On the trace formula for Hecke operators

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§1. Introduction and theorem.

1.1. Let H be the complex upper half plane, and $G=SL(2, R)$. We regard G as a group of transformations on H. Let Γ be a subgroup of G operating on H discontinuously, with a fundamental domain of finite volume. Assume that Γ contains the element $\begin{pmatrix} -1, & 0 \\ 0, & -1 \end{pmatrix}$. We fix once for all an element α in G such that $\alpha\Gamma\alpha^{-1}$ is commensurable with Γ, and denote by Γ' the subgroup of G generated by Γ and α. Let χ be a representation of Γ' by unitary matrices of degree $\nu<\infty$. We assume that

(i) $\chi\left(\begin{pmatrix} -1, & 0 \\ 0, & -1 \end{pmatrix}\right)=1_\nu$,

(ii) the kernel Γ^0 of χ in Γ is of finite index in Γ.

Let $T=T(\Gamma\alpha\Gamma)$ be the Hecke operator acting on the space of cusp forms with respect to Γ and χ, and of dimension $-k$ (see below). The trace of T has been explicitly calculated in most of cases, but as far as we know, not yet for the case of $k=2$ and $\nu>1$. In this note, we follow the method of A. Selberg [8], and calculate the trace for the above remaining case, making reference to the method of H. Shimizu [10] and of T. Kubota. The result is as follows.

1.2. By a cusp form with respect to Γ and χ of dimension $-k$, we understand a function $F(z)$ on H taking values in the representation space of χ, which satisfies the following conditions:

(i) $F(z)$ is holomorphic on H,

(ii) $F(\gamma z)=\chi(\gamma)j(\gamma, z)^{-1}F(z)$, for $\gamma\in\Gamma$,

(iii) in the case $\Gamma\backslash H$ is non-compact, $F(z)$ is regular at every parabolic point p of Γ^0, and a constant term in the Fourier expansion of F at p vanishes.

Here, $j(g, z)$ denotes $(cz+d)^{-k}$ for $g=\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}\in G$ and $z\in H$. The linear space consisting of all $F(z)$ is denoted by $S(\Gamma, k, \chi)$. We now define the Hecke operator $T(\Gamma\alpha\Gamma)$ in $S(\Gamma, k, \chi)$. Let $\Gamma\alpha\Gamma=\bigcup_{\mu=1}^{d}\alpha_\mu\Gamma$ be the right Γ-coset decomposition of $\Gamma\alpha\Gamma$. For $F\in S(\Gamma, k, \chi)$, we set

$$T(\Gamma\alpha\Gamma)F(z)=\sum_{\mu=1}^{d}\chi(\alpha_\mu)j(\alpha_\mu^{-1}, z)F(\alpha_\mu^{-1}z)\ . \tag{1.1}$$

THEOREM. *In the case $k=2$, the trace of $T(\Gamma\alpha\Gamma)$ is given by the following formula*:

$$\begin{aligned}(1.2)\qquad \operatorname{Tr} T(\Gamma\alpha\Gamma) = & \delta\frac{1}{4\pi}v(\Gamma\backslash H)\operatorname{tr}\chi(g_0)+\sum_{[g]\in\mathfrak{S}_1}\frac{1}{[\Gamma(g):Z(\Gamma)]}\frac{\bar{\zeta}^2}{1-\bar{\zeta}^2}\operatorname{tr}\chi(g) \\ & +\sum_{\mu=1}^{d}\operatorname{tr}{}_{M_0}\chi(\alpha_\mu)-\sum_{[g]\in\mathfrak{S}_3}\frac{\min(|\lambda|,|\lambda^{-1}|)}{|\lambda-\lambda^{-1}|}\operatorname{tr}\chi(g) \\ & -\sum_{i=1}^{h}\left\{\frac{1}{2r_i}\sum_{\{g\}\in B_i/\Gamma_i^0}\operatorname{tr}\chi(g)+\frac{1}{2\sqrt{-1}r_i}\sum_{\substack{\{g\}\in B_i/\Gamma_i^0\\ \{g\}\neq\Gamma_i^0}}\cot\left(\frac{\mu(g)\pi}{r_i}\right)\operatorname{tr}\chi(g)\right\}.\end{aligned}$$

The notations used in this formula are defined as follows;

$$\delta=\begin{cases}1\cdots\alpha\in\Gamma,\\ 0\cdots\alpha\notin\Gamma,\end{cases}$$

$M_0=\{c\in C^\nu\,|\,\chi(\gamma)c=C,\ \text{for } \gamma\in\Gamma\}$,

g_0; an element of the group $Z(\Gamma)$, which consists of the elements $\begin{pmatrix}1, & 0\\ 0, & 1\end{pmatrix}$ and $\begin{pmatrix}-1, & 0\\ 0, & -1\end{pmatrix}$,

$v(\Gamma\backslash H)$; the volume of a fundamental domain of Γ in H relative to the invariant measure $dz=\dfrac{dxdy}{y^2}$ $\quad(z=x+iy)$,

$\zeta,\ \bar{\zeta}$; the eigenvalues of an elliptic element g of G and supposed that $\dfrac{gz-z_0}{gz-\bar{z}_0}=\zeta^2\dfrac{z-z_0}{z-\bar{z}_0}$ $\quad$ ($z_0\in H$ is the fixed point of g),

$\lambda,\ \lambda^{-1}$; the eigenvalues of a hyperbolic element g of G,

d; the number of right Γ-cosets in $\Gamma\alpha\Gamma$,

$[g]$; the equivalence class of g in $\Gamma\alpha\Gamma$, where the equivalence relation is defined by: $g\sim g' \Longleftrightarrow g'=\pm\gamma g\gamma^{-1}$, for some $\gamma\in\Gamma$,

$\mathfrak{S}_1$ (resp. $\mathfrak{S}_3$); a complete system of inequivalent elliptic elements (resp. hyperbolic elements leaving a parabolic point of Γ fixed) in $\Gamma\alpha\Gamma$,

$\Gamma(g)=\{\gamma\in\Gamma\,|\,g=\pm\gamma g\gamma^{-1}\}$,

$\kappa_1,\cdots,\kappa_h$; representatives of all Γ-inequivalent cusps.

$B_i=\left\{g\in\Gamma\alpha\Gamma\,|\,g\kappa_i=\kappa_i,\ \text{parabolic or } \pm\begin{pmatrix}1, & 0\\ 0, & 1\end{pmatrix}\right\}$,

$\Gamma_i=\{g\in\Gamma\,|\,g\kappa_i=\kappa_i\}$,

$\Gamma_i^0=\ker\chi\cap\Gamma_i$,

$r_i=[\Gamma_i:\Gamma_i^0]$,

$\mu(g)$ is defined by $g=\pm\sigma_i\begin{pmatrix}1, & \mu(g)\\ 0, & 1\end{pmatrix}\sigma_i^{-1}$, where σ_i is an element of G such that $\sigma_i\infty=\kappa_i$, $\sigma_i^{-1}\Gamma_i\sigma_i=\left\{\pm\begin{pmatrix}1, & n\\ 0, & 1\end{pmatrix}\middle|\, n\in Z\right\}$.

We shall prove this theorem in the following sections. Especially, in the case $\nu=1$, we regain the results of M. Eichler [1] and of H. Saito [7]. The author wishes to express his sincere thanks to Prof. Y. Ihara and to Prof. H. Shimizu who encouraged him with many suggestions.

§2. The Hecke operators.

2.1. Let H be the complex upper half plane, and $\tilde{H}=H\times(R/2\pi Z)$ with elements (z, ϕ), where we will identify ϕ and $\phi+2\pi$. Let $G=SL(2, R)$ be the special linear group and $\tilde{G}=G\times(R/2\pi Z)$ with elements (g, θ), where g is a matrix $\begin{pmatrix} a, & b \\ c, & d \end{pmatrix}$ with determinant 1 and θ a real number, and it act on the space (z, ϕ) as:

$$(g, \theta)(z, \phi)=\left(\frac{az+b}{cz+d}, \phi+\arg(cz+d)-\theta\right).$$

Let Γ be a subgroup of G operating on H discontinuously with a fundamental domain of finite volume and consisting the element $\begin{pmatrix} -1, & 0 \\ 0, & -1 \end{pmatrix}$. Then the subgroup $\tilde{\Gamma}=\Gamma\times\{0\}$ of $\tilde{G}$ acts on $\tilde{H}$ discontinuously with a fundamental domain of finite volume. We identify Γ with $\tilde{\Gamma}$; so we shall write Γ instead of $\tilde{\Gamma}$. We assume that the fundamental domain of Γ is non-compact. We fix once and for all an element α in G such that $\alpha\Gamma\alpha^{-1}$ is commensurable with Γ, and denote by Γ' the subgroup of G generated by Γ and α. Let χ be a representation of Γ' by unitary matrices of degree ν, satisfying the conditions given in §1.

Let $L^2(\tilde{H}, \Gamma)$ be the space of functions $F(z, \phi)$ on $\tilde{H}$ taking values in the representation space of χ and satisfying the following conditions:

(i) $F(z, \phi)=\begin{pmatrix} f_1(z, \phi) \\ \vdots \\ f_\nu(z, \phi) \end{pmatrix}$, each $f_i(z, \phi)$ is a measurable function on $\tilde{H}$ taking values in C;

(ii) $F(\gamma(z, \phi))=\chi(\gamma)F(z, \phi)$, for $\gamma\in\Gamma$;

(iii) $\int_{\Gamma\backslash\tilde{H}} {}^tF(z, \phi)\overline{F(z, \phi)}dz\, d\phi<\infty$, where $dz=\dfrac{dxdy}{y^2}$ is a G-invariant measure on H $(z=x+iy)$.

Let P be the set of all parabolic points of Γ^0, and put

$$\Gamma_p^0=\{g\in\Gamma | gp=p\}\cap\ker\chi, \quad (p\in P).$$

Define the subspace $L_0^2(\tilde{H}, \Gamma)$ of $L^2(\tilde{H}, \Gamma)$ by the additional condition:

(iv) $\int_0^1 F(\sigma_p(z, \phi))dx=0$, for all $p\in P$, where $\sigma_p \in G$ satisfies that $\sigma_p \infty = p$ and that

$$\sigma_p^{-1} \Gamma_p^0 \sigma_p = \left\{ \pm \begin{pmatrix} 1, & n \\ 0, & 1 \end{pmatrix} \middle| n \in Z \right\} .$$

Let $C^\infty(\tilde{H})$ be the space of C^∞-class functions on $\tilde{H}$ taking values in C. As is well-known that the ring of all $\tilde{G}$-invariant differential operators on $\tilde{H}$ is generated by

$$\frac{\partial}{\partial \phi}, \quad \tilde{\Delta}_1 = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + y \frac{\partial}{\partial x} \frac{\partial}{\partial \phi} .$$

Generally, for a $\tilde{G}$-invariant differential-integral operator L in $C^\infty(\tilde{H})$ and $F(z, \phi) \in L^2(\tilde{H}, \Gamma)$, we define $LF(z, \phi)$ simply by

$$LF(z, \phi) = \begin{pmatrix} Lf_1(z, \phi) \\ \vdots \\ Lf_\nu(z, \phi) \end{pmatrix} .$$

When $LF(z, \phi) \in L^2(\tilde{H}, \Gamma)$, we can regard L as an operator in $L^2(\tilde{H}, \Gamma)$. Thus $\frac{\partial}{\partial \phi}$, $\tilde{\Delta}_1$ and etc. will also be considered as operators in $L^2(\tilde{H}, \Gamma)$. From now on, for simplicity, let us write simply as "an operator" instead of "an operator in $L^2(\tilde{H}, \Gamma)$", unless otherwise specified.

2.2. The classification of eigen spaces in $L_0^2(\tilde{H}, \Gamma)$ is given by Kuga [6] in the compact fundamental case. We follow the method of Gel'fand and Pyateckiĭ-Šapiro [2], who treated this in the compact fundamental case, with the aid of the representation theory of groups, and see that it is true in the non-compact fundamental case. To do this, we need some propositions.

PROPOSITION 1. (Bargmann). *Irreducible unitary representations of the group $G = SL(2, R)$ are of the following types.*

(i) *Principal series: $\mathscr{H}_s^+$ (s = purely imaginary). This representation is realized in the space of functions on the real line with summable square. The representation operators are defined by:*

$$T(g)\varphi(x) = \varphi\left(\frac{ax+c}{bx+d}\right)|bx+d|^{s-1} , \quad \text{for} \quad g = \begin{pmatrix} a, & b \\ c, & d \end{pmatrix} \in G .$$

The inner product is defined by:

$$(\varphi_1, \varphi_2) = \int_{-\infty}^{\infty} \varphi_1(x)\overline{\varphi_2(x)} dx .$$

(ii) *Second Principal series: $\mathscr{H}_s^-$ (s = purely imaginary). This representation is realized in the same space as in* (i) *with the same inner product. The representation operators are defined by:*

$$T(g)\varphi(x)=\varphi\left(\frac{ax+c}{bx+d}\right)|bx+d|^{s-1} \quad \text{sign } (bx+d) .$$

(iii) *Supplementary series:* $\mathscr{H}'_s$ $(-1<s<0)$. *The representation is realized in the space as above, but with another inner product given by:*

$$(\varphi_1, \varphi_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|x_1-x_2|^{-s-1}\varphi_1(x_1)\overline{\varphi_2(x_2)}dx_1dx_2 .$$

The representation operators are defined as same as in (i).

(iv) *Discrete series:* $\mathscr{H}_n^+$ $(n\geqq 0$, *rational integer*). *The representation is realized in the space of analytic functions on the upper half-plane with finite norm. The representation operators are given by:*

$$T(g)\varphi(z)=\varphi\left(\frac{az+c}{bz+d}\right)(bz+d)^{-n-1} .$$

The inner product is defined by:

$$(\varphi_1, \varphi_2)=\int_{\mathrm{Im}\, z>0}\varphi_1(z)\overline{\varphi_2(z)}y^{n-1}dxdy , \quad (n\geqq 1) ,$$

$$(\varphi_1, \varphi_2)=\lim_{\varepsilon\to 0}\int_{\mathrm{Im}\, z>0}\varphi_1(z)\overline{\varphi_2(z)}y^{\varepsilon-1}dxdy , \quad (n=0) .$$

(v) *Second Discrete series:* $\mathscr{H}_n^-$ $(n\geqq 0$, *rational integer*). *The representation is realized in the space of analytic functions on the lower half-plane with finite norm. The representation operators are given by the same as in* (iv). *The inner product is also defined by an analogus formula.*

PROPOSITION 2. (Godement). $L_0^2(\tilde{H}, \Gamma)$ *decomposes into the sum of a countable number of irreducible unitary representations. Each irreducible representation enters into* $L_0^2(\tilde{H}, \Gamma)$ *with a finite multiplicity.*

Now, to each element $g\in G$, we correspond a unitary operator T_g in $L_0^2(\tilde{H}, \Gamma)$ of the following kind:

$$T_g\varphi(\pi^{-1}g')=\varphi(\pi^{-1}(g'g)) ;$$

where $\pi : g'=\begin{pmatrix}1, & x\\ 0, & 1\end{pmatrix}\begin{pmatrix}y^{1/2}, & 0\\ 0\ , & y^{-1/2}\end{pmatrix}\begin{pmatrix}\cos\phi, & -\sin\phi\\ \sin\phi, & \cos\phi\end{pmatrix}\to(z=x+iy, \phi)$ is the canonical isomorphism of G onto $\tilde{H}$. Let K be $SO(2, R)$ and denote a one-dimensional irreducible representation of K by

$$\sigma_m : k=\begin{pmatrix}\cos\theta, & -\sin\theta\\ \sin\theta, & \cos\theta\end{pmatrix}\to\exp(-im\theta), \text{ for } k\in K, \ (m\in Z).$$

The next proposition is well-known for $G=SL(2, R)$.

PROPOSITION 3. *Let $(T, \mathscr{H})$ be an irreducible unitary representation of G. Put*

$$\mathscr{H}(\sigma_m)=\{\varphi \in \mathscr{H} \mid T(k)\varphi=\sigma_m(k)\varphi\ , \quad \text{for all} \quad k\in K\}\ .$$

Then $\mathscr{H}_0=\sum_m \mathscr{H}(\sigma_m)$ is a dense set of analytic vectors in $\mathscr{H}$.

Denote by $\mathfrak{G}$ the Lie algebra of G and by $U(\mathfrak{G})$ the universal envelopping algebra. Then as is known, we can give the differential representation of T in $\mathscr{H}_0$ by:

$$T(X)\varphi=\left(\frac{d}{dt}T_{\exp(tX)}\varphi\right)_{t=0}, \quad \text{for} \quad X\in\mathfrak{G}\ .$$

It is well-known that its representation is uniquely extended to the representation of $U(\mathfrak{G})$ and more to that of $U(\mathfrak{G})_C=U(\mathfrak{G})\otimes C$ C-linearly. Choose a basis of $\mathfrak{G}$ as follows: $X_1=\begin{pmatrix}1, & 0\\ 0, & -1\end{pmatrix}$, $X_2=\begin{pmatrix}0, & 1\\ 1, & 0\end{pmatrix}$ and $X_3=\begin{pmatrix}0, & 1\\ -1, & 0\end{pmatrix}$. Put $V^+=X_1+iX_2$, $V^-=X_1-iX_2$ and $D=X_1^2+X_2^2-X_3^2$. Let φ be an element of $\mathscr{H}(\sigma_m)$; we get

$$T(X_3)\varphi=\mathrm{im}\,\varphi\ ,$$

$$T(V^{\pm})\varphi\in\mathscr{H}(\sigma_{m\pm 2})\ .$$

For representations of the second principal series and of the discrete and the second discrete series with even n, $T_{g_0}F=-F$, for $F\in\mathscr{H}$ $\left(g_0=\begin{pmatrix}-1, & 0\\ 0, & -1\end{pmatrix}\right)$; so that they cannot appear in the decomposition of $L_0^2(H,\Gamma)$ into irreducible representations. Moreover if $\mathscr{H}$ is an irreducible component of $L_0^2(\tilde{H},\Gamma)$, $\mathscr{H}(\sigma_m)=\{0\}$ for all odd m. Firstly, let T be a representation of principal or supplymentary series and $\mathscr{H}=\mathscr{H}_s^+$ or $\mathscr{H}_s$ respectively. For $\varphi\in\mathscr{H}_0$, the following equation comes from Gel'fand [2]:

$$T(D)\varphi=(s^2-1)\varphi\ .$$

With this, we get

$$T(V^+V^-)\varphi=(s^2-(m-1)^2)\varphi\ ,$$

$$T(V^-V^+)\varphi=(s^2-(m+1)^2)\varphi\ ,$$

for $\varphi\in\mathscr{H}(\sigma_m)$. As s is purely imaginary or belongs to $(-1,0)$, $s^2-(m-1)^2\neq 0$ and $s^2-(m+1)^2\neq 0$. On the other hand, $\mathscr{H}(\sigma_0)\neq\{0\}$; for instance, $\varphi(x)=(x^2+1)^{\frac{s-1}{2}}$ belongs to $\mathscr{H}(\sigma_0)$. Therefore $\mathscr{H}(\sigma_m)\neq\{0\}$ for all even m. Secondly, let T be a representation of discrete series and $\mathscr{H}=\mathscr{H}_n^+$. Again it follows from Gel'fand that, for $\varphi\in\mathscr{H}_0$,

$$T(D)\varphi=(n^2-1)\varphi\ .$$

Then, we get

$$T(V^+V^-)\varphi=(n^2-(m-1)^2)\varphi\ ,$$

$$T(V^-V^+)\varphi=(n^2-(m+1)^2)\varphi\ ,$$

for $\varphi\in\mathscr{H}(\sigma_m)$. On the other hand, it is easily seen that $\mathscr{H}(\sigma_0)=\{0\}$, $\mathscr{H}(\sigma_{-(n+1)})=\{0\}$ and $\mathscr{H}(\sigma_{n+1})\ni(z+i)^{-(n+1)}\neq 0$. Therefore, $\mathscr{H}(\sigma_m)\neq 0$ for all $m\geqq n+1$, $m\equiv 0$ (2). Finally, let T be a representation of the second discrete series and $\mathscr{H}=\mathscr{H}_n^-$. By the same argument as above, we obtain $\mathscr{H}(\sigma_m)\neq\{0\}$ for all $m\leqq -(n+1)$, $m\equiv 0$ (2).

Now, we define the subspace $M(m,\lambda)$ of $L_0^2(\tilde{H},\Gamma)$ consisting of φ which satisfies the following conditions:

(i) $T_k\varphi=\sigma_m(k)\varphi$, for $k\in K$,

(ii) $T(D)\varphi=4\lambda\varphi$.

Let $F(z,\phi)\in M\left(k,\frac{1}{4}k^2-\frac{1}{2}k\right)$ and put $F(z)$ as follows:

$$F(z,\phi)=\exp(-ik\phi)y^{k/2}F(z)\ . \tag{2.1}$$

As $V^-M\left(k,\frac{1}{4}k^2-\frac{1}{2}k\right)=M\left(k-2,\frac{1}{4}k^2-\frac{1}{2}k\right)=\{0\}$, we get $V^-F(z,\phi)=0$, namely, $\frac{\partial}{\partial\bar{z}}F(z)=0$. Besides

$$F(\gamma z)=\chi(\gamma)j(\gamma,z)^{-1}F(z)\ ,\quad\text{for}\quad\gamma\in\Gamma\ .$$

Then $F(z)$ belongs to $S(\Gamma,k,\chi)$. Conversely, since $F(z)$ belongs to $S(\Gamma,k,\chi)$ it is clear that $F(z,\phi)$ is contained in $M\left(k,\frac{1}{4}k^2-\frac{1}{2}k\right)$; hence $M\left(k,\frac{1}{4}k^2-\frac{1}{2}k\right)$ is isomorphic to $S(\Gamma,k,\chi)$. By the same argument as above, $M\left(-k,\frac{1}{4}k^2-\frac{1}{2}k\right)$ is isomorphic to $S(\Gamma,k,\chi)$ (anti-linear). Since D and X_3 have the forms $4\tilde{\Delta}_1$, $-\frac{\partial}{\partial\phi}$ respectively, as the differential operators on $\tilde{H}=\pi(G)$, we now obtain the following proposition.

PROPOSITION 4. *If Γ and χ satisfy the conditions given in* §1, *the classification of the eigen spaces in $L_0^2(\tilde{H},\Gamma)$ for each eigenvalue-pair $(-ki,\lambda)$ of $\left(\frac{\partial}{\partial\phi},\tilde{\Delta}_1\right)$ is given by the following table. In this table, λ_l ranges over all eigenvalues of $\Delta=y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$ on $L^2(\tilde{H},\Gamma)$ satisfying $M(0,\lambda_l)\neq\{0\}$, except $\lambda_0=0$; $M\left(k,\frac{1}{4}k^2-\frac{1}{2}k\right)$ is isomorphic to $S(\Gamma,k,\chi)$, $M\left(-k,\frac{1}{4}k^2-\frac{1}{2}k\right)$ is also isomorphic to $S(\Gamma,k,\chi)$ (anti-linear).*

Series	Eigenvalues	Eigen spaces
A_{λ_l} $(l\neq 0)$	$(-2ki,\ \lambda_l)$ $k\in Z$	$M(2k,\lambda_l)=V^{+k}M(0,\lambda_l),\quad k\geqq 0$ $M(2k,\lambda_l)=V^{-\|k\|}M(0,\lambda_l),\quad k<0$
C_k	$\left(-(k+2m)i,\frac{1}{4}k^2-\frac{1}{2}k\right)$ $m=0,1,2,\cdots$	$M\left(k+2m,\frac{1}{4}k^2-\frac{1}{2}k\right)=V^{+m}M\left(k,\frac{1}{4}k^2-\frac{1}{2}k\right)$
C_k	$\left((k+2m)i,\frac{1}{4}k^2-\frac{1}{2}k\right)$ $m=0,1,2,\cdots$	$M\left(-(k+2m),\frac{1}{4}k^2-\frac{1}{2}k\right)$ $=V^{-m}M\left(-k,\frac{1}{4}k^2-\frac{1}{2}k\right)$

2.3. In order to calculate the trace of the Hecke operator acting on $S(\Gamma,2,\chi)$, we shall write down the action of the Hecke operator carried over to the space $M(2,0)$ by the canonical isomorphism $(2,1): M(2,0)\simeq S(\Gamma,2,\chi)$, and extend it to the space $L^2(\tilde{H},\Gamma)$. Thus,

$$T(\Gamma\alpha\Gamma)F(z,\phi)=\sum_{\mu=1}^{d}\chi(\alpha_\mu)F(\alpha_\mu^{-1}(z,\phi)) , \tag{2.2}$$

for $F\in L^2(\tilde{H},\Gamma)$. For the calculation of its trace, we consider a $\tilde{G}$-invariant integral operator k_s in $C^\infty(\tilde{H})$ defined by a point pair invariant kernel: for $s>0$,

$$k_s(z,\phi,z',\phi')=\exp(-2i(\phi-\phi'))\left[\frac{(yy')^{1/2}}{(z-\bar{z}')/2i}\right]^2\frac{(yy')^{s/2}}{|(z-\bar{z}')/2i|^s} . \tag{2.3}$$

By the general theory, the eigenvalues of k_s only depend on (k,λ). For $k=2$, using the special eigenfunction:

$$f(z,\phi)=\exp(-2i\phi)y^\delta ,\quad \lambda=\delta(\delta-1) ,$$

for an eigenvalue-pair $(-2i,\lambda)$ of $\left(\frac{\partial}{\partial\phi},\tilde{\Delta}_1\right)$ in $C^\infty(\tilde{H})$ (with $\lambda\neq 0$), we obtain the eigenvalue $h_s(2,\lambda)$ of k_s given by:

$$h_s(2,\lambda)=-8\pi 2^s\frac{c(s)}{\Gamma(1+s)}\Gamma\left(\frac{s}{2}+\delta\right)\Gamma\left(\frac{2+s}{2}-\delta\right) , \tag{2.4}$$

where
$$c(s)=\frac{s}{2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1+s}{2}\right)}{\left(2+\frac{s}{2}\right)} .$$

Note that the integral operator k_s, considered as an operator in $L^2(\tilde{H},\Gamma)$, vanishes on $M(k,\lambda)$ for all $k\neq 2$.

We can express the operator $T(\Gamma\alpha\Gamma)$, restricted to $M(2,\lambda)$, by k_s in the following way:

$$T(\Gamma\alpha\Gamma)F(z,\phi)=h_s(2,\lambda)^{-1}\sum_{\mu=1}^{d}\int_{\tilde{H}}\chi(\alpha_\mu)k_s(\alpha_\mu^{-1}(z,\phi)\,z',\phi')F(z',\phi')dz'd\phi' .$$

But for $s>0$, the kernel $k_s(z,\phi,z',\phi')$ is of (a)-(b) type in the sense of Selberg [8]; therefore

$$\sum_{\gamma\in\Gamma}\chi(\gamma)k_s(z,\phi,\gamma(z',\phi')) ,$$

is absolutely convergent for all (z,ϕ), $(z',\phi')\in\tilde{H}$ and uniformly, if (z,ϕ) and (z',ϕ') are contained in some compact subregion of $\tilde{H}$. Now, we have

$$\sum_{\mu=1}^{d}\int_{\tilde{H}}\chi(\alpha_\mu)k_s(\alpha_\mu^{-1}(z,\phi),z',\phi')dz'd\phi'=\int_{\Gamma\backslash\tilde{H}}\sum_{g\in\Gamma\alpha\Gamma}\chi(g)k_s(z,\phi,g(z',\phi'))dz'd\phi' .$$

With this equation, we may define $K_s(z,\phi,z',\phi')$ by:

(2.5) $$K_s(z,\phi,z',\phi')=\sum_{g\in\Gamma\alpha\Gamma}\chi(g)k_s(z,\phi,g(z',\phi')) .$$

But if the fundamental domain of Γ is non-compact, the operator K_s with the kernel $K_s(z,\phi,z',\phi')$ is not generally completely continuous. So we must modify the operator K_s by a certain integral operator H_s, with which $K_s^*=K_s-H_s$ is completely continuous (§4).

§3. Eisenstein series.

In this section, to construct an operator H_s which will be given in §4, we shall give some preparations on the Eisenstein series related to Γ and χ. Since a matter of the Eisenstein series is treated by T. Kubota in detail (for instance; [5], but it is in the case χ=trivial), we only recall fundamental definitions and facts for $\nu\geqq 1$.

3.1. Let $\kappa_1,\cdots,\kappa_h$ be representatives of all Γ-inequivalent cusps, put $\Gamma_i=\{\gamma\in\Gamma|\gamma\kappa_i=\kappa_i\}$ and denote by σ_i an element of G such that $\sigma_i\infty=\kappa_i$ and that

$$\sigma_i^{-1}\Gamma_i\sigma_i=N_\infty=\left\{\pm\begin{pmatrix}1, & n\\ 0, & 1\end{pmatrix}\middle| n\in Z\right\} .$$

Put $\Gamma^0=\ker\chi\cap\Gamma$ and $\Gamma_i^0=\ker\chi\cap\Gamma_i$. By our assumption on χ, Γ_i^0 is of finite index in Γ_i, which is denoted by r_i. Put

$$P_i=\frac{1}{r_i}\sum_{g\in\Gamma_i/\Gamma_i^0}\chi(g) .$$

Now define the Eisenstein series attached to the cusp κ_i by:

$$(3.1)\qquad E_i(z,\phi,\delta)=\sum_{(\sigma)\in\Gamma_i\backslash\Gamma}\exp(-2\sqrt{-1}(\phi+\arg(cz+d)))(\mathrm{Im}(\sigma_i^{-1}\sigma z))^{\delta}\chi(\sigma)^{-1}P_i\ ,$$

where δ is a complex number with $\mathrm{Re}(\delta)>1$, $\sigma_i^{-1}\sigma=\begin{pmatrix}*, & *\\ c, & d\end{pmatrix}$, and σ runs over a complete representative of $\Gamma_i\backslash\Gamma$. Its definition is independent of the choice of σ, so that this series is well-defined. We shall prove that it is absolutely convergent for $\mathrm{Re}(\delta)>1$. Further, E_i is automorphic, and is an eigenfunction of $\tilde{\Delta}_1$ and $\dfrac{\partial}{\partial\phi}$; namely,

(i) $E_i(\sigma(z,\phi),\delta)=\chi(\sigma)E_i(z,\phi,\delta)$, for $\sigma\in\Gamma$.

(ii) $\tilde{\Delta}_1E_i(z,\phi,\delta)=\lambda E_i(z,\phi,\delta)$, $\lambda=\delta(\delta-1)$.

(iii) $\dfrac{\partial}{\partial\phi}E_i(z,\phi,\delta)=-2\sqrt{-1}E_i(z,\phi,\delta)$.

3.2. Since the function $E_i(\sigma_j(z,\phi),\delta)$ is invariant under $z\to z+r_j$, we can consider the Fourier expansion:

$$E_i(\sigma_j(z,\phi),\delta)=\exp(-2\sqrt{-1}\phi)\sum_{m=-\infty}^{\infty}a_{ij}^m(y,\delta)\exp(2\pi m\sqrt{-1}x/r_j)\ .$$

LEMMA (i) *The constant term of this Fourier expansion is given by:*

$$(3.2)\qquad a_{ij}^0(y,\delta)=\delta_{ij}y^{\delta}P_i+y^{1-\delta}\varphi_{ij}(\delta)\ ,$$

where

$$\varphi_{ij}(\delta)=\frac{(1-\delta)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\delta-\frac{1}{2}\right)}{\delta\Gamma(\delta)}\sum_{(\sigma')_{ij}}\frac{1}{|c|^{2\delta}}P_j\chi_{ij}(c,d)^{-1}P_i\ ,$$

and $\chi_{ij}(c,d)=\chi(\sigma_i\sigma'\sigma_j^{-1})$ *for* $\sigma'=\begin{pmatrix}*, & *\\ c, & d\end{pmatrix}\in\sigma_i^{-1}\Gamma\sigma_j$, *and* $\delta_{ij}=1$ *or* 0 *according to* $i=j$ *or not, and* σ' *runs over a full representative of double cosets* $N_\infty\backslash\sigma_i^{-1}\Gamma\sigma_j/N_\infty$, *except* $c=0$.

(ii) *For* $m\neq0$,

$$(3.3)\qquad a_{ij}^m(y,\delta)=y^{1-\delta}\varphi_{ij}(\delta,m)w(my/r_j,\delta)\ ,$$

where

$$\varphi_{ij}(\delta,m)=\sum_{(\sigma')_{ij}}\frac{1}{|c|^{2\delta}}\exp(2m\pi\sqrt{-1}d/cr_j)P_j\chi_{ij}(c,d)^{-1}P_i\ ,$$

and

$$w(u,\delta)=\int_{-\infty}^{\infty}\frac{\exp(-2\pi\sqrt{-1}ut)}{(t^2+1)^{\delta}}\ \frac{t-\sqrt{-1}}{t+\sqrt{-1}}\,dt\ .$$

This can be proved by the direct calculation.

REMARK. Let K_δ be a modified Bessel function defined by the equality:

$$\int_{-\infty}^{\infty}\frac{\exp(2\pi\sqrt{-1}ut)}{(t^2+1)^{\delta}}\,dt=2\pi^{\delta}u^{\delta-(1/2)}\Gamma(\delta)^{-1}K_{\delta-(1/2)}(2\pi u)\ ,\quad(u>0)\ .$$

By a simple calculation, we get

$$w(u, \delta)=2\pi^{\delta}u^{\delta-(1/2)}\Gamma(\delta)^{-1}\left\{K_{\delta-(1/2)}(2\pi u)\left(1-\frac{2\pi u}{\delta}\right)-k_{\delta-(1/2)}(2\pi u)\frac{2\pi u}{\delta}\right\}.$$

With the help of an asymptotic expansion of modified Bessel functions, we can show that $w(u, \delta)$ converges to zero as u tends towards infinity.

It follows from the general theory of Eisenstein series that $E_i(z, \phi, \delta)$ has an analytic continuation to the whole δ-plane and is a single-valued meromorphic function.

§4. An operator H_s.

4.1. Now we shall construct an operator H_s to make $K_s^* = K_s - H_s$ completely continuous. Put $h_s(\delta)=h_s(2, \lambda)$ for simplicity, where $\lambda=\delta(\delta-1)$ and $\lambda\neq 0$. Define a function $g_s(u)$ on a real line by:

$$(4.1)\qquad g_s(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-iru)h_s(\delta)dr\ ,\quad \left(\delta=\frac{1}{2}+ir\right).$$

Now, the kernel of H_s will be defined by:

$$(4.2)\qquad H_s(z, \phi, z', \phi')=\frac{1}{8\pi^2}\sum_{i=1}^{h}\sum_{\mu=1}^{d}\chi(\alpha_\mu)\int_{\mathrm{Re}(\delta)=\frac{1}{2}}h_s(\delta)E_i(\alpha_\mu^{-1}(z, \phi), \delta)\,{}^t\overline{E_i(z', \phi', \delta)}dr\ .$$

We shall study the behaviour of the kernel $H_s(z, \phi, z', \phi')$ as z and z' tend simultaneously towards the parabolic point of Γ^0. In doing so, we can assume that $\kappa_1=\infty$ and $\Gamma_1=N_\infty$, so that $E_1(z, \phi, \delta)$ is the Eisenstein series attached to the cusp ∞. Then we estimate the following integral;

$$(4.3)\qquad \int_{\mathrm{Re}(\delta)=\frac{1}{2}}h_s(\delta)(E_1(z, \phi, \delta)-\exp(-2i\phi)(y^{\delta}P_1+\varphi_{11}(\delta)y^{1-\delta}))\exp(2i\phi')y'^{\bar{\delta}}P_1dr\ .$$

The Fourier expansion gives:

$$(4.3)=\int_{\mathrm{Re}(\delta)=\frac{1}{2}}h_s(\delta)\left(\sum_{m\neq 0}a_{11}^{m}(y, \delta)\exp(2\pi m\sqrt{-1}x/r_1)y'^{\bar{\delta}}dr\right)P_1\ .$$

If $y\gg 0$, $a_{11}^{m}(y, \delta)$ $(m\neq 0)$ is bounded by (constant)$\cdot|\Gamma(\delta)^{-1}|\exp(-\pi y|m|/r_1)$, (§3). It follows that $\sum_{m\neq 0}a_{11}^{m}(y, \delta)$ is bounded by (constant)$\cdot|\Gamma(\delta)^{-1}|\cdot\exp(-\pi y/r_1)$. On the other hand, $h_s(\delta)$ converges to zero by the order of $\exp(-\pi|r|)$ if r tends towards infinity. Then, if $y\gg 0$, we have

$$\left|\int_{-\infty}^{\infty}h_s(\delta)\left(\sum_{m\neq 0}a_{11}^{m}(y, \delta)y'^{\frac{1}{2}-\sqrt{-1}r}\right)dr\right|,$$

$$(4.4)\qquad \sim \exp(-\pi y/r_1)y'^{1/2}\int_{-\infty}^{\infty}\exp\left(-\frac{\pi}{2}|r|\right)\exp(-ir\log(y'))dr\ ,$$

$$\sim \exp(-\pi y/r_1)y'^{1/2}\exp\left(-\left(\frac{\pi}{2}+\varepsilon\right)\log(y')\right),\qquad {}^{\exists}(\varepsilon>0)\ .$$

It follows that the integral (4.3) converges to zero if z tends towards ∞. By the same way,

(4.5)

$$\int_{\mathrm{Re}(\delta)=\frac{1}{2}} h_s(\delta)(E_1(z,\phi,\delta)-\exp(-2i\phi)(y^{\delta}P_1+\varphi_{11}(\delta)y^{1-\delta}))\exp(2i\phi')y'^{1-\bar{\delta}}\ {}^t\overline{\varphi_{11}(\delta)}\,dr\ ,$$

also converges to zero if z tends towards ∞. Now we estimate the kernel $H_s(z,\phi,z',\phi')$, both z and z' tending towards κ_i. Put

$$E_j^*(z,\phi,\delta)=E_j(z,\phi,\delta)-\exp(-2\sqrt{-1}(\phi+\arg(-c_iz+a_i)))$$
$$\times\{\delta_{ij}(\mathrm{Im}\,\sigma_i^{-1}z)^{\delta}P_i+(\mathrm{Im}\,\sigma_i^{-1}z)^{1-\delta}\varphi_{ji}(\delta)\}\ ,$$

where $\sigma_i=\begin{pmatrix}a_i, & b_i\\ c_i, & d_i\end{pmatrix}$ and $\alpha_\mu\sigma_i=\begin{pmatrix}a_{i,\mu}, & b_{i,\mu}\\ c_{i,\mu}, & d_{i,\mu}\end{pmatrix}$. Then, the part of $H_s(z,\phi,z',\phi')$ which may be tending towards infinity is

$$(1)\quad \frac{1}{8\pi^2}\sum_{\alpha_\mu\kappa_i=\kappa_i}\chi(\alpha_\mu)\int_{\mathrm{Re}(\delta)=\frac{1}{2}} h_s(\delta)\ \Big\{E_i^*(\alpha_\mu^{-1}(z,\phi),\delta)\,{}^t\overline{E_i^*(z',\phi',\delta)}$$

$$(2)\quad +E_i^*(\alpha_\mu^{-1}(z,\phi),\delta)\exp(2\sqrt{-1}(\phi'+\arg(-c_iz'+a_i)))$$
$$\times((\mathrm{Im}\,\sigma_i^{-1}z')^{\bar\delta}P_i+(\mathrm{Im}\,\sigma_i^{-1}z')^{1-\bar\delta}\,{}^t\overline{\varphi_{ii}(\delta)})$$

$$(3)\quad +\exp(-2\sqrt{-1}(\phi+\arg(-c_{i,\mu}z+a_{i,\mu})))$$
$$\times((\mathrm{Im}\,\sigma_i^{-1}\alpha_\mu^{-1}z)^{\delta}P_i+(\mathrm{Im}\,\sigma_i^{-1}\alpha_\mu^{-1}z)^{1-\delta}\varphi_{ii}(\delta))\,{}^t\overline{E_i^*(z',\phi',\delta)}$$

$$(4)\quad +\exp(-2\sqrt{-1}Q(i,\mu))((\mathrm{Im}\,\sigma_i^{-1}\alpha_\mu^{-1}z)^{\delta}P_i+(\mathrm{Im}\,\sigma_i^{-1}\alpha_\mu^{-1}z)^{1-\delta}\varphi_{ii}(\delta))$$
$$\times((\mathrm{Im}\,\sigma_i^{-1}z')^{\bar\delta}+(\mathrm{Im}\,\sigma_i^{-1}z')^{1-\bar\delta}\,{}^t\overline{\varphi_{ii}(\delta)})$$

$$(5)\quad +\sum_{j\neq i}E_j^*(\alpha_\mu^{-1}(z,\phi),\delta)\,{}^t\overline{E_j^*(z',\phi',\delta)}$$

$$(6)\quad +\sum_{j\neq i}E_j^*(\alpha_\mu^{-1}(z,\phi),\delta)\exp(2\sqrt{-1}(\phi'+\arg(-c_iz'+a_i)))$$
$$\times(\mathrm{Im}\,\sigma_i^{-1}z')^{1-\bar\delta}\,\overline{\varphi_{ji}(\delta)}$$

$$(7)\quad +\sum_{j\neq i}\exp(-2\sqrt{-1}(\phi+\arg(-c_{i,\mu}z+a_{i,\mu})))$$
$$\times(\mathrm{Im}\,\sigma_i^{-1}\alpha_\mu z)^{1-\delta}\varphi_{ji}(\delta)\,{}^t\overline{E_j^*(z',\phi',\delta)}$$

$$(8)\quad +\sum_{j\neq i}\exp(-2\sqrt{-1}Q(i,\mu))(\mathrm{Im}\,\sigma_i^{-1}\alpha_\mu^{-1}z)^{1-\delta}$$
$$\times(\mathrm{Im}\,\sigma_i^{-1}z')^{1-\bar\delta}\varphi_{ji}(\delta)\,{}^t\overline{\varphi_{ji}(\delta)}\Big\}\,dr\ .$$

By the argument of §3, (1) and (5) are bounded; (2), (3), (6) and (7) are also bounded by (4.4). A part which is not actually bounded comes from (4) and (8), and is given by

$$(4.6)\qquad \frac{1}{2\pi}\sum_{\alpha_\mu\kappa_i=\kappa_i}\chi(\alpha_\mu)P_i\exp(-2\sqrt{-1}Q(i,\mu))\{(\operatorname{Im}\sigma_i^{-1}\alpha_\mu^{-1}z)(\operatorname{Im}\sigma_i^{-1}z')\}^{1/2}$$
$$\times g_s(\log(\operatorname{Im}\sigma_i^{-1}\alpha_\mu^{-1}z)-\log(\operatorname{Im}\sigma_i^{-1}z'))\ .$$

Here, $Q(i,\mu)=\phi-\phi'+\arg(-c_{i,\mu}z+a_{i,\mu})-\arg(-c_iz'+a_i)$.

4.2. Let us seek for a part which is not bounded in the kernel $K_s(z,\phi,z',\phi')$ when z and z' tend towards the fixed cusp κ_i. Since the argument of this section is treated by Kubota in [5] (Chapter V), we apply this to our case.

$$K_s(z,\phi,z',\phi')=\sum_{g\in\Gamma\alpha\Gamma}k_s(z,\phi,g(z',\phi'))\chi(g)$$

in §2. Most parts of K_s are bounded for all (z,ϕ), $(z',\phi')\in\tilde{H}$, and a partial sum which is unbounded is given by:

$$\sum_{\alpha_\mu\kappa_i=\kappa_i}\sum_{g\in\Gamma_i}k_s(z,\phi,\alpha_\mu g(z',\phi'))\chi(\alpha_\mu g)\ .$$

The above kernel is approximately equal to:

$$(4.7)\qquad \frac{1}{2\pi}\sum_{\alpha_\mu\kappa_i=\kappa_i}\chi(\alpha_\mu)P_i\exp(-2\sqrt{-1}Q(i,\mu))\{(\operatorname{Im}\sigma_i^{-1}\alpha_\mu^{-1}z)(\operatorname{Im}\sigma_i^{-1}z')\}^{1/2}$$
$$\times g_s(\log(\operatorname{Im}\sigma_i^{-1}\alpha_\mu^{-1}z)-\log(\operatorname{Im}\sigma_i^{-1}z'))\ .$$

It follows that

$$K_s^*(z,\phi,z',\phi')=K_s(z,\phi,z',\phi')-H_s(z,\phi,z',\phi')\ ,$$

is bounded for all (z,ϕ), $(z',\phi')\in\tilde{H}$; therefore, an integral operator K_s^* with the kernel $K_s^*(z,\phi,z',\phi')$ turns to be completely continuous.

4.3. Let $F(z,\phi)\in L^2(\tilde{H},\Gamma)$ be an eigenfunction of $\frac{\partial}{\partial\phi}$ and $\tilde{\Delta}$ with an eigenvalue-pair $(-2i,\lambda)$, where

$$\tilde{\Delta}=y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+y\frac{\partial}{\partial x}\frac{\partial}{\partial\phi}+\frac{5}{4}\frac{\partial^2}{\partial\phi^2}\ .$$

We shall check that an eigenvalue of F for the intergral operator K_s^* becomes equal to that for K_s.

When once (z,ϕ) is fixed, $H_s(z,\phi,z',\phi')$ is bounded if z' tends towards the cusp κ_i; so we can make H_s operate F. Moreover, the inner product

$$\int_{\Gamma\backslash\tilde{H}}{}^t\overline{E_i(z,\phi,\delta)}F(z,\phi)dzd\phi\ ,$$

is finite, since $E_i(z, \phi, \delta)$ increases at the order of $(\operatorname{Im}\sigma_i^{-1}z)^{1/2}$, and $F(z, \phi)$ increases at the order which is lower than (constant)$\times(\operatorname{Im}\sigma_i^{-1}z)^{1/2}$ by (i) when z tends towards κ_i. As $\tilde{\Delta}$ is self-adjoint, it follows that:

$$(\delta(\delta-1)-5)\int_{\Gamma\backslash\tilde{H}} {}^t\overline{E_i(z, \phi, \delta)}F(z, \phi)dzd\phi$$
$$=\lambda\int_{\Gamma\backslash\tilde{H}} {}^t\overline{E_i(z, \phi, \delta)}F(z, \phi)dzd\phi .$$

Therefore, for almost all δ,

$$\int_{\Gamma\backslash\tilde{H}} {}^t\overline{E_i(z, \phi, \delta)}F(z, \phi)dzd\phi=0_{\nu,1}\left(=\left.\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}\right\}\nu\right).$$

It follows that:

$$\int_{\Gamma\backslash\tilde{H}} H_s(z, \phi, z', \phi')F(z', \phi')dz'd\phi'=0_{\nu,1} .$$

In the result, an eigenvalue of F for K_s^* is equal to that for K_s.

4.4. We shall see that the image of K_s^* is contained in $L_0^2(\tilde{H}, \Gamma)$. It is as follows.

As K_s^* is completely continuous, the image of K_s^* consists of all eigenfunctions of K_s^*; moreover an eigenfunction, which has not the eigenvalue 0, belongs to a subspace that consists of all eigenfunctions of both $\frac{\partial}{\partial\phi}$ and $\tilde{\Delta}_1$ in $L^2(\tilde{H}, \Gamma)$. Let F be in the image of K_s^* and $F_0(y, \phi)$ the constant term of the Fourier expansion of F at the cusp κ_i, so that

$$F_0(y, \phi)=\frac{1}{r_i}\int_0^{r_i} F(\sigma_i^{-1}(z+x, \phi))dx$$
$$=\frac{1}{r_i}\int_0^{r_i} F\left(\sigma_i^{-1}\begin{pmatrix}1, & x\\ 0, & 1\end{pmatrix}(z, \phi)\right)dx .$$

Since differential operators $\frac{\partial}{\partial\phi}$ and $\tilde{\Delta}_1$ commute with an action of G, $F_0(y,\phi)$ is also an eigenfunction of $\frac{\partial}{\partial\phi}$ and $\tilde{\Delta}_1$ with an eigenvalue $-2i$ and λ, respectively. Therefore, we have

$$F_0(y, \phi)=c_1\exp(-2i\phi)y^{\delta}+c_2\exp(-2i\phi)y^{1-\delta} ,$$

where c_1 and c_2 are in C. Suppose that $F(z, \phi)\in L_0^2(\tilde{H}, \Gamma)$. As we have $\operatorname{Re}(\delta)=\operatorname{Re}(1-\delta)=\frac{1}{2}$, $F(z,\phi)$ does not belong to $L^2(\tilde{H}, \Gamma)$. This is a contradiction.

Consequently, F belongs to $L_0^2(\tilde{H}, \Gamma)$.

Let t_l be the trace of $T(\Gamma\alpha\Gamma)$ on $M(2, \lambda_l)$ for $l \geqq 0$. In the case $l=0$, we regard $\lambda_0=0$ and $h_s(2, 0)$ as the eigenvalue of k_s for the eigenfunctions belonging to $M(2, 0)$. Considering the trace of K_s^* in $L_0^2(\tilde{H}, \Gamma)$, we obtain:

$$(4.8) \qquad \sum_{l=0}^{\infty} h_s(2, \lambda_l) t_l = \operatorname{tr} \int_{\Gamma\backslash\tilde{H}} K_s^*(z, \phi, z, \phi) dz d\phi \,.$$

By the definition of K_s^*, the right hand side of (4.8) is equal to

$$\operatorname{tr} \int_{\Gamma\backslash\tilde{H}} \Big(\sum_{g \in \Gamma\alpha\Gamma} \chi(g) k_s(z, \phi, g(z, \phi)) - H_s(z, \phi, z, \phi) \Big) dz d\phi \,.$$

Let $[g]$ denote an equivalence class in $\Gamma\alpha\Gamma$ by the equivalence relation defined in §1.2., and put

$$\Gamma(g) = \{\gamma \in \Gamma \mid g = \pm \gamma g \gamma^{-1}\} \,.$$

H^* being a subregion of H obtained by subtracting the neighbourhood of each parabolic point of Γ from H, we can rewrite:

$$\sum_{g \in \Gamma\alpha\Gamma} 2\pi \operatorname{tr} \chi(g) \int_{\Gamma\backslash H^*} k_s(z, 0, g(z, 0)) dz$$
$$= \sum_{[g], g \in \Gamma\alpha\Gamma} 2\pi \operatorname{tr} \chi(g) \int_{\Gamma(g)\backslash H^*} k_s(z, 0, g(z, 0)) dz \,.$$

For simplicity, we put

$$A^*(g, s) = 2\pi \operatorname{tr} \chi(g) \int_{\Gamma(g)\backslash H^*} k_s(z, 0, g(z, 0)) dz \,,$$

Particularly, if there exists

$$\int_{\Gamma(g)\backslash H} k_s(z, 0, g(z, 0)) dz \,,$$

we put

$$A(g, s) = 2\pi \operatorname{tr} \chi(g) \int_{\Gamma(g)\backslash H} k_s(z, 0, g(z, 0)) dz \,.$$

§5. **An explicit formula for** tr $T(\Gamma\alpha\Gamma)$.

5.1. In this section, we shall calculate the trace of $T(\Gamma\alpha\Gamma)$ in $L_0^2(\tilde{H}, \Gamma)$. Firstly, we classify an element in $\Gamma\alpha\Gamma$ and afterwards we calculate "$A^*(g, s)$", for each class.

$g_0 \in \Gamma\alpha\Gamma$ is of one of the following types;

(i) $g_0 \in Z(\Gamma)$.

(ii) g_0 is elliptic.

(iii) g_0 is hyperbolic and no fixed point of g_0 is a parabolic point of Γ.

(iv) g_0 is hyperbolic and one of the fixed point of g_0 is a parabolic point of Γ.

(v) g_0 is parabolic.

Let $\mathfrak{S}_1$ (resp.: $\mathfrak{S}_2$; $\mathfrak{S}_3$) denote a complete system of inequivalent elliptic elements (resp.: hyperbolic elements leaving no parabolic point of Γ fixed; hyperbolic elements leaving a parabolic point of Γ fixed) in $\Gamma\alpha\Gamma$ with respect to the equivalence relation defined in §1.2.

5.2. *Case* i) Suppose that $\Gamma\alpha\Gamma\cap Z(\Gamma)\neq\phi$ and let g_0 be an element of $Z(\Gamma)$. We have

$$A(g_0, s)=2\pi\int_{\Gamma(g_0)\backslash H} dz\ \mathrm{tr}\ \chi(g_0)=2\pi v(\Gamma\backslash H)\ \mathrm{tr}\ \chi(g_0)\ , \tag{5.1}$$

where $v(\Gamma\backslash H)$ denotes the volume of a fundamental domain of Γ in H relative to dz.

Case ii) g_0 is elliptic. Let φ be a linear transformation that maps H into a unit circle, and a fixed point of g_0 to the origin of the circle. Let ζ, ζ^{-1} be the eigenvalues of g_0 and suppose that $\dfrac{g_0z-z_0}{g_0z-\bar{z}_0}=\zeta^2\dfrac{z-z_0}{z-\bar{z}_0}$ ($z_0\in H$ is the fixed point of g_0). By a simple calculation, we obtain:

$$A(g_0, s)=\frac{16\pi^2\bar{\zeta}^2}{[\Gamma(g_0):Z(\Gamma)]}\int_0^1\frac{(1-\rho^2)^s\rho d\rho\ \mathrm{tr}\ \chi(g_0)}{(1-\bar{\zeta}^2\rho^2)^2|1-\bar{\zeta}^2\rho^2|^s}\ .$$

It follows

$$\lim_{s\to 0} A(g_0, s)=\frac{8\pi^2}{[\Gamma(g_0):Z(\Gamma)]}\ \frac{\bar{\zeta}^2}{1-\bar{\zeta}^2}\ \mathrm{tr}\ \chi(g_0)\ . \tag{5.2}$$

Case iii) g_0 is hyperbolic and no fixed point of g_0 is a parabolic point of Γ. Let λ be an eigenvalue of g_0 and $\lambda_0>1$ be that of a generator of $\Gamma(g_0)$. We have

$$A(g_0, s)=8\pi^2 2^s c(s)\frac{\log\lambda_0^2|\lambda|^{2+s}}{(1+\lambda^2)^{1+s}|1-\lambda^2|}\ \mathrm{tr}\ \chi(g_0)\ , \tag{5.3}$$

because

$$\int_0^\pi\frac{\{(1+\lambda^2)^2\sin^2\theta-(1-\lambda^2)^2\cos^2\theta\}\sin^s\theta}{\{(1-\lambda^2)^2\cos^2\theta+(1+\lambda^2)^2\sin^2\theta\}^{2+s/2}}\,d\theta=\frac{c(s)}{(1+\lambda^2)^{1+s}|1-\lambda^2|}\ .$$

Here, $c(s)=\dfrac{s}{2}\ \dfrac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\frac{s}{2}\right)}{\Gamma\left(2+\frac{s}{2}\right)}$ (§2.3).

Case iv) g_0 is hyperbolic and leaving a parabolic point of Γ fixed. Let λ, λ^{-1} be the eigenvalues of g_0 $(\lambda>0)$. In this case, $\Gamma(g_0)\backslash H=H$ since $\Gamma(g_0)=Z(\Gamma)$. Let g_0 leave κ_i fixed. Take $Y\gg0$, and let V be a neighbourhood of κ_i obtained as follows:

$$\sigma_i^{-1}V=\{z\in H|\operatorname{Im} z>Y\}\ .$$

Put $H^*=H-\bigcup_{\sigma\in\Gamma}\sigma V$. Considering the integral on H^*, we get

$$\begin{aligned}A^*(g_0, s)&=g_s(-2\log\lambda)\frac{\lambda}{|\lambda^2-1|}\operatorname{tr}\chi(g_0)\log Y\\&-8\pi2^s\lambda^{2+s}\int_0^\pi\log((\sin\theta)^{-1})\frac{(\sin\theta)^s d\theta\operatorname{tr}\chi(g_0)}{(1-\lambda^2)^2\cos^2\theta+(1+\lambda^2)^2\sin^2\theta}\ .\end{aligned}$$

As the second term of the above equation is independent of Y, the limite of that, where s tends to zero, is given by:

$$-8\pi^2\frac{\min(\lambda,\lambda^{-1})}{|\lambda-\lambda^{-1}|}\operatorname{tr}\chi(g_0)\ . \tag{5.4}$$

Case v) B_i, Γ_i, Γ_i^0, r_i and $\mu(g)$ are in accordance with the definitions in §1.2. We can choose a set of a finite number of $\alpha_\mu\in B_i$ such that $\mu(\alpha_\mu)=a_\mu$ belongs to an interval $[0, 1)$ and that B_i is the disjoint union of cosets $\alpha_\mu\Gamma_i$. Take $Y\gg0$ and let V be a neighbourhood of the cusp κ_i defined as in case iv). Put $H^*=H-\bigcup_{\sigma\in\Gamma}\sigma V$ and consider the integral on H^*. Choosing $\gamma_i\in\Gamma_i$ such that $\Gamma_i=\langle\gamma_i\rangle$, we define $\chi_i(j)$ by $\chi(\gamma_i)^j$. With these notations, we get:

$$\begin{aligned}&\lim_{s\to0}\lim_{Y\to\infty}\left\{\sum_{g\in B_i}A^*(g, s)-\sum_{\alpha_\mu\kappa_i=\kappa_i}g_s(0)\log Y\operatorname{tr}(\chi(\alpha_\mu)P_i)\right\}\\&=-4\pi^2\sum_{\alpha_\mu\kappa_i=\kappa_i}\operatorname{tr}(\chi(\alpha_\mu)P_i)\\&-4\pi\sqrt{-1}\sum_{\alpha\kappa_i=\kappa_i}\sum_{j=1}^{r_i-1}\left\{\left(\gamma\left(\frac{j-a_\mu}{r_i}\right)+\frac{r_i}{j-a_\mu}\right)\frac{1}{r_i}\operatorname{tr}(\chi(\alpha_\mu)\chi_i(-j))\right.\\&\left.-\left(\gamma\left(\frac{j+a_\mu}{r_i}\right)+\frac{r_i}{j+a_\mu}\right)\frac{1}{r_i}\operatorname{tr}(\chi(\alpha_\mu)\chi_i(j))\right\}\\&-4\pi\sqrt{-1}\sum_{\alpha_\mu\kappa_i=\kappa_i}\left\{\gamma\left(\frac{-a_\mu}{r_i}\right)-\gamma\left(\frac{a_\mu}{r_i}\right)-\delta_\mu\frac{1}{a_\mu}\right\}\operatorname{tr}\chi(\alpha_\mu)\ ,\end{aligned}$$

where $\gamma(t)$ denotes $\lim_{n\to\infty}\left\{\sum_{m=1}^n\frac{1}{m+t}-\log(n+t)\right\}$, and δ_μ is 0 or 1 according to $a_\mu=0$, $a_\mu\neq0$ respectively. In the above calculation, we have used the following equalities:

$$\int_0^\infty\log t\exp(-t)dt=-\gamma\ \text{(Euler's constant)},$$

$$\int_0^\infty\left(\frac{\exp(-t)}{t}-\frac{\exp(-t(1+ir))}{1-\exp(-t)}\right)dt=\psi(1+ir)\,,$$

$$\int_{-\infty}^\infty\left(B\left(\frac{1}{2}+ir,\frac{1}{2}-ir\right)-B\left(\frac{3}{2}+ir,\frac{3}{2}-ir\right)\right)\psi(1+ir)dr$$

$$=-\frac{\pi}{2}\int_0^\infty\frac{\operatorname{sech}(\pi r)}{1+r^2}dr=-\pi+\frac{\pi^2}{4}\,,$$

where we denote by $\psi(z)$ the digamma function and by $B(z,z')$ the beta function. Now by a direct calculation of the right hand side, we obtain:

$$-\frac{4\pi^2}{r_i}\sum_{\{g\}\in B_i/\Gamma_i^0}\operatorname{tr}\chi(g)-\frac{4\pi^2}{\sqrt{-1}}\frac{1}{r_i}\sum_{\substack{\{g\}\in B_i/\Gamma_i^0\\ \{g\}\neq\Gamma_i^0}}\cot\left(\frac{\mu(g)}{r_i}\pi\right)\operatorname{tr}\chi(g)\,,\tag{5.5}$$

where $\{g\}$ denotes the Γ_i^0-coset of g.

5.3. Secondly, we shall calculate the trace of H_s. To do this, put

$$F_i(z,\phi,\delta)=\sum_{\mu=1}^{d}\chi(\alpha_\mu)E_i(\alpha_\mu(z,\phi),\delta)\,.$$

By §3, F_i is defined and is meromorphic in the whole δ-plane. As $F_i(\sigma_i(z,\phi),\delta)$ is invariant under an action of N_∞, we have the Fourier expansion at κ_i; its constant term $\exp(-2\sqrt{-1}\phi)\ a_i^*(y,\delta)$ is given by:

$$a_i^*(y,\delta)=\sum_{\alpha_\mu\kappa_i=\kappa_i}\frac{y^\delta}{|d_\mu|^\delta}\chi(\alpha_\mu)P_i+y^{1-\delta}\varphi_i^*(\delta)\,,$$

where
$$\varphi_i^*(\delta)=\sum_{\substack{\{\sigma\}\in N_\infty\backslash\sigma_i^{-1}\Gamma\alpha\Gamma\sigma_i/N_\infty\\ c\neq0}}\frac{1}{|c|^{2\delta}}\frac{1-\delta}{\delta}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\delta-\frac{1}{2}\right)}{\Gamma(\delta)}\chi(\sigma_i\sigma\sigma_i^{-1})P_i\,,$$

and $\alpha_\mu=\begin{pmatrix}*, & *\\ *, & d_\mu\end{pmatrix}$. Taking $Y\gg0$, define H^* as in §5.2. case v) and put $\tilde{H}^*=H^*\times(R/2\pi Z)$. The Fourier expansion of E_i and F_i gives:

$$\frac{1}{8\pi^2}\int_{\Gamma\backslash\tilde{H}^*}\int_{-\infty}^\infty h_s(\delta)\operatorname{tr}(F_i(z,\phi,\delta)^t\overline{E_i(z,\phi,\delta)})drdzd\phi$$

$$=\frac{1}{4\pi}\int_0^Y\int_{-\infty}^\infty h_s(\delta)\Big(\sum_{\alpha_\mu\kappa_i=\kappa_i}\frac{y^\delta}{|d_\mu|^\delta}\operatorname{tr}(\chi(\alpha_\mu)P_i)+y^{1-\delta}\varphi^*(\delta)\Big)drdy$$

$$-\frac{1}{8\pi^2}\int_Y^\infty\int_0^1\int_0^{2\pi}\int_{-\infty}^\infty h_s(\delta)\operatorname{tr}(F_i(\sigma_i(z,\phi),\delta)({}^t\overline{E_i(\sigma_i(z,\phi),\delta)}$$

$$-\exp(2\sqrt{-1}\phi)\,y^{\bar\delta}P_i))drdzd\phi$$

$$=\sum_{\alpha_\mu\kappa_i=\kappa_i}\frac{1}{|d_\mu|}g_s(2\log|d_\mu|)\operatorname{tr}(\chi(\alpha_\mu)P_i)\log Y+\frac{1}{4}h_s\Big(\frac{1}{2}\Big)\operatorname{tr}\Big(\varphi_i^*\Big(\frac{1}{2}\Big)\Big)$$

$$-\frac{1}{8\pi}\int_{-\infty}^{\infty} h_s(\delta)\left\{\operatorname{tr}\left(\varphi_i^{*\prime}(\delta)\,{}^t\overline{\varphi_i(\delta)}+\varphi_i^*(\delta)\,{}^t\overline{\varphi_i'(\delta)}\right)\right.$$

$$\left.+\sum_{\alpha_\mu\kappa_i\sim\kappa_i}\frac{\log|d_\mu|}{|d_\mu|^{1+2\sqrt{-1}r}}\operatorname{tr}(\chi(\alpha_\mu)P_i)\right\}dr+0(1)\,,$$

where $\varphi_i^{*\prime}(\delta)$, $\varphi_i'(\delta)$ denote $\frac{\partial}{\partial\sigma}(\varphi_i^*(\delta))$, $\frac{\partial}{\partial\sigma}(\varphi_i(\delta))$ respectively ($\delta=\sigma+\sqrt{-1}r$). Therefore,

$$\lim_{s\to 0}\lim_{Y\to\infty}\left\{\operatorname{tr} H_s-\sum_{i=1}^{h}\sum_{\alpha_\mu\kappa_i\sim\kappa_i}\frac{1}{|d_\mu|}g_s(2\log|d_\mu|)\operatorname{tr}(\chi(\alpha_\mu)P_i)\log Y\right\}=0\,.$$

In general, it is known that $\sum_{l=0}^{\infty} h_s(2,\lambda_l)t_l$ is absolutely convergent for $s>0$, hence $\sum_{l=1}^{\infty}\Gamma\left(\frac{s}{2}+\delta_l\right)\Gamma\left(\frac{2+s}{2}-\delta_l\right)t_l$ is absolutely convergent for $s>0$. By an estimation of Γ-function in a belt parallel to the imaginary line by Stirling's formula, we see that the above series is also absolutely and uniformly convergent for all $s\geqq 0$. Since $\lim_{s\to 0} c(s)=0$, we get

$$\lim_{s\to 0}\sum_{l=1}^{\infty} h_s(\delta_l)t_l=0\,.$$

On the other hand, noting that $\lim_{s\to 0} h_s(2,0)t_0=8\pi^2 t_0$, we obtain:

$$t_0=\delta\frac{1}{4\pi}v(\Gamma\backslash H)\operatorname{tr}\chi(g_0)+\sum_{[g]\in\mathfrak{S}_1}\frac{1}{[\Gamma(g):Z(\Gamma)]}\frac{\zeta^2}{1-\zeta^2}\operatorname{tr}\chi(g)$$
$$+\lim_{s\to 0}\sum_{[g]\in\mathfrak{S}_2}A(g,s)-\sum_{[g]\in\mathfrak{S}_3}\frac{\min(|\lambda|,|\lambda^{-1}|)}{|\lambda-\lambda^{-1}|}\operatorname{tr}\chi(g)$$
$$-\sum_{i=1}^{h}\left\{\frac{1}{2r_i}\sum_{\{g\}\in B_i/\Gamma_i^0}\operatorname{tr}\chi(g)-\frac{1}{2\sqrt{-1}r_i}\sum_{\substack{\{g\}\in B_i/\Gamma_i^0\\ \{g\}\neq\Gamma_i^0}}\cot\left(\frac{\mu(g)}{r_i}\pi\right)\operatorname{tr}\chi(g)\right\},\tag{5.6}$$

where $\delta=\begin{cases}1\cdots\alpha\in\Gamma\\ 0\cdots\alpha\notin\Gamma\end{cases}$, $g_0\in Z(\Gamma)$.

5.4. Finally, in order to calculate $\sum_{[g]\in\mathfrak{S}_2}A(g,s)$ definitely, we consider an integral operator $\hat{k}_s$ in $C^\infty(\tilde{H})$ which has an integral kernel defined by:

$$\hat{k}_s(z,\phi,z',\varphi')=\frac{(yy')^{\frac{2+s}{2}}}{|(z-\bar{z}')/2i|^{2+s}}\,.\tag{5.7}$$

Then an eigenvalue $\hat{h}_s(0,\lambda)$ for $\hat{k}_s$ of $f\in C^\infty(\tilde{H})$, that is an eigenfunction of $\frac{\partial}{\partial\phi}$ and $\tilde{\Delta}_1$ with an eigenvalue 0 and λ respectively, is given by:

(5.8) $$\hat{h}_s(0, \lambda)=8\pi 2^s c'(s)\frac{\Gamma\left(\frac{s}{2}+\delta\right)\Gamma\left(\frac{2}{s+2}-\delta\right)}{\Gamma(1+s)},$$

where δ satisfies $\lambda=\delta(\delta-1)$, $\lambda\neq 0$ and $c'(s)=\dfrac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(1+\frac{s}{2}\right)}$. For simplicity, we write $\hat{h}_s(\delta)$ instead of $\hat{h}_s(0, \lambda)$ after the manner of the above case. Now, by the remark in §2, we regard $\hat{k}_s$ as the operator in $L^2(\tilde{H}, \Gamma)$, and we can express the Hecke operator restricted to $M(0, \lambda_l)$ by $\hat{k}_s$ in the following way:

$$\begin{aligned} &T(\Gamma\alpha\Gamma)F(z, \phi)\\ &=h_s(\delta_l)^{-1}\sum_{\mu=1}^{d}\int_{\tilde{H}}\chi(\alpha_\mu)\hat{k}_s(\alpha_\mu^{-1}(z, \phi), z', \phi')F(z', \phi')dz'd\phi'\\ &=h_s(\delta_l)^{-1}\sum_{g\in\Gamma\alpha\Gamma}\int_{\Gamma\backslash\tilde{H}}\chi(g)\hat{k}_s(z, \phi, g(z', \phi'))F(z', \phi')dz'd\phi', \end{aligned}$$

for $F(z, \phi)\in M(0, \lambda_l)$. On the analogy in §2, we put

$$\hat{K}_s(z, \phi, z', \phi')=\sum_{g\in\Gamma\alpha\Gamma}\hat{k}_s(z, \phi, g(z', \phi'))\chi(g).$$

We define an Eisenstein series $E_i(z, \delta)$ as follows:

$$E_i(z, \delta)=\sum_{\sigma\in\Gamma_i\backslash\Gamma}(\operatorname{Im}\sigma_i^{-1}\sigma z)^\delta\chi(\sigma)^{-1}P_i.$$

It is also prolonged analytically in the whole δ-plane and becomes to a meromorphic function. Put

$$\hat{H}_s(z, \phi, z', \phi')=\frac{1}{8\pi^2}\int_{\operatorname{Re}(\delta)=\frac{1}{2}}\hat{h}_s(\delta)\left(\sum_{\mu=1}^{d}\chi(\alpha_\mu)E_i(\alpha_\mu^{-1}z, \delta)^t\overline{E_i(z', \delta)}\right)dr.$$

By the same argument as in §4, we can prove that $\hat{K}_s^*=\hat{K}_s-\hat{H}_s$ is completely continuous, and that its image is contained in $L_0^2(\tilde{H}, \Gamma)$, and that an eigenvalue for $\hat{K}_s^*$ of F, which is an eigenfunction of $\dfrac{\partial}{\partial\phi}$ and $\tilde{\Delta}_1$ in $L^2(\tilde{H}, \Gamma)$, is equal to that for $\hat{K}_s$. Therefore,

(5.9) $$\sum_{l=0}^{\infty}\hat{h}_s(0, \lambda_l)\hat{t}_l=\operatorname{tr}\int_{\Gamma\backslash\tilde{H}}\left\{\sum_{g\in\Gamma\alpha\Gamma}\hat{k}_s(z, \phi, g(z, \phi))\chi(g)-\hat{H}_s(z, \phi, z, \phi)\right\}dzd\phi.$$

Here, $\hat{t}_l$ denotes a trace of $T(\Gamma\alpha\Gamma)$ in $M(0, \lambda_l)$, and we have

$$\hat{t}_0=\sum_{\mu=1}^{d}\operatorname{tr}_{M_0}\chi(\alpha_\mu), \qquad \hat{t}_l=t_l\ (l>0),$$

where $M_0=\{c\in C^\nu\,|\,\chi(\gamma)c=c, \text{ for } \gamma\in\Gamma\}$.

Now, g_0 is hyperbolic and leaving no fixed point of Γ. Put

$$\hat{A}(g_0, s) = 2\pi \int_{\Gamma(g_0)\backslash H} \hat{k}_s(z, 0, g_0(z, 0))dz \operatorname{tr} \chi(g_0) ,$$

and λ, λ_0 as in §5.2. case iii). By the same calculation as in case iii), we get

$$\hat{A}(g_0, s) = 8\pi 2^s c'(s) \frac{\log \lambda_0^2 |\lambda|^{2+s}}{(1+\lambda^2)^{2+s}|1-\lambda^2|} \operatorname{tr} \chi(g_0) . \tag{5.10}$$

It follows

$$A(g_0, s) = \frac{s}{2} \frac{\Gamma\left(\frac{2+s}{2}\right)}{\Gamma\left(2+\frac{s}{2}\right)} \hat{A}(g_0, s) .$$

Multipling both sides of (5.9) by $\frac{s}{2} \frac{\Gamma\left(\frac{2+s}{2}\right)}{\Gamma\left(2+\frac{s}{2}\right)}$, and tending s towards zero, we get

$$8\pi^2 \hat{t}_0 = \lim_{s\to 0} \sum_{\{g\} \in \mathfrak{S}_2} A(g, s) , \tag{5.11}$$

Now, we obtain the theorem (§1).

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