therefore that $R\subseteq K$. Hence either the lemma holds or $S\subseteq K$, in which case $M=K=O^2(M)$.

Consider the latter case. Then as M has no normal subgroups of index 2 and $S=R\langle b_1\rangle$ with R maximal in S, Thompson's fusion lemma implies that b_1 must be conjugate in M to some involution r of R. Certainly r is not conjugate in S to b_1 or b_1z, so by Lemma 3.5 (iv), r is conjugate to rz in S. Since $r^m=b_1$ for some m in M, we have $(rz)^m=r^mz^m=b_1z$ as m centralizes z. Thus $b_1\sim r\sim rz\sim b_1z$. But by Lemma 3.12, b_1 is conjugate to z_1, while b_1z is conjugate to z. Since z and z_1 are not conjugate in G, we have a contradiction and the lemma is proved.

REMARK. R is also a Sylow 2-subgroup of $O^2(M)$ when $n=2$, as follows directly from Proposition 3.2 (i).

LEMMA 3.16. *The following conditions hold:*

(i) $z_1\sim t\sim b_1b_2\sim a_1b_1b_2$ *in* M;

(ii) *Either* M *is of type* $PS_p(4, q)$ *for some odd* q *or* $C_G(z_1)$ *involves* A_7.

PROOF: Let $K=O^2(M)$ and $\bar{M}=M/O(M)$ and set $\tilde{M}=\bar{M}/\langle\bar{z}\rangle$. Then $\tilde{R}$ is a Sylow 2-subgroup of $\tilde{K}$, $O(\tilde{K})=1$ and $O^2(\tilde{K})=\tilde{K}$. But as $R\cong Q_{2^{n+1}}*Q_{2^{n+1}}$, $\tilde{R}\cong D_{2^n}\times D_{2^n}$. Since $n\geqq 3$, it follows from the main result of [10] that $\tilde{R}\subseteq\tilde{K}'$ and that $\tilde{K}'=\tilde{L}_1\times\tilde{L}_2$, where $\tilde{L}_i\cong PSL(2, q_i)$, q_i odd, or A_7 $i=1$, 2. If $\bar{L}_i$ denotes the inverse image of $\tilde{L}_i$ in $\bar{K}$, then by the structure of $\bar{R}$ $(\cong R)$, $\bar{L}_i$ does not split and so by the results of Schur $\bar{L}_i\cong SL(2, q_i)$ or $\hat{A}_7$, where $\hat{A}_7$ denotes the unique perfect central extension of A_7 by Z_2, $i=1$, 2. Since $\bar{R}$ is a Sylow 2-subgroup of $\bar{L}_1\bar{L}_2$, it follows that $\bar{R}$ is the central product of $\bar{R}\cap\bar{L}_1$ and $\bar{R}\cap\bar{L}_2$, each of which is generalized quaternion. But $R=Q_1Q_2$ is the unique representation of R as a central product of generalized quaternion subgroups. Hence for a suitable choice of the numbering of $\bar{L}_1$, $\bar{L}_2$ we have $\bar{Q}_i=\bar{R}\cap\bar{L}_i$, $i=1$, 2. Since b_1 interchanges Q_1 and Q_2, it follows that $\bar{b}_1$ interchanges $\bar{L}_1$ and $\bar{L}_2$. Thus $\bar{L}_1\cong\bar{L}_2$ and so either $\bar{L}_i\cong\hat{A}_7$ or $SL(2, q)$ for some odd q $(=q_1=q_2)$, $i=1$, 2. In the latter case, we conclude at once from the definition that M is of type $PS_p(4, q)$. On the other hand, if $\bar{L}_i\cong\hat{A}_7$, we see that $C_{\bar{L}_1\bar{L}_2}(\bar{b}_1)\cong Z_2\times A_7$ inasmuch as $\bar{b}_1$ interchanges $\bar{L}_1$, $\bar{L}_2$ and so centralizes the "diagonal" of $\bar{L}_1\bar{L}_2$. Thus it follows in this case that $C_M(b_1)$ involves A_7. Since $b_1\sim z_1$, we conclude that $C_G(z_1)$ involves A_7. We have therefore established (ii).

Finally in either case it follows from the structure of $\bar{L}_1\bar{L}_2$ that all noncentral involutions of $\bar{R}$ are conjugate in $\bar{M}$. Indeed, every involution of $\bar{R}-\langle\bar{z}\rangle$ is contained in a subgroup $\bar{V}$ of $\bar{R}$ isomorphic to Q_8*Q_8 and, moreover, $|N_{\bar{M}}(\bar{V})/C_{\bar{M}}(\bar{V})|$ is divisible by 9. Hence all involutions of $\bar{V}-\langle\bar{z}\rangle$ are conjugate in $\bar{M}$. From this, we easily obtain our assertion. In particular, we have $\bar{z}_1\sim\bar{t}\sim\bar{b}_1\bar{b}_2\sim\bar{a}_1\bar{b}_1\bar{b}_2$ in $\bar{M}$, which implies (i).

We have now established a major step in the proof of Proposition 3.4.

LEMMA 3.17. *G has the involution fusion pattern of $PS_p(4, q)$ for some odd q, namely*

$$z\sim b_1z|z_1\sim b_1\sim t\sim u\sim tu\sim b_1b_2\sim a_1b_1b_2 .$$

PROOF: Given the assumed normalization of the notation for the elements of S, Lemmas 3.12, 3.13 and 3.16 together show that G has the specified involution fusion pattern. In particular, this involution fusion pattern is uniquely determined. On the other hand, we can choose q so that $G^*=PS_p(4, q)$ has a Sylow 2-subgroup S^* isomorphic to S. Since the preceding discussion applies as well to G^* as to G, we see that, again for a suitable choice of notation, G and G^* have the "same" involution fusion pattern. Hence G has the involution fusion pattern of $PS_p(4, q)$ for some odd q, as asserted.

We next prove

LEMMA 3.18. *Setting $H_i=N_G(A_i)$ and $\bar{H}_i=H_i/C_G(A_i)$, $i=1, 2$, we have*

(i) $\bar{H}_i\cong S_5$, $i=1, 2$;

(ii) *$\bar{H}_i$ contains a subgroup $\bar{X}_i$ of order 3, $i=1, 2$, such that $\bar{X}_1$ and $\bar{X}_2$ centralize $\langle z_1, b_1z\rangle$ and normalize, but do not centralize $\langle b_2, z_2\rangle$ and $\langle a_2b_2, z_2\rangle$ respectively;*

(iii) *Any subgroup of $\bar{H}_i$ of order 3 invariant under a four subgroup of $\overline{S\cap H_i}$ centralizes some involution of Z, $i=1, 2$;*

(iv) *$u\in O^2(H_1)$ and $tu\in O^2(H_2)$;*

(v) *$\langle z_1, b_1z\rangle$ and $\langle b_1, z\rangle$ are conjugate in H_1.*

PROOF: The proofs being entirely similar for A_1 and A_2, we treat only the case of A_1. Set $C_1=C_G(A_1)$, so that $\bar{H}_1=H_1/C_1$. By Lemma 3.10, $\bar{H}_1$ is isomorphic to S_5 or to a Sylow 3-normalizer in A_8. To prove (i), we need only rule out the latter possibility. However, in this case, a Sylow 3-subgroup of $\bar{H}_1$ is elementary of order 9 and acts on $A_1^\#$ in orbits of length 3, 3, and 9. On the other hand, it follows from the preceding lemma that z has five conjugates in $A_1^\#$ and z_1 has ten conjugates in $A_1^\#$. Since z and z_1 are not conjugate, this is clearly impossible. Thus $\bar{H}_1\cong S_5$, as asserted.

Since z_1 has ten conjugates in H_1, it follows now that $C_{\bar{H}_1}(z_1)$ contains a 3-element x_1 with $x_1\notin C_1$. Since z has five conjugates in $A_1^\#$, x_1 must centralize two of them. Thus $C_{A_1}(x_1)$ is a four group containing z_1 and two involutions conjugate to z. One checks that the only possibility is $\langle z_1, b_1z\rangle$.

Setting $D_1=\langle b_1, a_1^{2^{n-3}}\rangle\times\langle b_2, a_2^{2^{n-3}}\rangle$ and $S_1=D_1\langle u\rangle$, we have that $S_1=N_S(A_1)$ $\cong D_8\wr Z_2$ and that S_1 is a Sylow 2-subgroup of H_1. Then $\langle z\rangle=Z(S_1)$. Since z_1 is not conjugate to z in H_1 and z_1 is in $Z(D_1)$, it follows that D_1 is a Sylow 2-subgroup of $C_{H_1}(z_1)$. But $D_1 \cong D_8 \times D_8$ and so $N_{H_1}(D_1)/C_{H_1}(D_1)$ is a 2-group. We conclude therefore by the Frattini argument that $C_{\bar{H}_1}(\bar{z}_1)$ cannot be isomorphic to A_4. Since $\bar{x}_1 \in C_{\bar{H}_1}(\bar{z}_1)$, this in turn implies that $\langle \bar{x}_1 \rangle \cong Z_3$ is a normal 2-complement in $C_{\bar{H}_1}(\bar{z}_1)$.

Now $A_1 = \langle z_1, b_1 z \rangle \times B_1$, where $B_1 = [A_1, \bar{x}_1]$ is a four group. Since $\bar{D}_1$ normalizes $\langle \bar{x}_1 \rangle$ by the preceding paragraph, $\bar{D}_1$ also leaves B_1 invariant and consequently $B_1 \lhd D_1$. But all involutions of B_1 are clearly conjugate in H_1. Furthermore, we have $z \sim b_1 z \sim b_1 z_2 \sim b_2 z_1 \sim b_2 z$ with the remaining elements of $A_1^{\#}$ all conjugate to z_1 as $N_G(A_1)$ has exactly two orbits on $A_1^{\#}$. This implies that B_1 necessarily contains a conjugate of z_1 and so B_1 contains no conjugate of z. But $B_1 \cap Z(D_1) \neq 1$ and $Z(D_1) = \langle z_1, z_2 \rangle$. Since $z_1 \notin B_1$ by the given decomposition of A_1, it follows that $z_2 \in B_1$. We conclude now that $\langle b_2, z_2 \rangle$ is the unique possibility for B_1. Thus (ii) also holds.

Next let $\bar{V}$ be a four subgroup of $\bar{S}_1$ and let V be the subgroup of $S_1 = S \cap H_1$ containing A_1 which maps on $\bar{V}$. Since $S_1 \cong D_8 \int Z_2$, one checks that either $V \cong D_8 \times D_8$ or that V is of type A_8. Since $Z = Z(D_1)$, it also follows in either case that $Z(V) \subseteq Z$.

Suppose now that $\bar{V}$ normalizes a subgroup $\bar{Y}$ of order 3. Since $\bar{H}_1$ acts intransitively on $A_1^{\#}$, $B_1 = C_{A_1}(\bar{Y})$ is a four group. Since $\bar{V}$ normalizes $\bar{Y}$, we see that $B_1 \lhd V$, whence $B_1 \cap Z(V) \neq 1$. Since $B_1 \cap Z(V) \subseteq Z$, we conclude that (iii) holds.

Assume next that $\bar{V} \subseteq \bar{H}_1' \cong A_8$. Then $\bar{V}$ is normalized, but not centralized by a 3-element of $\bar{H}_1'$ and so V is normalized, but not centralized by a 3-element of H_1 by the Frattini argument. Since $\mathrm{Aut}(D_8 \times D_8)$ is a 2-group, it follows in this case that V is of type A_8. However, one checks directly that S_1 possesses a unique subgroup of type A_8 containing A_1: namely, $\langle b_1, z_1, b_2, z_2, a_1^{2^{n-3}} a_2^{2^{n-3}}, u \rangle$. In particular, $u \in V$. Since $V \subseteq H_1'$, we obtain (iv).

Finally $z_1^y = b_1$ for some y in H_1. Then the 3-element x_1^y centralizes b_1 and reasoning as with x_1, we check that $C_{A_1}(x_1^y)$ is a four group containing b_1 and two involutions conjugate to z. The only possibility is $\langle b_1, z \rangle$. Hence $\langle z_1, b_1 z^y \rangle = \langle b_1, z \rangle$ and (v) also holds.

REMARK. Part (v) of the lemma also holds when $n=2$. Even though $\bar{H}_1 \cong A_6$ in this case, z_1 and z still have 10 and 5 conjugate respectively in H_1, so the same proof applies.

We set $N = C_G(z_1)$ and fix this notation for the balance of the paper. *We now drop the assumption that* $n \geqq 3$.

We next prove

LEMMA 3.19. *The following conditions hold:*

(i) *M and N are of type $PS_p(4, q)$ for the same odd $q \geqq 5$;*

(ii) *If $\bar{N}=N/O(N)$, $\bar{N}=(\langle \bar{b}_1\bar{z}, \bar{a}_1\rangle \times O^2(\bar{N}))\langle \bar{t}\rangle$ and $\langle \bar{a}_2, \bar{b}_2\rangle$ is a Sylow 2-subgroup of $O^2(\bar{N})$.*

PROOF: We first treat the case $n=2$. By Proposition 3.2, G has the involution fusion pattern of $PS_p(4, q)$ and M is of type $PS_p(4, q)$ for some $q \equiv 3$, 5 (mod 8). From this information and for a suitable choice of the generators of S (note that $z_i=a_i$, $i=1$, 2, in this case), we can assume that

$$z \sim b_1 z | z_1 \sim b_1 \sim t \sim u \sim tu \sim b_1 b_2 \sim z_1 b_1 b_2 .$$

We also have that $R \cong Q_8 * Q_8$ is a Sylow 2-subgroup of $O^2(M)$ and that $M=O^2(M)\langle b_1\rangle$. Setting $\bar{M}=M/O(M)$, we know that $\bar{M}$ possesses a normal subgroup of the form $\bar{L}_1\bar{L}_2$, where $\bar{L}_i \cong SL(2, q)$, $i=1$, 2, and $[\bar{L}_1, \bar{L}_2]=1$ with $\bar{b}_1$ interchanging $\bar{L}_1$, $\bar{L}_2$ under conjugation. Hence $C_{\bar{L}_1\bar{L}_2}(\bar{b}_1) \cong Z_2 \times PSL(2, q)$ and consequently $C_G(b_1, z)=C_M(b_1)$ involves $PSL(2, q)$. Since $b_1 \sim z_1$, we conclude that also N involves $PSL(2, q)$.

Setting $A=\langle z_1, z_2, b_1, b_2\rangle$ and $T=A\langle t\rangle$, we see that $T \cong Z_2 \times Z_2 \wr Z_2$ and that T is a Sylow 2-subgroup of N. Hence by [7, Lemma 4.4], N has a normal subgroup K with Sylow 2-subgroup A. Since z and z_1 have 5 and 10 conjugates in $N_G(A)$ respectively, $|N_G(A)/C_G(A)|$ is not divisible by 9. Since K involves $PSL(2, q)$, it does not have a normal 2-complement and consequently $O^2(K)$ is of index 4 in K. Hence by the main theorem of [3] or [16] we have that $\bar{K}=K/O(K)$ possesses a normal subgroup $\bar{F} \cong PSL(2, r)$ for some $r \geqq q$ and that $\bar{F}$ centralizes a four subgroup $\bar{B}_1$ of $\bar{A}$. We have $\bar{A}=\bar{B}_1 \times \bar{B}_2$, where $\bar{B}_2=\bar{A} \cap \bar{F}$.

We know the fusion pattern of involutions of $A^{\#}$: namely, $z \sim b_1 z \sim b_1 z_2 \sim b_2 z \sim b_2 z_1$ with the remaining elements of $A^{\#}$ conjugate to z_1. It follows therefore as in the preceding lemma that $\bar{B}_1=\langle \bar{z}_1, \bar{b}_1\bar{z}\rangle$ and $\bar{B}_2=\langle \bar{b}_2, \bar{z}_2\rangle$. Hence (ii) holds. Furthermore, we have that $C_G(b_1 z)$ involves $PSL(2, r)$. Since $b_1 z \sim z$, M also involves $PSL(2, r)$. But $r \geqq q$ and we conclude at once from the structure of M that $r=q$. It follows at once now from the definition that N is of type $PS_p(4, q)$, so (i) also holds in this case.

Next assume $n \geqq 3$. By Lemma 3.18 (ii), $\langle b_2, z_2\rangle \subseteq O^2(C_G(\langle z_1, b_1 z\rangle))$ and consequently $\langle b_2, z_2\rangle \subseteq K=O^2(N)$. Since $T \subseteq N$ and $\langle a_2, b_2\rangle$ is the normal closure of $\langle b_2, z_2\rangle$ in T, it follows that $\langle a_2, b_2\rangle \subseteq K$. On the other hand, $T \cong D_{2^{n+1}} \wedge D_{2^{n+1}}$ is a Sylow 2-subgroup of N as $T \subseteq N$ and z_1 is not conjugate to z. But now [11, Lemma 8.5] implies that N has a normal subgroup H of index 2 with Sylow 2-subgroup $D=\langle a_1, b_1\rangle \times \langle a_2, b_2\rangle$. Clearly $O^2(H)=K$ and so $D \cap K=D_1 \times \langle a_2, b_2\rangle$, where

$D_1=\langle a_1, b_1\rangle \cap K$. By [10, Lemma 3.13], D_1 cannot be cyclic, otherwise K would have a normal subgroup of index 2. On the other hand, if $D_1 \cong D_{2^{n_1}}$, then by [10, Propositions 3.2 and 3.7] K does not possess an isolated involution. However, this is impossible since $z_1 \in Z(N)$ and $z_1 \in K$ as D_1 is noncyclic. We conclude that $\langle a_2, b_2\rangle$ is a Sylow 2-subgroup of K.

Setting $\bar{N}=N/O(N)$, it follows now from [12] that either $\bar{K}\cong A_7$ or $\bar{K}$ contains a normal subgroup $\bar{F}$ of odd index with $\bar{F}\cong PSL(2, r)$ for some odd r. However, $\bar{t}$ acts on $\bar{F}$ and $\langle \bar{a}_2, \bar{b}_2, \bar{t}\rangle \cong D_{2^{n+1}}$. This shows that $\bar{F}$ cannot be isomorphic so A_7, otherwise $\bar{F}\langle \bar{t}\rangle \cong S_7$, contrary to the fact that S_7 has Sylow 2-subgroups isomorphic to $Z_2 \times D_8$. Thus $\bar{F}\cong PSL(2, r)$.

We next determine the structure $C_{\bar{N}}(\bar{F})$. Since $\langle \bar{a}_1, \bar{b}_1\rangle$ centralizes the Sylow 2-subgroup $\langle \bar{a}_2, \bar{b}_2\rangle$ of $\bar{F}$, it follows from the structure of $P\Gamma L(2, r)$ that any involution of $\bar{D}$ not contained in $\bar{F}C_{\bar{N}}(\bar{F})$ necessarily induces a nontrivial field automorphism of $\bar{F}$. Suppose $\bar{b}_1 \notin \bar{F}C_{\bar{N}}(\bar{F})$, in which case $C_{\bar{F}}(\bar{b}_1)\cong PGL(2, r_1)$, where $r_1^2=r$. But then we see that there exists a 3-element of $N_N(A_i)-C_N(A_i)$ which centralizes $\langle z_1, b_1\rangle$, $i=1$ or 2. However, the involutions of $\langle z_1, b_1\rangle$ are all conjugate to z_1. On the other hand, by the preceding lemma, every 3-element of $N_G(A_i)-C_G(A_i)$ centralizes some involution conjugate to z. This contradiction shows that $\bar{b}_1 \in \bar{F}C_{\bar{N}}(\bar{F})$.

We argue next that $\bar{b}_1\bar{z}\in C_{\bar{N}}(\bar{F})$. Obviously $\bar{z}_1$ centralizes $\bar{F}$. Since $\langle \bar{z}_2, \bar{b}_2\rangle \subseteq \bar{F}$, it follows from the preceding paragraph that $\bar{A}_1=\langle \bar{z}_1, \bar{z}_2, \bar{b}_1, \bar{b}_2\rangle \subseteq \bar{F}C_{\bar{N}}(\bar{F})$. Setting $\bar{B}=C_{\bar{A}_1}(\bar{F})$, we see that $\bar{B}$ is a four group containing $\bar{z}_1$. Hence if B denotes the inverse image of $\bar{B}$ in A_1 and $H_1=N_G(A_1)$, we conclude that $C_{H_1}(z_1)$ contains a 3-subgroup Y which centralizes B, but does not centralize A_1. On the other hand, if $\bar{H}_1=H_1/C_G(A_1)$, Lemma 3.18 implies that $C_{\bar{H}_1}(\bar{z}_1)$ is isomorphic to $Z_2\times S_3$ and its unique subgroup of order 3 centralizes b_1z. Hence $C_{A_1}(Y)=\langle z_1, b_1z\rangle=B$, which shows that $\bar{b}_1\bar{z}\in C_{\bar{N}}(\bar{F})$.

Observe next that as $\bar{S}_1=\langle \bar{a}_1, \bar{b}_1\bar{z}\rangle$ is the normal closure of $\langle \bar{z}_1, \bar{b}_1\bar{z}\rangle$ in $\bar{T}$, it follows that $\bar{S}_1 \subseteq C_{\bar{N}}(\bar{F})$. We have $\bar{S}_1\cong D_{2^n}$ and that $\bar{S}_1$ is a Sylow 2-subgroup of $C_{\bar{N}}(\bar{F})$. Since $\bar{z}_1$ is isolated in $C_{\bar{N}}(\bar{F})$, this implies that $C_{\bar{N}}(\bar{F})$ has a normal 2-complement. Since $O(\bar{N})=1$, we obtain that $C_{\bar{N}}(\bar{F})=\bar{S}_1$. Since $\bar{K}/\bar{F}$ is of odd order, we conclude that $\bar{K}$ centralizes $\bar{S}_1$.

Since N does not involve A_7, Lemma 3.16 (ii) yields that M is of type $PS_p(4, q)$ for some odd q. Furthermore, to complete the proof that N is of type $PS_p(4, q)$, it remains only to show that $r=q$. Since $\bar{b}_1\bar{z}$ centralizes $\bar{F}$, $C_G(b_1z)$ involves $PSL(2, r)$. But $b_1z\sim z$ and so M involves $PSL(2, r)$, whence $r\leqq q$. On the other hand, as in the case $n=2$, $C_G(b_1)$ and hence N involves $PSL(2, q)$, so $q\leqq r$. Thus

$q=r$ and all parts of the lemma hold.

The preceding lemmas establish Proposition 3.4. Indeed, G has the involution fusion pattern of $PS_p(4, q)$ for some odd q by assumption if $n=2$ and by Lemma 3.17 if $n\geqq 3$. Furthermore, M and N are of type $PS_p(4, q)$ for the same odd q by the preceding lemma. Thus G has the centralizer of involution pattern of $PS_p(4, q)$. Moreover, if $n=2$, then $q\geqq 5$ by assumption. Finally if A is an elementary abelian subgroup of G of order 16, $N_G(A)/C_G(A)\cong A_5$ if $n=2$ by Lemma 3.3, while $N_G(A)/C_G(A)\cong S_5$ by Lemmas 3.5 (ix) and 3.18.

Proposition 3.4 has a number of elementary consequences which we shall need for our further analysis.

LEMMA 3.20. *The following conditions hold:*

(i) *If A is an elementary abelian subgroup of S of order* 16, $N_M(A)/C_M(A)$ $\cong A_4$ *or* S_4 *according as* $n=2$ *or* $n\geqq 3$;

(ii) *Every T-invariant subgroup of M of odd order lies in* $O(C_M(Z))O(M)$;

(iii) $C_M(Z)$ *has a normal 2-complement*;

(iv) *Every T-invariant subgroup of N of odd order lies in* $O(C_N(Z))O(N)$;

(v) $\bar{M}=M/O(M)=\bar{L}_1\bar{L}_2O(C_{\bar{M}}(\bar{R}))\bar{S}$, *where* $\bar{L}_i\cong SL(2, q)$, $i=1, 2$, $\bar{L}_1$ *centralizes* $\bar{L}_2$, *and* $\bar{L}_1\bar{L}_2$ *is normal in* $\bar{M}$;

(vi) $\bar{N}=N/O(N)=\bar{F}O(C_{\bar{N}}(\bar{T}))\bar{T}$, *where* $\bar{F}\cong PSL(2, q)$ *and* $\bar{F}$ *is normal in* $\bar{N}$;

(vii) *The normal closure of* $\bar{z}_1$ *in* $\bar{M}$ *contains* $\bar{L}_1\bar{L}_2$;

(viii) *The normal closure of* $\bar{z}$ *in* $\bar{N}$ *contains* $\bar{F}$.

PROOF: We know that M is of type $PS_p(4, q)$ and that R is a Sylow 2-subgroup of $O^2(M)$. Moreover, the structure of N is given in Lemma 3.19. The various parts of this lemma follow directly from the specified structures of M and N.

We also have

LEMMA 3.21. *Every prime divisor of* $|O(M)|$ *is also a prime divisor of* $|O(N)|$.

PROOF: Set $d=|C_{O(M)}(z_1)|$ and $e=|\bar{M}:\bar{L}_1\bar{L}_2\bar{S}|$, so that d and e are both odd. The proof will depend upon the following two assertions:

(a) Every prime divisor of $|O(M)|$ is a prime divisor of d;

(b) $O(C_{\bar{N}}(\bar{Z}))$ has order dividing $e(q+\delta)$, where $\delta=\pm 1$ and $q\equiv\delta \pmod 4$.

Indeed, suppose (a) and (b) hold. Since $\bar{L}_1\bar{L}_2\cong SL(2, q)*SL(2, q)$, $C_{\bar{L}_1\bar{L}_2}(\bar{z}_1)$ has a normal 2-complement of order $(q+\delta)^2/2^{2n}$. Hence by Lemma 3.20 (v), $C_{\bar{M}}(\bar{z}_1)$ has a normal 2-complement of order $e(q+\delta)^2/2^{2n}$ and consequently $C_M(z_1)=M\cap N=C_G(Z)$ has a normal 2-complement of order $de(q+\delta)^2/2^{2n}$. But $O(C_G(Z))$ maps onto $O(C_{\bar{N}}(\bar{Z}))$, which by (b) has order dividing $e(q+\delta)/2^n$. These conditions thus imply that d divides $|O(N)|$. Hence by (a), every prime divisor of $|O(M)|$ is a prime divisor of

$|O(N)|$.

To prove (a), let P be an S-invariant Sylow p-subgroup of $O(M)$ for some prime p dividing $O(M)$, so that $P\neq 1$. Our assertion is obvious if z_1 centralizes P, so assume the contrary. By the Frattini argument $M_0=N_M(P)$ covers $M/O(M)$. Since $z_1\notin C_M(P)$, we conclude at once from the structure of M, and hence of M_0, that the image $\bar{z}_1$ of z_1 in $\bar{M}_0=M_0/C_M(P)$ is not contained in $Z(\bar{M}_0)$. This clearly implies that $\bar{z}_1$ does not invert P and consequently $C_P(z_1)=C_P(Z)\neq 1$, as required.

As for (b), it will suffice to prove that $|\bar{N}:\bar{F}\bar{T}|=e_0$, where e_0 divides e. Indeed, if this is the case, then as $\bar{F}\cong PSL(2,q)$, it will follow from Lemma 3.20 (vi) that $C_{\bar{N}}(\bar{Z})$ has a normal 2-complement of order $e_0(q+\delta)/2^n$ and so (b) will hold. Observe now that $N_N(A_1)$ contains a nontrivial 3-element which centralizes $\langle z_1, b_1z\rangle$. Moreover, Lemma 3.18 (v) together with the remark following it implies that $\langle z_1, b_1z\rangle^w=\langle z, b_1\rangle$ for some w in $N_G(A_1)$. Since z_1 and b_1 are the only non-central involutions of $\langle z_1, b_1z\rangle$ and $\langle z, b_1\rangle$ respectively, we have $z_1^w=b_1$. Hence if $N_1=C_G(b_1)$, then $N_1=N^w$ and so if $\bar{N}_1=N_1/O(N_1)$, then we have $\bar{N}_1\cong\bar{N}$. Thus we need only show that $|\bar{N}_1|=2^{n+1}e_0|PSL(2,q)|$ with $e_0|e$. Moreover, by the structure of N and N_1, this will follow if we prove that $|O^2(\bar{N}_1)|=e_0|PSL(2,q)|$ with $e_0|e$.

By Lemma 3.19, $\langle\bar{z}_1, \bar{b}_1\bar{z}\rangle$ centralizes $O^2(\bar{N})$ and consequently $\langle\bar{z}, \bar{b}_1\rangle$ centralizes $O^2(\bar{N}_1)$. Hence if we set $C=C_G(\langle z, b_1\rangle)$, we have that $C\subseteq N_1$ and that C covers $O^2(\bar{N}_1)$. This in turn then implies that $O^2(C/O(C))\cong O^2(\bar{N}_1)$. However, by the structure of $\bar{M}$, $\bar{b}_1$ interchanges $\bar{L}_1$ and $\bar{L}_2$. Hence $C_{\bar{L}_1\bar{L}_2}(\bar{b}_1)=\langle\bar{z}_1\rangle\times\bar{L}_0$, where $\bar{L}_0\cong PSL(2,q)$. Furthermore, if we set $\bar{E}=O(C_{\bar{M}}(\bar{R}))$, Lemma 3.20 (v) together with the definition of e implies that $|\bar{E}|=e$ as $\bar{E}\cap\bar{L}_1\bar{L}_2=1$. Therefore if we set $\bar{E}_0=C_{\bar{E}}(\bar{b}_1)$, then $\bar{E}_0$ is of order e_0 dividing e. Moreover, no element of $\bar{E}_0^{\#}$ centralizes $\bar{L}_0$ and consequently $O(C_{\bar{M}}(\bar{b}_1))=1$. But clearly $C\subseteq M$ and $\bar{C}=C_{\bar{M}}(\bar{b}_1)$. We conclude therefore that $O(\bar{C})=1$ and that $O^2(\bar{C})=\bar{L}_0\bar{E}_0$ is of order $e_0|PSL(2,q)|$ with $e_0|e$. Hence $O^2(C/O(C))\cong O^2(\bar{N}_1)$ has the same order and (b) is proved. This completes the proof of the lemma.

Finally we prove

LEMMA 3.22. *The following conditions hold:*

(i) *The centralizer of every involution of G is 2-generated;*

(ii) *If $q>9$, then $N=\langle C_N(B)|B\in\mathscr{E}_3(A_1)\rangle T$.*

PROOF: Since $O(M)=\langle C_{O(M)}(B)|B\in\mathscr{E}_3(A_1)\rangle$ with a similar generation for $O(N)$, it will clearly suffice to prove that $\bar{M}$ and $\bar{N}$ are 2-generated and that $\bar{N}=\langle C_{\bar{N}}(\bar{B})|\bar{B}\in\mathscr{E}_3(\bar{A}_1)\rangle\bar{T}$ when $q>9$. Since $C_{\bar{L}_1\bar{L}_2}(\bar{b}_1)=C_{\bar{L}_1\bar{L}_2}(\langle\bar{z}, \bar{b}_1\rangle)=\langle\bar{z}\rangle\times\bar{L}_0$, where $\bar{L}_0\cong PSL(2,q)$, we conclude at once, using Lemma 3.20 (v), that $\bar{M}=\langle N_{\bar{M}}(\bar{R}), C_{\bar{L}_1\bar{L}_2}(\langle\bar{z}, \bar{b}_1\rangle)\rangle$ and so $\bar{M}$ is 2-generated. Likewise $\bar{N}=O^2(\bar{N})\bar{T}$ and

$\langle \bar{z}_1, \bar{b}_1\bar{z}\rangle$ centralizes $O^2(\bar{N})$ by Lemma 3.19, so $\bar{N}$ is also 2-generated. Thus (i) holds.

Suppose finally that $q \geqq 9$. By Lemmas 3.19 and 3.20 (iv), we have $\bar{N}=\bar{F}O(C_{\bar{N}}(\bar{A}_1))\bar{T}$ and so to establish the desired conclusion, we need only show that $\bar{F}\subseteq \langle C_{\bar{N}}(\bar{B})|\bar{B}\in \mathscr{E}_3(\bar{A}_1)\rangle$. However, $\bar{A}_1=\langle \bar{z}_1, \bar{b}_1\bar{z}\rangle\times\langle \bar{z}_2, \bar{b}_2\rangle$, where the first factor centralizes $\bar{F}$ and the second is a four subgroup of $\bar{F}$. Hence if we set $\bar{X}=\langle z_2, b_2\rangle$, it will suffice to prove that $\bar{F}=\langle C_{\bar{F}}(\bar{x})|\bar{x}\in \bar{Z}^{\#}\rangle$. But as $\bar{F}\cong PSL(2, q)$ with $q>9$, this follows from standard properties of the groups $PSL(2, q)$.

REMARK. Actually part (i) of the lemma is needed only in the case that q is a Fermat or Mersenne prime or 9 and, in particular, when $q \leqq 9$.

4. Subgroup structure of G. Henceforth we assume that our fusion-simple group G with Sylow 2-subgroup S of type $PS_p(4, q)$, q odd, is a minimal counterexample to Theorem B. If follows then that G has the centralizer of involution pattern of $PS_p(4, q)$ for some odd $q \geqq 5$. In particular, $M=C_G(z)$ is of type $PS_p(4, q)$. Since G does not satisfy the conclusion of Theorem B, we must therefore have that $O(M)\neq 1$. Our goal in the balance of the paper will be to derive a contradiction from these conditions.

In this section, we shall obtain such further information concerning the subgroup structure of G as we shall need to show that G is 2-balanced.

We begin with a result about subgroups H of G having $\langle D, u\rangle$ or $\langle D, tu\rangle$ as Sylow 2-subgroup. By Lemma 3.5 (xv), these groups are isomorphic to $D_{2^n}\int Z_2$. Even though there exist simple groups with Sylow 2-subgroups of type $D_{2^n}\int Z_2$ (the groups $PSL(4, q)$, $q\equiv -1 \pmod 4$ and $PSU(4, q)$, $q\equiv 1 \pmod 4$, for example), we do not require any general results about groups with such Sylow 2-subgroups in the present situation. Instead we utilize the fact that H is a subgroup of a group G with Sylow 2-subgroups of type $PS_p(4, q)$.

We shall prove

PROPOSITION 4.1. *If H is a subgroup of G with Sylow 2-subgroup $\langle D, u\rangle$ or $\langle D, tu\rangle$, then $H/O(H)$ is not fusion-simple.*

PROOF: The proofs being entirely similar, we treat only the case that $F=\langle D, u\rangle$ is a Sylow 2-subgroup of H. If $n=2$, $F\cong Z_2\times Z_2\int Z_2$ and the result follows from another application of [7, Lemma 4.4]. Hence we can assume that $n \geqq 3$ and that H is fusion-simple.

We shall study the structure of $N_H(A_2)$. We set $K=N_H(A_2)$ and $C=C_H(A_2)$. We claim first that $N_F(A_2)=\langle b_1, a_1^{2^{n-3}}\rangle\times\langle a_2b_2, a_2^{2^{n-3}}\rangle\cong D_8\times D_8$ is a Sylow 2-subgroup of K. Indeed, F has two conjugacy classes in $S=\langle F, t\rangle$ of elementary subgroups of order 16, represented by A_1 and A_2. Moreover, $|N_F(A_1)|>|N_F(A_2)|$. If $N_F(A_2)$

were not a Sylow 2-subgroup of K, it would then follow that A_2 and A_1 are conjugate in $\langle H, t\rangle$. But then they would be conjugate in S by Lemma 3.8, which is not the case. This proves our assertion.

Since $N_G(A_2)/C_G(A_2)\cong S_5$ by Lemma 3.18 (i) and since $\mathrm{Aut}(D_8\times D_8)$ is a 2-group, we see that either $K/C\cong Z_2\times Z_2$ or $Z_2\times S_3$. In the first case, it is immediate that b_1z is not conjugate in K to any element of $\langle z_1, z_2, a_2b_2\rangle$. We argue that we can reduce to the same situation in the second case as well. Indeed, in this case Lemma 3.18 (iii) implies that a 3-element x of $K-C$ centralizes some element of $Z^{\sharp}$. But $N_M(A_2)/C_M(A_2)\cong S_4$ by Lemma 3.20 (ii). Since $K/C\cong Z_2\times S_3$, it follows that $x\notin M$ and so x centralizes z_1 or z_2. But z_1 and z_2 are conjugate by the element tu of $N_S(A_2)$ as $N_G(A_2)/C_G(A_2)\cong S_5$. Moreover, F is a Sylow 2-subgroup of H^{tu} as $F=\langle D, u\rangle\lhd S$. Hence replacing H by H^{tu}, if necessary, we can assume without loss that x centralizes z_1. Now Lemma 3.18 (ii) yields that x also centralizes b_1z and again we conclude that b_1z is not conjugate in K to any element of $\langle z_1, z_2, b_1b_2\rangle$.

Observe next that if we use Alperin's fusion lemma as we did in Lemma 3.8, we easily obtain that any two elementary subgroups of F of order 16 are conjugate in H if and only if they are conjugate in F. Hence as in Lemma 3.9 if z were conjugate to b_1z in H, it would be conjugate to b_1z in K. Thus b_1z and z are not conjugate in H.

We shall now contradict this last conclusion. Set $E=\langle a_1, a_2, b_1b_2, u\rangle$, so that E is a maximal subgroup of F. We check that u, b_1b_2, $b_1b_2a_1$, $b_1b_2a_1a_2$, z_1, z are representatives of the conjugacy classes in F of the involutions of E. Since H is fusion-simple, Thompson's lemma implies that the involution b_1z of $F-E$ is conjugate to one of the involutions listed. However, $u\sim b_1b_2\sim b_1b_2a_1\sim b_1b_2a_1a_2\sim z_1$ in G, while $b_1z\sim z$ in G, by Lemma 3.17. Since z and z_1 are not conjugate in G, we must have $b_1z\sim z$ in H, giving the desired contradiction.

Recall now that $T=\langle D, t\rangle\cong D_{2^{n+1}}\wedge D_{2^{n+1}}$. We next prove

PROPOSITION 4.2. *If H is a proper subgroup of G containing T, then one of the following holds:*

(i) *H contains an isolated involution;*

(ii) *$n=2$, D is elementary of order 16, and $H=O(H)N_H(D)$; or*

(iii) *H contains a Sylow 2-subgroup of G and $H/O(H)$ possesses a normal subgroup of odd index isomorphic to $PS_p(4, r)$ for some odd r.*

PROOF: Suppose first that T is a Sylow 2-subgroup of H. If $n\geqq 3$, [11, Lemma 8.5] implies that H has a normal subgroup K of index 2 with Sylow 2-subgroup D. On the other hand, if $n=2$, we reach the same conclusion by [7, Lemma 4.4] as then $T\cong Z_2\times Z_2\int Z_2$. Consider the case that $n=2$, whence D is elementary of order 16. It follows from Proposition 3.4 that $N_G(D)/O(N_G(D)) \cong A_5 \cdot E_{16}^{(1)}$, which implies that $N_G(D)$ does not possess a 3-element which acts regularly on D. Hence by the classification of groups with abelian Sylow 2-subgroups [3], [16], either K has an isolated involution or $O(H)D$ is normal of index 5 in K. Correspondingly, we conclude that (i) or (ii) holds.

Hence we may suppose that $n \geqq 3$. We may assume that z_1 and z_2 are not isolated in K, otherwise (i) holds. Since $N_K(D)=DC_K(D)$ as $D \cong D_{2^n} \times D_{2^n}$, it follows from a result of Burnside that no two involutions of $Z=Z(D)$ are conjugate in K. Hence there exist elements x_i in D with $x_i \notin Z$ such that $x_i \sim z_i$ in K, $i=1, 2$. Clearly there then exist elementary abelian subgroups X_i of order 16 in D containing $\langle Z, x_i \rangle$, $i=1, 2$. As shown in [10, Lemma 3.1], two elementary abelian subgroups of D of order 16 are conjugate in K if and only if they are conjugate in D. We conclude from this exactly as in the proof of Lemma 3.11 (as $z_i \in Z(D)$) that x_i is conjugate to z_i in $N_K(X_i)$, $i=1, 2$. It also follows that $N_D(X_i) \cong D_8 \times D_8$ is a Sylow 2-subgroup of $K_i=N_K(X_i)$, $i=1, 2$.

On the other hand, we know that $N_G(X_i)/C_G(X_i) \cong S_6$ by Proposition 3.4, $i=1$, 2. Comparing the structure of a Sylow 2-subgroup of K_i with that of $N_G(X_i)$, we see that there is only one possibility for the structure of $\bar{K}_i=K_i/C_K(X_i)$, namely $\bar{K}_i \cong Z_2 \times S_3$ (cf. proof of Lemma 3.18). Hence $O(\bar{K}_i)$ is of order 3 and is invariant under $\overline{D \cap K_i}$, which is a four group. It follows therefore from Lemma 3.18 (iii) that $O(\bar{K}_i)$ centralizes some involution of Z. If $O(\bar{K}_i)$ centralized z, then $M \cap K_i$ would cover $\bar{K}_i \cong Z_2 \times S_3$. However, this is impossible as $N_M(X_i)/C_M(M_i) \cong S_4$ by Lemma 3.20 (i). Furthermore, $O(\bar{K}_i)$ does not centralize z_i as then z_i would not be conjugate to x_i in K_i. Hence, in fact, $O(\bar{K}_i)$ centralizes z_j, $i \neq j$, $1 \leqq i$, $j \leqq 2$. We conclude therefore that $C_K(z_j)$ covers $\bar{K}_i$, $i \neq j$, $1 \leqq i$, $j \leqq 2$.

The structure of $N=C_G(z_1)$ is given in Lemma 3.19. Since $z_2=z_1^y$, we also have the structure of $C_G(z_2)$. In particular, $\langle a_i, b_i \rangle$ is a Sylow 2-subgroup of $O^2(C_G(z_j))$, $i \neq j$, and it follows from this that $O(\bar{K}_i)$ normalizes but does not centralize $Y_i=X_i \cap \langle a_i, b_i \rangle$, $i=1, 2$. Thus the four group $Y_i \subseteq O^2(K)$, $i=1, 2$. However, it is immediate from the structure of T that $\langle Y_1, Y_2 \rangle^T=D$. Since $T \cap O^2(K) \lhd T$, this implies that $D \subseteq O^2(K)$. Since D is a Sylow 2-subgroup of K, we conclude that $K=O^2(K)$.

But now [10, Theorem A*] is applicable to K and it is a consequence of this result that $N_K(A)/C_K(A)$ is divisible by 9 for any elementary abelian subgroup A of order 16 in K. In particular, this is true of X_1, contrary to the fact that $N_G(X_1)/C_G(X_1) \cong S_6$. This completes the proof of the proposition when T is a Sylow 2-subgroup of H.

Suppose next that T is not a Sylow 2-subgroup of H, in which case H contains a Sylow 2-subgroup of G. Since T is the only maximal subgroup of S with its structure by Lemma 3.5 (xii), we can assume without loss that S itself is a Sylow 2-subgroup of H. We can also suppose that H does not satisfy (i) or (ii). If A is an elementary abelian subgroup of S of order 16, we know from Proposition 3.4 that $N_G(A)/C_G(A)\cong A_5$ or S_5. In addition, $N_H(A)$ acts intransitively on $A^\sharp$ and so $N_H(A)-C_H(A)$ does not possess a 3-element which acts regularly on A. Since H has no isolated involution, we conclude therefore from the remark following Lemma 3.3, if $n=2$, that $N_H(A)/C_H(A)\cong A_5$ and from Lemmas 3.10 and 3.11, if $n\geqq 3$, that $N_H(A)/C_H(A)\cong S_5$. Hence in either case, $N_H(A)$ contains a 5-element which acts regularly on A, which implies that $A\subseteq K=O^2(H)$. But as $D\cong D_{2^n}\times D_{2^n}$, D is generated by its elementary abelian subgroups of order 16 and consequently $D\subseteq K$.

Suppose D is a Sylow 2-subgroup of K. If $n\geqq 3$, we again use [10, Theorem A^*] and reach a contradiction as above. On the other hand, if $n=2$, it follows as in the first paragraph of the proof that $O(H)D$ is normal of index 5 in K as K has no isolated involution and so (ii) holds in this case.

Hence we can suppose that D is not a Sylow 2-subgroup of K. Since $O^2(K)=K$, it follows once again from [11, Lemma 8.5] and [7, Lemma 4.4] that $T=\langle D, t\rangle$ is not a Sylow 2-subgroup of K. For the same reason, the preceding proposition shows that $\langle D, u\rangle$ or $\langle D, tu\rangle$ are not Sylow 2-subgroups of K. The only possibility is therefore that S is a Sylow 2-subgroup of K, in which case $H=K$. Setting $\bar{H}=H/O(H)$, we see that $\bar{H}$ is fusion-simple. Furthermore, the involution fusion pattern in $\bar{H}$ is the same as that in H. Since the involution fusion pattern of G is that of $PS_p(4, q)$ for some odd q, $\bar{H}$ is not isomorphic to A_8, A_9, $A_5\cdot E_{16}^{(1)}$ or $GL(3, 2)\cdot E_8^{(1)}$ if $n=2$. ($A_5\cdot E_{16}^{(1)}$ is excluded by our assumption that H is not of the form (ii).) Since G is a minimal counterexample to Theorem B, it follows therefore that part (ii) of Theorem B holds for $\bar{H}$. Thus $C_{\bar{H}}(\bar{z})$ is of type $PS_p(4, r)$ for some odd r and $O(C_{\bar{H}}(\bar{z}))=1$. But now Theorem 2.5 yields that H contains a normal subgroup of odd index isomorphic to $PS_p(4, r)$ and so (iii) holds.

As a corollary we have

LEMMA 4.3. *If H is a proper subgroup of G containing T, then we have*

(i) *Every T-invariant subgroup of H of odd order lies in $O(H)O(C_H(Z))$;*

(ii) *Any two maximal T-invariant p-subgroups of H, p odd, are conjugate by an element of $C_H(T)$.*

PROOF: Clearly (ii) will follow immediately from (i). Let then X be a T-invariant subgroup of H of odd order. Setting $\bar{H}=H/O(H)$, we thus need only show that $\bar{X}\subseteq O(C_{\bar{H}}(\bar{Z}))$. Suppose $\bar{H}$ has an isolated involution $\bar{z}'$, whence $\bar{H}=C_{\bar{H}}(\bar{z}')$. We have that $\bar{z}'=\bar{z}_1$, $\bar{z}_2$, or $\bar{z}$. Furthermore, if z' denotes the inverse image of $\bar{z}'$ in S, then $X_0=C_X(z')$ maps onto $\bar{X}$. Thus X_0 is a T-invariant subgroup of $J=C_H(z')$ of odd order. But $z'=z_1$, z_2, or z and $z_1\sim z_2$ by an element of S which normalizes T. It follows therefore from Lemma 3.20 that $X_0\subseteq O(J)O(C_J(Z))$. On the other hand, $J=C_H(z')$ maps onto $\bar{H}=C_{\bar{H}}(\bar{z}')$ and consequently $\overline{O(J)}=1$. Thus $\bar{X}=\bar{X}_0\subseteq\overline{O(C_J(Z))}\subseteq O(C_{\bar{H}}(\bar{Z}))$, as required. Hence (i) holds in this case.

Assume next that H satisfies part (ii) of Proposition 4.2, in which case $\bar{D}\triangleleft\bar{H}$ and $C_{\bar{H}}(\bar{D})=\bar{D}$. Since $\bar{X}$ is $\bar{D}$-invariant, it follows that $\bar{X}$ centralizes $\bar{D}$, whence $\bar{X}=1$ and again (i) holds.

Suppose finally that part (iii) of Proposition 4.2 holds for H, in which case $\bar{H}$ contains a normal subgroup $\bar{K}$ isomorphic to $PS_p(4, r)$ for some odd r. In particular, $\bar{H}$ satisfies the conclusion of Theorem B, so $C_{\bar{H}}(\bar{z})$ is of type $PS_p(4, r)$ with $O(C_{\bar{H}}(\bar{z}))=1$. Since $\bar{H}$ is fusion-simple with the involution fusion pattern of $PS_p(4, r)$, Lemma 3.20 applies to $\bar{H}$ and, as $O(C_{\bar{H}}(\bar{z}))=1$, we conclude that $C_{\bar{H}}(\bar{Z})$ has a normal 2-complement which contains every $\bar{T}$-invariant subgroup of $C_{\bar{H}}(\bar{z})$ of odd order. In particular, $C_{\bar{X}}(\bar{z})\subseteq O(C_{\bar{H}}(\bar{Z}))$. Since $\bar{X}=\langle C_{\bar{X}}(\bar{z}), C_{\bar{X}}(\bar{z}_1), C_{\bar{X}}(\bar{z}_2)\rangle$, it remains to show that $\bar{X}_i=C_{\bar{X}}(\bar{z}_i)\subseteq O(C_{\bar{H}}(\bar{Z}))$, $i=1, 2$.

Again as $\bar{z}_1\sim\bar{z}_2$ by an element of $\bar{S}$ which normalizes $\bar{T}$, we need only treat the case $i=1$. Setting $\bar{N}_1=C_{\bar{H}}(\bar{z}_1)$, we apply Lemma 3.20 once again to conclude that $\bar{X}_1\subseteq O(\bar{N}_1)O(C_{\bar{H}}(\bar{Z}))$. Thus it suffices to prove that $O(\bar{N}_1)$ centralizes $\bar{Z}$. But as $\bar{H}$ is isomorphic to a subgroup of $P\Gamma S_p(4, r)$ containing $PS_p(4, r)$, we know that $O(\bar{N}_1)$ is cyclic. Furthermore, the structure of $\bar{N}_1$ is described in Lemma 3.19, and, in particular, $\langle\bar{a}_2, \bar{b}_2\rangle\subseteq O^2(\bar{N}_1)$. Since $O(\bar{N}_1)$ is cyclic, it follows at once that $\langle\bar{a}_2, \bar{b}_2\rangle$ centralizes $O(\bar{N}_1)$. Since $\bar{z}_2\in\langle\bar{a}_2, \bar{b}_2\rangle$ and $\bar{z}_1$ obviously centralizes $O(\bar{N}_1)$, we conclude that $\bar{Z}=\langle\bar{z}_1, \bar{z}_2\rangle$ centralize $O(\bar{N}_1)$ and the proof is complete.

As a corollary, we have

PROPOSITION 4.4. *Any two maximal T-invariant p-subgroups of G with a nontrivial intersection, p odd, are conjugate by an element $C_G(T)$.*

PROOF: In view of the preceding proposition, we obtain this result by exactly the same argument as established [11, Proposition 5.5].

Finally we prove

LEMMA 4.5. *Let H be a p-local subgroup of G, p odd, with the following properties:*

(a) *H contains T and covers $M/O(M)$ or $N/O(N)$;*

(b) *H is p-constrained and $O_{p'}(H)\subseteq O(H)$.*

If P is a maximal T-invariant p-subgroup of H, then

$$H=O_{p'}(H)N_H(Z(J(P)))\,.$$

PROOF: The proof is essentially identical to that of [11, Lemma 5.6]. Furthermore, this time no exceptional case arises as both "components" of $M/O(M)$ are isomorphic to $SL(2, q)$ for the *same* value of q in the present situation. Again we argue that H is p-stable with respect to P and then apply the extended form of Glauberman's ZJ-theorem, Theorem 2.2 above. We shall limit ourselves to a few comments.

The verification that P satisfies conditions (a) and (b) in the definition of p-stability with respect to P goes through without change. We note only that assumption (a) of the lemma together with Proposition 4.2 imply that either z or z_1 is isolated in H or else $\bar{H}=H/O(H)$ contains a normal subgroup of odd index isomorphic to $PS_p(4, r)$ for some odd r. Likewise we use Lemma 4.3 in place of [11, Lemma 5.4]. Moreover, if z or z_1 is isolated in H, we have that $\bar{H}\cong M/O(M)$ or $N/O(N)$ respectively and it follows correspondingly from Lemma 3.20 (v) or (vi) that $\bar{H}=\bar{L}_1\bar{L}_2O(C_{\bar{H}}(\bar{Z}))\bar{S}$ or $\bar{H}=\bar{F}O(C_{\bar{H}}(\bar{Z}))\bar{T}$, where $\bar{L}_i\cong SL(2, q)$, $i=1, 2$, $[\bar{L}_1, \bar{L}_2]=1$, $\bar{L}_1\bar{L}_2\lhd\bar{H}$ or $\bar{F}\cong PSL(2, q)$ and $\bar{F}\lhd\bar{H}$. This result replaces the use of [11, Lemma 4.7 (i)].

Next we verify condition (c) of the corrected definition of relative p-stability: namely, that $H=JN_H(J\cap P)$ for any normal subgroup J of H. If $J\cap P$ is a Sylow p-subgroup of J, (c) is immediate by the Frattini argument. In particular, this holds if $J\subseteq O_{2',2}(H)$; so assume the contrary. Then our conditions imply that $\bar{J}$ contains a normal subgroup $\bar{L}\cong SL(2, q)*SL(2, q)$, $PSL(2, q)$, or $PS_p(4, r)$, r odd, with $C_{\bar{H}}(\bar{L})\subseteq\bar{L}$. In each case, we easily check that $\bar{H}=\bar{L}N_{\bar{H}}(\bar{Z})$, whence $\bar{H}=\bar{J}N_{\bar{H}}(\bar{Z})$. But $\bar{P}$ is a Sylow p-subgroup of $O(C_{\bar{H}}(\bar{Z}))$ by Lemma 4.3, so $\bar{P}\cap\bar{J}$ is one of $\bar{J}\cap O(C_{\bar{H}}(\bar{Z}))$. Since the latter group is $N_{\bar{H}}(\bar{Z})$-invariant, it follows, again by the Frattini argument, that $\bar{H}=\bar{J}N_{\bar{H}}(\bar{J}\cap\bar{P})$. As in the proof of the lemma of [18, Section 4], this equality suffices to yield (c).[#)]

Thus we are again reduced to showing that $AC_H(P_0)/C_H(P_0)\subseteq O_p(N_H(P_0)/C_H(P_0))$ for each nontrivial subgroup of P_0 of P for which $O_{p'}(H)P_0\lhd H$ and each subgroup A of P for which $[P_0, A, A]=1$. If the desired conclusion is false, then, as usual, $SL(2, p)$ must be involved in H. Since N does not involve $SL(2, p)$ by its structure, $\bar{H}$ is not isomorphic to $N/O(N)$. At this point, we reduce to the case

[#)] Using [11, Lemma 5.4] in place of Lemma 4.3, we similarly verify that condition (c) holds for the subgroups H and H_0 of [11, Lemma 5.6]. Hence those subgroups are p-stable with respect to the given subgroups P, in accordance with the corrected definition of this term, and so that lemma holds, as stated.

We take this opportunity to correct an error in [11, Lemma 5.6].

that z is isolated in H in the same way as in [11]. (This reduction is possible because the core of the centralizer of a *central* involution in $PS_p(4, r)$ is trivial; see [11, Lemma 5.6]). Hence z must be isolated in H and so $\bar{H}\cong M/O(M)$. Now we reach a contradiction exactly as in the corresponding argument of [11].

5. **Covering p-local subgroups.** Since G is a minimal counterexample to Theorem B, we have $O(M)\neq 1$, as noted in the preceding section. We let π be the set of primes dividing $|O(M)|$. If $p\in\pi$, we shall say that a p-local subgroup K_p of G is a *covering p-local* subgroup provided:

(a) K_p contains S, an S-invariant Sylow p-subgroup of $O(M)$, and a maximal T-invariant p-subgroup of G;

(b) K_p covers $M/O(M)$;

(c) $K_p/O(K_p)$ is fusion-simple.

Our goal in this section will be to prove that covering p-local subgroups exist for each p in π. We fix a prime p in π. We let P_1 be a maximal T-invariant p-subgroup of M and set $P_0=P_1\cap O(M)$. Then P_0 is a T-invariant Sylow p-subgroup of $O(M)$. Moreover, we can choose P_1 so that P_0 is S-invariant. We have already argued in the proof of Lemma 3.21 that $C_{P_0}(z_1)\neq 1$. We let Q_1 be a maximal T-invariant p-subgroup of N containing $C_{P_0}(z_1)$ and we set $Q_0=Q_1\cap O(N)$. Thus Q_0 is a T-invariant Sylow p-subgroup of $O(N)$. Furthermore, by Lemma 3.21, we have that also $Q_0\neq 1$. We fix the notation P_1, P_0, Q_1, Q_0.

We first prove

LEMMA 5.1. *One of the following holds:*

(i) *P_1 is maximal T-invariant p-subgroup of G; or*

(ii) *$N_G(P_0)$ is p-constrained and $O_{p'}(N_G(P_0))\subseteq O(N_G(P_0))$.*

PROOF: The proof is essentially identical to that of [11, Lemma 5.2]. The key preliminary result for that proof was [11, Lemma 5.4]. However, our present Lemma 4.3 (i) is the direct analogue of [11, Lemma 5.4] and so the argument goes through without change. We omit the details.

We consider the two possibilities of Lemma 5.1 separately.

LEMMA 5.2. *If P_1 is a maximal T-invariant p-subgroup of G, then $N_G(P_0)$ is a covering p-local subgroup of G.*

PROOF: Set $K=N_G(P_0)$, so that K is a p-local subgroup of G which covers $M/O(M)$ and contains both S and P_1. We need only show that z is not isolated in K, for then $K/O(K)$ will be fusion-simple by Proposition 4.2 and hence K will be a covering p-local subgroup.

Let Q_2 be a maximal T-invariant p-subgroup of $H=N_G(Q_0)$ containing Q_1 and let Q be a maximal T-invariant p-subgroup of G containing Q_2. Then $Q \supseteq Q_1 \supseteq C_{P_0}(z_1) \neq 1$ and so $P_1 \cap Q \neq 1$. It follows therefore from Proposition 4.4 that $Q = P_1^c$ for some c in $C_G(T)$. In particular, z centralizes Q. We shall argue now that z_2 also centralizes Q. By Thompson's $A \times B$-lemma, it will be enough to prove that z_2 centralizes Q_2 (cf. [8, Lemma 8.7]). But $Q_2 \subseteq O(H) C_H(Z)$ by Lemma 4.3 as Q_2 is T-invariant. Hence, in fact, it will suffice to show that z_2 centralizes $Q_3 = Q_2 \cap O(H)$. But Q_3 is a T-invariant Sylow p-subgroup of $O(H)$, so $N_H(Q_3)$ contains T and covers $H/O(H)$. However, H covers $N/O(N)$ as Q_0 is a Sylow p-subgroup of $O(N)$. Hence $N_H(Q_3)$ covers $N/O(N)$. By Lemmas 3.19 and 3.20 (viii), $\langle a_2, b_2 \rangle$ is contained in the normal closure of z in N. Since z centralizes Q_3 and $z_2 \in \langle a_2, b_2 \rangle$, we conclude that z_2 does as well. Thus z_2 centralizes Q, as asserted.

We therefore obtain that $Z = \langle z, z_2 \rangle$ centralizes Q and so also z_1 centralizes Q. Thus $Q \subseteq N$ and consequently $Q = Q_1$. Since $c \in C_G(T) \subseteq N$, we can assume on replacing Q_1 by a suitable conjugate that $P_1 = Q_1$. By Lemma 3.15 and the remark following it, R is a Sylow 2-subgroup of $O^2(M)$. It follows therefore from Lemma 3.20 (vii) that the normal closure of z_1 in M contains R. Since $N_M(P_0)$ contains S and covers $M/O(M)$, we see that R centralizes P_0. Since $S = \langle R, b_1 \rangle = \langle R, b_1 z \rangle$, this implies that $P_0 = [P_0, b_1 z] C_{P_0}(S)$. But by Lemma 3.19, $[P_0, b_1 z] \subseteq O(N)$, whence $[P_0, b_1 z] \subseteq Q_0$. We thus conclude that $P_0 \subseteq Q_0 C_{P_0}(S)$.

Finally set $E = C_{P_0}(S)$, $N_0 = N_N(Q_0)$, and $C_0 = C_N(Q_0)$. Then $C_0 \lhd N_0$, $E \subseteq N_0$, and N_0 covers $N/O(N)$. Set $\bar{N}_0 = N_0/O(N_0)$. Since z centralizes Q_0, the normal closure of $\bar{z}$ in $\bar{N}_0$ is contained in $\bar{C}_0$. It follows therefore from Lemmas 3.19 and 3.20 (viii) that $\bar{C}_0$ contains $\bar{F}$, where $\bar{F} \cong PSL(2, q)$, $\bar{F} \lhd \bar{N}_0$, and $\langle \bar{a}_2, \bar{b}_2 \rangle$ is a Sylow 2-subgroup of $\bar{F}$. Since $\bar{E}$ centralizes $\langle \bar{a}_2, \bar{b}_2 \rangle$ and acts on $\bar{F}$, we conclude now from standard properties of $P\Gamma L(2, q)$ that $\bar{F}_0 = C_{\bar{F}}(\bar{E}) \cong PSL(2, q_0)$ for some q_0 dividing q and that $\langle \bar{a}_2, \bar{b}_2 \rangle$ is a Sylow 2-subgroup of $\bar{F}_0$. In particular, it follows that $\bar{z}$ is not isolated in $\langle \bar{z}_1 \rangle \times \bar{F}_0$. On the other hand, it is not difficult to see that $C_1 = C_{C_0}(E)$ covers $\langle \bar{z}_1 \rangle \times \bar{F}_0$. Thus z is not isolated in C_1. But $C_1 \subseteq C_N(Q_0 E) \subseteq C_N(P_0)$ as $P_0 \subseteq Q_0 E$ and consequently $C_1 \subseteq N_G(P_0) = K$. Therefore z is not isolated in K and the lemma is proved.

We next prove

LEMMA 5.3. *If $N_G(P_0)$ is p-constrained with $O_{p'}(N_G(P_0)) \subseteq O(N_G(P_0))$, then also $N_G(Q_0)$ is p-constrained with $O_{p'}(N_G(Q_0)) \subseteq O(N_G(Q_0))$.*

PROOF: Again set $H = N_G(Q_0)$ and let Q be a maximal T-invariant p-subgroup of G such that $Q_2 = Q \cap H$ is a maximal T-invariant p-subgroup of H. Let $\tilde{P}_0$ be a T-invariant Sylow p-subgroup of $O_{p',p}(N_G(P_0))$. Then $P_0 \subseteq \tilde{P}_0$ and by our hypotheses on $N_G(P_0)$, no involution of T centralizes $\tilde{P}_0$. However, as $P_0 \cap Q \neq 1$ and

$P_0 \subseteq \tilde{P}_0$, we have that $\tilde{P}_0^c \subseteq Q$ for some c in $C_G(T)$ by Proposition 4.4. It follows therefore that no involution of T centralizes Q. But now we conclude from Thompson's $A \times B$ lemma that no involution of T centralizes Q_2. But $Q_2 = Q_3 C_{Q_2}(Z)$ by Lemma 4.3, where, as before, $Q_3 = Q_2 \cap O(H)$. Thus no involution of Z centralizes Q_3. Since $Z = Z(T)$, this in turn implies that no involution of T centralizes Q_3, whence $C_H(Q_3)$ is of odd order. But Q_3 is a Sylow p-subgroup of $O(H)$ and so $H = O(H) N_H(Q_3)$. Since $|C_H(Q_3)|$ is odd, this yields that $C_H(Q_3) \subseteq O(H)$. The lemma now follows by a standard argument.

Finally we have

LEMMA 5.4. *If* $N_G(P_0)$ *is p-constrained with* $O_{p'}(N_G(P_0)) \subseteq O(N_G(P_0))$, *then* $N_G(Z(J(P)))$ *is a covering p-local subgroup of G for some maximal T-invariant p-subgroup P of G.*

PROOF: The proof is entirely similar to the corresponding results established in [7], [8], [9] and [11]. Hence we shall limit ourselves to a sketch of the proof. First, setting $H = N_G(P_0)$, we let P^* be a TP_1-invariant Sylow p-subgroup of $O_{p',p}(O(H))$. Then $N_G(P^*)$ contains P_1 and covers $M/O(M)$.

We let K be a p-local subgroup of G such that

(a) K covers $M/O(M)$ and contains T;

(b) $O_p(K) \supseteq P^*$ and $K \supseteq P_1$;

(c) $C_T(O_p(K)) = 1$;

(d) Subject to (a), (b), (c), a maximal T-invariant p-subgroup of K has maximal order.

If P is a maximal T-invariant p-subgroup of K containing P_1, the argument of [11, Lemma 6.5] applies with no essential change and yields that P is a maximal T-invariant p-subgroup of G and that $N_G(Z(J(P)))$ covers $M/O(M)$. In carrying through the proof, we make use of Lemma 4.5 (for $M/O(M)$), which is the direct analogue of [11, Lemma 5.6]. Likewise in analyzing the structure of $J = N_G(Z(J(P)))$, we make use of Lemma 4.3, which is the analogue of [11, Lemma 5.4], to conclude that $[T, O_p(K)] \subseteq O(J)$.

Next we let $H^* = N_G(Q_0)$ and let Q^* be a TQ_1-invariant Sylow p-subgroup of $O_{p',p}(O(H^*))$. Then $N_G(Q^*)$ contains Q_1 and covers $N/O(N)$.

This time we let K^* be a p-local subgroup of G such that

(a) K^* covers $N/O(N)$;

(b) $O_p(K^*) \supseteq Q^*$ and $K \supseteq Q_1$;

(c) $C_T(O_p(K^*)) = 1$;

(d) Subject to (a), (b), (c) a maximal T-invariant p-subgroup of K^* has maximal order.

By the preceding lemma, H^* is p-constrained and $O_{p'}(H^*)\subseteq O(H^*)$. Hence if Q denotes a maximal T-invariant p-subgroup of K^* containing Q_1, the argument of [11, Lemma 6.5] yields similarly that $J^*=N_G(Z(J(Q)))$ covers $N/O(N)$. This time, we use Lemma 4.5 for $N/O(N)$ and again use Lemma 4.3.

Finally as $P_1\cap Q_1\supseteq C_{P_0}(z_1)\neq 1$, we have $P\cap Q\neq 1$ and consequently $Q^c=P$ for some c in $C_G(T)$ by Proposition 4.4. Thus $Z(J(Q))^c=Z(J(P))$ and consequently $(J^*)^c=J$. But as J^* covers $N/O(N)$ and contains T, z is not isolated in J^*. Since c centralizes z, it follows that z is not isolated in J. However, J covers $M/O(M)$ and contains T, so $J/O(J)$ is fusion-simple by Proposition 4.2. Moreover, J contains the T-invariant Sylow p-subgroup P_0 of $O(M)$.

Our conditions imply that a Sylow 2-subgroup $\tilde{S}$ of J containing T is a Sylow 2-subgroup of G. Then T is maximal in $\tilde{S}$ and so $\tilde{S}\subseteq N_G(T)\subseteq N_G(Z)$. By Lemma 3.6, $N_G(Z)$ has a normal 2-complement. Since $S\subseteq N_G(Z)$ and also $S\supset T$, we conclude that $\tilde{S}^x=S$ for some element x in $O(N_G(T))$ with x centralizing T.

Hence $J_1=N_G(Z(J(P^x)))$ contains S and $J_1/O(J_1)$ is also fusion-simple. Furthermore, as x centralizes T, $x\in M$ and so J_1 also covers $M/O(M)$. In addition, $P_0^x\subseteq J_1$ and P_0^x is a Sylow p-subgroup of $O(M)$. Since $S\subseteq J_1$, an S-invariant Sylow p-subgroup of $J_1\cap O(M)$ exists and is an S-invariant Sylow p-subgroup of $O(M)$. Hence J_1 is a covering p-local subgroup of G. Since P^x is also a maximal T-invariant p-subgroup of G, the lemma is proved.

Lemmas 5.1, 5.2, and 5.4 together yield as a corollary

PROPOSITION 5.5. *G possesses a covering p-local subgroup for each prime p in π.*

Because G has two conjugacy classes of involutions, we require one further property of covering p-local subgroups.

LEMMA 5.6. *A covering p-local subgroup of G also covers $N/O(N)$ and contains a T-invariant Sylow p-subgroup of $O(N)$.*

PROOF: Let $K=K_p$ be a covering p-local subgroup for p in π. Then $S\subseteq K\subseteq G$ and K covers $M/O(M)$. Thus Theorem B and hence also Theorem A holds for K, so $K/O(K)$ contains a normal subgroup of odd index isomorphic to $PS_p(4, q)$ for some odd q. In particular, K has the same involution fusion pattern as G and $N_K(A_1)/C_K(A_1)\cong N_G(A_1)/C_G(A_1)\cong A_5$ or S_5 according as $n=2$ or 3. Hence by Lemma 3.18 (v) and the remark following it, $\langle b_1, x\rangle^k=\langle z_1, b_1z\rangle$ for some k in K.

Clearly K covers $C_M(b_1)/O(C_M(b_1))=C_G(\langle b_1, z\rangle)/O(C_M(b_1))$. It follows therefore from the preceding equality that K covers $C_G(\langle z_1, b_1z\rangle)/O(C_G(\langle z_1, b_1z\rangle))=C_N(b_1z)/O(C_N(b_1z))$. However, by Lemma 3.19 (ii), $O(C_N(b_1z))\subseteq O(N)$ and $N=C_N(b_1z)TO(N)$. Since $T\subseteq S\subseteq K$, we conclude that K covers $N/O(N)$.

If P is a maximal T-invariant p-subgroup of K containing a Sylow p-subgroup P_0 of $O(M)$, we have shown above that $C_{P_0}(z_1)=P_0\cap O(N)\neq 1$. It follows therefore from Proposition 4.4 that P contains a maximal T-invariant p-subgroup of N and hence a T-invariant Sylow p-subgroup of $O(N)$, as asserted.

6. Proof of Theorem B. In this section we shall derive a contradiction from the fact that $O(M)\neq 1$. This will show that no counterexample to Theorem B exists and will thus establish the theorem.

We first treat the case that q is either a Fermat or Mersenne prime or 9.

PROPOSITION 6.1. *If the characteristic power q of G is a Fermat or Mersenne prime or* 9, *then G is balanced.*

PROOF: We proceed essentially as in [11, Proposition 7.1]. Let x, y be two commuting involutions of G. If $F=O(C_G(x))\cap C_G(y)$, we need only prove that a Sylow p-subgroup of F is contained in $O(C_G(y))$ for each prime p dividing $|F|$. Without loss we can assume that $x=z$ or z_1 and that $y\in S$. By Proposition 5.5, G possesses a covering p-local subgroup $K=K_p$. Setting $\bar{K}=K/O(K)$, we have that $\bar{K}$ is fusion-simple. Since K covers $M/O(M)$ and contains S, we see that the characteristic power of $\bar{K}$ is also q. By our minimal choice of G, $\bar{K}$ satisfies the conclusion of Theorem B. Since q is a Fermat or Mersenne prime or 9, it follows therefore from Theorem 2.5 that $\bar{K}\cong PS_p(4, q)$. This in turn implies that $O(C_{\bar{K}}(\bar{x}))=1$.

By definition of a covering p-local subgroup and Lemma 5.6, K contains an S-invariant Sylow p-subgroup of $O(M)$ and a T-invariant Sylow p-subgroup of $O(N)$. According as $x=z$ or z_1, let P_0 be such a subgroup of $O(M)$ or $O(N)$. Note that $y\in T$ if $x=z_1$ as $y\in C_S(x)$, so y leaves P_0 invariant in either case. Then $F_0=C_{P_0}(y)$ is clearly a Sylow p-subgroup of F.

Since $O(C_{\bar{K}}(\bar{x}))=1$, it follows that $F_0\subseteq O(K)$. On the other hand, if we set $H=C_G(y)$, we see by the structure of $\bar{K}$ (and the fact that q is a Fermat or Mersenne prime or 9) that $H/O(H)\cong C_{\bar{K}}(\bar{y})$. Hence $C_K(y)$ covers $H/O(H)$, whence $H=O(H)(K\cap H)$. Since $F_0\subseteq O(K)\cap H$, this implies that $F_0\subseteq O(H)=O(C_G(y))$, as required.

As a consequence, we have

PROPOSITION 6.2. *The characteristic power q of G is not a Fermat or Mersenne prime or* 9. *In particular, $q>9$.*

PROOF: Assume the contrary, in which case G is balanced by the preceding proposition. Moreover, the centralizer of every involution of G is 2-generated by Lemma 3.22. Since S is connected, as noted at the beginning of Section 3, and since $O(G)=1$, we conclude therefore from Theorem 2.1 that $O(C_G(z))=O(M)=1$, which is a contradiction.

We next prove

PROPOSITION 6.3. *G is 2-balanced.*

PROOF: Let X be a four subgroup of G and y an involution of G which centralizes X. We must show that $F=\Delta_G(X)\cap C_G(y)\subseteq O(C_G(y))$. Again we need only prove that a Sylow p-subgroup of F is contained in $O(C_G(y))$ for each prime p dividing $|F|$. We can suppose that $\langle X, y\rangle\subseteq S$ and that z or z_1 is contained in X. Again we consider a covering p-local subgroup $K=K_p$ of G and set $\bar{K}=K/O(K)$. We conclude now as in the preceding proposition, with the aid of Theorem 2.5, that $\bar{K}$ possesses a normal subgroup $\bar{L}$ of odd index isomorphic to $PS_p(4, q)$. Since $O(\bar{K})=1$, we have that $C_{\bar{K}}(\bar{L})=1$. But now the discussion in the introduction yields the important conclusion that $\Delta_{\bar{K}}(\bar{X})=1$.

If $z\in X$, let P_0 be an S-invariant Sylow p-subgroup of $O(M)$ contained in K, while in the contrary case, let P_0 be a T-invariant Sylow p-subgroup of $O(N)$ contained in K. In the latter case, $z_1\in X$ and so $\langle X, y\rangle\subseteq C_S(x_1)=T$. Thus $\langle X, y\rangle$ leaves P_0 invariant in either case. Next let F_0 be an $\langle X, y\rangle$-invariant Sylow p-subgroup of F. Then $F_0\subseteq O(C_G(x))$ for each x in $X^\#$ and consequently $F_0\subseteq O(M)$ or $O(N)$ according as $P_0\subseteq O(M)$ or $O(N)$. Since P_0 is correspondingly an $\langle X, y\rangle$-invariant Sylow p-subgroup of $O(M)$ or $O(N)$, we see that $F_0^c\subseteq P_0$ for some c in $C_G(\langle X, y\rangle)$. Since $C_G(X)$ leaves $\Delta_G(X)$ invariant, $C_G(\langle X, y\rangle)$ leaves F invariant and so F_0^c is also an $\langle X, y\rangle$-invariant Sylow p-subgroup of F. Hence without loss we can suppose to begin with that $F_0\subseteq P_0$.

Clearly $O(C_G(x))\cap K\subseteq O(C_K(x))$ for x in $X^\#$ and consequently $F_0\subseteq\Delta_K(X)$. But clearly $\Delta_K(X)$ maps into $\Delta_{\bar{K}}(\bar{X})$. Since the latter group is trivial, we conclude that $F_0\subseteq O(K)$. Again we set $H=C_G(y)$. Since K has the same involution fusion pattern as G, $y\sim z$ or z_1 in K. But K covers $M/O(M)$ by definition of a covering p-local subgroup and covers $N/O(N)$ by Lemma 5.6. Hence in either case, it follows that K also covers $H/O(H)$. Thus $H=O(H)(K\cap H)$ and we obtain the desired conclusion $F_0\subseteq O(H)$.

The preceding arguments yield a further conclusion:

LEMMA 6.4. *We have* $\Delta_G(Z)\neq 1$.

PROOF: Choose p in π, let K be a covering p-local subgroup of G, and let P_0 be a T-invariant Sylow p-subgroup of $O(M)$ contained in K. Setting $\bar{K}=K/O(K)$, it follows once again from the structure of $\bar{K}$ that $O(C_{\bar{K}}(\bar{z}))=1$, which implies that $P_0\subseteq O(K)$. Furthermore, the argument at the beginning of Section 5 shows that $F_0=C_{P_0}(z_1)=C_{P_0}(Z)\neq 1$. But now we conclude exactly as in the final paragraph of the preceding proposition, with $y=z_1$ or z_2, that $F_0\subseteq O(C_G(z_i))$, $i=1, 2$. Hence

$F_0 \subseteq \bigcap_{z' \in Z^\#} O(C_G(z')) = \Delta_G(Z)$. Since $F_0 \neq 1$, the lemma follows.

We next prove

PROPOSITION 6.5. *If we set $W_1 = \langle \Delta_G(T_1) \mid T_1 \in \mathscr{E}_2(A_1) \rangle$, then W_1 is a nontrivial subgroup of G of odd order.*

PROOF: Since G is 2-balanced and A_1 is of rank 4, W_1 is of odd order by Theorem 2.3. Since $Z \in \mathscr{E}_2(A_1)$, $\Delta_G(Z) \subseteq W_1$ and so $W_1 \neq 1$ by the preceding lemma.

Finally we prove

PROPOSITION 6.6. *$N_G(W_1)$ is strongly embedded in G.*

PROOF: Set $H = N_G(W_1)$. By Theorem 2.4, $N_G(B) \subseteq H$ for any subgroup B of A_1 of order at least 8. In particular, $N_G(A_1) \subseteq H$. Moreover, as $q > 9$, we have by Lemma 3.22

$$N = \langle C_N(B) \mid B \in \mathscr{E}_3(A_1) \rangle T$$

and consequently $O^2(N) \subseteq H$. In particular, $\langle a_2, b_2 \rangle \subseteq K = O^2(H)$. Since $u \in N_G(A_1) \subseteq H$ and $\langle a_2, b_2 \rangle^u = \langle a_1, b_1 \rangle$, it follows that $\langle a_1, b_1 \rangle \subseteq K$, whence $D = \langle a_1, b_1 \rangle \times \langle a_2, b_2 \rangle \subseteq K$. If $n = 2$, then $O^2(N_G(A_1)) = N_G(A_1)$ by Lemma 3.3, so $u \in O^2(N_G(A_1))$. On the other hand, if $n \geqq 3$, we reach the same conclusion by Lemma 3.18 (iv). Hence in either case, $\langle D, u \rangle \subseteq K$. Since $N_G(A_1)/C_G(A_1) \cong A_5$ or S_5, $O^2(N_G(A_1))$ contains no isolated involution and hence neither does K. Thus $\bar{K} = K/O(K)$ is fusion-simple. We conclude therefore from Proposition 4.1 that $\langle D, u \rangle$ is not a Sylow 2-subgroup of K. Hence K must contain a Sylow 2-subgroup of G, which without loss we can take to be S. Moreover, we have $K = H$ and $N = O^2(N) T \subseteq H$.

We conclude now, as usual, from our minimal choice of G that $\bar{H}$ possesses a normal subgroup $\bar{L}$ of odd index isomorphic to $PS_p(4, r)$ for some odd r with $C_{\bar{H}}(\bar{L}) = 1$. Since $N \subseteq H$, we must have $r = q$, whence $C_{\bar{L}}(\bar{z})' \cong (O^2(M/O(M)))'$. Hence $H \cap M$ covers $(O^2(M/O(M)))'$ and so, by Lemma 3.20 (v), we have

$$M = O(M)(H \cap M) C_M(R) S\,.$$

But $C_M(R) \subseteq H$ as $A_1 \cap R$ is of order 8. Likewise

$$O(M) = \langle C_{O(M)}(B) \mid B \in \mathscr{E}_3(A) \rangle \subseteq H\,.$$

We conclude that $M \subseteq H$.

Since z and z_1 are clearly representatives of the two conjugacy classes of involutions in H, it follows now that H contains the centralizer in G of each of its involutions. Furthermore, it follows from Lemma 3.5 (xiii) that $N_G(S) = SC_G(S)$, so also $N_G(S) \subseteq H$. But $H = N_G(W_1)$ is a proper subgroup of G as $O(G) = 1$ and W_1 is a nontrivial subgroup of odd order. We conclude therefore from the definition that H is strongly embedded in G.

The proposition yields a final contradiction at once. For as H is strongly embedded in G, it has only one conjugacy class of involutions by [5, Theorem 9.2.1]. But then z is conjugate to z_1 in H, contrary to the fact that they are not conjugate in G. This completes the proof of Theorem B and hence also of Theorem A.

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