On finite Moore graphs

By Eiichi BANNAI*) and Tatsuro ITO

1. Introduction

A Moore graph of type (d, k) is a regular graph of valence (degree) d and diameter k which contains the maximal number $n=n(d, k)$ of nodes, where

$$n(d, k)=1+d\sum_{r=1}^{k}(d-1)^{r-1}=\frac{d(d-1)^k-2}{d-2}, \quad (d>2).$$

The existence of Moore graphs of type (d, k) has generally been an open question. In [4] A. J. Hoffman and R. R. Singleton have settled the question for diameters 2 and 3 by showing that there is only one graph for each type (3, 2), (7, 2) and no other Moore graphs (for $d>2$) with the possible exception of type (57, 2).

The purpose of this note is to settle the remaining case of $k\geqq 4$. That is, we will prove the following Theorem 1.

THEOREM 1. *There exist no Moore graphs of type* (d, k) *with diameter* $k\geqq 4$ *and valence* $d>2$.

Consequently, the existence (and the uniqueness) problem about finite Moore graphs is now open only in the case of type (57, 2)[1].

The following corollary is immediately obtained from Theorem 1.

COROLLARY 2. *Let* (G, Ω) *be a primitive permutation group of rank* $k+1$ *with subdegrees* $1, d, d(d-1), \cdots, d(d-1)^{k-1}$, *and let* $k\geqq 4$. *Then* $d=2$, $|\Omega|=2k+1$, $2k+1$ *being prime, and* G *is the dihedral group of order* $2(2k+1)$.

The method of the proof of Theorem 1 is based on the method previously used in Hoffman and Singleton [4], Feit Higman [1] and Higman [3]. That is, we obtain the non-existence proof through the the consideration about eigenvalues and their multiplicities in the incidence matrix of the graph. However, our method given below is a little more elaborate than that which was used before, and it will be possible to obtain many non-existence theorems for wider classes of graphs other than the Moore graphs.

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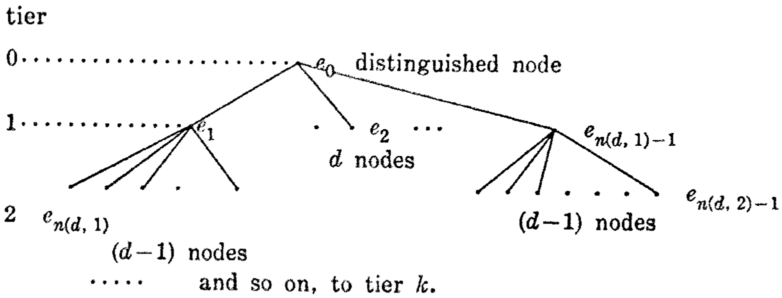
1) The non-existence of Moore graph of type (57, 2) which admits rank 3 automorphism group has been recently proved by M. Aschbacher, J. of Algebra 19 (1971), 538-540.

2. Preliminary results.

Let us recall some fundamental properties about the incidence matrix of the Moore graph of type (d, k). (For more detailed introductory articles about the finite Moore graphs, see e.g. [4] and [2].)

Choosing a node e_0, we fix a linear order in all the nodes of the graph as follows.



For a node e, we denote by $\Lambda_i(e)$ the set of nodes in the tier i from the distinguished node e. Clearly $\Lambda_0(e)=\{e\}$ and $|\Lambda_i(e)|=d(d-1)^{i-1}$ for $i\geqq 1$. Let us define the incidence matrix $A=(a_{\alpha\beta})_{0\leqq\alpha\leqq n-1, 0\leqq\beta\leqq n-1}$ by $a_{\alpha\beta}=1$ if $e_\alpha\in\Lambda_1(e_\beta)$ and $a_{\alpha\beta}=0$ otherwise. Let us define the submatrix A_{ij} of A by $A_{ij}=(a_{\alpha\beta})$ with $e_\alpha\in\Lambda_i(e_0)$, $e_\beta\in\Lambda_j(e_0)$, and so the A_{ij} give the following block decomposition of the incidence matrix A:

$$A=\begin{pmatrix} A_{00} & A_{01} & \cdots & A_{0k}\\ A_{10} & A_{11} & \cdots & A_{1k}\\ \vdots & \vdots & & \vdots\\ A_{k0} & A_{k1} & \cdots & A_{kk}\end{pmatrix}$$

Now, from the fact that the graph under consideration is a Moore graph, we can immediately conclude that any column sum of A_{ij} for each fixed pair of i and j is equal, and we denote this column sum by $\mu_{ij}(0\leqq i\leqq k, 0\leqq j\leqq k)$. If we set $M=(\mu_{ij})_{0\leqq i\leqq k, 0\leqq j\leqq k}$,

$$M=\begin{pmatrix} 0 & 1 & & & 0\\ d & 0 & 1 & & \\ & d-1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1\\ & & & \ddots & 0 & 1\\ & 0 & & & d-1 & d-1\end{pmatrix}.$$

We call M the intersection matrix of the graph.

Now we state the four propositions from A to D which are used in our proof of Theorem 1. The assertions of these Propositions are known except possibly the formulas (I) and (II).

PROPOSITION A. 1) *The minimal polynomial of A coincides with that of M.*

2) *Any eigenvalues of M are simple and real-valued, hence the characteristic polynomial of M coincides with the minimal polynomial of M.*

3) *The minimal polynomial of M is given by $(x-d)F_k(x)$, where $F_k(x)$ is the polynomial of degree k defined inductively by $F_0(x)=1$, $F_1(x)=x+1$ and $F_{i+1}(x)=xF_i(x)-(d-1)F_{i-1}(x)$.*

4) *If we set $d-1=s^2$ with $s>0$ and $x=2s\cos\varphi$, then we obtain*

$$F_k(x)=\frac{s^k}{\sin\varphi}\left\{\sin(k+1)\varphi+\frac{1}{s}\sin k\varphi\right\}\dots\dots\dots\dots\dots\dots(\mathrm{I}).$$

PROOF. 1) The assertion is immediately proved from the construction of M, by a result of Wielandt (cf. [3], § 4.)

2) The assertion is immediately proved from a result of [3], § 6, because the matrix M is tri-diagonal. This is also immediately proved from 4).

3) The assertion is immediately proved by [3], § 6.

4) Let us define the polynomials $G_k(x)$ as follows: $G_0(x)=1$, $G_1(x)=x$ and $G_{\nu+1}(x)=xG_\nu(x)-s^2G_{\nu-1}(x)$. Then clearly we have $F_k(x)=G_k(x)+G_{k-1}(x)$. If we set $T=\begin{pmatrix}0 & -s^2\\ 1 & x\end{pmatrix}$, then $(G_k(x), G_{k+1}(x))=(G_{k-1}(x), G_k(x))T=\cdots=(G_0(x), G_1(x))T^k$. Now, the eigenvalues of T are $\lambda=\frac{x+\sqrt{x^2-4s^2}}{2}$ and $\mu=\frac{x-\sqrt{x^2-4s^2}}{2}$, and the corresponding eigen-vectors are $(1, \lambda)$ and $(1, \mu)$ respectively. Thus we have

$$(G_0(x), G_1(x))=\frac{\lambda}{\sqrt{x^2-4s^2}}(1, \lambda)-\frac{\mu}{\sqrt{x^2-4s^2}}(1, \mu)\ .$$

Therefore,

$$(G_k(x), G_{k+1}(x))=\frac{\lambda^{k+1}}{\sqrt{x^2-4s^2}}(1, \lambda)-\frac{\mu^{k+1}}{\sqrt{x^2-4s^2}}(1, \mu)\ .$$

If we set $x=2s\cos\varphi$, then we have $\lambda=se^{\sqrt{-1}\varphi}$ and $\mu=se^{-\sqrt{-1}\varphi}$. Now, putting these values of λ and μ in the right hand side of the above formula for $(G_k(x), G_{k+1}(x))$, we have $G_k(x)=\frac{s^k\sin(k+1)\varphi}{\sin\varphi}$, and so we also have the desired formula (I) for $F_k(x)$.

PROPOSITION B. *Let θ_i $(i=0, 1, \cdots, k)$ be an eigenvalue of the incidence* *matrix A, and let us set* $\theta_i = 2s\cos\varphi_i$. *Let us denote by* $m(\theta_i)$ *the multiplicity of the eigenvalue* θ_i *in A. Then we obtain*

$$\frac{m(\theta_i)}{n} = \frac{\operatorname{trace} f_i(A)}{n\cdot f_i(\theta_i)} \quad (i=1,2,\cdots,k,\ \theta_0=d)$$

$$= \frac{-2\sin\varphi_i \sin k\varphi_i}{(1-(2s/1+s^2)\cos\varphi_i)\{(k+1)\cos(k+1)\varphi_i+(k/s)\cos k\varphi_i\}} \quad \ldots\text{(II)}.$$

where $f(x)=(x-\theta_0)(x-\theta_1)\cdots(x-\theta_k)$ *is the characteristic polynomial of M and* $f_i(x)=f(x)/(x-\theta_i)$.

PROOF. By Lemma 3.4 in Feit and Higman [1], $m(\theta_i)=\dfrac{\operatorname{trace} f_i(A)}{f_i(\theta_i)}$. In the following we fix the valence d. Let M_k be the intersection matrix of the graph with diameter k, and let $f_i^{(k)}(x)=f_i(x)=(x-d)(x-\theta_1)\cdots(x-\theta_{i-1})(x-\theta_{i+1})\cdots(x-\theta_k)=a_{i0}^{(k)}+a_{i1}^{(k)}x+\cdots+a_{ik}^{(k)}x^k$. Let $\eta_q^{(k)}=(\eta_{q0}^{(k)},\eta_{q1}^{(k)},\cdots,\eta_{qk}^{(k)})$ be the $(k+1)$-dimensional vector defined by $\eta_0^{(k)}=(1,0,\cdots,0)$ and $\eta_q^{(k)}=\eta_{q-1}^{(k)}M_k$. Then, clearly the values of $\eta_{q0}^{(k)}(0\leqq q\leqq k+1)$ are independent of k, and so we denote simply η_{q0} for $\eta_{q0}^{(k)}$. Since trace $A^q=n\cdot\eta_{q0}^{(k)}$, we have

$$\frac{m(\theta_i)}{n}=\frac{a_{i0}^{(k)}\eta_{00}^{(k)}+a_{i1}^{(k)}\eta_{10}^{(k)}+\cdots+a_{ik}^{(k)}\eta_{k0}^{(k)}}{f_i^{(k)}(\theta_i)}. \quad \text{(c.f. Higman [3])}$$

Now, let us set $F_k(x)=b_0^{(k)}+b_1^{(k)}x+\cdots+b_k^{(k)}x^k$. Since $(x-\theta_i)(a_{i0}^{(k)}+a_{i1}^{(k)}x+\cdots+a_{ik}^{(k)}x^k)=(x-d)(b_0^{(k)}+b_1^{(k)}x+b_k^{(k)}x^k)$, we have

$$(a_{i0}^{(k)},a_{i1}^{(k)},\cdots,a_{ik}^{(k)})=(b_0^{(k)},b_1^{(k)},\cdots,b_k^{(k)})\begin{pmatrix}1 & & & & 0\\ \theta_i-d & \ddots & & & \\ \theta_i^2-d\theta_i & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \\ \theta_i^k-d\theta_i^{k-1} & \cdots & \theta_i^2-d\theta_i & \theta_i-d & 1\end{pmatrix}$$

Now, if we set

$$h^{(k)}(\theta)=(b_0^{(k)},b_1^{(k)},\cdots,b_k^{(k)})\begin{pmatrix}1 & & & & 0\\ \theta-d & \ddots & & & \\ \theta^2-d\theta & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \\ \theta^k-d\theta^{k-1} & \cdots & \theta^2-d\theta & \theta-d & 1\end{pmatrix}\begin{pmatrix}\eta_{00}\\ \eta_{10}\\ \vdots\\ \vdots\\ \eta_{k0}\end{pmatrix}$$

then we have $h^{(k)}(\theta_i)=a_{i0}^{(k)}\eta_{00}+a_{i1}^{(k)}\eta_{10}+\cdots+a_{ik}^{(k)}\eta_{k0}$. Noting that $f_i^{(k)}(\theta_i)=(\theta_i-d)F_k'(\theta_i)$, where $F_k'(x)=\dfrac{dF_k(x)}{dx}$ (i.e., the derived function of $F_k(x)$), we have

$$\frac{m(\theta_i)}{n}=\frac{h^{(k)}(\theta_i)}{(\theta_i-d)F_k'(\theta_i)}\ .$$

Next, we will show that $h^{(k)}(\theta)=F_k(\theta)-dG_{k-1}(\theta)$, where $G_{k-1}(x)$ is the polynomial defined in the proof of Proposition A, 4). In order to prove this, we have only to show that the following three equations hold.

$$h^{(\nu+1)}(\theta)=\theta h^{(\nu)}(\theta)-(d-1)h^{(\nu-1)}(\theta)\ ,$$
$$h^{(1)}(\theta)=F_1(\theta)-dG_0(\theta)\ ,$$
$$h^{(2)}(\theta)=F_2(\theta)-dG_1(\theta)\ .$$

The last two assertions are clear. Now, we will show the first equality. We have

$$\theta h^{(\nu)}(\theta)=(0,b_0^{(\nu)},b_1^{(\nu)},\cdots,b_\nu^{(\nu)})\begin{pmatrix}1 & & & & 0\\ \theta-d & \ddots & & & \\ \theta^2-d\theta & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ \theta^{\nu+1}-d\theta^\nu & \cdots & \theta^2-d\theta & \theta-d & 1\end{pmatrix}\begin{pmatrix}\eta_{00}\\ \eta_{10}\\ \vdots\\ \eta_{\nu+1,0}\end{pmatrix}$$

$$-(0,b_0^{(\nu)},b_1^{(\nu)},\cdots,b_\nu^{(\nu)})\begin{pmatrix}1 & & & 0\\ -d & \ddots & & \\ & \ddots & \ddots & \\ 0 & & -d & 1\end{pmatrix}\begin{pmatrix}\eta_{00}\\ \eta_{10}\\ \vdots\\ \eta_{\nu+1,0}\end{pmatrix}.$$

However, the second term is equal to 0, because $(x-d)F_\nu(x)=(x-d)(b_\nu^{(\nu)}x^\nu+b_{\nu-1}^{(\nu)}x^{\nu-1}+\cdots+b_0^{(\nu)})$ is the characteristic polynomial of M_ν and because $\eta_{q0}(=\eta_{q0}^{(\nu)})$ is the $(0,0)$-component of the matrix $(M_\nu)^q$. While

$$h^{(\nu-1)}(\theta)=(b_0^{(\nu-1)},b_1^{(\nu-1)},\cdots,b_{\nu-1}^{(\nu-1)},0,0)\begin{pmatrix}1 & & & & 0\\ \theta-d & \ddots & & & \\ \theta^2-d\theta & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ \theta^{\nu+1}-d\theta^\nu & \cdots & \theta^2-d\theta & \theta-d & 1\end{pmatrix}\begin{pmatrix}\eta_{00}\\ \eta_{10}\\ \vdots\\ \eta_{\nu+1,0}\end{pmatrix}$$

Comparing the coefficients of the equality $F_{\nu+1}(x)=xF_\nu(x)-(d-1)F_{\nu-1}(x)$, we conclude that $(b_0^{(\nu+1)},b_1^{(\nu+1)},\cdots,b_{\nu+1}^{(\nu+1)})=(0,b_0^{(\nu)},b_1^{(\nu)},\cdots,b_\nu^{(\nu)})-(d-1)(b_0^{(\nu-1)},b_1^{(\nu-1)},\cdots,b_{\nu-1}^{(\nu-1)},0,0)$. Therefore $h^{(\nu+1)}(\theta)=\theta h^{(\nu)}(\theta)-(d-1)h^{(\nu-1)}(\theta)$ holds. Therefore, we obtain

$$\frac{m(\theta_i)}{n}=\frac{F_k(\theta_i)-dG_{k-1}(\theta_i)}{(\theta_i-d)F_k'(\theta_i)}=\frac{-dG_{k-1}(\theta_i)}{(\theta_i-d)F_k'(\theta_i)}\ .$$

Putting $\theta_i=2s\cos\varphi_i$ in this formula, we immediately have the desired formula (II) for $m(\theta_i)/n$.

PROPOSITION C. *Let θ_i and θ_j be eigenvalues of the incidence matrix A. If θ_i and θ_j are algebraically conjugate over the rational field, then $m(\theta_i)=m(\theta_j)$.* (Proof of Proposition C is clear.)

PROPOSITION D. ([4], Theorem 12.) *If there exists a Moore graph of type (d, k) with $k>2$ and $d>2$, then the polynomial $F_k(x)$ defined in Proposition A is not irreducible in $Z[x]$ (=the ring of polynomials with the rational integral coefficients.)*

3. Proof of Theorem 1.

In the following, we assume that there exists a Moore graph of type (d, k) with $d>2$ and $k\geqq 4$, and we will drive a contradiction.

Let $\theta_i=2s\cos\varphi_i\ (i=1, \cdots, k)$ be the roots of the equation $F_k(x)=0$, where $s^2=d-1$ with $s>0$. Here we may assume that $0<\varphi_1<\cdots<\varphi_k<\pi$. (Cf. Proposition A.)

The proof of Theorem 1 is completed through the following Steps from 1 to 7.

Step 1. If $s\geqq\sqrt{2}$ and $k\geqq 4$, then

$$(1)\qquad -2s<\theta_k<-2s+s\left(\frac{\pi}{k+1}\right)^2,$$

$$(2)\qquad 2s\cos\frac{k\pi}{k+1}-\left(\frac{\pi}{k+1}\right)^2<\theta_k<2s\cos\frac{k\pi}{k+1},$$

$$2s\cos\frac{\pi}{k+1}-\frac{5}{4}\left(\frac{\pi}{k+1}\right)^2<\theta_1<2s\cos\frac{\pi}{k+1}.$$

Especially

$$-1<\theta_1+\theta_k<0.$$

We easily have the assertion of (1), because

$$0<2s+2s\cos\varphi_k<2s+2s\cos\frac{k\pi}{k+1}$$
$$=2s\left(1-\cos\frac{\pi}{k+1}\right)<2s\times\frac{1}{2}\left(\frac{\pi}{k+1}\right)^2=s\left(\frac{\pi}{k+1}\right)^2.$$

Clearly

$$\frac{k\pi}{k+1}<\varphi_k<\frac{k\pi}{k+1}+\frac{\pi}{2s(k+1)}\quad\text{and}\quad\frac{\pi}{k+1}<\varphi_1<\frac{\pi}{k+1}+\frac{\pi}{2s(k+1)}$$

(Consider the intersection of the graphs of $\sin(k+1)\varphi$ and $(-1)/s\sin k\varphi$.) Since

$$0<2s\cos\frac{k\pi}{k+1}-2s\cos\varphi_k<2s\cos\frac{k\pi}{k+1}-2s\cos\left(\frac{k\pi}{k+1}+\frac{\pi}{2s(k+1)}\right)$$
$$=2s\cos\frac{k\pi}{k+1}\left(1-\cos\frac{\pi}{2s(k+1)}\right)+2s\sin\frac{k\pi}{k+1}\sin\frac{\pi}{2s(k+1)}$$
$$<2s\sin\frac{\pi}{k+1}\sin\frac{\pi}{2s(k+1)}<\left(\frac{\pi}{k+1}\right)^2,$$

we have the first inequality of (2). Since

$$0<2s\cos\frac{\pi}{k+1}-2s\cos\varphi_1<2s\cos\frac{\pi}{k+1}-2s\cos\left(\frac{\pi}{k+1}+\frac{\pi}{2s(k+1)}\right)$$
$$=2s\cos\frac{\pi}{k+1}\left(1-\cos\frac{\pi}{2s(k+1)}\right)+2s\sin\frac{\pi}{k+1}\sin\frac{\pi}{2s(k+1)}$$
$$<2s\cdot\frac{1}{2}\left(\frac{\pi}{2s(k+1)}\right)^2+2s\cdot\frac{\pi}{k+1}\cdot\frac{\pi}{2s(k+1)}$$
$$=\left(\frac{1}{4s}+1\right)\left(\frac{\pi}{k+1}\right)^2<\frac{5}{4}\left(\frac{\pi}{k+1}\right)^2,$$

we also have the second assertion of (2). The last assertion is immediately obtained by summing these two inequalities.

Step 2. If $s\geqq\sqrt{2}$ and $k\geqq 4$, then we have

(1) $$\frac{i\pi}{k+1}<\varphi_i<\frac{i\pi}{k+1}+\frac{\lambda i\pi}{k(k+1)}\quad\left(\text{for the case }\frac{i\pi}{k+1}\leqq\frac{\pi}{2}\right),$$

where $\lambda=\dfrac{1}{s+1}$, and

(2) $$\frac{i\pi}{k+1}<\varphi_i<\frac{i\pi}{k+1}+\frac{\mu(k+1-i)\pi}{k(k+1)}\quad\left(\text{for the case }\frac{i\pi}{k+1}\geqq\frac{\pi}{2}\left(2-\frac{(s-1)k}{k+1}\right)\right),$$

where $\mu=\dfrac{1}{s-1}$.

This is immediately proved by applying the mean value theorem for the function $g(\varphi)=\sin(k+1)\varphi+\dfrac{1}{s}\sin k\varphi$. That is, $g\left(\dfrac{i\pi}{k+1}\right)=-\dfrac{1}{s}\cos i\pi\sin\dfrac{i\pi}{k+1}$ has the opposite sign to that of $\cos i\pi$, and

$$g\left(\frac{i\pi}{k+1}+\frac{\lambda i\pi}{k(k+1)}\right)=\cos i\pi\left\{\sin\frac{\lambda i\pi}{k}-\frac{1}{s}\sin\frac{(1-\lambda)i\pi}{k+1}\right\}$$

and

$$g\left(\frac{i\pi}{k+1}+\frac{\mu(k+1-i)\pi}{k(k+1)}\right)=\cos i\pi\left\{\sin\frac{\mu(k+1-i)\pi}{k}-\frac{1}{s}\sin\frac{(\mu+1)(k+1-i)\pi}{k+1}\right\}$$

have the same sign as that of $\cos i\pi$ in the range under consideration.

Step 3. If $s\geqq 5$, and $k\geqq 4$, then

$$m(\theta_1)<m(\theta_2),\ m(\theta_3),\ \cdots,\ m(\theta_{k-1})\ .$$

By Step 2, we have immediately

$$\frac{i\pi}{k+1}<\varphi_i<\frac{i\pi}{k+1}+\frac{i\pi}{6k(k+1)}\qquad\left(\frac{i\pi}{k+1}\leqq\frac{\pi}{2}\right),$$

$$\frac{i\pi}{k+1}<\varphi_i<\frac{i\pi}{k+1}+\frac{(k+1-i)\pi}{4k(k+1)}\qquad\left(\frac{i\pi}{k+1}\geqq\frac{\pi}{2}\right).$$

Let us set

$$\alpha'=\left|\frac{\sin\varphi_i}{\sin\varphi_1}\right|,\qquad\beta'=\left|\frac{\sin k\varphi_i}{\sin k\varphi_1}\right|,\qquad\gamma'=\left|\frac{1-\dfrac{2s}{1+s^2}\cos\varphi_1}{1-\dfrac{2s}{1+s^2}\cos\varphi_i}\right|$$

and

$$\delta'=\left|\frac{\cos(k+1)\varphi_1+\dfrac{k}{s(k+1)}\cos k\varphi_1}{\cos(k+1)\varphi_i+\dfrac{k}{s(k+1)}\cos k\varphi_i}\right|.$$

Then, by the formula (II) in Proposition B, we have $\dfrac{m(\theta_i)}{m(\theta_1)}=\alpha'\cdot\beta'\cdot\gamma'\cdot\delta'$.

(i) Let us assume that $\varphi_i\leqq\dfrac{\pi}{2}$ $\left(\text{i.e. } i\leqq\dfrac{k}{2}\right)$. Then

$\beta'=\left|\dfrac{\sin k\varphi_i}{\sin k\varphi_1}\right|>1$ (Consider the intersection of the graphs of $\sin(k+1)\varphi$ and $(-1)/s\sin k\varphi$).

$$\delta'=\frac{|\cos(k+1)\varphi_1|+\dfrac{k}{s(k+1)}|\cos k\varphi_1|}{|\cos(k+1)\varphi_i|+\dfrac{k}{s(k+1)}|\cos k\varphi_i|}>1\ ,$$

because $|\cos(k+1)\varphi_i|<|\cos(k+1)\varphi_1|$ and $|\cos k\varphi_i|<|\cos k\varphi_1|$ (i.e. $\beta'>1$).

$$\alpha'\cdot\gamma'=\left|\frac{\sin\varphi_i}{\sin\varphi_1}\right|\left|\frac{1-\dfrac{2s}{1+s^2}\cos\varphi_1}{1-\dfrac{2s}{1+s^2}\cos\varphi_i}\right|>1\ ,$$

because if we set $v(\varphi)=\dfrac{\sin\varphi}{1-\dfrac{2s}{1+s^2}\cos\varphi}$, then

$$v(\varphi_1)<v\left(\frac{\pi}{k+1}+\frac{\pi}{6k(k+1)}\right)\leqq v\left(\frac{\pi}{5}+\frac{\pi}{6\times 20}\right)<1=v\left(\frac{\pi}{2}\right)$$

(Consider the derived function $v'(\varphi)$). Therefore, $\alpha'\cdot\beta'\cdot\gamma'\cdot\delta'>1$.

(ii) Let us assume that $\varphi_i>\frac{\pi}{2}$ $\left(\text{i.e. } i>\frac{k}{2}\right)$. Then

$$\alpha'=\left|\frac{\sin\varphi_i}{\sin\varphi_1}\right|>\frac{\sin\left(\frac{2\pi}{k+1}-\frac{2\pi}{4k(k+1)}\right)}{\sin\left(\frac{\pi}{k+1}+\frac{\pi}{6k(k+1)}\right)}\geqq\frac{\sin\left(2-\frac{1}{8}\right)\frac{\pi}{k+1}}{\sin\left(1+\frac{1}{24}\right)\frac{\pi}{k+1}}$$

$$\geqq\frac{\sin\left(2-\frac{1}{8}\right)\frac{\pi}{5}}{\sin\left(1+\frac{1}{24}\right)\frac{\pi}{5}}>1.5\,.$$

$$\beta'=\left|\frac{\sin k\varphi_i}{\sin k\varphi_1}\right|>\frac{\min\left\{\sin\frac{2\pi}{k+1},\ \sin\frac{3}{8}\pi\right\}}{\sin\frac{\pi}{k+1}}\geqq\min\left\{2\cos\frac{\pi}{5},\ \frac{\sin\frac{3}{8}\pi}{\sin\frac{\pi}{5}}\right\}>1.5\,,$$

because

$$i\pi-\frac{i\pi}{k+1}<k\varphi_i<i\pi-\frac{i\pi}{k+1}+\frac{(k+1-i)\pi}{4(k+1)}$$

and so

$$|\sin k\varphi_i|>\min\left\{\sin\frac{2\pi}{k+1},\ \sin\left(\frac{\pi}{2}-\frac{1}{4}\cdot\frac{\pi}{2}\right)\right\}\,,$$

and because

$$\pi-\frac{\pi}{k+1}<k\varphi_1<\pi-\frac{\pi}{k+1}+\frac{\pi}{6(k+1)}\quad\text{and so}\quad|\sin k\varphi_1|<\sin\frac{\pi}{k+1}\,.$$

$$\gamma'=\frac{1-\frac{2s}{1+s^2}|\cos\varphi_1|}{1+\frac{2s}{1+s^2}|\cos\varphi_i|}>\frac{1-\frac{2s}{1+s^2}}{1+\frac{2s}{1+s^2}}=\left(\frac{s-1}{s+1}\right)^2\geqq\frac{4}{9}\,.$$

$$\delta'\geqq\frac{|\cos(k+1)\varphi_1|+\frac{k}{s(k+1)}|\cos k\varphi_1|}{|\cos(k+1)\varphi_i|+\frac{k}{s(k+1)}|\cos k\varphi_i|}>1$$

$\left(\text{because } \beta'=\left|\frac{\sin k\varphi_i}{\sin k\varphi_1}\right|=\left|\frac{\sin(k+1)\varphi_i}{\sin(k+1)\varphi_1}\right|>1 \text{ and hence } \left|\frac{\cos k\varphi_1}{\cos k\varphi_i}\right|>1,\ \left|\frac{\cos(k+1)\varphi_1}{\cos(k+1)\varphi_i}\right|>1\right)$.

Therefore, we have $\alpha'\cdot\beta'\cdot\gamma'\cdot\delta'>1.5\times1.5\times\frac{4}{9}=1$.

Step 4. If $\sqrt{2}\leqq s\leqq\sqrt{5}$ and $k\geqq300$, then $m(\theta_k)<m(\theta_2)$, $m(\theta_3)$, $\cdots$, $m(\theta_{k-1})$. By step 2, we have immediately

$$\frac{i\pi}{k+1}<\varphi_i<\frac{i\pi}{k+1}+\frac{i\pi}{2k(k+1)} \qquad \left(\frac{i\pi}{k+1}\leqq\frac{\pi}{2}\right),$$

$$\frac{i\pi}{k+1}<\varphi_i<\frac{i\pi}{k+1}+\frac{2.5(k+1-i)\pi}{k(k+1)} \qquad \left(\frac{i\pi}{k+1}\geqq0.8\pi\right).$$

Let us set

$$\alpha=\left|\frac{\sin\varphi_i}{\sin\varphi_k}\right|,\qquad \beta=\left|\frac{\sin k\varphi_i}{\sin k\varphi_k}\right|,\qquad \gamma=\left|\frac{1-\dfrac{2s}{1+s^2}\cos\varphi_k}{1-\dfrac{2s}{1+s^2}\cos\varphi_i}\right|$$

and

$$\delta=\left|\frac{\cos(k+1)\varphi_k+\dfrac{k}{s(k+1)}\cos k\varphi_k}{\cos(k+1)\varphi_i+\dfrac{k}{s(k+1)}\cos k\varphi_i}\right|.$$

Then, by the formula (II) in Proposition B, we have

$$\frac{m(\theta_i)}{m(\theta_k)}=\alpha\cdot\beta\cdot\gamma\cdot\delta\,.$$

(i) Let us assume that $\varphi_i\leqq\frac{\pi}{2}$, i.e. $i\leqq\frac{k}{2}$.

(i-a) If $i\leqq5$, then

$$a=\left|\frac{\sin\varphi_i}{\sin\varphi_k}\right|>\frac{\sin\dfrac{2\pi}{k+1}}{\sin\dfrac{k\pi}{k+1}}>\frac{\sin\dfrac{2\pi}{300}}{\sin\dfrac{\pi}{300}}=2\cos\frac{\pi}{300}>2\times0.99\,.$$

$$\beta=\left|\frac{\sin k\varphi_i}{\sin k\varphi_k}\right|\geqq\left|\frac{\sin k\varphi_2}{\sin k\varphi_k}\right|>\frac{\sin k\dfrac{\pi}{k(k+1)}}{\sin k\dfrac{(1+2.5)\pi}{k(k+1)}}=\frac{\sin\dfrac{\pi}{k+1}}{\sin\dfrac{3.5\pi}{k+1}}>\frac{1}{3.5}\,.$$

$$\gamma=\frac{1+\dfrac{2s}{1+s^2}|\cos\varphi_k|}{1-\dfrac{2s}{1+s^2}|\cos\varphi_i|}>\frac{1+\dfrac{2s}{1+s^2}\times0.99}{1-\dfrac{2s}{1+s^2}\times0.99}>\frac{s^2+1.9s+1}{s^2-1.9s+1}\,,$$

because $|\cos\varphi_k|>\cos\dfrac{\pi}{300}>0.99$ and $|\cos\varphi_i|>\cos\dfrac{6\pi}{300}>0.99$.

$$\delta=\frac{|\cos(k+1)\varphi_k|-\dfrac{k}{s(k+1)}|\cos k\varphi_k|}{|\cos(k+1)\varphi_i|+\dfrac{k}{s(k+1)}|\cos k\varphi_i|}>|\cos(\mathrm{k}+1)\varphi_k|\frac{1-\dfrac{k}{s(k+1)}}{1+\dfrac{k}{s(k+1)}}>0.99\frac{s-1}{s+1}\ .$$

Here note that $|\cos(k+1)\varphi_k|>\cos\dfrac{2.5}{k}\pi\geqq\cos\dfrac{2.5}{300}\pi>0.99$ and that the relation $|\cos(k+1)\varphi_i|>|\cos k\varphi_i|$ holds, because

$$\begin{aligned}\cos^2(k+1)\varphi_i&=1-\sin^2(k+1)\varphi_i\\&=1-\frac{1}{s^2}\sin^2 k\varphi_i>1-\sin^2 k\varphi_i=\cos^2 k\varphi_i\ .\end{aligned}$$

Since $\gamma\cdot\delta>0.99\dfrac{s^3+0.9s^2-0.9s-1}{s^3-0.9s^2-0.9s+1}>0.99\times2$, we have

$$\alpha\cdot\beta\cdot\gamma\cdot\delta>(2\times0.99)\times\left(\frac{1}{3.5}\right)\times(2\times0.99)>1\ .$$

(i-b) If $i\geqq6$, then

$$\alpha>\frac{\sin\dfrac{6\pi}{k+1}}{\sin\dfrac{\pi}{k+1}}\geqq\frac{\sin\dfrac{6\pi}{300}}{\sin\dfrac{\pi}{300}}>5.9$$

$\beta>1$ (Consider the intersection of the graphs of $\sin(k+1)\varphi$ and $(-1)/s$ $\sin k\varphi$.)

$\gamma>1$

$$\delta=\frac{|\cos(k+1)\varphi_k|-\dfrac{k}{s(k+1)}|\cos k\varphi_k|}{|\cos(k+1)\varphi_i|+\dfrac{k}{s(k+1)}|\cos k\varphi_i|}>\left|\frac{\cos(k+1)\varphi_k}{\cos(k+1)\varphi_i}\right|\cdot\frac{1-\dfrac{k}{s(k+1)}}{1+\dfrac{k}{s(k+1)}}$$

$$>\frac{s-1}{s+1}\geqq\frac{\sqrt{2}-1}{\sqrt{2}+1}>0.17$$

$$\left(\text{because } \beta=\left|\frac{\sin(k+1)\varphi_1}{\sin(k+1)\varphi_k}\right|>1 \text{ and hence } \left|\frac{\cos(k+1)\varphi_k}{\cos(k+1)\varphi_1}\right|>1\right).$$

Therefore, $\alpha\cdot\beta\cdot\gamma\cdot\delta>5.9\times0.17>1$.

(ii) Let us assume that $\varphi_i>\dfrac{\pi}{2}$, i.e. $i>\dfrac{k}{2}$.

(ii-a) If $i\leqq(k+1)-10$, then

$$\alpha>\frac{\sin\frac{9\pi}{k+1}}{\sin\frac{\pi}{k+1}}\geqq\frac{\sin\frac{9\pi}{300}}{\sin\frac{\pi}{300}}>8$$

$$\beta>1\quad\left(\text{because }\beta=\left|\frac{\sin(k+1)\varphi_1}{\sin(k+1)\varphi_k}\right|>1\text{ and hence }\left|\frac{\cos(k+1)\varphi_k}{\cos(k+1)\varphi_1}\right|>1\right)$$

$$\gamma>1$$

$$\delta\geqq\frac{|\cos(k+1)\varphi_k|-\frac{k}{s(k+1)}|\cos k\varphi_k|}{|\cos(k+1)\varphi_i|+\frac{k}{s(k+1)}|\cos k\varphi_i|}>0.17\text{ (as in (i-b))}.$$

Therefore $\alpha\cdot\beta\cdot\gamma\cdot\delta>8\times0.17>1$.

(ii-b) If $i\geqq(k+1)-9$, then

$$\alpha>\frac{\sin\left(\frac{2\pi}{k+1}-\frac{2.5\times2\pi}{k(k+1)}\right)}{\sin\frac{\pi}{k+1}}\geqq\frac{\sin\left(2-\frac{1}{60}\right)\frac{\pi}{k+1}}{\sin\frac{\pi}{k+1}}\geqq\frac{\sin\left(2-\frac{1}{60}\right)\frac{\pi}{300}}{\sin\frac{\pi}{300}}>1.9$$

$\beta>1$ (Consider the intersection of the graphs of $\sin(k+1)\varphi$ and $(-1)/s\sin k\varphi$.)

$\gamma>1$

$$\delta=\frac{|\cos(k+1)\varphi_k|-\frac{k}{s(k+1)}|\cos k\varphi_k|}{|\cos(k+1)\varphi_i|-\frac{k}{s(k+1)}|\cos k\varphi_i|}>|\cos(k+1)\varphi_k|\frac{1-\frac{k}{s(k+1)}}{1-\frac{k}{s(k+1)}|\cos k\varphi_i|}$$

$$>|\cos(k+1)\varphi_k|\frac{\sqrt{2}-1}{\sqrt{2}-|\cos k\varphi_i|}>0.9\frac{\sqrt{2}-1}{\sqrt{2}-0.9}>0.9\times0.8,$$

because $|\cos(k+1)\varphi_k|>\cos\frac{2.5\pi}{k}\geqq\cos\frac{2.5\pi}{300}>0.9$, and because

$$|\cos k\varphi_i|\geqq|\cos k\varphi_{(k+1)-9}|>\cos(1+2.5)\frac{9\pi}{k+1}\geqq\cos3.5\frac{9\pi}{300}>0.9.$$

Therefore $\alpha\cdot\beta\cdot\gamma\cdot\delta>1.9\times0.9\times0.8>1$.

Step 5. If $s\geqq\sqrt{6}$ and $k=4$, then

$$m(\theta_4)<(m(\theta_2),\ m(\theta_3)).$$

(i) If $i=2$, then

$$\alpha>\frac{\sin\frac{2}{5}\pi}{\sin\frac{\pi}{5}}>2\times0.8.$$

Because

$$\sin 5\left(\frac{4}{5}\pi+\frac{1}{2}\cdot\frac{1}{20}\pi\right)+\frac{1}{s}\sin 4\left(\frac{4}{5}\pi+\frac{1}{2}\cdot\frac{1}{20}\pi\right)$$
$$=\sin\frac{\pi}{8}-\frac{1}{s}\sin\frac{3}{10}\pi>0\ .$$

$\beta>1$ (Consider the intersection of the graphs of $\sin(k+1)\varphi$ and $(-1)/s\sin k\varphi$.)

$$\gamma>1+\frac{2s}{1+s^2}\cos\frac{\pi}{5}>1+\frac{2s}{1+s^2}0.8=\frac{s^2+1.6s+1}{s^2+1}\ ,$$

$$\delta=\left|\frac{|\cos 5\varphi_4|-\frac{4}{5s}|\cos 4\varphi_4|}{|\cos 5\varphi_2|+\frac{4}{5s}|\cos 4\varphi_2|}\right|>\left|\frac{\cos 5\varphi_4}{\cos 5\varphi_2}\right|\cdot\frac{1-\frac{4}{5s}}{1+\frac{4}{5s}}>\frac{1-\frac{4}{5s}}{1+\frac{4}{5s}}=\frac{s-0.8}{s+0.8}$$

$$\left(\beta>1 \text{ and hence } \left|\frac{\cos 5\varphi_4}{\cos 5\varphi_2}\right|>1\right).$$

Since $\gamma\cdot\delta>\frac{s^3+0.8s^2-0.28s-0.8}{s^3+0.8s^2+s+0.8}>0.7$, we have $\alpha\cdot\beta\cdot\gamma\cdot\delta>(2\times 0.8)\times 0.7>1$.

(ii) If $i=3$, then

$$\alpha=\left|\frac{\sin\varphi_3}{\sin\varphi_4}\right|>\frac{\sin\left(\frac{2}{5}\pi-\frac{\pi}{20}\right)}{\sin\frac{\pi}{5}}>1.5\ ,$$

$\beta>1$

$$\gamma=\frac{1+\frac{2s}{1+s^2}|\cos\varphi_4|}{1+\frac{2s}{1+s^2}|\cos\varphi_3|}>\frac{1+\frac{2s}{1+s^2}0.8}{1+\frac{2s}{1+s^2}0.5}=\frac{s^2+1.6s+1}{s^2+s+1}\ ,$$

because $|\cos\varphi_4|>\cos\frac{\pi}{5}>0.8$ and $|\cos\varphi_3|<\cos\left(\frac{2\pi}{5}-\frac{\pi}{20}\right)<0.5$

$$\delta\geqq\frac{|\cos 5\varphi_4|-\frac{4}{5s}|\cos 4\varphi_4|}{|\cos 5\varphi_3|+\frac{4}{5s}|\cos 4\varphi_3|}>\frac{0.92-\frac{4}{5s}0.81}{1+\frac{4}{5s}0.31}>\frac{0.92s-0.65}{s+0.25}\ ,$$

because

$$|\cos 5\varphi_4|>\cos\frac{\pi}{8}>0.92\ ,\quad |\cos 4\varphi_4|<\cos\frac{\pi}{5}<0.81\ ,$$

$$|\cos 5\varphi_3|<1 \quad\text{and}\quad |\cos 4\varphi_3|<\sin\frac{\pi}{10}<0.31\ .$$

Since

$$\gamma\cdot\delta>\frac{0.92s^3+0.822s^2-0.12s-0.65}{s^3+1.25s^2+1.25s+0.25}>0.67\ ,$$

we have $\alpha\cdot\beta\cdot\gamma\cdot\delta>1.5\times0.67>1$.

Step 6. If $s\geqq\sqrt{6}$ and $k\geqq5$, then $m(\theta_k)<m(\theta_2),\ m(\theta_3),\ \cdots,\ m(\theta_{k-1})$.

By Step 2 we have immediately

$$\frac{i\pi}{k+1}<\varphi_i<\frac{i\pi}{k+1}+\frac{i\pi}{3k(k+1)}\qquad\left(\frac{i\pi}{k+1}\leqq\frac{\pi}{2}\right),\ \text{and}$$

$$\frac{i\pi}{k+1}<\varphi_i<\frac{i\pi}{k+1}+\frac{0.7(k+1-i)\pi}{k(k+1)}\qquad\left(\frac{i\pi}{k+1}\geqq\frac{\pi}{2}\right).$$

(i) First we consider the case $\varphi_i\leqq\dfrac{\pi}{2}$ $\left(\text{i.e. } i\leqq\dfrac{k}{2}\right)$. Clearly

$$\alpha>\frac{\sin\dfrac{2\pi}{k+1}}{\sin\dfrac{\pi}{k+1}}=2\cos\frac{\pi}{k+1}\geqq2\cos\frac{\pi}{6}>2\times0.866\ .$$

(i-a) In the first place, we consider the case $i=2$. Then

$$\beta=\left|\frac{\sin(k+1)\varphi_2}{\sin(k+1)\varphi_k}\right|>\left|\frac{\sin\dfrac{\pi}{2k}}{\sin\dfrac{0.7\pi}{k}}\right|>\frac{5}{7}$$ (Let $\beta<1$ and consider the intersection of the graphs of $\sin(k+1)\varphi$ and $(-1)/s\sin k\varphi$).

$$\gamma=\frac{1+\dfrac{2s}{1+s^2}|\cos\varphi_k|}{1-\dfrac{2s}{1+s^2}|\cos\varphi_2|}>\frac{1+\dfrac{2s}{1+s^2}\times0.866}{1-\dfrac{2s}{1+s^2}\times0.4}>\frac{s^2+1.7s+1}{s^2-0.8s+1}\ ,$$

because

$$|\cos\varphi_k|>\cos\frac{\pi}{k+1}\geqq\cos\frac{\pi}{6}>0.866$$

and

$$|\cos\varphi_2|>\cos\left(\frac{2\pi}{k+1}+\frac{2\pi}{3k(k+1)}\right)\geqq\cos\left(\frac{2\pi}{6}+\frac{2\pi}{90}\right)>0.4\ .$$

$$\delta=\frac{|\cos(k+1)\varphi_k|-\dfrac{k}{s(k+1)}|\cos k\varphi_k|}{|\cos(k+1)\varphi_2|+\dfrac{k}{s(k+1)}|\cos k\varphi_2|}>|\cos(k+1)\varphi_k|\frac{1-\dfrac{k}{s(k+1)}}{1+\dfrac{k}{s(k+1)}}>0.9\frac{s-1}{s+1}\ ,$$

because

$$|\cos(k+1)\varphi_k| > \cos\frac{0.7\pi}{k} \geqq \cos\frac{0.7}{5}\pi > 0.9 .$$

Therefore, we conclude that

$$\alpha\cdot\beta\cdot\gamma\cdot\delta > (2\times 0.866)\times\frac{5}{7}\times 0.9\times\frac{s^3+0.7s^2-0.7s-1}{s^3+0.2s^2+0.2s+1} > (2\times 0.866)\times\frac{5}{7}\times 0.9\times 0.9 > 1 .$$

(i-b) Let us assume that $i \geqq 3$. Then

$\beta > 1$ (consider the intersection of the graphs of $\sin(k+1)\varphi$ and $(-1)/s \sin k\varphi$),

$$\gamma > |\cos\varphi_k|\left(1+\frac{2s}{1+s^2}\right) > 0.866\times\frac{1+2s+s^2}{1+s^2} ,$$

$$\delta > \left|\frac{\cos(k+1)\varphi_k}{\cos(k+1)\varphi_i}\right|\cdot\frac{s-1}{s+1} > \frac{s-1}{s+1} \quad \left(\beta>1 \text{ and hence } \left|\frac{\cos(k+1)\varphi_k}{\cos(k+1)\varphi_i}\right| > 1\right) .$$

Therefore

$$\alpha\cdot\beta\cdot\gamma\cdot\delta > (2\times 0.866)\times\left(0.866\times\frac{s^2-1}{s^2+1}\right) \geqq (2\times 0.866)\times\left(0.866\times\frac{5}{7}\right) > 1 .$$

(ii) Next, let us consider the case $\varphi_i > \frac{\pi}{2}\left(\text{i.e. } i > \frac{k}{2}\right)$.

(ii-a) Let us assume that $i=k-1$. Then

$$\alpha > \frac{\sin\left(\frac{2\pi}{k+1}-\frac{0.7\times 2\pi}{k(k+1)}\right)}{\sin\frac{\pi}{k+1}} \geqq \frac{\sin 1.72\frac{\pi}{k+1}}{\sin\frac{\pi}{k+1}} \geqq \frac{\sin 1.72\frac{\pi}{6}}{\sin\frac{\pi}{6}} > 1.56 ,$$

$$\beta > \frac{\sin\frac{2\pi}{k+1}}{\sin\left(\frac{\pi}{k+1}+\frac{0.7}{k+1}\pi\right)} \geqq \frac{\sin\frac{2}{6}\pi}{\sin\frac{1.7\pi}{6}} > 1.1 ,$$

because $i\pi-\frac{i\pi}{k+1} < k\varphi_i < i\pi-\frac{i\pi}{k+1}+\frac{0.7(k+1-i)\pi}{k+1}$,

$\gamma > 1$,

$$\delta = \frac{|\cos(k+1)\varphi_k| - \frac{k}{s(k+1)}|\cos k\varphi_k|}{|\cos(k+1)\varphi_{k-1}| - \frac{k}{s(k+1)}|\cos k\varphi_{k-1}|} > \left|\frac{\cos(k+1)\varphi_k}{\cos(k+1)\varphi_{k-1}}\right|\left(1-\frac{k}{s(k+1)}\right)$$

$$>1-\frac{1}{s}>0.59 \quad \left(\beta>1 \text{ and hence } \left|\frac{\cos(k+1)\varphi_k}{\cos(k+1)\varphi_{k-1}}\right|>1\right).$$

Therefore, $\alpha\cdot\beta\cdot\gamma\cdot\delta>1.56\times 1.1\times 0.59>1$.

(ii-b) Let us assume that $i\leqq k-2$. Then

$$\alpha>\frac{\sin\left(\dfrac{3\pi}{k+1}-\dfrac{0.7\times 3\pi}{k(k+1)}\right)}{\sin\dfrac{\pi}{k+1}}\geqq\frac{\sin 2.58\dfrac{\pi}{k+1}}{\sin\dfrac{\pi}{k+1}}\geqq\frac{\sin 2.58\dfrac{\pi}{6}}{\sin\dfrac{\pi}{6}}>2\times 0.9759\,,$$

$$\beta>1\,,$$

and

$$\gamma>1\,.$$

If

$$\delta=\frac{|\cos(k+1)\varphi_k|-\dfrac{k}{s(k+1)}|\cos k\varphi_k|}{|\cos(k+1)\varphi_i|-\dfrac{k}{s(k+1)}|\cos k\varphi_i|}\,,$$

then $\delta>1-\dfrac{1}{s}>0.59$, and this shows that $\alpha\cdot\beta\cdot\gamma\cdot\delta>1$. Thus we may assume that

$$\delta=\frac{|\cos(k+1)\varphi_k|-\dfrac{k}{s(k+1)}|\cos k\varphi_k|}{|\cos(k+1)\varphi_i|+\dfrac{k}{s(k+1)}|\cos k\varphi_i|}\,.$$

Now, we have $|\cos k\varphi_i|<\sin\dfrac{k}{k+1}\operatorname{Arcsin}\dfrac{1}{s}<\dfrac{1}{s}$ $\left(0<\varphi_i-\dfrac{i\pi}{k+1}<\dfrac{1}{k+1}\operatorname{Arcsin}\dfrac{1}{s}\right.$ because $\varphi_i>\dfrac{\pi}{2}$ and $\left.\cos(k+1)\varphi_i\cos k\varphi_i>0\right)$. Therefore,

$$\delta>\frac{|\cos(k+1)\varphi_k|-\dfrac{k}{s(k+1)}|\cos(k+1)\varphi_k|}{|\cos(k+1)\varphi_k|+\dfrac{k}{s(k+1)}\dfrac{1}{s}}>\frac{0.9-0.9\dfrac{k}{s(k+1)}}{0.9+\dfrac{1}{s}\cdot\dfrac{k}{s(k+1)}}\,.$$

If $k=5$ or 6, then $\delta>\dfrac{0.9-0.9\dfrac{6}{\sqrt{6}\cdot 7}}{0.9+\dfrac{1}{\sqrt{6}}\cdot\dfrac{6}{\sqrt{6}\cdot 7}}>0.56$. Therefore, since $\alpha>2\times 0.9759$, $\beta>1$ and $\gamma>1$, $\alpha\cdot\beta\cdot\gamma\cdot\delta>1$. If $k\geqq 7$, then

$$\delta>\frac{0.9-0.9\dfrac{1}{\sqrt{6}}}{0.9+\dfrac{1}{\sqrt{6}}\cdot\dfrac{1}{\sqrt{6}}}>0.499\,,$$

and

$$\alpha > \frac{\sin\left(\frac{3\pi}{k+1} - \frac{0.7\times 3\pi}{k(k+1)}\right)}{\sin\frac{\pi}{k+1}} \geqq \frac{\sin 2.7\frac{\pi}{k+1}}{\sin\frac{\pi}{k+1}} \geqq \frac{\sin 2.7\frac{\pi}{8}}{\sin\frac{\pi}{8}} > 2.2 .$$

Since $\beta>1$ and $\gamma>1$, we conclude $\alpha\cdot\beta\cdot\gamma\cdot\delta>1$.

Step 7. (Completion of the proof of Theorem 1).

(i) First let us assume that $s\geqq 5$ (and $k\geqq 4$). Then $m(\theta_1)<m(\theta_2), \cdots, m(\theta_{k-1})$, and $m(\theta_k)<m(\theta_2), \cdots, m(\theta_{k-1})$, by Steps 3, 5 and 6. Thus by Proposition C, $(x-\theta_1)(x-\theta_k)$ must be a factor of $F_k(x)$ in $Z[x]$. Therefore, $\theta_1+\theta_k$ must be an integer. But this is a contradiction, because $-1<\theta_1+\theta_k<0$ by Step 1, (2).

(ii) Let us assume that $\sqrt{6}\leqq s<5$ (and $k\geqq 4$). Then by Steps 5 and 6, $m(\theta_k)<m(\theta_2), \cdots, m(\theta_{k-1})$. Therefore, by Proposition C, we have either $\theta_1+\theta_k\in Z$ or $\theta_k\in Z$. The first case is impossible, because $-1<\theta_1+\theta_k<0$ by Step 1, (2). Let us assume that the second alternative holds. Then, for a fixed value of s, there exist only a finite number of k's such that the open interval from $-2s$ to $-2s+s\left(\frac{\pi}{k+1}\right)^2$ contains an integral point.

These are:

s^2	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
upper bound of k	4	8	5	4	8	6	5	10	7	6	5	12	8	6	6	15	10	8	6

Now we can easily show that for any integer θ in the open interval $-2s$ to $-2s+s\left(\frac{x}{k+1}\right)^2$ $F_k(\theta)\neq 0$. This is a contradiction.

(iii) Let us assume that $\sqrt{2}\leqq s<\sqrt{6}$ and $k>300$. Then, by Step 4, we have $m(\theta_k)<m(\theta_2), \cdots, m(\theta_{k-1})$. Now, quite the same argument as in the proof in (ii) immediately proves the assertion.

(iv) The case $\sqrt{2}\leqq s<\sqrt{6}$ and $k\leqq 300$ was already eliminated in Friedman [2]. (The possibility of type (5, 7) is stated in [2]. However, $F_7(x)$ $(d=5)$ is irreducible over $Z/5Z$ and this case is excluded by Proposition D).

Thus the proof of Theorem 1 is completed.

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Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Tokyo
113 Japan

Added in Proof.

After we had completed this work, we were informed that the same result was also proved by R. Damerell (University of London) independently. Damerell's paper is to appear in Proc. Cambridge Phil. Soc.

Partial solution of Theorem 1 was also informed to us: Juraj Bosák, Moore graphs, where the assertion of Theorem 1 is proved for some special values of d (for example, $d \leqq 64$), by extending the method of [2].