On the 3-rank of the ideal class groups of certain pure cubic fields

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Introduction

For a finite algebraic number field k, let C_k be the ideal class group of k and $d^{(l)}C_k$ be the rank of C_k w.r.t. a prime number l. $d^{(l)}C_k$ is known to be closely connected with $t_{k/Q}^{(l)}$, where $t_{k/Q}^{(l)}$ is the number of those prime numbers every prime factor of which in k has a ramification index divisible by l, or equivalently, those prime numbers of the form $p=\mathfrak{a}^l$ for an ideal $\mathfrak{a}$ in k. (cf. [6]).

In some cases, on the other hand, $d^{(l)}C_k$ is explicitly given for $l|[k:\boldsymbol{Q}]$, e.g. when $k/\boldsymbol{Q}$ is quadratic or cubic cyclic ([1], [2]), and in fact for certain pure cubic fields. To show this, let m be a cubic free positive rational integer which contains no prime factors of the type $p\equiv+1 \pmod 3$, and put $\Omega=\boldsymbol{Q}(\sqrt[3]{m})$, $K=\Omega(\zeta_3)$. Then we have the following.

THEOREM. *Let m, Ω and K be as above.*

Case (A): *every prime factor $p\neq3$ of m satisfies $p^2\equiv1 \pmod{3^2}$. Then*

$$d^{(3)}C_\Omega=t_{\Omega/Q}^{(3)}-1 .$$

Case (B): *otherwise. Then*

$$d^{(3)}C_\Omega=t_{\Omega/Q}^{(3)}-2 .$$

In both cases, $d^{(3)}C_K=2d^{(3)}C_\Omega$.

Remark. See Remark at the end of [5] for the value of $t_{\Omega/Q}^{(3)}$.

In a previous paper [5], we obtained the theorem in Case (B) rather as a lucky by-product. Our new method is applicable to both cases, depending neither on Theorem 1 in [5], nor on an estimation of $d^{(3)}C_\Omega$ of the type as is given in [6], so we prove the whole theorem here.

In §1, we list some propositions used in the sequel and give a characterization of elements of the principal genus $C_K^{1-\sigma}$. Of course the latter is a well-known result. Using this we compute $(C_K^{1-\sigma}:C_K^{(1-\sigma)^2})$ in §2 which settles the problem for $d^{(3)}C_K$. Now let $\tilde{K}$ (resp. $\tilde{\Omega}$) be the unramified class field over K (resp. Ω) corresponding to the ideal group C_K^3 (resp. C_Ω^3). In §3, we study the structure of the Galois group $G(\tilde{K}/k)$, especially its canonical generators. Finally in §4 we construct a matrix representation of $G(\tilde{K}/\Omega)$. This enables us to compute $[G(\tilde{K}/\Omega), G(\tilde{K}/\Omega)]$. The fixed field of this subgroup is $\tilde{\Omega}K$ and we get the theorem.

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§1. Some results on the extension K/k

Let m, Ω and K be as defined in Introduction, and put $k=\boldsymbol{Q}(\zeta)$, where $\zeta=\zeta_3$ denotes a primitive cube root of 1. K/k is a cyclic extension of degree 3. Let σ be a fixed generator of $G=G(K/k)$, C_K^G be the subgroup of C_K of G-invariant classes of K and D_K be the subgroup of C_K generated by G-invariant ideals of K. We need the following propositions.

PROPOSITION 1 ([3] Ia, Satz 13). $|C_K^G|=3^\nu$, $|D_K|=3^{\nu'}$, *where* $\nu=t+q^*-2$ *and* $\nu'=t+q-2$. *Here* t *is the number of primes in* k *which are ramified in* K, q^* *and* q *are defined respectively by* $(E_k\cap \mathrm{N}_{K/k}(K^\times):E_k^3)=3^{q^*}$ *and* $(\mathrm{N}_{K/k}(E_K):E_k^3)=3^q$, E_K *(resp.* E_k*) being the unit group of* K *(resp.* k*).*

Note that $t=t_{\Omega/Q}^{(3)}$ and $q^*=1$ in Case (A), $q^*=0$ in Case (B). (See [4], 3, for the value of q^*.)

PROPOSITION 2 ([5] Prop. 1). $C_K^3=C_K^{(1-\sigma)^2}$.

PROPOSITION 3 ([5] Prop. 2). *Let* K_0 *be the unramified class field over* K *corresponding to the ideal group* $C_K^{1-\sigma}$. *Then* K_0 *is the maximal unramified extension of* K *which is abelian over* k, *and* $G(K_0/k)$ *is an elementary abelian group of exponent* 3.

By Proposition 2, $(C_K:C_K^3)=(C_K:C_K^{1-\sigma})(C_K^{1-\sigma}:C_K^{(1-\sigma)^2})=3^\nu|C_K^G\cap C_K^{1-\sigma}|$, since $1-\sigma$ maps $C_K^{1-\sigma}$ onto $C_K^{(1-\sigma)^2}$. So $d^{(3)}C_K$ will be determined if we know $|C_K^G\cap C_K^{1-\sigma}|$. As in the case of quadratic fields, the elements of the principal genus $C_K^{1-\sigma}$ are characterized by genus characters given by the norm residue symbol. (See [3], II, §7, §11 for the definition and the properties of the norm residue symbol.) More precisely we have the following.

PROPOSITION 4. *An ideal* $\mathfrak{A}$ *in* K *belongs to the principal genus* $C_K^{1-\sigma}$ *if and only if for a suitable generator* $\alpha\in k^\times$ *of the principal ideal* $\mathrm{N}_{K/k}(\mathfrak{A})$,

$$\left(\frac{\alpha, m}{\mathfrak{p}}\right)=1,$$

for all (finite or infinite) primes $\mathfrak{p}$ *in* k.

Moreover, it is sufficient to check the condition only for the primes $\mathfrak{p}\neq(\sqrt{-3})$ *which are ramified in* K.

PROOF. For the first part, let α satisfy the condition (for all primes). Then

$$N_{K/k}(\mathfrak{A})=(\alpha)=(N_{K/k}(\beta)) \quad \text{for} \quad {}^{\exists}\beta\in K^{\times} ,$$

hence $N_{K/k}(\mathfrak{A}/(\beta))=(1)$, $\mathfrak{A}/(\beta)=\mathfrak{B}^{1-\sigma}$ for an ideal $\mathfrak{B}$ in K. This means that $\mathfrak{A}$ belongs to the principal genus. The converse is clear.

For the second part of the assertion, note that $(\alpha)=N_{K/k}(\mathfrak{A})$ has the following form of prime ideal decomposition:

$$N_{K/k}(\mathfrak{A})=\prod_{\mathfrak{p}}\mathfrak{p}^{3x_{\mathfrak{p}}}\prod_{\mathfrak{p}'}\mathfrak{p}'^{y_{\mathfrak{p}'}}\prod_{\mathfrak{p}''}\mathfrak{p}''^{z_{\mathfrak{p}''}} ,$$

where $\mathfrak{p}$ (resp. $\mathfrak{p}'$, $\mathfrak{p}''$) denotes the primes in k which remain prime (resp. decompose, ramify) in K. So the symbol is equal to 1 except for the primes $\mathfrak{p}''$. (It is clear that the symbol is equal to 1 for infinite primes.) Finally we can drop $(\sqrt{-3})$ by virtue of the product-formula. q.e.d.

§2. Computation of $(C_K^{1-\sigma}:C_K^{(1-\sigma)^2})$

Let $p_i (i=1,\cdots,t)$ be the rational primes which are totally ramified in Ω and $\mathfrak{P}_i$ be the prime factor of p_i in K. Then we have

LEMMA 1. *$\mathfrak{P}_i$ $(i=1,\cdots,t)$ belong to the principal genus $C_K^{1-\sigma}$. Hence $D_K\subset C_K^{1-\sigma}$.*

PROOF. We take p_i (resp. $\sqrt{-3}$) if $p_i\neq 3$ (resp. $p_i=3$) as a generator of $N_{K/k}(\mathfrak{P}_i)$. Put $p_j^{a_j}\|m$.

Let $p_i\neq 3$. Then for $p_j\neq 3$,

$$\left(\frac{p_i,m}{(p_j)}\right)=\left(\frac{m,p_i}{(p_j)}\right)^{-1}=\left(\frac{p_i}{(p_j)}\right)^{a_j}=1 , \qquad (j\neq i) ,$$

$$\left(\frac{p_i,m}{(p_i)}\right)=\left(\frac{p_i,p_i}{(p_i)}\right)^{a_i}\prod_{r\neq i}\left(\frac{p_i,p_r}{(p_i)}\right)^{a_r}=\prod_{r\neq i}\left(\frac{p_r}{(p_i)}\right)^{-a_r}=1 , \qquad (j=i) ,$$

since $(Z/p_jZ)^{\times}$ is of order $p_j-1\not\equiv 0 \pmod 3$ and $\left(\frac{p_i,p_i}{(p_i)}\right)=\left(\frac{p_i,p_i}{(p_i)}\right)^{-1}=1$.

Let $p_i=3$. Then for $p_j\neq 3$,

$$\left(\frac{\sqrt{-3},m}{(p_j)}\right)=\left(\frac{m,\sqrt{-3}}{(p_j)}\right)^{-1}=\left(\frac{\sqrt{-3}}{(p_j)}\right)^{a_j}=1 ,$$

since $\left(\frac{\sqrt{-3}}{(p_j)}\right)^2=\left(\frac{-3}{(p_j)}\right)=1$ as above. q.e.d.

Now we can compute $(C_K^{1-\sigma}:C_K^{(1-\sigma)^2})=|C_K^{\Omega}\cap C_K^{1-\sigma}|$.

Case (B): $q^*=q=0$, hence $C_K^{\Omega}=D_K\subset C_K^{1-\sigma}$, and

$$(C_K^{1-\sigma}:C_K^{(1-\sigma)^2})=3^{t-2} ,$$

$$d^{(3)}C_K=2(t-2) .$$

Case (A): This is further divided into two cases.

(a) $\zeta\in N_{K/k}(E_K)$: $q^*=q=1$, hence $C_K^G=D_K\subset C_K^{1-\sigma}$, and

$$(C_K^{1-\sigma}:C_K^{(1-\sigma)^2})=3^{t-1},$$

$$d^{(3)}C_K=2(t-1).$$

(b) $\zeta\notin N_{K/k}(E_K)$: $q^*=1$, $q=0$, and $N_{K/k}(E_K)=\{\pm1\}$.

As $(C_K^G:D_K)=3$, there exists a class $c\in C_K^G$, $c\notin D_K$. We choose a prime ideal $\mathfrak{P}$ in c which is of absolute degree 1, and put $\mathfrak{P}^{1-\sigma}=(\alpha)$, $\alpha\in K^\times$. Then $N_{K/k}(\alpha)=\pm\zeta^e$, $e=1$ or 2. For, if not, $N_{K/k}(\alpha)=\pm1$, and $\alpha=\alpha_1^{1-\sigma}$ for an element $\alpha_1\in K^\times$, hence $(\mathfrak{P}/(\alpha_1))^{1-\sigma}=(1)$ and $\mathfrak{P}\sim\mathfrak{P}/(\alpha_1)$, a contradiction.

Put $\mathfrak{A}=\mathfrak{P}\mathfrak{P}^\tau$, where τ denotes the complex conjugation. Note that $\tau\sigma=\sigma^2\tau$. Then $\mathfrak{A}^{1-\sigma}=\mathfrak{P}^{1-\sigma}\mathfrak{P}^{\tau(1-\sigma)}=(\alpha^{1+(1+\sigma)\tau})$. On the other hand, if $\mathfrak{A}$ belongs to D_K and $\mathfrak{A}=\mathfrak{B}(\beta)$, $\mathfrak{B}^{1-\sigma}=(1)$, $\beta\in K$, then $\mathfrak{A}^{1-\sigma}=(\beta^{1-\sigma})=(\alpha^{1+(1+\sigma)\tau})$, hence

$$\varepsilon\beta^{1-\sigma}=\alpha^{1+(1+\sigma)\tau}, \quad \exists\varepsilon\in E_K.$$

Taking the norms to k, we get $\pm1=N_{K/k}(\alpha)^{1+2\tau}=\pm\zeta^{-e}$, a contradiction. Hence the class containing $\mathfrak{A}$ belongs to C_K^G but not to D_K.

Now $N_{K/k}(\mathfrak{A})=N_{K/Q}(\mathfrak{P})=(p)$, p being a rational prime. Taking p as a generator of $N_{K/k}(\mathfrak{A})$, we get

$$\left(\frac{p,m}{(p_j)}\right)=\left(\frac{m,p}{(p_j)}\right)^{-1}=\left(\frac{p}{(p_j)}\right)^{a_j}=1, \quad \text{for} \quad p_j\neq3,$$

as before. So $\mathfrak{A}$ belongs to $C_K^{1-\sigma}$ and again $C_K^G\subset C_K^{1-\sigma}$, hence

$$(C_K^{1-\sigma}:C_K^{(1-\sigma)^2})=3^{t-1},$$

$$d^{(3)}C_K=2(t-1).$$

REMARK. In Case (B) and (a) of Case (A), our theorem is an easy consequence of Theorem 1 in [5] and we can go without the results of the next two sections. But there exist fields for which (b) occurs. For example, put $m=3\cdot89\cdot809=60^3+3$. According to a result of [7], $\varepsilon=(\sqrt[3]{m}-60)^3/3$ is a unit in $\Omega=\boldsymbol{Q}(\sqrt[3]{m})$, and we see that the prime factors of 3, 89 and 809 in Ω generate a subgroup of order 3. Then D_K must also be of order 3 and $(C_K^G:D_K)=3$. We can find many more examples of this type.

§3. The group structure of $G(\tilde{K}/k)$

Let $\tilde{K}$ be as defined in Introduction.

PROPOSITION 5. *$G(\tilde{K}/k)$ is of nilpotent class $\leqq2$ (i.e. the commutator sub-*

group is contained in the center) and of exponent 3.

PROOF. Proposition 3 means that the commutator subgroup of $G(\tilde{K}/k)$ is $G(\tilde{K}/K_0)$. The elements of $G(\tilde{K}/K_0)$ correspond to the elements of $C_K^{1-\sigma}/C_K^{(1-\sigma)^2}$ under the Artin's reciprocity map : $C_K/C_K^{(1-\sigma)^2} \xrightarrow{\sim} G(\tilde{K}/K)$. So for each $\rho \in G(\tilde{K}/K_0)$, we can find an element $c \in C_K$ such that

$$\left(\frac{\tilde{K}/K}{c^{1-\sigma}}\right)=\rho .$$

Then for an arbitrary extension $\tilde{\sigma}$ of σ to $\tilde{K}$,

$$\tilde{\sigma}^{-1}\rho\tilde{\sigma}=\left(\frac{\tilde{K}/K}{c^{(1-\sigma)\sigma}}\right)=\left(\frac{\tilde{K}/K}{c^{(1-\sigma)-(1-\sigma)^2}}\right)=\rho ,$$

which proves the first part of the proposition.

As for the second part, note that $G(\tilde{K}/k)$ is of an exponent at most 3^2. Assume that an element $\rho \in G(\tilde{K}/k)$ has the order 3^2, and let $\mathfrak{P}$ be a prime ideal in $\tilde{K}$ unramified in $\tilde{K}/k$ and having ρ as the Frobenius substitution. Then $\mathfrak{P}$ has degree 3 both in $\tilde{K}/K$ and K/k. Put $\mathfrak{p}_K=\mathfrak{P}\cap K$, $\mathfrak{p}_k=\mathfrak{P}\cap k$. Then by the definition of $\mathfrak{P}$,

$$\omega^{\rho}\equiv\omega^{N\mathfrak{p}_k} \quad (\mathrm{mod}\ \mathfrak{P}) \quad \forall\omega\in\mathfrak{O}_{\tilde{K}} ,$$

$$\omega^{\rho^3}\equiv\omega^{N\mathfrak{p}_K} \quad (\mathrm{mod}\ \mathfrak{P}) .$$

This means that

$$\left(\frac{\tilde{K}/K}{\mathfrak{p}_K}\right)=\rho^3 .$$

But $\mathfrak{p}_K=\mathfrak{p}_k\sim 1$ in K, hence $\rho^3=1$ and we get a contradiction. q.e.d.

Now for each p_i, let T_i be the inertia group of p_i in $G(K_0/k)$. The fixed field of $T_1 \cdots T_t$ is unramified over k, hence it coincides with k and $G(K_0/k)=T_1 \cdots T_t$. In Case (A), $T_1, \cdots, T_t$ are independent and we put $n=t$. In Case (B), $T_1, \cdots, T_t$ are not independent and one of them can be omitted in $G(K_0/k)=\prod_i T_i$. We put $n=t-1$ and assume that T_t is omitted.

Choose a generator σ_i of T_i for $i=1, \cdots, n$. They make a basis of $G(K_0/k)$. The subgroup $G(K_0/K)$ is an $(n-1)$-dimensional subspace of $G(K_0/k)$, so that it is given by a linear equation, namely, there exists $c_i \in \boldsymbol{Z}$ such that

$$\prod_{i=1}^{n} \sigma_i^{x_i} \in G(K_0/K) \Longleftrightarrow \sum_{i=1}^{n} c_i x_i \equiv 0 \quad (\mathrm{mod}\ 3) .$$

Here none of the $c_i\equiv 0$ (mod 3). For if $c_i\equiv 0$ (mod 3), $\sigma_i \in G(K_0/K)$, which is impossible, since K_0/K is unramified. So we can replace σ_i by $\sigma_i^{c_i}$, and then

$$\prod_{i=1}^{n} \sigma_i^{x_i} \in G(K_0/K) \Longleftrightarrow \sum_{i=1}^{n} x_i \equiv 0 \pmod 3 .$$

Therefore we can take as a basis of $G(K_0/K)$ the $n-1$ elements

$$\sigma_i \sigma_{i+1}^2 , \quad i=1, \cdots, n-1 .$$

Next we extend σ_i to $\tilde{K}$ and denote them again by σ_i. By the theory of p-groups, $G(\tilde{K}/k)$ is generated by $\sigma_1, \cdots, \sigma_n$ and the commutator subgroup $G(\tilde{K}/K_0)$ is generated by $\{[\sigma_i, \sigma_j]\}_{i<j}$. (The latter assertion is also easily verified by direct computation in the present case.) For $i<r<j$, however,

$$\begin{aligned}[] [\sigma_i, \sigma_r][\sigma_r, \sigma_j] &= \sigma_i^{-1}\sigma_r^{-1}\sigma_i\sigma_r\sigma_r^{-1}\sigma_j^{-1}\sigma_r\sigma_j \\ &= \sigma_i^{-1}\sigma_j^{-1}\sigma_r\sigma_r^{-1}\sigma_i\sigma_j = [\sigma_i, \sigma_j] , \end{aligned}$$

since $\sigma_r^{-1}\sigma_i$, $\sigma_j^{-1}\sigma_r \in G(\tilde{K}/K)$ and are commutative. Hence $G(\tilde{K}/K_0)$ is generated by the $n-1$ elements

$$[\sigma_i, \sigma_{i+1}] , \qquad i=1, \cdots, n-1 ,$$

and we can take them as a basis of $G(\tilde{K}/K_0)$. Putting together, the $2(n-1)$ elements defined above make a basis of $G(\tilde{K}/K)$.

§4. The group structure of $G(\tilde{K}/\Omega)$ and the proof of Theorem

$\tilde{K}/\Omega$ is clearly a Galois extension. $\tilde{\Omega}$ being as defined in Introduction, $\tilde{\Omega}K$ is easily seen to be the maximal abelian extension of Ω contained in $\tilde{K}$, hence it is the fixed field of $[G(\tilde{K}/\Omega), G(\tilde{K}/\Omega)]$. We want to determine the order of this subgroup.

As $G(\tilde{K}/K)$ is normal in $G(\tilde{K}/\Omega)$, the complex conjugation τ operates on $G(\tilde{K}/K)$ by inner automorphism $\rho \mapsto \tau\rho\tau^{-1}$, and $G(\tilde{K}/\Omega) = G(\tilde{K}/K) \cdot \langle\tau\rangle$ (semi-direct) w.r.t. this operation. Choose any basis of $G(\tilde{K}/K) \simeq \boldsymbol{F}_3^{2(n-1)}$ and let X be the matrix representing the action of τ on $G(\tilde{K}/K)$ w.r.t. this basis. Put (denoting by I_r the identity matrix of size r)

$$\bar{H} = \left\{ \begin{pmatrix} I_{2(n-1)} & \boldsymbol{a} \\ 0 & 1 \end{pmatrix} \middle| \boldsymbol{a} \in G(\tilde{K}/K) \right\} ,$$

$$\bar{X} = \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix} .$$

Then it is easily verified that $\bar{H} \simeq G(\tilde{K}/K)$ and

$$\bar{X} \begin{pmatrix} I_{2(n-1)} & \boldsymbol{a} \\ 0 & 1 \end{pmatrix} \bar{X}^{-1} = \begin{pmatrix} I_{2(n-1)} & X\boldsymbol{a} \\ 0 & 1 \end{pmatrix} .$$

So $\bar{G}=\bar{H}.\langle\bar{X}\rangle$ is a semi-direct product and $\bar{G}\simeq G(\tilde{K}/\Omega)$.

Now take the $2(n-1)$ elements

$$\sigma_i\sigma_{i+1}^2 , \qquad [\sigma_i, \sigma_{i+1}] , \qquad i=1, \cdots, n-1 ,$$

as the basis of $G(\tilde{K}/K)$. We need the following.

LEMMA 2. $\tau\sigma_i\tau^{-1}=\sigma_i^2 g_i$, $g_i\in G(\tilde{K}/K_0)$, $i=1, \cdots, n$.

PROOF. By the assumption on m, $\tau T_i\tau^{-1}=T_i$, so that $\tau\sigma_i\tau^{-1}=\sigma_i^{x_i}g_i$, $g_i\in G(\tilde{K}/K_0)$. Apply this on $\sqrt[3]{m}$. Since $T_i\not\subset G(K_0/K)$, σ_i is non-trivial on K, hence $\sigma_i(\sqrt[3]{m})=\zeta^e\sqrt[3]{m}$, $e=1$ or 2. Then

$$\tau\sigma_i\tau^{-1}(\sqrt[3]{m})=\zeta^{-e}\sqrt[3]{m}=\sigma_i^2(\sqrt[3]{m}) ,$$

and hence $x_i=2$. q.e.d.

Therefore we have

$$\tau(\sigma_i\sigma_{i+1}^2)\tau^{-1}=\sigma_i^2 g_i(\sigma_{i+1}^2 g_{i+1})^2\equiv(\sigma_i\sigma_{i+1}^2)^2 \mod G(\tilde{K}/K_0) ,$$

$$\tau[\sigma_i, \sigma_{i+1}]\tau^{-1}=[\sigma_i^2 g_i, \sigma_{i+1}^2 g_{i+1}]=[\sigma_i, \sigma_{i+1}] ,$$

since $G(\tilde{K}/k)$ being of nilpotent class 2, $[gg', h]=[g, h][g', h]$ and similarly for $[g, hh']$ for any $g, g', h, h'\in G(\tilde{K}/k)$. This yields

$$X=\begin{pmatrix} 2I_{n-1} & 0 \\ * & I_{n-1} \end{pmatrix} .$$

We see that the eigenvalues of X are in $\boldsymbol{F}_3$, so that we can choose a basis of $G(\tilde{K}/K)$ w.r.t. which X is of the Jordan's normal form. But as τ is of order 2, the coefficients just below the diagonal must be 0, and hence $X=\begin{pmatrix} 2I_{n-1} & 0 \\ 0 & I_{n-1} \end{pmatrix}$. Then for $\boldsymbol{a}=\begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \end{pmatrix}\in G(\tilde{K}/K)$,

$$\left[\bar{X}, \begin{pmatrix} I_{2(n-1)} & \begin{matrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \end{matrix} \\ 0 & 1 \end{pmatrix}\right]=\begin{pmatrix} I_{2(n-1)} & \begin{matrix} -\boldsymbol{a}_1 \\ 0 \end{matrix} \\ 0 & 1 \end{pmatrix} .$$

As the commutator of any two elements of $\bar{G}$ is of this form, we have now proved that

$$|G(\tilde{K}/\tilde{\Omega}K)|=3^{n-1} ,$$

hence

$$|G(\tilde{\Omega}K/K)|=|G(\tilde{\Omega}/\Omega)|=3^{n-1} ,$$

$$d^{(3)}C_\Omega=n-1 .$$

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