A note on geometrically reductive groups

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Introduction.

Let k be a field. An affine algebraic k-group G is said *geometrically reductive* if for any surjective homomorphism of k-G-modules $V \rightarrow k \rightarrow 0$, where k is a trivial 1-dimensional k-G-module, there is an integer $r>0$ such that the induced homomorphism $S^r V \rightarrow S^r k \rightarrow 0$ splits, where $S^r V$ denotes the r^{th} symmetric power of V. The following are examples of geometrically reductive k-groups:

(1) $GL(2)$ (Seshadri [8]).

(2) Finite groups (2.2). (An affine group G is *finite* if and only if its Hopf algebra $O(G)$ is finite dimensional.)

(3) Linearly reductive groups. (An affine group G is *linearly reductive* if and only if any k-G-module is a direct sum of irreducible k-G-modules.)

(4) An affine algebraic k-group which is isogeneous to some geometrically reductive k-group (2.4).

(5) The A-subgroups of a geometrically reductive k-group (Theorem 1.7). (A closed subgroup H of an affine algebraic k-group G is called an *A-subgroup* if the homogeneous space G/H is affine.) (cf. Bialynicki-Birula [1] and Popov [6])

Suppose that k is algebraically closed. An affine algebraic k-group G is called *reductive* if it is reduced and its unipotent radical is $\{e\}$. Mumford [4] conjectures that reductive k-groups are geometrically reductive. Nagata and Miyata [5] prove that reduced geometrically reductive k-groups are reductive. In view of the properties (4) and (5) above, the Mumford conjecture will follow from the following two conjectures:

C_1. $GL(n)$ are geometrically reductive.

C_2. Any almost simple affine algebraic k-group is isogeneous to an A-subgroup of $GL(n)$ for some $n>0$.

In this short note we shall *not* treat the conjecture C_1 but solve partially the conjecture C_2. As is well-known the almost simple k-groups are characterized up to isogeny by their Dynkin diagrams. We shall prove that the almost simple k-groups of types A_l, B_l, C_l and D_l satisfy the condition C_2. The case of groups of exceptional types is still open.

§1. The A-subgroups of a geometrically reductive k-group

In the following k denotes a fixed ground field.

1.1 DEFINITION. Let C be a k-coalgebra, A a k-algebra and W a left C-comodule and a right A-module. We say that W is a *(C, A)-bimodule* if one of the following equivalent conditions holds:

(i) The module structure map $W\otimes A \to W$, $w\otimes a \mapsto wa$ is a left C-comodule map.

(ii) The comodule structure map $\rho: W \to C\otimes W$ is a right A-module map.

1.2 DEFINITION. Let C be a k-coalgebra. Let V be a right C-comodule and W a left C-comodule. Let $\rho_V: V \to V\otimes C$ and $\rho_W: W \to C\otimes W$ denote the comodule structure maps. The kernel of the diagram

$$V\otimes W \underset{1_V\otimes\rho_W}{\overset{\rho_V\otimes 1_W}{\rightrightarrows}} V\otimes C\otimes W$$

is denoted $V\square_C W$ and called the *co-tensor* product of V and W.

1.3 Let C be a k-coalgebra and A a k-algebra. Let V be a right C-comodule and W a (C, A)-bimodule. Since

$$(V\square_C W)\otimes A \cong V\square_C(W\otimes A)$$

clearly, we can well-define a map

$$(V\square_C W)\otimes A \cong V\square_C(W\otimes A) \xrightarrow{V\square_C\omega} V\square_C W$$

where $\omega: W\otimes A \to W$, $w\otimes a \to wa$, which makes $V\square_C W$ into a right A-module as is easily checked.

Let U be a left A-module. Then the composite

$$W\otimes_A U \xrightarrow{\rho_W\otimes 1} (C\otimes W)\otimes_A U \cong C\otimes(W\otimes_A U)$$

where $\rho_W: W \to C\otimes W$ denotes the comodule structure map, makes $W\otimes_A U$ into a left C-comodule.

PROPOSITION. *With the notations as above we have*

$$(V\square_C W)\otimes_A U \cong V\square_C(W\otimes_A U)$$

if U is a flat *A-module.*

PROOF. Apply the exact functor $?\otimes_A U$ to the following exact sequence:

$$V\square_C W \longrightarrow V\otimes W \underset{1_V\otimes\rho_W}{\overset{\rho_V\otimes 1_W}{\rightrightarrows}} V\otimes C\otimes W\,.$$

1.4 Let C and D be two k-coalgebras. Let W be a left C-comodule and a right D-comodule. We say that W is a (C, D)-*bicomodule* if one of the following equivalent conditions holds:

(i) The left C-comodule structure map $\rho_l: W\to C\otimes W$ is a right D-comodule map.

(ii) The right D-comodule structure map $\rho_r: W\to W\otimes D$ is a left C-comodule map.

If W is a (C, D)-bicomodule and V a right C-comodule, then the composite

$$V\square_C W \xrightarrow{1\square_C\rho_r} V\square_C(W\otimes D)\cong(V\square_C W)\otimes D$$

makes $V\square_C W$ into a right D-comodule.

Let $\pi: D\to C$ be a coalgebra map. Then D becomes a (C, D)-bicomodule via

$$\rho_r=\Delta: D\to D\otimes D \qquad \text{and}$$

$$\rho_l: D\xrightarrow{\Delta} D\otimes D\xrightarrow{\pi\otimes 1} C\otimes D$$

where Δ is the comultiplication of D. Hence $V\square_C D$ is a right D-comodule for any right C-comodule V. On the other hand the exact sequence

$$V\xrightarrow{\rho_V} V\otimes C \underset{1_V\otimes\Delta}{\overset{\rho_V\otimes 1_C}{\rightrightarrows}} V\otimes C\otimes C$$

implies an isomorphism of right C-comodules

$$V\xrightarrow{\cong} V\square_C C\,.$$

The composite

$$V\square_C C\xrightarrow{1\square_C\pi} V\square_C C\cong V$$

is a right C-comodule map and satisfies the following UMP: For any right D-comodule U we have

$$\mathrm{Comod}_D(U, V\square_C D)\xrightarrow{\cong}\mathrm{Comod}_C(U, V)\,.$$

This can be easily verified.

1.5 Let G be an affine algebraic k-group. We do *not* assume that G is reduced. Let $O(G)$ denote the Hopf algebra of G. By definition a (left) k-G-module means a *right* $O(G)$-comodule. We view always k as a trivial left and right $O(G)$-comodule. If V is a k-G-module with $\rho_V: V\to V\otimes O(G)$ the $O(G)$-comodule structure map, then V^G the space of G-invariants in V is by definition

$$V\square_{O(G)}k=\{v\in V\mid \rho_V(v)=v\otimes 1\}\ .$$

Let H be a closed subgroup of G. Suppose that the quotient k-sheaf $G\tilde{/}H$ (see [3, III, §3] for the definition) is affine. (Such a subgroup is called an *A-subgroup* of G.) Let $\pi: O(G)\to O(H)$ be the canonical projection of Hopf algebras. By (1.4) $O(G)$ is naturally an $(O(H), O(G))$-bicomodule. Since $G\tilde{/}H \xrightarrow{\cong} H\tilde{\backslash}G$, $g\mapsto g^{-1}$, $H\tilde{\backslash}G$ is also affine and its k-algebra $O(H\tilde{\backslash}G)$ clearly equals

$$k\square_{O(H)}O(G)=\{x\in O(G)\mid \rho_l(x)=1\otimes x\}$$

where $\rho_l: O(G)\to O(H)\otimes O(G)$ denotes the left $O(H)$-comodule structure on $O(G)$. Notice that $O(G)$ is a faithfully flat $O(H\tilde{\backslash}G)$-algebra [3, III, §3, 2.5]. If we view $O(G)$ as a right $O(H\tilde{\backslash}G)$-module, then $O(G)$ becomes an $(O(H), O(H\tilde{\backslash}G))$-bimodule in the sense of (1.1). Observe the following isomorphism of k-schemes:

$$H\times G \xrightarrow{\cong} G\times_{H\tilde{\backslash}G}G,\ (h, g)\mapsto hg, g)\ .$$

If we let H act on $H\times G$ and $G\times_{H\tilde{\backslash}G}G$ from the left via

$$H\times(H\times G)\to H\times G,\ (h, (h', g))\mapsto(hh', g)\quad\text{and}$$
$$H\times(G\times_{H\tilde{\backslash}G}G)\to G\times_{H\tilde{\backslash}G}G,\ (h, (g, g'))\mapsto(hg, g')\ ,$$

then the above isomorphism commutes with the H-action. This means that the representing isomorphism of k-algebras

$$O(H)\otimes O(G) \xleftarrow{\cong} O(G)\otimes_{O(H\tilde{\backslash}G)}O(G)$$

is a left $O(H)$-comodule isomorphism, where the both hand sides are left $O(H)$-comodules via

$$\Delta\otimes 1: O(H)\otimes O(G)\to O(H)\otimes(O(H)\otimes O(G))\quad\text{and}$$
$$\rho_l\otimes 1: O(G)\otimes_{O(H\tilde{\backslash}G)}O(G)\to O(H)\otimes(O(G)\otimes_{O(H\tilde{\backslash}G)}O(G))\ .$$

PROPOSITION. *The left $O(H)$-comodule $O(G)$ is* coflat, *that is the functor $?\square_{O(H)}O(G)$ is exact.*

PROOF. Let V be a right $O(H)$-comodule. It follows from Proposition 1.3 that

$$\begin{aligned}(V\square_{O(H)}O(G))\otimes_{O(H\tilde{\backslash}G)}O(G)&\cong V\square_{O(H)}(O(G)\otimes_{O(H\tilde{\backslash}G)}O(G))\\ &\cong V\square_{O(H)}(O(H)\otimes O(G))\cong(V\square_{O(H)}O(H))\otimes O(G)\\ &\cong V\otimes O(G)\ .\end{aligned}$$

Since $O(G)$ is a faithfully flat $O(H\tilde{\backslash}G)$-algebra, this means that the functor $?\square_{O(H)}O(G)$ is exact.

1.6 COROLLARY. *Let G be an affine algebraic k-group and H an A-subgroup of G. Then for any right $O(H)$-comodule V, the canonical homomorphism*

$$V\square_{O(H)}O(G)\to V\square_{O(H)}O(H)\cong V$$

is surjective.

PROOF. Let $O(H\tilde{\backslash}G)$ act on $O(H)$ from the right trivially. Then $O(H)$ becomes an $(O(H), O(H\tilde{\backslash}G))$-bimodule in the sense of (1.1) and the canonical projection $\pi: O(G)\to O(H)$ is easily seen to be a homomorphism of $(O(H), O(H\tilde{\backslash}G))$-bimodules. Since $O(G)$ is a faithfully flat $O(H\tilde{\backslash}G)$-algebra, it is enough to show that the induced homomorphism

$$(V\square_{O(H)}O(G))\otimes_{O(H\tilde{\backslash}G)}O(G)\to(V\square_{O(H)}O(H))\otimes_{O(H\tilde{\backslash}G)}O(G)\ ,$$

is surjective. But since we have a commutative diagram

$$\begin{array}{ccc} H\times G & \cong & G\times_{H\tilde{\backslash}G}G \\ \cup & & \cup \\ H\times H & \cong & H\times_{H\tilde{\backslash}G}G=H\times H\ , \end{array}$$

it follows that

$$\begin{array}{ccc} O(H)\otimes O(G) & \cong & O(G)\otimes_{O(H\tilde{\backslash}G)}O(G) \\ {\scriptstyle 1\otimes\pi}\downarrow & & \downarrow{\scriptstyle \pi\otimes 1} \\ O(H)\otimes O(H) & \cong & O(H)\otimes_{O(H\tilde{\backslash}G)}O(G)\ . \end{array}$$

Hence we have

$$\begin{array}{ccc} (V\square_{O(H)}O(H))\otimes O(G) & \cong & (V\square_{O(H)}O(G))\otimes_{O(H\tilde{\backslash}G)}O(G) \\ {\scriptstyle 1\otimes\pi}\downarrow & & \downarrow{\scriptstyle (1\square\pi)\otimes 1} \\ (V\square_{O(H)}O(H))\otimes O(H) & \cong & (V\square_{O(H)}O(H))\otimes_{O(H\tilde{\backslash}G)}O(G)\ . \end{array}$$

Since $1\otimes\pi$ is surjective, the proof is done.

1.7 The definition of geometrically reductive k-groups is given in Introduction.

THEOREM. *The A-subgroups of an affine algebraic geometrically reductive k-group are geometrically reductive.*

FROOF. Let G be an affine algebraic geometrically reductive k-group and H

an A-subgroup of G. Let $\phi: V \to k \to 0$ be a surjective homomorphism of k-H-modules. Then we have a commutative diagram

$$\begin{array}{ccc} V\square_{O(H)}O(G) & \xrightarrow{\phi\square 1} & k\square_{O(H)}O(G) \\ \downarrow{\scriptstyle p_V} & & \downarrow{\scriptstyle p_k} \\ V & \xrightarrow{\phi} & k \end{array}$$

where p_V and p_k denote the canonical projections. By Proposition 1.5 and Corollary 1.6, the diagram consists of surjective homomorphisms. The map $\phi\square 1$: $V\square_{O(H)}O(G) \to k\square_{O(H)}O(G)$ is a k-G-module map. Notice that

$$k\square_{O(H)}O(G) \cong O(H\tilde{\backslash}G) \qquad \text{(as right } O(G)\text{-comodules)}.$$

It is easy to see that the composite

$$O(H\tilde{\backslash}G) \cong k\square_{O(H)}O(G) \xrightarrow{p_k} k$$

coincides with the restriction of the structure map

$$\varepsilon: O(G) \to k\ .$$

Since the element $1 \in O(H\tilde{\backslash}G)$ is G-invariant and $\varepsilon(1)=1$, it follows that there is a G-invariant element e of $k\square_{O(H)}O(G)$ such that $p_k(e)=1$. Put

$$\bar{V} = (\phi\square 1)^{-1}(ke)\ .$$

Then $\bar{V}$ is a sub-k-G-module of $V\square_{O(H)}O(G)$ and we have a commutative diagram

$$\begin{array}{ccccc} \bar{V} & \xrightarrow{\phi\square 1} & ke & \longrightarrow & 0 \\ \downarrow{\scriptstyle p_V} & & \wr\downarrow{\scriptstyle p_k} & & \\ V & \xrightarrow{\phi} & k & \longrightarrow & 0\,. \end{array}$$

Since G is geometrically reductive, there exists an integer $r>0$ such that the induced homomorphism of k-G-modules

$$S^r\bar{V} \xrightarrow{S^r(\phi\square 1)} S^r(ke) \longrightarrow 0$$

splits. This implies immediately that the induced homomorphism of k-H-modules

$$S^rV \xrightarrow{S^r\phi} S^rk \longrightarrow 0$$

splits. Therefore H is geometrically reductive.

§2. A program for solving the Mumford conjecture

In the following k is a fixed ground field.

2.1 PROPOSITION. *Let G be an affine algebraic k-group and N a closed* normal *subgroup of G. Then G is geometrically reductive if and only if N and $G\tilde{/}N$ are both geometrically reductive.*

PROOF. Suppose that G is geometrically reductive. Then N is geometrically reductive, since it is an A-subgroup of G. On the other hand $G\tilde{/}N$ is clearly geometrically reductive. Conversely suppose that N and $G\tilde{/}N$ are geometrically reductive. Let $V\to k\to 0$ be a surjective homomorphism of k-G-modules. There is an integer $r>0$ such that the induced homomorphism $S^rV\to S^rk\to 0$ splits as a k-N-homomorphism. Let $\bar{V}$ be the N-invariant elements in S^rV. Then the induced homomorphism $\bar{V}\to S^rk\to 0$ can be seen as a k-$G\tilde{/}N$-homomorphism. There exists an integer $s>0$ such that the induced homomorphism $S^s\bar{V}\to S^s(S^rk)\to 0$ splits. This means that the homomorphism of k-G-modules $S^{rs}V\to S^{rs}k\to 0$ splits. Hence G is geometrically reductive.

2.2 An affine algebraic k-group G is said to be *finite* if the Hopf algebra $O(G)$ is finite dimensional.

PROPOSITION. *Finite k-groups are geometrically reductive.*

PROOF. Let $K|k$ be a field extension. Then if $G\otimes K$ is a geometrically reductive K-group, then the k-group G is clearly a geometrically reductive k-group. Hence we can assume that k is algebraically closed. Let G be a finite k-group. Let G^0 denote the connected component of G at 1. We have only to prove that G^0 and $G\tilde{/}G^0$ are geometrically reductive. Since G^0 is infinitesimal, it is geometrically reductive by Lemma 2.3. Since k is algebraically closed, $G\tilde{/}G^0$ is a constant k-group scheme Γ_k for some finite (abstract) group Γ. Since k-Γ_k-modules are the same as k-Γ-modules, it follows easily that Γ_k is geometrically reductive. Hence G is geometrically reductive.

2.3 An affine algebraic k-group G is said *infinitesimal* if the algebra $O(G)$ is finite dimensional and local.

LEMMA. *Infinitesimal k-groups are geometrically reductive.*

PROOF. We can assume that $\mathrm{char}(k)=p>0$. Let G be infinitesimal and put $I=\mathrm{Ker}(\varepsilon: O(G)\to k)$, where ε is the structure map. Then there is an integer $r>0$ such that $x^{p^r}=0$ for all $x\in I$. Let V be a k-G-module with $\rho: V\to V\otimes O(G)$ the $O(G)$-comodule structure map. Then the element v^{p^r} of $S^{p^r}V$ is G-invariant for any $v\in V$, since

$$\rho(v^{p^r})=\rho(v)^{p^r}\in S^{p^r}V\otimes k\ .$$

Hence if $V\to k\to 0$ is a surjective homomorphism of k-G-modules, then the induced homomorphism $S^{p^r}V \to S^{p^r}k \to 0$ splits. Therefore G is geometrically reductive.

2.4 Two affine algebraic k-groups G and G' are said to be *isogeneous* if there are an affine algebraic k-group G'' and two *finite* closed normal subgroups N and N' of G'' such that $G \cong G''\tilde{/}N$ and $G' \cong G''\tilde{/}N'$.

COROLLARY. *An affine algebraic k-group which is isogeneous to some geometrically reductive k-group is geometrically reductive.*

2.5 An affine k-group G is said *unipotent* if the Hopf algebra $O(G)$ is irreducible.

PROPOSITION. *A unipotent algebraic k-group G is geometrically reductive if and only if it is finite.*

PROOF. Enough to prove the "only if" part. Suppose that a unipotent algebraic k-group G is geometrically reductive. G has a central series of closed subgroups

$$G = G_0 \supset G_1 \supset \cdots \supset G_N = \{e\}$$

such that each quotient $G_{i-1}\tilde{/}G_i$ is isomorphic to a subgroup of $\boldsymbol{\alpha}_k$ the additive k-group. Since G is geometrically reductive, it follows that $G_{i-1}\tilde{/}G_i$ are all geometrically reductive. But it is easy to show that $\boldsymbol{\alpha}_k$ is *not* geometrically reductive. Since any proper subgroup of $\boldsymbol{\alpha}_k$ is finite [3, IV, §2, 1.1], $G_{i-1}\tilde{/}G_i$ are all finite. Hence G is finite.

2.6 COROLLARY. *If k is algebraically closed, then any reduced geometrically reductive k-group has no unipotent radical.*

2.7 Recall that the symplectic k-group $Sp(2n)$ and the orthogonal k-group $O(n)$ are defined as follows: Let R be a k-model (, that is a small commutative k-algebra). Let $\phi_R^{(n)}$ and $q_R^{(n)}$ be an alternate bilinear form and a quadratic form on the R-modules R^{2n} and R^n respectively defined by

$$\phi_R^{(n)}(x, y) = \Sigma_{i=1}^n x_i y_{i+n} - \Sigma_{i=1}^n x_{i+n} y_i, \quad x \text{ and } y \in R^{2n}$$

$$q_R^{(n)}(z) = \begin{cases} \Sigma_{i=1}^{\nu} z_i z_{i+\nu} & \text{if } n = 2\nu \\ \Sigma_{i=1}^{\nu} z_i z_{i+\nu} + z_n^2 & \text{if } n = 2\nu+1, \ z \in R^n . \end{cases}$$

Then $Sp(2n)(R)$ and $O(n)(R)$ are the groups of R-automorphisms of R^{2n} and R^n which leave fixed the alternate bilinear form $\phi_R^{(n)}$ and the quadratic form $q_R^{(n)}$ respectively.

PROPOSITION. *$Sp(2n)$ and $O(n)$ are A-subgroups of $GL(2n)$ and $GL(n)$ respectively.*

PROOF. For any k-model R, let $X(R)$ and $Y(R)$ denote the set of all alternate bilinear forms on R^{2n} and the set of all quadratic forms on R^n respectively. Then the correspondences $X: R\mapsto X(R)$ and $Y: R\mapsto Y(R)$ are k-functors (cf. [2, page 57, Prop. 3]). We claim that X and Y are affine algebraic k-smooth k-schemes. Let $\{e_1, \cdots, e_n\}$ be the canonical basis of k^n. It follows from [2, page 55, Prop. 2] that the map

$$Y(R)\to R^n\times R^{\frac{n(n-1)}{2}}$$
$$Q\mapsto ((Q(e_i))_i, (Q(e_i+e_j)-Q(e_i)-Q(e_j))_{i<j})$$

is bijective. Hence $Y\cong(k^{\frac{n(n+1)}{2}})_a$ is affine algebraic k-smooth. On the other hand $X(R)$ is clearly isomorphic to $R^{\frac{n(n-1)}{2}}$, naturally in R. Hence $X\cong(k^{\frac{n(n-1)}{2}})_a$ is also affine algebraic k-smooth. For any k-model R, let $X'(R)$ and $Y'(R)$ denote the set of non-degenerate alternate bilinear forms on R^{2n} and the set of non-degenerate quadratic forms on R^n respectively. Then the k-functors $X': R\mapsto X'(R)$ and $Y': R\mapsto Y'(R)$ are affine algebraic k-smooth k-schemes, since they are principal open subfunctors of X and Y respectively. Now the k-groups $GL(2n)$ and $GL(n)$ acts naturally on X' and Y' respectively and the k-groups $Sp(2n)$ and $O(n)$ are the stabilizer groups of the elements $\phi_k^{(n)}\in X'(k)$ and $q_k^{(n)}\in Y'(k)$ respectively. Let K be a k-model which is an algebraically closed field. Then the action of $GL(2n)(K)$ on $X'(K)$ and of $GL(n)(K)$ on $Y'(K)$ are transitive by [2, page 80, Cor. 1] and by [2, page 70, Cor. 2] respectively. Since X' and Y' are algebraic k-smooth, it follows from [3, III, §3, 2.1] that we have

$$GL(2n)\tilde{/}Sp(2n)\xrightarrow{\cong}X' \quad\text{and}\quad GL(n)\tilde{/}O(n)\xrightarrow{\cong}Y'.$$

Since X' and Y' are affine, the assertion follows.

2.8 Suppose that k is algebraically closed. It is well-known [7, page 50] that the k-groups $SL(n+1)$, $SO(2n+1)$, $Sp(2n)$ and $SO(2n)$ are almost simple k-groups of types A_n, B_n, C_n and D_n respectively. Since these are A-subgroups of $GL(n+1)$, $GL(2n+1)$, $GL(2n)$ and $GL(2n)$ respectively, we have proved

THEOREM. *Any almost simple k-group of type A_l, B_l, C_l or D_l is isogeneous to some A-subgroup of $GL(n)$ for some $n>0$.*

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