Deformation of Milnor fiberings

By Mutsuo OKA

§1. Introduction

Let $f(z)$ be a polynomial in C^{n+1} and assume that $f(z)$ has an isolated critical point at the origin and $f(O)=0$. There is a Milnor fibering

$$\varphi = \arg(f) : S_\varepsilon^{2n+1} - K_\varepsilon \to S^1 ,$$

where S_ε^{2n+1} is a sphere of radius ε centered at the origin and K_ε is the neighborhood boundary $f^{-1}(0) \cap S_\varepsilon^{2n+1}$. A polynomial $f(z)$ is called a weighted homogeneous polynomial if there are positive integers $q_0, \cdots, q_n, d$ such that $\tilde{f}(z)=f(z_0^{q_0}, z_1^{q_1}, \cdots, z_n^{q_n})$ is homogeneous of degree d. In this paper, we shall prove that Milnor fibering is stable under a certain "deformation" and that Milnor fiberings of weighted homogeneous polynomials are uniquely determined by its weight (Theorem 1 and Theorem 2).

§2. General facts, main results

Let $f(z)$ be a polynomial as in §1. Local degree of f at the origin, L.D. (f, O), is defined by the degree of the map:

$$df/\|df\| : S_\varepsilon^{2n+1} \to S^{2n+1} ,$$

where $df(a)=\left(\frac{\partial f}{\partial z_0}(a), \cdots, \frac{\partial f}{\partial z_n}(a)\right)$. Assume further that f has only isolated critical points, say $\{P_1, \cdots, P_s\}$, as a function from C^{n+1} to C. Global degree of f, G.D. (f), is defined by $\sum_{i=1}^{s}$ L.D. (f, P_i).

PROPOSITION 1. *Global degree G.D.* (f) *is equal to the degree of the map*:

$$df/\|df\| : S_R^{2n+1} \to S^{2n+1}$$

where R is a large enough positive number.

For the proof, see [2], Appendix B. Now, assume that we have a family $\{f_t\}_{t\in U}$ of polynomials such that each f_t has only isolated critical points which are contained in D_R^{2n} for some R and $f_t(O)=0$ where U is an open interval and $\{f_t\}$ depends smoothly on t. Then we have:

PROPOSITION 2. *G.D.* (f_t) *is constant under the above deformation.*

This is an easy consequence of the Proposition 1. On the contrary, assume that there is a positive number ε such that the origin is the only critical point of each f_t in D_ε^{2n+2}. Then the following is easily obtained by the definition.

PROPOSITION 3. *L.D.(f_t, O) is constant under the above deformation.*

But the above two deformations are too weak for the stableness of the Milnor fibering. We assume that $\{f_t\}_{t\in U}$ satisfies the following condition.

(A): There exists a positive number ε such that $f_t^{-1}(0)$ and S_η^{2n+1} are transverse for each $t\in U$ and $0<\eta\leqq\varepsilon$. Then we have

THEOREM 1. *Milnor fibering of f_t at the origin is stable under the above deformation.*

As a corollary of Theorem 1, we have:

THEOREM 2. *Assume that two weighted homogeneous polynomials f and g have the same weight and have an isolated critical point at the origin. Then their Milnor fiberings are equivalent.*

§ 3. Proof of Theorem 1

Let $\{f_t\}$ be a deformation as in Theorem 1 and let $\tilde{K}=\{(z, t)\in S_\varepsilon^{2n+1}\times I; f_t(z)=0\}$ $(I\subset U)$. Let $\pi: S_\varepsilon^{2n+1}\times I\to I$ be the projection map. Then $\pi|\tilde{K}$ is non-degenerate and therefore $\tilde{K}$ is a product cobordism. Let δ be a small positive number such that $f_t^{-1}(\eta)$ and S_ε^{2n+1} are transverse for each $t\in I$ and η $(|\eta|\leqq\delta)$. Let $E=\{(z, t)\in D_\varepsilon^{2n+2}\times I | f_t(z)=\delta\}$ and define $\varphi: E\to S_\delta^1$ by $\varphi(z, t)=f_t(z)$. Then φ is a fibre bundle. Let $\tilde{\pi}$ be the projection of E into I. Then it is easy to see that $\tilde{\pi}$ is non-degenerate on each fibre of φ. Thus using a fibre-preserving connection vector field for $\tilde{\pi}$, we have a diffeomorphism ψ and a commutative diagram

$$\begin{tikzcd}
f_0^{-1}(S_\delta^1)\cap D_\varepsilon^{2n+2} \arrow[rr, "\psi"] \arrow[dr, "f_0"'] & \cap & f_1^{-1}(S_\delta^1)\cap D_\varepsilon^{2n+2} \arrow[dl, "f_1"] \\
& S_\delta^1 . &
\end{tikzcd}$$

By Theorem 5.11 of [2], there is a fibre-preserving diffeomorphism φ_i $(i=0, 1)$

$$\begin{tikzcd}
f_i^{-1}(S_\delta^1)\cap D_\varepsilon^{2n} \arrow[rr, "\varphi_i"] \arrow[dr, "f_i"'] & \cap & S_\varepsilon^{2n+1}-f_i^{-1}(\mathrm{Int}\, D_\delta^2) \arrow[dl, "f_i/|f_i|"] \\
& S_\delta^1\cong S^1 . &
\end{tikzcd}$$

Combining these diffeomorphisms, we obtain a fibre-preserving diffeomorphism $\tilde{\Psi}$.

$$\begin{tikzcd} S_\varepsilon^{2n+1}-K_{0,\varepsilon} \arrow[rr, "\tilde{\Psi}"] \arrow[dr, "f_0/|f_0|"'] & \cap & S_\varepsilon^{2n+1}-K_{1,\varepsilon} \arrow[dl, "f_1/|f_1|"] \\ & S^1 & \end{tikzcd}$$

$(K_{i,\varepsilon}=f_i^{-1}(0)\cap S_\varepsilon^{2n+1},\ i=0,\ 1.)$ Q.E.D.

§4. Proof of the Theorem 2 and an application

Assume that f and g are weighted homogeneous polynomials of a weight $w=(q_0, \cdots, q_n\,; d)$, having an isolated singularity at the origin.

Let $S(w)=\{\nu=(\nu_0, \cdots, \nu_n)\in(\boldsymbol{Z})^{n+1}\,;\ \sum_{j=0}^{n} q_j\nu_j=d, \nu_i\geqq 0, i=0, \cdots, n\}$ and let $W_t(z)=\sum_{\nu\in S(w)} t_\nu z_0^{\nu_0}z_1^{\nu_1}\cdots z_n^{\nu_n}$ ($\boldsymbol{t}=(t_\nu)$). We consider a set $V(w)$, defined by $V(w)=\{\boldsymbol{t}=(t_\nu)\in \boldsymbol{C}^\rho\,;$ $W_t(z)$ has non-isolated singularities at the origin$\}$ ($\rho=$the number of elements of $S(w)$).

LEMMA. *$V(w)$ is an algebraic set.*

PROOF. We consider the branched covering $\varphi: \boldsymbol{C}^{n+1}\to\boldsymbol{C}^{n+1}$ defined by $\varphi(z_0, \cdots, z_n)=(z_0^{q_0}, \cdots, z_n^{q_n})$. Then $W_t(\varphi(z))$ and $\dfrac{\partial W_t}{\partial z_k}(\varphi(z))$ are homogeneous polynomials of degree d and $d-q_k$ respectively. Let $\tilde{V}(w)=\left\{([z_0:z_1:\cdots:z_n], \boldsymbol{t})\in \boldsymbol{CP}^n\times\boldsymbol{C}^\rho\,;\ \dfrac{\partial W_t}{\partial z_k}(\varphi(z))=0,\ k=0, \cdots, n\right\}$ and let $\pi:\boldsymbol{CP}^n\times\boldsymbol{C}^\rho\to\boldsymbol{C}^\rho$ be the projection. Then we have $V(w)=\pi(\tilde{V}(w))$ and therefore by the Remmert's proper mapping theorem (see, [1], p. 162), $V(w)$ is an analytic set. Now it is easy to see that $V(w)$ is a cone at the origin, and this implies by Chow's theorem $V(w)$ is an algebraic set.

Q.E.D.

PROOF OF THEOREM 2. Let $t(f)$ and $t(g)$ be the corresponding points in $\boldsymbol{C}^\rho$ to f and g i.e. $W_{t(f)}=f$ and $W_{t(g)}=g$. By the assumption, $t(f), t(g)\notin V(w)$. Thus $V(w)$ is a proper algebraic subset of $\boldsymbol{C}^\rho$ and we can find a smooth path $p:I\to \boldsymbol{C}^\rho-V(w)$ such that $p(0)=t(f)$ and $p(1)=t(g)$. Now we define a deformation $\{f_t\}_{t\in I}$ by $f_t=W_{p(t)}$. We must prove that $\{f_t\}$ satisfies the condition (A): Assume that $\lambda\cdot\boldsymbol{z}=\operatorname{grad} f_t(\boldsymbol{z})$ for some $\boldsymbol{z}$ and $\lambda\neq 0$. Because $f_t(\boldsymbol{z})$ satisfies the functional equation

$$f_t(\boldsymbol{z})=\sum_{k=0}^{n}\frac{q_k}{d}\cdot z_k\cdot\frac{\partial f_t}{\partial z_k}(\boldsymbol{z})\,,$$

this means $\sum_{k=0}^{n}\dfrac{q_k}{d}|z_k|^2=0$. This means $\boldsymbol{z}=0$.

Therefore $\{f_t\}$ is a deformation satisfying (A) and Theorem 2 is a result of Theorem 1. Q.E.D.

Now, we have an immediate application. Assume that we have a projective hypersurface V of $\boldsymbol{C}P^n$ which is non-singular and defined by a homogeneous polynomial f of degree d. By the above lemma, the isotopy class of V depends only on d. Therefore we may assume that $f(z)=z_0^d+z_1^d+\cdots+z_n^d$.

Let $K=f^{-1}(0)\cap S^{2n+1}$. Then $H^*(K)$ is easily computed by using the Wang sequence of the Milnor fibering and therefore using the Gysin sequence of the Hopf bundle: $K \xrightarrow{S^1} V$, we can compute $H^*(V)$ as follows.

$$H^j(V)=\begin{cases} \boldsymbol{Z} & j:\ \text{even},\ \neq n-1,\ 0\leqq j\leqq 2n-2 \\ \left\{\varepsilon(n-1)+\dfrac{1}{d}\{(d-1)^{n+1}+(-1)^{n+1}(d-1)\}\right\}\boldsymbol{Z} & j=n-1 \\ 0 & \text{otherwise} \end{cases}$$

where $\varepsilon(k)=\begin{cases} 0 & k:\ \text{odd} \\ 1 & k:\ \text{even}\ . \end{cases}$

This method can be extended to the general hypersurface. For the detail, see [4].

References

[1] Gunning, R.C. and H. Rossi, Analytic Functions of Several Complex Variables, Prentice Hall, 1965.

[2] Milnor, J., Singular points of complex hypersurfaces, Ann. of Math. Studies 61, 1968.

[3] Oka, M., On the homotopy types of hypersurfaces defined by weighted homogeneous polynomials, Topology **12** (1973), 19-32.

[4] Oka, M., On the Cohomology Structure of Projective Varieties, to appear in Proc. Tokyo Manifold Conference.

[5] Orlik, P. and P. Wagreich, Isolated singularities of algebraic surfaces with $\boldsymbol{C}^*$-action, Ann. of Math. **93** (1971), 205-228.

(Received June 13, 1973)

Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Tokyo
113 Japan