On a class of deformations of holomorphic functions

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§ 0. Introduction

Throughout this paper, we shall denote by U an open set in a complex $(n+1)$-space $\boldsymbol{C}^{n+1}$ which contains the origin 0, and by S an open interval of reals $\boldsymbol{R}$.

Suppose we are given a funcition

$$F(z, s): U\times S \longrightarrow \boldsymbol{C}\ ,$$

which depends on $z\in U$ holomorphically, and on $s\in S$ real analytically. It will be always assumed that $F(0, s)=0$ for any $s\in S$. Such a function $F(z, s)$ is called a deformation of $F(z, s_0)$ to $F(z, s_1)$, for any fixed $s_0, s_1\in S$. (Cf. [4].)

In this paper, we shall concern ourselves with the problem; under what conditions have the functions $F(z, s_0)$ and $F(z, s_1)$ mutually isomorphic Milnor fiberings? This problem is studied by Oka for polynomials with isolated singularities, [4]. We shall study the general case.

Let $W=\{(z, s)\in U\times S|F(z, s)=0\}$. For a fixed $s\in S$, grad_s denotes the gradient operator in the domain $U\times\{s\}$. Let

$$\Delta=\left\{(z, s)\in U\times S \left| \begin{array}{l} F(z, s)\neq 0, \text{ and the vector} \\ \mathrm{grad}_s F(z, s)\left(=\left(\dfrac{\partial F}{\partial z_1}(z, s), \cdots, \dfrac{\partial F}{\partial z_{n+1}}(z, s)\right)\right) \\ \text{is a complex multiple of the vector } z. \end{array}\right.\right\}.$$

DEFINITION 0.1. A deformation $F(z, s)$ is called *a strong deformation with a support* $D_\varepsilon\times J$ if it satisfies the following conditions (SD1) and (SD2).

(SD1) *There are closed interval* $J\subset S$ *and a small closed disk* $D_\varepsilon\subset U$ *centered at* 0, *so that* $\mathrm{grad}_s F(z, s)\neq 0$ *for any* $(z, s)\in D_\varepsilon\times J-W$.

Let $G(t): [0, \delta)\to\boldsymbol{C}^m$ (or $\boldsymbol{R}^m$) be an analytic curve (or a function) with the Taylor expansion

$$G(t)=\boldsymbol{a}t^{\alpha}+\boldsymbol{a}_1t^{\alpha+1}+\cdots,\quad (\boldsymbol{a}\neq 0,\ \alpha\in\boldsymbol{Z})\ .$$

We shall denote the leading exponent α by l.e. $\{G(t)\}$. Put l.e. $\{0\}=+\infty$ for the

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sake of convenience.

(SD2) *If* $p(t)=(q(t), s(t)) : [0, \delta) \to D_\varepsilon \times J$ *is an analytic curve with* $q(0)=0$, $s(0) \in J$, $p(t) \in \Delta$ *for* $t>0$, *then we should have*

$$\text{l.e.}\left\{\frac{\partial F}{\partial s}(p(t))\right\} \geqq \text{l.e.}\,\{F(p(t))\}\ .$$

Our main result in this paper is the following

THEOREM 0.2. *If* $F(z, s)$ *is a strong deformation with a support* $D_\varepsilon \times J$, *the Milnor fiberings of* $F(z, s_0)$ *and* $F(z, s_1)$ *are smoothly isomorphic to each other, for any* s_0, $s_1 \in \operatorname{Int} J$.

This will be proved in §2 after a few lemmas. Some applications will be given in §3.

§1. Local stability of Milnor fiberings.

First, we shall recall the concept of the Milnor fibering [3]: Let $f : U \to C$ be a holomorphic function with $f(0)=0$.

The Fibration Theorem. There exists a positive number $\varepsilon_0>0$ *so that for any positive number* $\varepsilon \leqq \varepsilon_0$ *the complement* $S_\varepsilon - K_\varepsilon$ *is a smooth fiber bundle over a circle* S^1, *with projection mapping* $\varphi(z)=\dfrac{f(z)}{|f(z)|}$; ($S_\varepsilon$ *denotes an* ε*-sphere centered at the origin.* K_ε *is the intersection* $S_\varepsilon \cap V$, *where* $V=\{z \in U \mid f(z)=0\}$.)

Milnor proved the Fibration Theorem for polynomial functions. However, his proof is also valid for holomorphic functions by virtue of the real analytic version of the curve selection lemma, refer to [1], Lemma 2.1.

The bundle $\varphi : S_\varepsilon - K_\varepsilon \to S^1$ is called a *Milnor fibering*, and a fiber $\varphi^{-1}(e^{i\theta})$ is said to be a *Milnor fiber*. $\varphi^{-1}(e^{i\theta})$ is denoted by F_θ^ε (or simply by F_θ); ε stands for the radius of S_ε.

The following stability theorem is observed by several mathematicians independently (Cf. [1], [4], [5]).

THEOREM 1.1. *There exists a positive number* $\eta>0$ *such that for any* α, β *with* $0<\alpha<\beta<\eta$, *there is a diffeomorphism* $h : S_\alpha - K_\alpha \to S_\beta - K_\beta$ *which makes the diagram commute*:

$$\begin{array}{ccc} h\ :\ S_\alpha - K_\alpha & \longrightarrow & S_\beta - K_\beta \\ & \searrow^{\varphi} \quad {}^{\varphi}\swarrow & \\ & S^1 & \end{array}$$

REMARK. Such a number $\eta>0$ is called a *stable radius*, and a Milnor fibering $\varphi: S_\alpha - K_\alpha \to S^1$ with $0<\alpha<\eta$ is said to be a *stable Milnor fibering*.

The rest of this section is devoted to the outline of the proof of Theorem 1.1.

LEMMA 1.2 ([3]). *If* ε (>0) *is small enough*, $\operatorname{grad} f(z) \neq 0$ *for each* $z \in D_\varepsilon - V$.

This is proved by the curve selection lemma.

Define an analytic set Γ as follows:

$$\Gamma = \left\{ z \in U \,\middle|\, \begin{array}{l} f(z)\neq 0, \text{ and } \operatorname{grad} f(z) \text{ is a complex} \\ \text{multiple of the vector } z \end{array} \right\} .$$

LEMMA 1.3 ([3], [5]). *For a point* $z \in S_\varepsilon - K_\varepsilon$, *the following four conditions are equivalent*:

(i) $z \in \Gamma$.

(ii) *z is not a critical point of a smooth function* $a_\theta^\varepsilon(z) = |\log f(z)|$ *on* F_θ^ε, *where* $\theta = \arg f(z)$.

(iii) *z is not a critical point of a smooth function* $\|z\|^2$ *on* $f^{-1}(c)$, *where* $c = f(z)$.

(iv) $f|S_\varepsilon - K_\varepsilon : S_\varepsilon - K_\varepsilon \to C - \{0\}$ *is of full rank* $(=2)$ *at z*.

PROOF OF LEMMA 1.3. The equivalence of (i) and (ii) is proved by Milnor [3], Lemma 5.3. Note that the directional derivative of $\|z\|^2$ in a direction v is equal to

$$\mathscr{R}_e \langle 2z, v \rangle .$$

($\mathscr{R}_e$ means "the real part of", and $\langle\ ,\ \rangle$ is the standard Hermitian inner product of C^{n+1}.) So z is a critical point of $\|z\|^2$ on $f^{-1}(c)$ if and only if the vector z is orthogonal to the tangent space to $f^{-1}(c)$ at z, in other words, if and only if $\operatorname{grad} f(z)$ is a complex multiple of z. This proves the equivalence (i) $\Longleftrightarrow$ (iii). Let $T_z(X)$ denote the tangent space to X at z. Now $f|S_\varepsilon - K_\varepsilon$ fails to be a submersion at z if and only if

$$\dim_R \operatorname{Ker}(df_z : T_z(S_\varepsilon - K_\varepsilon) \to T_c(C - \{0\})) \geqq (2n+1) - 1 .$$

But the left-hand side is equal to $\dim_R (T_z f^{-1}(c) \cap T_z(S_\varepsilon - K_\varepsilon))$. Thus the inequality holds if and only if

$$T_z f^{-1}(c) \subset T_z(S_\varepsilon - K_\varepsilon) ,$$

because $\dim_R T_z f^{-1}(c) = 2n$. This is equivalent to $z \in \Gamma$. This completes the proof.

Q.E.D.

LEMMA 1.4 (Hamm [1], Korollar 3.8, Sakamoto [5]). *The closure* $\overline{\Gamma \cap S_\varepsilon}$ *does not intersect* K_ε.

For detailed proofs see [1], [5].

From Lemmas 1.3, 1.4 we deduce the following.

PROPOSITION 1.5 (Cf. Hamm [1], Satz 1.6). *Let* $S^1_r=\{z\in C \mid |z|=r\}$ *with* r *small enough. Then* M_r $(=f^{-1}(S^1_r)\cap D_\varepsilon)$ *is a smooth manifold with boundary, and it is a smooth fiber bundle over the circle* S^1_r, *with projection mapping* $\psi^\varepsilon_r=f|M^\varepsilon_r$. *Moreover, the "interior bundle"* $\psi^\varepsilon_r|\mathrm{Int}\,M^\varepsilon_r:\mathrm{Int}\,M^\varepsilon_r\to S^1_r$ *is isomorphic to the Milnor fibering* $\varphi:S_\varepsilon-K_\varepsilon\to S^1$.

We shall call such a bundle M^ε_r a *closed Milnor fibering.*

Note that the closure $\overline{\Gamma}$ admits a structure of a Whitney stratified set and hence it is locally a cone ([2], [6]).

Then Lemma 1.4 implies a sharper statement:

LEMMA 1.6. *There exists a small* δ *such that*

$$\overline{\Gamma}\cap V\cap D_\delta=\{0\}\,.$$

(Cf. [5].)

Now we are in a position to prove the stability Theorem 1.1.

PROOF OF THEOREM 1.1. Let η be a positive number with $0<\eta<\min\left\{\frac{\varepsilon_0}{2},\frac{\delta}{2}\right\}$, Here ε_0 (or δ) is a number mentioned in the Fibration Theorem (or in Lemma 1.6). Then for any α, β with $0<\alpha<\beta<\eta$, we have

$$(*)\qquad \Gamma\cap(\mathrm{Int}\,M^{2\beta}_r-M^{\alpha/2}_r)=\Gamma\cap f^{-1}(S^1_r)\cap(\mathrm{Int}\,D_{2\beta}-D_{\alpha/2})=\emptyset\ ,$$

provided that r is small enough. In fact, this follows from Lemma 1.6. Now we will construct a smooth vector field $\boldsymbol{v}(\boldsymbol{x})$ on $\mathrm{Int}\,M^{2\beta}_r-M^{\alpha/2}_r$ which satisfies

(i) $\boldsymbol{v}(\boldsymbol{x})$ is tangent to $f^{-1}(c)$ if $f(\boldsymbol{x})=c$, and

(ii) $m_*\boldsymbol{v}(\boldsymbol{x})=\frac{\partial}{\partial t}$, where $m:\mathrm{Int}\,M^{2\beta}_r-M^{\alpha/2}_r\to\left(\frac{\alpha}{2},2\beta\right)$ is defined by $m(\boldsymbol{x})=\|\boldsymbol{x}\|$ and $\frac{\partial}{\partial t}$ is the standard unit vector field on the open interval $\left(\frac{\alpha}{2},2\beta\right)$.

Note that by (*) together with Lemma 1.3 (iii), m is a proper submersion of $f^{-1}(c)\cap\{\mathrm{Int}\,D_{2\beta}-D_{\alpha/2}\}$ to $\left(\frac{\alpha}{2},2\beta\right)$ for each $c\in S^1_r$. Thus the required vector field is constructed first locally using this submersion and then globalized by a smooth partition of unity. The trajectories of the vector field $\boldsymbol{v}(\boldsymbol{x})$ provide us with a diffeomorphism $g:M^\alpha_r\to M^\beta_r$ which makes the diagram

$$\begin{array}{ccc} M^\alpha_r & \xrightarrow{\quad g\quad} & M^\beta_r \\ & \psi^\alpha_r\searrow \quad \swarrow\psi^\beta_r & \\ & S^1 & \end{array}$$

commute.

But by Proposition 1.5 the "interior bundle" of φ_r^α (resp. φ_r^β) is isomorphic to the Milnor fibering $S_\alpha - K_\alpha \to S^1$ (resp. $S_\beta - K_\beta \to S^1$). So Theorem 1.1 follows.

COROLLARY 1.7. *Let $\eta(>0)$ be a stable radius in the sense of Theorem 1.1. If $r(>0)$ is small enough, the interior bundle of the closed Milnor fibering $M_r^\eta \to S_r^1$ is isomorphic to the stable Milnor fibering.*

§ 2. Proof of the main theorem.

Our main theorem is stated more precisely as follows:

THEOREM 2.1. *If $F(z, s): U \times S \to \boldsymbol{C}$ is a strong deformation with a support $D_\varepsilon \times J$, for any s_0, $s_1 \in \operatorname{Int} J$ the stable Milnor fiberings of $F(z, s_0)$ and $F(z, s_1)$ are smoothly isomorphic to each other.*

We need several lemmas.

First, note that, since

$$\operatorname{grad}_s \log F(z, s) = \frac{\operatorname{grad}_s F(z, s)}{F(z, s)} ,$$

a point (z, s) belongs to the analytic set Δ (defined in § 0) if and only if the vector $\operatorname{grad}_s \log F(z, s)$ is a complex multiple of the vector z.

In the rest of this section, $F(z, s)$ is always assumed to be a strong deformation with a support $D_\varepsilon \times J$.

LEMMA 2.2. *Let $\boldsymbol{p}(t)\,(=(\boldsymbol{q}(t), s(t))): [0, \delta) \to D_\varepsilon \times J$ be an analytic curve with $\boldsymbol{p}(0) \in \{0\} \times J$, $\boldsymbol{p}(t) \in \Delta$ for $t>0$. By the above remark we may suppose that*

$$(*) \qquad \operatorname{grad}_{s(t)} \log F(\boldsymbol{p}(t)) = \lambda(t)\boldsymbol{q}(t) , \quad (\lambda(t) \in \boldsymbol{C})$$

for each $t>0$. Then the leading coefficient λ_0 of $\lambda(t)$ (in the Laurent expansion) is a positive real number.

PROOF OF LEMMA 2.2. (The proof is an analogy of Lemma 4.4 in [3].) Consider the Taylor expansion with non-zero leading coefficients:

$$\boldsymbol{q}(t) = \boldsymbol{a}t^\alpha + \boldsymbol{a}_1 t^{\alpha+1} + \cdots ,$$
$$F(\boldsymbol{p}(t)) = bt^\beta + b_1 t^{\beta+1} + \cdots ,$$
$$\operatorname{grad}_{s(t)} F(\boldsymbol{p}(t)) = \boldsymbol{c}t^\gamma + \boldsymbol{c}_1 t^{\gamma+1} + \cdots ,$$

Denote the set $\{(z, s) \in U \times S \mid F(z, s) = 0\}$ by W. The analytic set Δ does not contain any point of W, by the definition of Δ. So $F(\boldsymbol{p}(t)) \neq 0$ for $t>0$. On the other hand, we may assume $\operatorname{grad}_{s(t)} F(\boldsymbol{p}(t)) \neq 0$ for $t>0$ by the condition (SD1). The leading exponents are integers with $\alpha \geqq 1$, $\beta \geqq 1$, and $\gamma \geqq 0$.

From (*) we have

$$\mathrm{grad}_{z(t)} F(p(t)) = \lambda(t)\bar{F}(p(t))q(t) ,$$

and hence

$$(ct^{\gamma} + \cdots) = \lambda(t)(\bar{b}at^{\beta+\alpha} + \cdots) .$$

Comparing leading coefficients, we have

$$(**) \qquad c = \lambda_0 \bar{b} a .$$

Consider the power series expansion of the identity

$$\frac{d}{dt} F(p(t)) = \left\langle \frac{dq}{dt}, \mathrm{grad}_{z(t)} F(p(t)) \right\rangle + \frac{\partial F}{\partial s}(p(t)) \frac{ds}{dt} .$$

It is of the form

$$(\beta b t^{\beta-1} + \cdots) = (\alpha \langle a, c \rangle t^{\alpha-1+\gamma} + \cdots) + \frac{\partial F}{\partial s}(p(t)) \frac{ds}{dt} .$$

However, by the condition (SD2) the leading exponent l.e. $\left\{ \frac{\partial F}{\partial s}(p(t)) \right\}$ is strictly greater than $\beta-1$ which is equal to l.e. $\{F(p(t))\}-1$. Therefore, by comparing the corresponding leading coefficients, we obtain

$$\beta b = \alpha \langle a, c \rangle = \alpha \|a\|^2 \bar{\lambda}_0 b \quad \text{by } (**) .$$

So λ_0 is a positive number, completing the proof. Q.E.D.

The following is a consequence of Lemma 2.2 together with the curve selection lemma. (Cf. [3], pp. 37–40.)

PROPOSITION 2.3. *There exists a positive number* ε_0 *such that for any* $(z, s) \in D_{\varepsilon_0} \times J - W$ *the two vectors* $\mathrm{grad}_z \log F(z, s)$ *and* z *are either linearly independent over* C *or else* $\mathrm{grad}_z \log F(z, s) = \lambda z$ *where the argument of* $\lambda(\neq 0)$ *has absolute value less than* $\frac{\pi}{4}$.

Let $L_\delta = W \cap (S_\delta \times J)$.

PROPOSITION 2.4. *If a positive number* δ *is small enough, the closure* $\overline{\Delta \cap (S_\delta \times J)}$ *contains no point of* L_δ .

PROOF OF PROPOSITION 2.4. Suppose, on the contrary, that there were arbitrarily small numbers $\delta(>0)$ with $\overline{\Delta \cap (S_\delta \times J)} \cap L_\delta \neq \emptyset$. Let δ be one of those numbers. Then we would have a real analytic curve

$$p(t) = (q(t), s(t)) : [0, \varepsilon) \to S_\delta \times J$$

with $p(0) \in L_\delta$, $p(t) \in \Delta \cap (S_\delta \times J)$ for $t>0$, (thanks to the curve selection lemma applied to the situation that the closure $\overline{\Delta\cap(S_\delta\times J)}$ contains a point of L_δ). By the definition of Δ, we would have

$$(*)\qquad \mathrm{grad}_{s(t)}\log F(\boldsymbol{p}(t))=\lambda(t)\boldsymbol{q}(t),\ (\lambda(t)\in \boldsymbol{C}-\{0\})$$

for each $t>0$. From (*) it follows that

$$\frac{1}{\lambda(t)\bar{F}(\boldsymbol{p}(t))}\mathrm{grad}_{s(t)}F(\boldsymbol{p}(t))=\boldsymbol{q}(t)\ .$$

Applying $\mathscr{R}e\left\langle\frac{d\boldsymbol{q}}{dt},\cdots\right\rangle$ on both sides, we have

$$\begin{aligned}&\mathscr{R}e\left(\frac{1}{\overline{\lambda(t)F(\boldsymbol{p}(t))}}\left\langle\frac{d\boldsymbol{q}}{dt},\mathrm{grad}_{s(t)}F(\boldsymbol{p}(t))\right\rangle\right)\\&=\mathscr{R}e\left\langle\frac{d\boldsymbol{q}}{dt},\boldsymbol{q}(t)\right\rangle\\&=\frac{1}{2}\frac{d}{dt}\|\boldsymbol{q}(t)\|^2\\&=0\ .\end{aligned}$$
(Cf. Hamm [1], the proof of Lemma 3.7.)

Since

$$\frac{d}{dt}F(\boldsymbol{p}(t))=\left\langle\frac{d\boldsymbol{q}}{dt},\mathrm{grad}_{s(t)}F(\boldsymbol{p}(t))\right\rangle+\frac{\partial F}{\partial s}(\boldsymbol{p}(t))\frac{ds}{dt}\ ,$$

this implies

$$(**)\qquad \mathscr{R}e\left(\frac{1}{\overline{\lambda(t)F(\boldsymbol{p}(t))}}\left\{\frac{d}{dt}F(\boldsymbol{p}(t))-\frac{\partial F}{\partial s}(\boldsymbol{p}(t))\frac{ds}{dt}\right\}\right)=0\ .$$

Now we will prove that l.e. $\left\{\frac{d}{dt}F(\boldsymbol{p}(t))\right\}\geqq$ l.e. $\left\{\frac{\partial F}{\partial s}(\boldsymbol{p}(t))\frac{ds}{dt}\right\}$: (Since $\boldsymbol{p}(0)\in\{0\}\times J$, this inequality does not directly contradict (SD2).) Suppose, on the contrary, that the leading exponent l.e. $\left\{\frac{d}{dt}F(\boldsymbol{p}(t))\right\}<$ l.e. $\left\{\frac{\partial F}{\partial s}(\boldsymbol{p}(t))\frac{ds}{dt}\right\}$. Then the leading term of $\left\{\frac{d}{dt}F(\boldsymbol{p}(t))-\frac{\partial F}{\partial s}(\boldsymbol{p}(t))\frac{ds}{dt}\right\}$ would be equal to that of $\frac{d}{dt}F(\boldsymbol{p}(t))$. Thus considering the Taylor expansion

$$F(\boldsymbol{p}(t))=bt^\beta+b_1t^{\beta+1}+\cdots\ ,\qquad (b\neq 0,\ \beta\geqq 1)\ ,$$

we have

$$\mathscr{R}e\left(\frac{\beta}{\bar{\lambda}_0}t^{-\alpha-1}+\cdots\right)=0\ ,$$

from (**), where $\lambda_0t^\alpha+\cdots$ is the Laurent expansion of $\lambda(t)$. This implies that λ_0 is a pure imaginary number and that $|\arg\lambda(t)|\to\frac{\pi}{2}(t\to 0)$, contradicting Proposition 2.3. Therefore, $\text{l.e.}\left\{\frac{d}{dt}F(p(t))\right\}$ must be greater than or equal to $\text{l.e.}\left\{\frac{\partial F}{\partial s}(p(t))\frac{ds}{dt}\right\}$, as asserted.

We have

$$\text{l.e.}\{F(p(t))\}>\text{l.e.}\left\{\frac{d}{dt}F(p(t))\right\}\geqq\text{l.e.}\left\{\frac{\partial F}{\partial s}(p(t))\frac{ds}{dt}\right\}\geqq\text{l.e.}\left\{\frac{\partial F}{\partial s}(p(t))\right\}.$$

In particular, if $t(>0)$ is small enough, we have

$$\frac{|F(p(t))|}{\left|\frac{\partial F}{\partial s}(p(t))\right|}<\delta .$$

Since $p(t)$ is a curve on $S_\delta\times J$ with $p(t)\in\Delta\cap(S_\delta\times J)$ for $t>0$, this inequality proves that

$$A\cap\Delta\cap(S_\delta\times J)\neq\emptyset ,$$

where A is an open set defined by

$$A=\left\{(z, s)\in U\times S\,\middle|\,|F(z, s)|<\|z\|\cdot\left|\frac{\partial F}{\partial s}(z, s)\right|\right\}.$$

(A is considered to be defined by a real analytic inequality $|F(z, s)|^2<\|z\|^2\left|\frac{\partial F}{\partial s}(z, s)\right|^2$.) However, recall that we are supposing that there were arbitrarily small such a δ. Then by the curve selection lemma there would be an analytic curve, again denoted by $p(t)=(q(t), s(t)):[0, \varepsilon)\to D_\varepsilon\times J$, satisfying

(1) $p(0)\in\{0\}\times J$ and $p(t)\in\Delta$ for $t>0$.

(2) $p(t)\in A$ for $t>0$, in other words, $p(t)$ satisfies the inequality $|F(p(t))|<\|q(t)\|\cdot\left|\frac{\partial F}{\partial s}(p(t))\right|$ for each $t>0$. But this contradicts (SD2) which requires that $\text{l.e.}\{F(p(t))\}\leqq\text{l.e.}\left\{\frac{\partial F}{\partial s}(p(t))\right\}$. Now Proposition 2.4 follows by contradiction.

Q.E.D.

COROLLARY 2.4.1. *If $\delta(>0)$ is small enough, there exists a number $\eta_J^{(\delta)}>0$ so that, for any $s\in J$ the point (z, s) at which the restriction*

$$F(z, s)\big|(S_\delta\times\{s\}-K_{\delta,s}):S_\delta\times\{s\}-K_{\delta,s}\to C-\{0\}$$

fails to be a submersion all lie within the compact set $|F(z, s)|\geqq\eta_J^{(\delta)}$. ($K_{\delta,s}$ denotes $S_\delta\times\{s\}\cap W$.)

PROOF OF 2.4.1. This follows easily from Lemma 1.3, (iv) and Proposition 2.4.

Q.E.D.

Now we can prove Theorem 2.1.

PROOF OF THEOREM 2.1. Fix a positive number δ which is small enough. By (SD1) and Corollary 2.4.1, the intersection

$$N=\{(z, s) \mid |F(z, s)|=r\} \cap (D_\delta \times \mathrm{Int}\, J)$$

is a smooth manifold with boundary if r is such that $0<r<r_f^{(\delta)}$ with $r_f^{(\delta)}$ mentioned in Corallary 2.4.1. Moreover, N is a smooth fiber bundle over $S^1_r \times \mathrm{Int}\, J$ with the (proper) projection mapping $\Psi(z, s)=(F(z, s), s) : N \to S^1_r \times \mathrm{Int}\, J$. For $s \in \mathrm{Int}\, J$, let $M_s=N \cap (D_\delta \times \{s\})$, and let $\psi_s=\Psi | M_s : M_s \to S^1_r \times \{s\}$. ψ_s is a smooth fiber bundle over a circle (Proposition 1.5), and evidently the isomorphism class of ψ_s does not depend on the choice of s. However, for any fixed s_0, $s_1 \in \mathrm{Int}\, J$, one can find suitable δ and r so that the "interior bundles" of ψ_{s_i} are isomorphic to the corresponding stable Milnor fiberings of the functions $F(z, s_i)$ at the origin, $i=0$, 1 (Corollary 1.7).

The stable Milnor fiberings of $F(z, s_i)$ are, therefore, mutually isomorphic for $i=0$, 1. This completes the proof of Theorem 2.1. Q.E.D.

§ 3. Applications.

Let $u : U \to \boldsymbol{C}$ be a holomorphic function which does not vanish at the origin. Moreover, suppose $u(0)=1$ and $u(z)=1+v(z)$ with $v(0)=0$. Let $F(z, s) : U \times \boldsymbol{R} \to \boldsymbol{C}$ be defined by

$$F(z, s)=(1+sv(z))f(z) ,$$

where $f : U \to \boldsymbol{C}$ is a holomorphic function with $f(0)=0$. This is the first example of the strong deformation:

PROPOSITION 3.1. *$F(z, s)$ is a strong deformation with a support, say, $D_\varepsilon \times [-2, 2]$, where ε is a certain positive number.*

PROOF OF PROPOSITION 3.1. We have to check (SD1) and (SD2).

(SD1): If there were not such ε as is mentioned in (SD1), there would be an analytic curve $p(t)=(q(t), s(t)) : [0, \delta) \to U \times S$ with $p(0) \in \{0\} \times [-2, 2]$, $p(t) \notin W$ for $t>0$, $\mathrm{grad}_{s(t)} F(p(t))=0$ for each $t>0$. Since $\mathrm{grad}_{s(t)} F(p(t))=\mathrm{grad}\, f(q(t))+s(t)\, \mathrm{grad}\, v(q(t)) f(q(t))$, we have $\mathrm{grad}\, f(q(t))=-s(t)\, \mathrm{grad}\, v(q(t)) f(q(t))$. Supposing that t is small enough, $\mathrm{grad}\, f(q(t)) \neq 0$ for $t>0$ (Lemma 1.2). Moreover, $s(t)$ may be assumed to satisfy $|s(t)|<3$. Thus

$$\left| \frac{d}{dt} f(q(t)) \right| = \left| \left\langle \frac{dq}{dt}, \mathrm{grad}\, f(q(t)) \right\rangle \right|$$

$$<3\left|\left\langle \frac{dq}{dt}, \operatorname{grad} v(q(t))f(q(t))\right\rangle\right|$$
$$=3\left|\frac{d}{dt}v(q(t))f(q(t))\right| .$$

This is a contradiction. So (SD1) is verified.

(SD2): For any curve $p(t)=(q(t), s(t))$ with $q(0)=0$, we have

$$F(p(t))=(1+s(t)v(q(t)))f(q(t)) \text{ and } \frac{\partial F}{\partial s}(p(t))=v(q(t))f(q(t)) .$$

The requirement l.e. $\left\{\frac{\partial F}{\partial s}(p(t))\right\} \geqq$ l.e. $\{F(p(t))\}$ is obviously satisfied. Q.E.D.

COROLLARY 3.1.1. *If $u: U \to C$ does not vanish at the origin, the stable Milnor fibering of $u\cdot f$ is isomorphic to that of f.*

PROOF. Clearly f and $c\cdot f$ has the isomorphic Milnor fiberings, where c is a non-zero constant. By multiplying $\frac{1}{u(0)}$, if necessary, we may assume $u(0)=1$. The corollary follows from Proposition 3.1 and Theorem 2.1 by noting that $f(z)=F(z, 0)$, $u\cdot f(z)=F(z, 1)$. Q.E.D.

Let $h: U \to U$ be a holomorphic mapping with $h(0)=0$. Moreover, the Jacobian at 0, J_0h, is assumed to be the identity matrix I. Then

$$h(z)=z+(v_1(z), \cdots, v_{n+1}(z))=z+v(z) ,$$

where none of $v_i(z)$ contains linear terms.

Let $G(z, s): U\times \boldsymbol{R} \to \boldsymbol{C}$ be a deformation which is defined by

$$G(z, s)=f(z+sv(z)) .$$

PROPOSITION 3.2. *$G(z, s)$ is a strong deformation with a support, say, $D_\epsilon\times[-2, 2]$.*

PROOF OF PROPOSITION 3.2. Set

$$Y(z)=\begin{pmatrix} \overline{\frac{\partial v_1}{\partial z_1}}, & \cdots, & \overline{\frac{\partial v_1}{\partial z_{n+1}}} \\ \vdots & & \vdots \\ \overline{\frac{\partial v_{n+1}}{\partial z_1}}, & \cdots, & \overline{\frac{\partial v_{n+1}}{\partial z_{n+1}}} \end{pmatrix},$$

Then $Y(0)=0$ and

$$(*) \qquad \operatorname{grad}_s G(z, s)=\operatorname{grad} f\cdot(I+sY(z)) .$$

(SD1) follows from (*). The verification of (SD2) goes as follows: Let $p(t)=(q(t), s(t)): [0, \delta) \to U\times S$ be such a curve as is in (SD2). Since $p(t)\in \Delta$ for each

$t>0$, we have

$$(**)\qquad \operatorname{grad}_{s(t)} G(\boldsymbol{p}(t))=\lambda(t)\boldsymbol{q}(t)\ , \quad (\lambda(t)\in \boldsymbol{C}-\{0\})$$

for $t>0$. Consider the Taylor (or Laurent) expansions with non-zero leading coefficients:

$$\boldsymbol{q}(t)=\boldsymbol{a}t^{\alpha}+\boldsymbol{a}_1 t^{\alpha+1}+\cdots\ ,$$
$$\lambda(t)=\lambda_0 t^{\varepsilon}+\lambda_1 t^{\varepsilon+1}+\cdots\ ,$$
$$\operatorname{grad} f(z(t))=\boldsymbol{c}t^{\gamma}+\boldsymbol{c}_1 t^{\gamma+1}+\cdots\ ,$$

where $z(t)=\boldsymbol{q}(t)+s(t)\boldsymbol{v}(\boldsymbol{q}(t))$. From (*) we have

$$\begin{aligned}\operatorname{grad}_{s(t)} G(\boldsymbol{p}(t))&=\operatorname{grad} f(z(t))(I+s(t)Y(\boldsymbol{q}(t)))\\ &=\boldsymbol{c}t^{\gamma}+\text{higher terms.}\end{aligned}$$

Comparing the leading terms with (**) we obtain

$$\boldsymbol{c}=\lambda_0\boldsymbol{a}\ .$$

Let $A(t)=\left\langle \dfrac{d\boldsymbol{q}}{dt},\ \operatorname{grad}_{s(t)} G(\boldsymbol{p}(t))\right\rangle$. Then

$$\begin{aligned}A(t)&=\langle \alpha\boldsymbol{a}t^{\alpha-1}+\cdots,\ \boldsymbol{c}t^{\gamma}+\cdots\rangle\\ &=\alpha\langle \boldsymbol{a},\boldsymbol{c}\rangle t^{\alpha-1+\gamma}+\cdots\\ &=\alpha\bar{\lambda}_0\|\boldsymbol{a}\|^2 t^{\alpha-1+\gamma}+\cdots\ .\end{aligned}$$

So l.e. $\{A(t)\}=\alpha-1+\gamma$. On the other hand, note that

$$\frac{\partial G}{\partial s}(\boldsymbol{p}(t))=\langle \boldsymbol{v}\boldsymbol{q}(t)),\ \operatorname{grad} f(z(t))\rangle\ .$$

It follows that l.e. $\left\{\dfrac{\partial G}{\partial s}(\boldsymbol{p}(t))\right\}\geqq v+\gamma$, where $v=$l.e. $\{\boldsymbol{v}(\boldsymbol{q}(t))\}$. But $\boldsymbol{v}(z)$ contains no constant terms. So $v=$l.e. $\{\boldsymbol{v}(\boldsymbol{q}(t))\}\geqq$l.e. $\{\boldsymbol{q}(t)\}=\alpha$, and we have

$$(***)\qquad \text{l.e.}\left\{\frac{\partial G}{\partial s}(\boldsymbol{p}(t))\right\}\geqq \alpha+\gamma>\text{l.e. }\{A(t)\}\ .$$

Recall that

$$\frac{d}{dt}G(\boldsymbol{p}(t))=A(t)+\frac{\partial G}{\partial s}(\boldsymbol{p}(t))\frac{ds}{dt}\ .$$

(***) implies l.e. $\left\{\dfrac{d}{dt}G(\boldsymbol{p}(t))\right\}<$l.e. $\left\{\dfrac{\partial G}{\partial s}(\boldsymbol{p}(t))\right\}$. But l.e. $\left\{\dfrac{d}{dt}G(\boldsymbol{p}(t))\right\}=$l.e. $\{G(\boldsymbol{p}(t))\}$ -1, so l.e. $\{G(\boldsymbol{p}(t))\}\leqq$l.e. $\left\{\dfrac{\partial G}{\partial s}(\boldsymbol{p}(t))\right\}$ as required. Q.E.D.

COROLLARY 3.2.1. *Let $h:U\to U$ be a holomorphic mapping such that*

$h(0)=0$ and that the Jacobian at 0, J_0h, is non-singular. Then the stable Milnor fiberings of f and $f\circ h$ are isomorphic to each other.

Although this corollary looks quite trivial, the author does not know any "trivial" proof.

PROOF OF 3.2.1. Let $L=J_0h$. By the Corollary 3.3.1 below, $f\circ h(z)$ and $f\circ h\circ L^{-1}(z)$ have the isomorphic Milnor fiberings. So we may assume that $L=I$. Then Corollary 3.2.1 follows from Proposition 3.2. Q.E.D.

Let $L(s):(-3,3)\to GL_{n+1}(C)$ be an analytic path.

PROPOSITION 3.3. *A deformation $F(z,s)$ defined by $F(z,s)=f(z\cdot L(s))$ is a strong deformation with a support $D_\varepsilon\times[-2,2]$.*

PROOF OF PROPOSITION 3.3. First, note that

$$(*)\qquad \operatorname{grad}_s F(z,s)=\{\operatorname{grad} f(z\cdot L(s))\}L^*(s)\ ,$$

where $L^*(s)=\text{transpose } \overline{L(s)}$. (SD1) follows from this. Again let $p(t)=(q(t),s(t)):[0,\delta)\to U\times S$ be an analytic curve such that $p(0)\in\{0\}\times[-2,2]$ and that $p(t)\in\Delta$ for $t>0$. Then

$$(**)\qquad \operatorname{grad}_{s(t)} F(p(t))=\lambda(t)q(t)\ ,\quad (\lambda(t)\in C-\{0\})\ ,$$

for $t>0$. Consider the Taylor (or Laurent) expansions with non-zero leading coefficients: $q(t)=at^\alpha+\cdots$, $\lambda(t)=\lambda_0t^e+\cdots$, and $\operatorname{grad}_{s(t)}F(p(t))=ct^\gamma+\cdots$. From (**) we have $c=\lambda_0 a$. Let $A(t)=\left\langle\dfrac{dq}{dt},\ \operatorname{grad}_{s(t)}F(p(t))\right\rangle$. Then $A(t)=\langle\alpha at^{\alpha-1}+\cdots,\ ct^\gamma+\cdots\rangle=\alpha\langle a,c\rangle t^{\alpha-1+\gamma}+\cdots=\alpha\|a\|^2\bar\lambda_0t^{\alpha-1+\gamma}+\cdots$. So we have l.e. $\{A(t)\}=\alpha-1+\gamma$. On the other hand,

$$\frac{\partial F}{\partial s}(p(t))=\left\langle q(t)\cdot\frac{\partial L}{\partial s}(s(t)),\ \operatorname{grad} f(q\cdot L(s(t)))\right\rangle .$$

From (*) we have l.e. $\{\operatorname{grad} f(q\cdot L(s(t)))\}=$ l.e. $\{\operatorname{grad}_{s(t)}F(p(t))\}=\gamma$. So l.e. $\left\{\dfrac{\partial F}{\partial s}(p(t))\right\}\geqq\alpha+\gamma>$ l.e. $\{A(t)\}$. (SD2) follows from this, (Cf. the proof of Proposition 3.2).

Q.E.D.

COROLLARY 3.3.1. *Let $L\in GL_{n+1}(C)$. $f(z)$ and $f(z\cdot L)$ have mutually isomorphic stable Milnor fiberings.*

$GL_{n+1}(C)$ is a connected analytic manifold. So L is connected to the identity matrix I by a piecewise analytic curve. The corollary follows from Proposition 3.3.

References

[1] Hamm, H., Lokale toplogische Eigenshaften Komplex Räume, Math. Ann. **191** (1971),

235-252.
[2] Mather, J., Notes on topological stability, Lecture Note, Harvard University, (1970).
[3] Milnor, J., Singular points of complex hypersurfaces, Ann. of Math. Studies, **61** (1968).
[4] Oka, M., Local deformation of polynomials with isolated singularities, "Geometry of Manifolds" R. I. M. S. Symposium, Kyoto University, (1972).
[5] Sakamoto, K., On the Milnor fibering and its characteristic map, (to appear).
[6] Thom, R., Ensembles et morphismes stratifiés, Bull. Amer. Math. Soc., **75** (1969), 240-284.

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