Knot cobordism groups and surgery in codimension two

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CONTENTS

Introduction.

Knot cobordism groups were introduced by Fox and Milnor [3] in the classical knot theory, and were extended by Kervaire [11] and Levine [13] to the higher dimensional knots of (homeomorphy) spheres. Levine analyzed these groups extensively [13], [14].

In the present paper, we will study an analogy of knot cobordism in manifold pairs of codimension 2.

Let W^{m+2} be an oriented compact connected $(m+2)$-manifold, K^{m-1} a locally flat oriented $(m-1)$-submanifold in ∂W, the boundary of W. Suppose the pair (W^{m+2}, K^{m-1}) is simple homotopy equivalent to a finite Poincaré pair (X, Y) of formal dimension m. Then our problem is:

Under what conditions can we find *locally flat* submanifold L^m of W^{m+2} such that $\partial L^m = K^{m-1}$ and such that the inclusion $i:(L^m, K^{m-1}) \subset (W^{m+2}, K^{m-1})$ is a simple homotopy equivalence of the pairs?

We will call such a submanifold L^m a (locally flat) *spine* of the pair (W^{m+2}, K^{m-1}). (In [17], the terminology *homotopy spine* was used.)

For an illustration, consider the case where W^{m+2}, K^{m-1} and (X, Y) are an $m+2$-disk D^{m+2}, an $(m-1)$-sphere Σ^{m-1} and the pair (D^m, S^{m-1}), respectively. Then the pair (D^{m+2}, Σ^{m-1}) admits a locally flat spine if and only if the $(m-1, m+1)$-knot

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$(\partial D^{m+2}, \Sigma^{m-1})$ is null-cobordant (Levine [13]). Therefore, the $(m-1, m+1)$-knot cobordism group C_{m-1} may be taken as the obstruction group to finding a locally flat spine of the pair (D^{m+2}, Σ^{m-1}).

Consider another special case where $Y=\phi$ and $K^{m-1}=\phi$. This situation turns out to be a codimension 2 analogy of the Browder-Casson-Sullivan theorem [4] [6]. Kato first studied this case with $X=S^m$ [8].

In this paper we will introduce algebraically certain abelian groups $P_m(\mathscr{E})$ which are functorially determined by short exact sequences $\mathscr{E}$ (extensions),

$$\mathscr{E}: 1 \longrightarrow C \longrightarrow \pi \longrightarrow \pi' \longrightarrow 1 ,$$

where C is a (finite or infinite) cyclic group with a specified generator t in the centre of a finitely presented group π. In other words, our groups $P_m(\mathscr{E})$ are determined by elements in the 2-nd cohomology groups of π'

$$H^2(\pi'; C) ,$$

where the π'-action on C is trivial.

We will show that these groups $P_m(\mathscr{E})$ play the role of obstruction groups to finding locally flat spines.

Given a pair (W^{m+2}, K^{m-1}) which satisfies the conditions described above, we can canonically associate an extension $\mathscr{E}$ with it (§§ 1.4, 1.5). The following is our main result:

THEOREM (5.10). *Suppose $m\geqq 5$. A pair (W^{m+2}, K^{m-1}) admits a locally flat spine if and only if a well-defined obstruction element $\eta(W, K)\in P_m(\mathscr{E})$ equals zero. Any element of $P_m(\mathscr{E})$ with $m\geqq 6$ is geometrically realized in this way.*

Our groups are related to the (orientable) Wall groups and the knot cobordism groups C_{m-1} as follows.

THEOREM (cf. Concluding remarks in § 5, and also § 6.2.).

(*i*) $P_{2k+1}(1\longrightarrow C\longrightarrow \pi\longrightarrow \pi'\longrightarrow 1)=L_{2k+1}(\pi')$. (*This is rather considered as the definition of the groups $P_{odd}(\mathscr{E})$.*)

(*ii*) $P_m(1\longrightarrow 1\longrightarrow \pi\longrightarrow \pi\longrightarrow 1)=L_m(\pi)$, *in particular,*
$P_m(1\longrightarrow 1\longrightarrow 1\longrightarrow 1\longrightarrow 1)=L_m(1)$, *the Kervaire-Milnor group.*

(*iii*) $P_m(1\longrightarrow \boldsymbol{Z}\longrightarrow \boldsymbol{Z}\longrightarrow 1\longrightarrow 1)=C_{m-1}$ $(m\geqq 5)$.

Like the Wall groups and the knot-cobordism groups for $m\geqq 5$, the $P_m(\mathscr{E})$ have an algebraic periodicity of period 4, as one can easily see from the definition of $P_m(\mathscr{E})$. However, the periodicity can also be obtained geometrically by multiplying by a comlex projective plane CP_2:

THEOREM (5.12). *Suppose $m\geqq 5$. We have*

$$\eta(W^{m+2}\times CP_2, K^{m-1}\times CP_2)=\rho(\eta(W^{m+2}, K^{m-1}))\ ,$$

where $\rho: P_m(\mathscr{E}) \xrightarrow{\sim} P_{m+4}(\mathscr{E})$ *is the algebraic periodicity.*

Periodicity of this type for knot cobordism groups was conjectured by López de Medrano [15].

The algebraic calculation of these groups $P_m(\mathscr{E})$ seems very difficult, especially when $C\neq\{1\}$.

The following theorem naturally leads us to cojecture that if $C\neq\{1\}$ the groups $P_m(\mathscr{E})$ are very large for m even.

THEOREM (6.6). $P_{4k+2}(1\longrightarrow C\longrightarrow C\longrightarrow 1\longrightarrow 1)$ *is infinitely generated if* C *is a cyclic group of even (or infinite) order.*

REMARK. For any cyclic group C, $P_m(1\longrightarrow C\longrightarrow C\longrightarrow 1\longrightarrow 1)$ is a quotient group of the $(m-1, m+1)$-knot cobordism group C_{m-1}; so the group contains at most countably many elements (cf. Concluding remark B in §5.).

Our method is essentially the surgery technique; first we find a submanifold L_0^m of W^{m+2} such that $\partial L_0^m=K^{m-1}$, and then perform a Wall surgery on it. However, we must perform the surgery within the ambient manifold W^{m+2} (the ambient surgery in codimension 2). It is one of our main tasks to decide whether or not the desired Wall surgery can be carried out as the ambient surgery in codimension 2.

Now the problem is divided into two cases; the even and the odd dimensional cases.

The odd dimensional case is rather easy. In the simply-connected cases, it is essentially done by Kervaire [11] who showed that if L^{2k+1} is a locally flat $2k+1$-submanifold of a $2k+3$-disk D^{2k+3} with $\partial L^{2k+1}=\Sigma^{2k}$, a $2k$-sphere in ∂D^{2k+2}, then by a sequence of ambient surgeries, L^{2k+1} can be made contractible. This observation was extended to the non-simply connected ambient surgery by Kato and the author [10]; there we showed that in the odd-dimensional case there is no essential difference between ambient and abstract surgeries.

However, in the even dimensional case, the two types of surgeries display somewhat different features, and in this paper, we will be mainly concerned with this case. The difference of the two surgeries was first algebraically described in the case of knots by Levine [13] in terms of Seifert matrices. However, his method is too special to be applied directly to manifold pairs. In this paper, we will introduce an intersection form (λ, μ) associated with an even-dimensional submanifold in codimension 2. This form is strongly related to Wall's special Hermitian form of the submanifold, and it represents the obstruction to performing the "ambient" Wall surgery on the submanifold in codimension 2. In contrast to Wall's form, our form is not generally nonsingular, Roughly speaking, the deviation from nonsingularity measures the extent to which the submanifold is "knotted" — the principle of the Alexander polynomial.

We will call our form (λ, μ) a *Seifert form*, for (λ, μ) generalizes Levine's Seifert matrix on the one hand, and on the other hand the definition of λ (Formula (6)) is very similar to that of a form defined by Seifert [22].

The groups $P_m(\mathscr{E})$ are defined in terms of Seifert forms (§5).

The contents of the paper are as follows:

In §1, we define an element $\lambda(f, g)$ in the group-ring $\Lambda = \boldsymbol{Z}[\pi]$ and $\mu(f)$ in an abelian group $Q'_n(\pi) = \Lambda/\{a - (-1)^n \bar{a}t \mid a \in \Lambda\}$ which are defined by pathed "nice" immersions f, g. (Formulae (6), (9).)

§2 proves that these elements $\lambda(f, g)$ and $\mu(f)$ depend only on the homotopy classes of f and g. (Th. 2.5 and Th. 2.9.)

§3 gives some general properties of (λ, μ).

§4 describes necessary and sufficient conditions in terms of (λ, μ) so that (W^{m+2}, K^{m-1}) admits a locally flat spine (Th. 4.12). Lemma 4.5 will clarify the relationship between (λ, μ) and Wall's Hermitian forms.

§5 introduces the group $P_m(\mathscr{E})$ and reformulates the results of §4 (Th. 5.10).

§6 deals with the simply-connected cases.

Our results in the simply-connected cases were announced in [17].

REMARK. In [17], we assumed condition (H), but in fact it is not necessary.

Throughout this paper, we will work in the *PL* or Diff categories.

Finally, the author expresses his warmest thanks to Professor I. Tamura for many inspiring suggestions and constant encouragement. Discussions with Professor M. Kato were also very useful.

Added in proof:

Recently, Cappell and Shaneson have developed a different method of surgery in codimension two from homology surgery point of view, [31] [32]. The relationship between their theory and ours will be clarified in the furture.

CHAPTER I. SEIFERT FORMS.

— Intersection forms associated with submanifolds in codimension 2 —

§1. Definition of (λ, μ)

The purpose of this and the subsequent sections is to extend the definition of the Seifert matrix of knots [13] to the case of even-dimensional manifold pairs with codimension 2 which satisfy some reasonable conditions. It appears as a certain "quadratic form" (λ, μ) over a group-ring rather than a matrix.

To be precise, let E be the exterior of the submanifold, $\mathscr{F}N$ the frontier of a tubular neighbourhood N. Then λ, μ are maps

$$\lambda : \pi_{n+1}(E, \mathscr{F}N) \times \pi_{n+1}(E, \mathscr{F}N) \longrightarrow \Lambda \qquad (n+1=1/2 \dim E) ,$$

$$\mu : \pi_{n+1}(E, \mathscr{F}N) \longrightarrow Q'_n(\pi_1 \mathscr{F}N) ,$$

where $\Lambda = \boldsymbol{Z}[\pi_1 \mathscr{F}N]$ and $Q'_n(\pi_1 \mathscr{F}N) = \Lambda / \{\nu - (-1)^n \bar{\nu} t | \nu \in \Lambda\}$, t is a certain element of $\pi_1 \mathscr{F}N$ which will be defined later.

In this section we define elements $\lambda(f, g) \in \Lambda$ and $\mu(f) \in Q'_n(\pi_1 \mathscr{F}N)$ for pairs of certain pathed nice immersions $f, g : (D^{n+1}, S^n) \longrightarrow (E, \mathscr{F}N)$ (Formulae (6), (9)). In the next section, we will prove that these elements depend only on the homotopy classes of f and g.

Before defining $\lambda(f, g)$ and $\mu(f)$, we shall introduce some preliminary notions needed later, such as Poincaré thickening, exterior k-connectivity, the cyclic extension associated with a Poincaré thickening, etc. This occupies sections 1.1 through 1.8. In what follows, to simplify the situation we define (λ, μ) only for a closed submanifold of a Poincaré thickening (1.1). One may easily extend the definition in more general situations, for example in the case of submanifolds with boundary. In fact this general setting will appear after §4.

DEFINITION 1.1. A compact connected orientable $m+2$-manifold W^{m+2} is called an *m-Poincaré thickening* if it is simple homotopy equivalent to a (simple) Poincaré complex with the formal dimension m. An m-Poincaré thickening is *special* if an orientation $[W^{m+2}]$ (as an $m+2$-manifold) and a fundamental class $\mu \in H_m(W; \boldsymbol{Z})$ (as a Poincaré complex) are specified.

Let L^m be a locally flat closed submanifold of Int W^{m+2}. We use the following notations:

N: a regular (or tubular) neighbourhood of L^m in W^{m+2}. Since $\widetilde{PL}(2) \simeq 0(2)$ (Kato [7], Wall [28]), N has a structure of the total space of a 2-disk bundle over L^m.

E: the exterior of N, i.e., E=closure $(W-N)$.

$\mathscr{F}N$: the frontier of N, i.e., $\mathscr{F}N = N \cap E$. $\mathscr{F}N$ is the total space of an S^1-bundle over L^m.

ϖ: the projection map of the S^1-bundle $\mathscr{F}N \longrightarrow L$.

REMARK. These definitions can be used also for a proper submanifold L^m with a boundary (i.e. $L^m \cap \partial W = \partial L$).

DEFINITION 1.2. Let k be an integer $\geqq 0$. A submanifold L^m is *exterior k-connected* if $\pi_i(E, \mathcal{T}N)=0$ for any i, $0\leqq i\leqq k$.

LEMMA 1.3. *Let L^m be an exterior k-connected submanifold of W^{m+2} with $k\geqq 2$. Then we have*
(i) $\pi_1(\mathcal{T}N)\cong\pi_1(E)$,
(ii) the inclusion $i: L^m \longrightarrow W^{m+2}$ is k-connected in the usual sense, that is, $\pi_j(W, L)=0$ for $j\leqq k$.

PROOF OF 1.3. (i) is trivial by the definition of exterior k-connectivity. (ii) is proved in [10]. Here we give an indication of a proof. Suppose that $x\in\pi_j(W, L)$ is represented by an immersion $f: (D^j, S^{j-1})\longrightarrow(W, L)$. Let f be in general position with respect to $\mathcal{T}N$ and consider the inverse image $f^{-1}(E)\subset \mathrm{Int}\, D^j$, which is a submanifold of D^j. A handlebody decomposition of $f^{-1}(E)$ of the form

$$\partial f^{-1}(E)\times I\cup 0\text{-handles}\cup 1\text{-handles}\cup\cdots,$$

consists of i-handles with $i\leqq k$. Therefore by the exterior k-connectivity, the image of $f^{-1}(E)$ can be shrinked handlewise into N, which has L^m as a deformation retract. So we have proved that $x=0$. Q.E.D.

LEMMA 1.4. *Let L^m be a locally flat exterior 2-connected submanifold of a special Poincaré thickening W^{m+2} which represents the fundamental class μ of the m-Poincaré complex W^{m+2}. Then we have an exact sequence:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C & \longrightarrow & \pi_1(\mathcal{T}N) & \xrightarrow{\varpi_2} & \pi_1(L^m) & \longrightarrow & 1, \\ & & & & \| \wr & & \| \wr & & (1.3) \\ & & & & \pi_1(E) & & \pi_1(W) & & \end{array}$$

where C is a cyclic group in the centre of $\pi_1(\mathcal{T}N)$. Moreover a generator of C denoted by t is canonically specified. We shall call t the special generator.

PROOF OF 1.4. The first assertion is obvious from the exact sequence of the S^1-bundle:

$$\pi_2(L^m)\longrightarrow\pi_1(S^1)\longrightarrow\pi_1(\mathcal{T}N)\longrightarrow\pi_1(L^m)\longrightarrow 1,$$

We define C by $\mathrm{Coker}\,(\pi_2(L^m)\longrightarrow\pi_1(S^1))$. That C is in the centre of $\pi_1(\mathcal{T}N)$ is proved, for example, in [23, pp. 445–446]. N, E and L^m are oriented by the orientation $[W^{m+2}]$ and the fundamental class $\mu\in H_m(W; Z)$. (The orientations $[N]$, $[E]$ are induced from $[W]$, and the orientation $[L^m]$ is taken so that $i_*[L^m]=\mu$.)

To specify the generator of C, we will canonically choose the orientation of the S^1-fibres as follows:

Let v be a vector field on $\mathscr{F}N$ which is normal to $\mathscr{F}N$ and points inward into E.

$\mathscr{F}N$ is oriented by the formula

$$(1) \qquad [E]=[\mathscr{F}N]\times v ,$$

where $\times$ is homology cross product.

Let $*$ be a base point of $\mathscr{F}N$. Choose a local cross section $s: U \longrightarrow \mathscr{F}N$ which passes $*$, where U is an open set of L^m. Then the fibre S^1 passing $*$ is oriented by using the local orientation $[U]$:

$$(2) \qquad [\mathscr{F}N]=s_*[U]\times[S^1] .$$

In the following, the formula (2) is written more intuitively:

$$(2)' \qquad [\mathscr{F}N]=[L^m]\times[S^1] .$$

The special generator of C is defined to be the image of the "positive generator" of $\pi_1(S^1)$ under the map $\pi_1(S^1)\longrightarrow C$. Q.E.D.

LEMMA 1.5.

(*i*) *Any extension of finitely presented groups*

$$1\longrightarrow C\longrightarrow B\longrightarrow A\longrightarrow 1$$

such that C is a (finite or infinite) cyclic group contained in the centre of B, can be realized by an oriented S^1-bundle over an oriented manifold.

(*ii*) *Let L and L' be locally flat exterior 2-connected closed submanifold of W^{m+2} both of which represent the fundamental class μ. Then the corresponding extensions*

$$1\longrightarrow C\longrightarrow \pi_1(\mathscr{F}N)\longrightarrow \pi_1(L)\longrightarrow 1$$

and

$$1\longrightarrow C'\longrightarrow \pi_1(\mathscr{F}N')\longrightarrow \pi_1(L')\longrightarrow 1$$

are mutually isomorphic. The isomorphism sends the special generator of C to that of C'.

PROOF OF 1.5. (i) is observed by Wall [29, p. 125]. Let us prove (ii). Denote by $\mu^*\in H^2(W,\partial W; Z)$ the dual of the fundamental class $\mu\in H_m(W)$. Take f as a composition

$$W\longrightarrow W/\partial W\xrightarrow{\mu^*} K(Z,2)\simeq \mathrm{BSO}(2)\simeq \mathrm{MSO}(2) .$$

Then f is identified by the Pontrjagin-Thom construction starting with L (or L'):

$$W\longrightarrow W/E\longrightarrow \mathrm{BSO}(2)\simeq \mathrm{MSO}(2) .$$

Let ξ be the S^1-bundle over W which is induced by f. Then $\xi|L$ (or $\xi|L'$) is isomorphic to $\mathscr{T}N \longrightarrow L$ (or $\mathscr{T}N' \longrightarrow L'$), and we have the following diagram:

$$
\begin{array}{lccccccccc}
 & \pi_2(L) & \longrightarrow & \pi_1(S^1) & \longrightarrow & \pi_1(\mathscr{T}N) & \longrightarrow & \pi_1(L) & \longrightarrow & 1 \\
\text{(1.3 ii)} & \downarrow \text{onto} & & \downarrow \cong & & \downarrow & & \downarrow \cong & & \\
 & \pi_2(W) & \longrightarrow & \pi_1(S^1) & \longrightarrow & \pi_1(\xi) & \longrightarrow & \pi_1(W) & \longrightarrow & 1 \\
\text{(1.3 ii)} & \uparrow \text{onto} & & \uparrow \cong & & \uparrow & & \uparrow \cong & & \\
 & \pi_2(L') & \longrightarrow & \pi_1(S^1) & \longrightarrow & \pi_1(\mathscr{T}N') & \longrightarrow & \pi_1(L') & \longrightarrow & 1\,.
\end{array}
$$

By the five lemma, we have the desired conclusion. Q.E.D.

TERMINOLOGY 1.6. Let L^m be an exterior 2-connected submanifold of a special m-thickening W^{m+2} which represents the fundamental class μ. Then the extension

$$1 \longrightarrow C \longrightarrow \pi_1(\mathscr{T}N) \longrightarrow \pi_1(L) \longrightarrow 1$$

is called the *cyclic extension associated with* W^{m+2}, where $C = \mathrm{Coker}\,(\pi_2(L) \longrightarrow \pi_1(S^1))$. By Lemma 1.5, this does not depend on the choice of L^m.

Hereafter we shall assume that m is an even integer, $m = 2n \geqq 4$, and that L^m is an exterior 2-connected locally flat submanifold of a special m-thickening W^{m+2}.

Let D^{n+1} be an $n+1$-disk, S^n its boundary sphere.

DEFINITION 1.7. A map $f: (D^{n+1}, S^n) \longrightarrow (E^{2n+2}, \mathscr{T}N)$ is a *nice immersion* if it satisfies the conditions:

(i) f is a generic immersion in the sense of Haefliger [5],

(ii) $f|S^n : S^n \longrightarrow (\mathscr{T}N)^{2n+1}$ is an embedding, and

(iii) the composition $\varpi \circ (f|S^n): S^n \longrightarrow L^{2n}$ is a generic immersion.

A nice immersion f is said to be *pathed* when we specify a path in $\mathscr{T}N$, $\gamma(f)$ from the base point $* \in \mathscr{T}N$ to a point in the image $f(S^n)$.

DEFINITION 1.8. Two nice immersions $f,\ g: (D^{n+1}, S^n) \longrightarrow (E^{2n+2}, \mathscr{T}N)$ *intersect nicely* if

(i) $f(D^{n+1})$ and $g(D^{n+1})$ intersect in general position,

(ii) $f(S^n) \cap g(S^n) = \phi$ and

(iii) the image of the compositions $\varpi f(S^n)$ and $\varpi g(S^n)$ intersect in general position in L^{2n}.

Let $f,\ g: (D^{n+1}, S^n) \longrightarrow (E, \mathscr{T}N)$ be pathed nice immersions which intersect nicely.

DEFINITION OF $\lambda(f, g)$.

Before defining $\lambda(f, g)$ we need two auxiliary pairings $\alpha(f, g)$ and $\beta(f, g)$. Let $\{p_1, \cdots, p_r\}$ be the set of intersection points of $\varpi f(S^n)$ and $\varpi g(S^n)$ in L^{2n}. Let ε_i be the sign ± 1 of the intersection at p_i of $\varpi f(S^n)$ and $\varpi g(S^n)$. It will be convenient to write $\varepsilon_i = \dfrac{[\varpi f(S^n)]_{p_i} \times [\varpi g(S^n)]_{p_i}}{[L^{2n}]_{p_i}}$, where the horizontal bar denotes the incidence relation between the two orientations, and $[\ ]_{p_i}$ denotes the induced local orientation at p_i.

Let $g_i \in \pi_1(\mathscr{F}N)$ be represented by the following loop in $\mathscr{F}N$:

(3) $$g_i = \{* \xrightarrow{\gamma(f)} p_i^f \longrightarrow \text{(along the } S^1\text{-fibre } \varpi^{-1}(p_i) \text{ in the positive direction)} \longrightarrow p_i^g \xrightarrow{\gamma(g)^{-1}} *\}\ ,$$

where p_i^f (or p_i^g) is the point of $f(S^n)$ (or $g(S^n)$) over p_i, i.e., $\{p_i^f\} = f(S^n) \cap \varpi^{-1}(p_i)$ (or $\{p_i^g\} = g(S^n) \cap \varpi^{-1}(p_i)$). See Fig. 1.

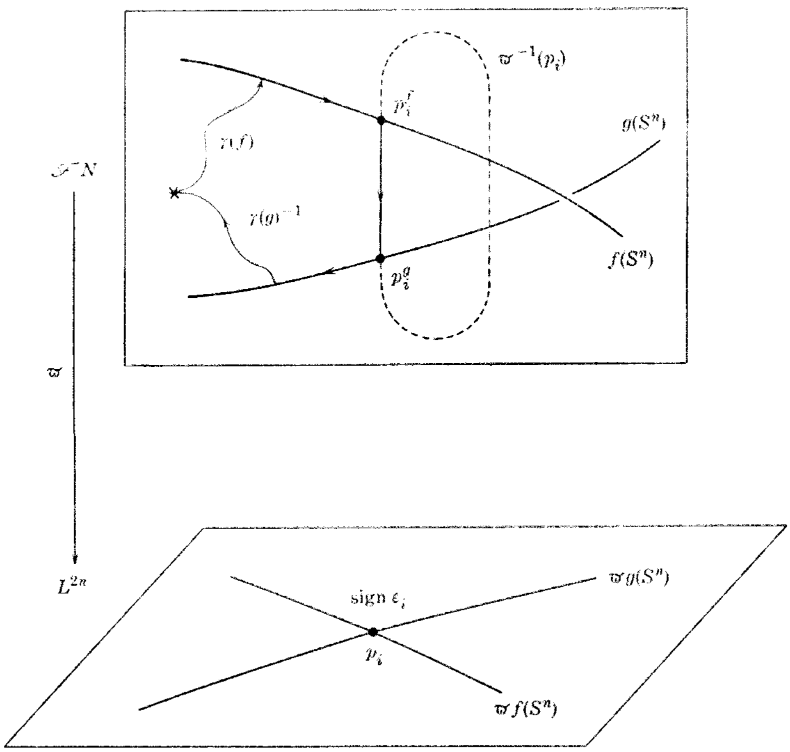


Fig. 1.

The pairing $\alpha(f, g)$ is defined by the summation in the integral group ring $\boldsymbol{Z}[\pi_1\mathscr{F}N]$ over all intersection points $\{p_1, \cdots, p_r\}$

$$\alpha(f, g) = \sum_{i=1}^{r} \varepsilon_i g_i\ .$$

Next the definition of $\beta(f, g)$ is as follows:

Let $\{q_1, \cdots, q_s\}$ be the set of intersection points of $f(D^{n+1})$ and $g(D^{n+1})$. ε'_i is the sign ± 1 of the intersection at q_i of $f(D^{n+1})$ and $g(D^{n+1})$, i.e.,

$$\varepsilon'_i = \frac{[f(D^{n+1})]_{q_i} \times [g(D^{n+1})]_{q_i}}{[E^{2n+2}]_{q_i}} ,$$

$g'_i \in \pi_1(E)$ $(\cong \pi_1(\mathscr{F}N))$ is defined by the following loop in E:

(4) $$g'_i = \{* \xrightarrow{\gamma(f)} q_i \xrightarrow{\gamma(g)^{-1}} *\} .$$

Then the pairing $\beta(f, g)$ is the summation $\sum_{i=1}^{s} \varepsilon'_i g'_i$ in the group ring $Z[\pi_1 \mathscr{F}N]$.

In the above, we assume that the orientations $[D^{n+1}]$ and $[S^n]$ are related to each other by

(5) $$[D^{n+1}] = [S^n] \times (\text{the radial inward direction of } D^{n+1}) .$$

We are now in a position to define a pairing $\lambda(f, g) \in Z[\pi_1 \mathscr{F}N]$:

(6) $$\lambda(f, g) = \alpha(f, g) + (-1)^{n+1}(1-t) \cdot \beta(f, g) ,$$

where $t \in \pi_1(\mathscr{F}N)$ denotes the special generator of C.

In the next section, we will see that the pairing $\lambda(f, g)$ depends only on the homotopy class of f and g (Th. 2.5).

LEMMA 1.9. *$\lambda(f, g) = (-1)^n \overline{\lambda(g, f)} \cdot t$, where $- : Z[\pi_1 \mathscr{F}N] \longrightarrow Z[\pi_1 \mathscr{F}N]$ is the anti-involution induced by the inverse:* $\overline{\sum_{g \in \pi_1 \mathscr{F}N} m_g g} = \sum m_g g^{-1}$.

PROOF OF 1.9. It is easy to see that

$$\beta(f, g) = (-1)^{n+1} \overline{\beta(g, f)} .$$

To obtain the analogous formula for α-pairing, we observe that $\varepsilon_i(f, g) = (-1)^n \varepsilon_i(g, f)$ and that

$$\begin{aligned} g_i(f, g) &= \{* \xrightarrow{\gamma(f)} p_i^f \xrightarrow{S^1\text{-fibre in the positive direction}} p_i^g \xrightarrow{\gamma(g)^{-1}} *\} \\ &= [\text{the inverse of } \{* \xrightarrow{\gamma(g)} p_i^g \xrightarrow{S^1\text{-fibre in the positive direction}} \\ &\longrightarrow p_i^f \xrightarrow{\gamma(f)^{-1}} \}] \cdot t \\ &= g_i(g, f)^{-1} \cdot t . \end{aligned}$$

Thus we have

$$\alpha(f, g) = (-1)^n \overline{\alpha(g, f)} \cdot t .$$

From the two formulae we obtain

$$\begin{aligned}\lambda(f, g) &= \alpha(f, g)+(-1)^{n+1}(1-t)\cdot\beta(f, g)\\ &=(-1)^{n}\overline{\alpha(g, f)}\cdot t+(-1)^{n+1}(1-t)\cdot(-1)^{n+1}\overline{\beta(g, f)}\\ &=(-1)^{n}\{\overline{\alpha(g, f)+(-1)^{n+1}(1-t)\cdot\beta(g, f)}\}\cdot t\\ &=(-1)^{n}\overline{\lambda(g, f)}\cdot t\,.\end{aligned}$$

(Recall that t is in the centre of $\pi_1(\mathscr{F}N)$.) Q.E.D.

Next we define the "self-intersection" $\mu(f)$ for a nice immersion $f:(D^{n+1}, S^n)\longrightarrow(E, \mathscr{F}N)$. For this we define an abelian group $Q'_n(\pi_1\mathscr{F}N)$ by

$$(7)\qquad Q'_n(\pi_1\mathscr{F}N)=\boldsymbol{Z}[\pi_1\mathscr{F}N]/\{\nu-(-1)^n\bar{\nu}\cdot t\,|\,\nu\in\boldsymbol{Z}[\pi_1\mathscr{F}N]\}\,.$$

DEFINITION OF $\mu(f)$. Let $f:(D^{n+1}, S^n)\longrightarrow(E^{2n+2}, \mathscr{F}N)$ be a pathed nice immersion, and $\{p_1, \cdots, p_r\}$ the set of self-intersection points of $\varpi f(S^n)$ in L^{2n}. Since each p_i is a double point, there are two points $p_i^{(1)}$, $p_i^{(2)}$ of $f(S^n)$ over p_i, i.e., $\{p_i^{(1)}, p_i^{(2)}\}=f(S^n)\cap\varpi^{-1}(p_i)$. Choose an order of the two points, $(p_i^{(1)}, p_i^{(2)})$ or $(p_i^{(2)}, p_i^{(1)})$, once for all p_i. Let θ_i denote the order for p_i, say, $\theta_i=(p_i^{(1)}, p_i^{(2)})$.

ε_i is the sign of the intersection with respect to θ_i

$$\varepsilon_i=\frac{\varpi_*[f(S^n)]_{p_i^{(1)}}\times\varpi_*[f(S^n)]_{p_i^{(2)}}}{[L^{2n}]_{p_i}}\,.$$

$g_i\in\pi_1(\mathscr{F}N)$ is represented by the loop in $\mathscr{F}N$

$$g_i=\{*\xrightarrow{\gamma(f)}p_i^{(1)}\longrightarrow(\text{along the } S^1\text{-fibre } \varpi^{-1}(p_i) \text{ in the positive direction})\longrightarrow p_i^{(2)}\xrightarrow{\gamma(f)^{-1}}*\}\,.$$

The summation $\alpha(f)=\sum_{i=1}^{r}\varepsilon_i g_i\in\boldsymbol{Z}[\pi_1\mathscr{F}N]$ depends on the choice of orders $\{\theta_i\}_{i=1,\dots,r}$, and if an order θ_i is reversed, $\alpha(f)$ is changed to $\alpha(f)-\varepsilon_i g_i+(-1)^n\varepsilon_i g_i^{-1}\cdot t$ (see the proof of 1.9). However, the term $-\varepsilon_i g_i+(-1)^n\varepsilon_i g_i^{-1}\cdot t$ is contained in the subgroup $\{\nu-(-1)^n\bar{\nu}\cdot t\,|\,\nu\in\boldsymbol{Z}[\pi_1\mathscr{F}N]\}$, so $\alpha(f)$ is well-defined in the group $Q'_n(\pi_1\mathscr{F}N)$ independently of $\{\theta_i\}$.

Next we define $\beta(f)$. Let $\{q_1, \cdots, q_s\}$ be the set of self-intersection points of $f(D^{n+1})$ in E^{2n+2}. ε'_i is the sign of the intersection at q_i, and $g'_i\in\pi_1(E)$ $(\cong\pi_1(\mathscr{F}N))$ is represented by the loop $\{*\xrightarrow{\gamma(f)}q_i\xrightarrow{\gamma(f)^{-1}}*\}$. ε'_i and g'_i are defined with respect to an order of two branches at q_i of $f(D^{n+1})$, and if it is reversed, $\beta(f)$ is changed to $\beta(f)-\varepsilon'_i g'_i+(-1)^{n+1}\varepsilon'_i g_i'^{-1}$. Therefore, $\beta(f)$ is well defined in the group $Q_{n+1}(\pi_1\mathscr{F}N)=\boldsymbol{Z}[\pi_1\mathscr{F}N]/\{\nu-(-1)^{n+1}\bar{\nu}\,|\,\nu\in\boldsymbol{Z}[\pi_1\mathscr{F}N]\}$, in Wall's notation [29]. Multiplication by $(1-t)$ induces a homomorphism: $Q_{n+1}(\pi_1\mathscr{F}N)\longrightarrow Q'_n(\pi_1\mathscr{F}N)$. Therefore, $(1-t)\cdot\beta(f)$ is a well-defined element in $Q'_n(\pi_1\mathscr{F}N)$. To give the definition of $\mu(f)$, we need another invariant $\mathscr{O}(f)\in\boldsymbol{Z}$. Let w be a "positive tangent vector field" over $\mathscr{T}N$ along the S^1-fibres. Since f is a nice immersion, $f(S^n)\subset\mathscr{T}N$ is transversal to each S^1-fibre, thus the restriction $w|f(S^n)$ is a non-zero normal field over $f(S^n)$. The normal bundle ν of $f: D^{n+1}\longrightarrow E^{2n+2}$ is a trivial $n+1$-vector (or block) bundle with fibre $\boldsymbol{R}^{n+1}$, and $w|f(S^n)$ gives a cross section of $\nu|S^n\times\boldsymbol{R}^{n+1}$. The obstruction to extend this section to a non-zero cross section over the whole of D^{n+1} is defined by $f(S^n)\xrightarrow{w} f(S^n)\times(\boldsymbol{R}^{n+1}-\{0\})\xrightarrow{\text{proj.}}\boldsymbol{R}^{n+1}-\{0\}$ and is an element of the group $\pi_n(\boldsymbol{R}^{n+1}-\{0\})$. We fix an orientation of $\boldsymbol{R}^{n+1}$ by the formula

$$(8) \qquad [\mathscr{T}N]=[f(S^n)]\times[\boldsymbol{R}^{n+1}]\ ,$$

the group $\pi_n(\boldsymbol{R}^{n+1}-\{0\})$ is identified with $\boldsymbol{Z}$, and the obstruction is denoted by $\mathscr{O}(f)$.

We define $\mu(f)\in Q'_n(\pi_1\mathscr{T}N)$ by

$$(9) \qquad \mu(f)=\alpha(f)+(-1)^{n+1}(1-t)\cdot\beta(f)+(-1)^{n+1}\cdot\mathscr{O}(f)\ .$$

In this formula, $\mathscr{O}(f)$ is added as the image under the mapping: $\boldsymbol{Z}=\{me|m\in\boldsymbol{Z}$, e is the neutral element of $\pi_1\mathscr{T}N\}\subset\boldsymbol{Z}[\pi_1\mathscr{T}N]\longrightarrow Q'_{n+1}(\pi_1\mathscr{T}N)$.

We shall prove in the next section that $\mu(f)$ depends only on the homotopy class of f (Th. 2.9).

§2. Homotopy invariance of (λ, μ).

DEFINITION 2.1. Let h_s $(s\in[0,1])$ be a regular homotopy between two pathed nice immersions $f, g:(D^{n+1},S^n)\longrightarrow(E^{2n+2},\mathscr{T}N)$. h_s is said to be *transversal (to the S^1-fibres)* if $\varpi\circ(h_s|S^n): S^n\longrightarrow L^{2n}$ is a regular homotopy, also f and g are *transversally regular homotopic.*

LEMMA 2.2. *Suppose that f, $h:(D^{n+1},S^n)\longrightarrow(E^{2n+2},\mathscr{T}N)$ are pathed nice immersions which are transversally regular homotopic. If the regular homotopy h_s is such that $h_s(S^n)\cap g(S^n)=\phi$ $(\forall s)$, where g is a pathed nice immersion, then*

$$\lambda(f,g)=\lambda(h,g)\ .$$

PROOF OF 2.2. By the hypothesis, $\varpi\circ(f|S^n)$ and $\varpi\circ(h|S^n): S^n\longrightarrow L^{2n}$ are regular homotopic. Thus the intersection points $\varpi h_s(S^n)$ and $\varpi g(S^n)$ appear or disappear in pairs. Let (p, p') be such a pair of intersection points. It is not difficult to see that $\varepsilon_p=-\varepsilon_{p'}$ and $g_p=g_{p'}$. So we have $\alpha(f,g)=\alpha(h,g)$. Under the assumption that $h_s(S^n)\cap g(S^n)=\phi$ $(\forall s)$, the only possible change of $h_s(D^{n+1})\cap g(D^{n+1})$ is a pairwise appearance or disappearance of intersections. Thus by a similar argument we have $\beta(f,g)=\beta(h,g)$, and $\lambda(f,g)=\lambda(h,g)$, completing the proof.

Q.E.D.

The following lemma is an essential step, which asserts that the condition $h_s(S^n)\cap g(S^n)=\phi$ ($\forall s$) is in fact not necessary.

LEMMA 2.3. *Let f, h, $g:(D^{n+1}, S^n)\longrightarrow(E^{2n+2}, \mathscr{T}N)$ be pathed nice immersions such that f and g, (resp. h and g) nicely intersect. If f and h are transversally regular homotopic, then*

$$\lambda(f, g)=\lambda(h, g)\ .$$

PROOF OF 2.3. We have only to show that λ is invariant if the image of S^n under the transversal regular homotopy h_s ($s\in[0, 1]$) meets $g(S^n)$ in its deformation. Without loss of generality we may suppose that $h_s(S^n)$ crosses $g(S^n)$ in general position at a finite number of points. Moreover, by decomposing h_s into some steps of small transversal regular homotopies, it will be sufficient to consider the following restricted case (*).

(*) $h_s(S^n)$ slides down along the S^1-fibres of $\mathscr{T}N\longrightarrow L^{2n}$ in the positive direction as s increases from 0 to 1, and it crosses $g(S^n)$ at a point q when $s=s_0$. In addition $h_s(S^n)\cap g(S^n)=\phi$ if $s\neq s_0$.

This situation is illustrated in Fig. 2. (h_0 and h_1 are identified with f and h respectively.)

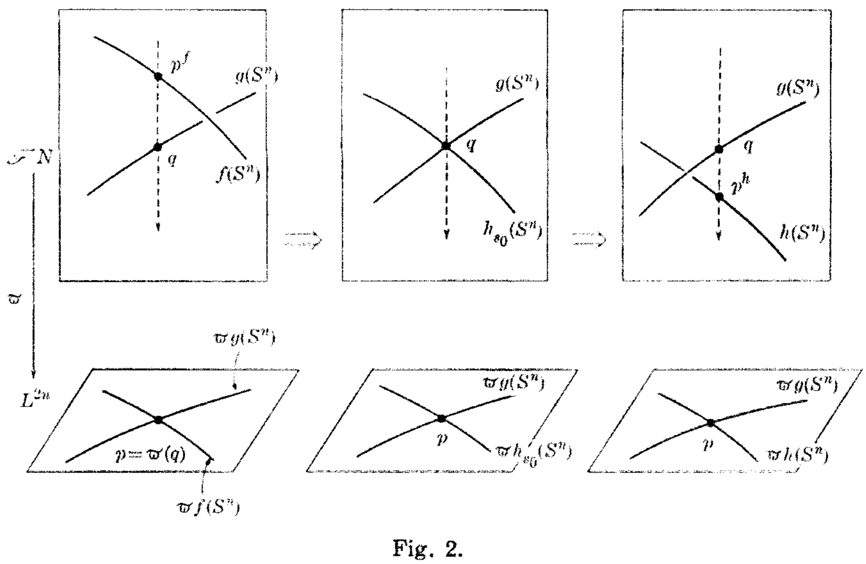


Fig. 2.

In the situation (*) we are assuming that

(i) the set of intersection points of $\varpi h_s(S^n)$ and $\varpi g(S^n)$ remains unchanged, and

(ii) the set $h_s(D^{n+1})\cap g(D^{n+1})$ changes by a point q at the moment when $h_s(S^n)$

crosses $g(S^n)$.

Two cases are possible: The point q is newly introduced to the intersection or q vanishes from the intersection. However by replacing the parameter s by $1-s$, if necessary, it will be sufficient to consider only the first case.

We now compare $\alpha(f, g)$ with $\alpha(h, g)$. Let $p=\varpi(q)$. Let $\varepsilon_p(f, g)$, $\mathrm{g}_p(f, g)$, etc. denote ε_p, g_p, etc. in the expression $\alpha(f, g)=\sum \varepsilon_i \mathrm{g}_i$. We get $\varepsilon_p(f, g)=\varepsilon_p(h, g)$, and

$$\begin{aligned}\mathrm{g}_p(h, g) &= \{* \xrightarrow{\gamma(h)} p^h \xrightarrow{\text{in the positive direction}} p^g \xrightarrow{\gamma(g)^{-1}} *\} \\ &= \{* \xrightarrow{\gamma(h)} p^h \xrightarrow{\text{in the negative direction}} p^g \xrightarrow{\gamma(g)^{-1}} *\}\cdot t \\ &= \{* \xrightarrow{\gamma(f)} p^f \xrightarrow{\text{in the positive direction}} p^g \xrightarrow{\gamma(g)^{-1}} *\}\cdot t \\ &= \mathrm{g}_p(f, g)\cdot t \, .\end{aligned}$$

Therefore, we have

$$\alpha(h, g)=\alpha(f, g)-\varepsilon_p \mathrm{g}_p+\varepsilon_p \mathrm{g}_p\cdot t \, . \tag{10}$$

Next $\beta(h, g)=\beta(f, g)+\varepsilon'_q \mathrm{g}'_q$, where g'_q is clearly equal to g_p. In the situation (*), we can suppose that the radial inward direction of $h(D^{n+1})$ near q is almost "parallel" to the negative direction of S^1-fibres (for we are assuming that $h_s(S^n)$ moves in the positive direction). So substituting in (5), the following equation holds in a neighbourhood of q:

$$[h(D^{n+1})]=[h(S^n)]\times(-[S^1_q]) \, , \tag{11}$$

where S^1_q denotes the fibre which passes q. Moreover near q, the radial inward direction of $g(D^{n+1})$ can be considered to coincide with the inward direction v of E (see Formula (1)). Thus near q, we have

$$[g(D^{n+1})]=[g(S^n)]\times v \, . \tag{12}$$

We now calculate ε'_q.

$$\begin{aligned}\varepsilon'_q &= \frac{[h(D^{n+1})]\times[g(D^{n+1})]}{[E^{2n+2}]} \\ &= \frac{[h(S^n)]\times(-[S^1_q])\times[g(S^n)]\times v}{[\mathscr{F}N]\times v} && \text{(Formulae (1), (11), (12))} \\ &= (-1)^{n+1}\frac{[h(S^n)]\times[g(S^n)]\times[S^1_q]\times v}{[L^{2n}]\times[S^1]\times v} && \text{(Formula (2)')} \\ &= (-1)^{n+1}\frac{[h(S^n)]\times[g(S^n)]}{[L^{2n}]} \\ &= (-1)^{n+1}\varepsilon_p \, .\end{aligned}$$

Consequently,

$$\beta(h, g)=\beta(f, g)+(-1)^{n+1}\varepsilon_p g_p .$$

This formula and (10) give the following:

$$\begin{aligned}\lambda(h, g)&=\{\alpha(f, g)-(1-t)\varepsilon_p g_p\}+(-1)^{n+1}(1-t)\{\beta(f, g)+(-1)^{n+1}\varepsilon_p g_p\}\\ &=\alpha(f, g)+(-1)^{n+1}(1-t)\beta(f, g)\\ &=\lambda(f, g) .\end{aligned}$$

This completes the proof of 2.3. Q.E.D.

The homotopy invariance of λ is established by studying the relationship between transversal regular homotopies and homotopies.

Suppose there is a homotopy h_s, $0\leqq s\leqq 1$, between pathed nice immersions $f(=h_0)$ and $h(=h_1)$. Approximate h_s by a generic homotopy (Haefliger [5, p. 58]) which is denoted by $\hat{h}_s$. Since $\dim \mathscr{F}N=2n+1$, $\hat{h}_s|S^n : S^n \longrightarrow \mathscr{F}N$ is a regular homotopy.

We can take $\hat{h}_s$ so that $\varpi\circ(\hat{h}_s|S^n): S^n \longrightarrow L^{2n}$ is also a generic homotopy. The set of singularities of $\hat{h}_s$ (respectively of $\varpi\circ(\hat{h}_s|S^n)$) is finite, and each singularity is an isolated point, where, as s varies, a Whitney's self-intersection (i.e., an ordinary double point) is locally introduced. More precisely, in a neighbourhood of a singular point of $\hat{h}_s(D^{n+1})$, $\hat{h}_s$ is described by the following equations (Whitney [30], Haefliger [5]):

$$\underset{(1/2-\varepsilon<s<1/2+\varepsilon)}{\hat{h}_s}\begin{cases}X_1=x_1(1-2Y_1) , & X_i=x_i \\ Y_1=\dfrac{s}{1+x_1^2} , & Y_i=x_1x_i\end{cases}\qquad 1<i\leqq n+1 ,$$

where $(x_1, \cdots, x_{n+1})$ and $(X_1, \cdots, X_{n+1}, Y_1, \cdots, Y_{n+1})$ are local coordinates of D^{n+1} and E^{2n+2}, respectively. In the above, $\hat{h}_s$ has no self-intersections for $s<1/2$, $\hat{h}_s$ has a singular point $X_1=\cdots=X_{n+1}=0$, $Y_1-1/2=Y_2=\cdots=Y_{n+1}=0$, when $s=1/2$, and $\hat{h}_s$ has a double point $X_1=\cdots=X_{n+1}=0$, $Y_1-1/2=Y_2=\cdots=Y_{n+1}=0$ for $s>1/2$.

Similarly, with respect to certain local coordinates $(x_1', \cdots, x_n')$ of S^n and $(X_1', \cdots, X_n', Y_1', \cdots, Y_n')$ of L^{2n}, $\varpi\circ(\hat{h}_s|S^n)$ can be writtdn as follows:

$$\underset{(1/2-\varepsilon<s<1/2+\varepsilon)}{\varpi\circ(\hat{h}_s|S^n)}\begin{cases}X_1'=x_1'(1-2Y_1') , & X_i'=x_i' \\ Y_1'=\dfrac{s}{1+x_1'^2} , & Y_i'=x_1'x_i'\end{cases}\qquad 1<i\leqq n .$$

As mentioned above, $\hat{h}_s|S^n$ is a regular homotopy, and of course has no singular points. However, the transversality to S^1-fibres fails at a finite number of points which corresponds to the singular points of $\varpi\circ(\hat{h}_s|S^n)$. See Fig. 3.

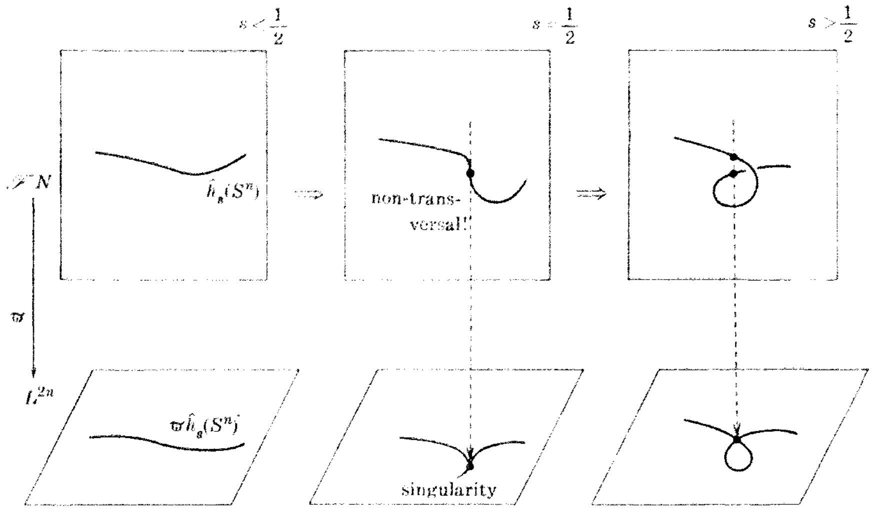


Fig. 3.

In a neighbourhood of a non-transversal point, one can find a local coordinate of $\mathscr{T}N$ $(X_1', \cdots, X_n', Y_1', \cdots, Y_n', Z)$, Z is along the S^1-fibres, with which $\hat{h}_s|S^n$ is written by equations

$$\underset{(1/2-\varepsilon \leqq s \leqq 1/2+\varepsilon)}{\hat{h}_s|S^n} \begin{cases} X_1' = x_1'(1-2Y_1')\,, & X_i' = x_i' \\ Y_1' = \dfrac{s}{1+x_1'^2}\,, & Y_i' = x_1' x_i' \qquad 1 < i \leqq n\,. \\ Z = x_1'\,. & \end{cases}$$

This situation is described as "a non-transversal movement of $\hat{h}_s|S^n$ introduces a Whitney self-intersection to $\varpi\circ(\hat{h}_s|S^n)$". Summarizing these observations we have the following lemma.

LEMMA 2.4. *Let $f, g : (D^{n+1}, S^n) \longrightarrow (E^{2n+2}, \mathscr{T}N)$ be pathed nice immersions which are homotopic to each other. Then f can be transformed into g by a sequence of the three kinds of deformations (or of their inverses);*

(I) a transversal regular homotopy,

(II) a generic homotopy introducing a Whitney self-intersection at a point q in $\hat{h}_s(D^{n+1})$, and

(III) a generic homotopy introducing a Whitney self-intersection at a point p in $\varpi(\hat{h}_s(S^n))$ by a non-transversal movement of $\hat{h}_s(S^n)$.

We now prove the homotopy invariance of λ.

THEOREM 2.5. *$\lambda(f, g)$ depends only on the homotopy class of pathed nice immersions f and g. Since any pathed continuous map $(D^{n+1}, S^n) \longrightarrow (E, \mathscr{T}N)$*

is approximated by a nice immersion, λ defines a pairing

$$\pi_{n+1}(E, \mathscr{F}N)\times\pi_{n+1}(E, \mathscr{F}N)\longrightarrow \boldsymbol{Z}[\pi_1\mathscr{F}N]$$

which will be donoted by the same symbol λ.

PROOF OF 2.5. By 1.9, it will be sufficient to show that if g is fixed, $\lambda(f, g)$ depends only on the homotopy class of f. By 2.4, we have only to show the invariance of λ under each of the deformations (I), (II) or (III). The invariance under (I) is proved in 2.3. The invariance under (II) or (III) is obvious, since to calculate the value of λ we count only the intersection points of $f(D^{n+1})$ (or $\varpi f(S^n)$) and $g(D^{n+1})$ (or $\varpi g(S^n)$), ignoring their self-intersection points; One has only to choose the generic approximation of a homotopy h_s carefully so that the local introduction of Whitney self-intersections does not occur at any intersection point of f and g, and this is always possible. Q.E.D.

HOMOTOPY INVARIANCE OF $\mu(f)$.

LEMMA 2.6. *$\mu(f)$ is a transversal regular homotopy invariant.*

PROOF OF 2.6. The method in the proof of 2.3 is also applicable in this case, and that $\alpha(f)+(-1)^{n+1}(1-t)\beta(f)$ is a transversal regular homotopy invariant is proved. By the definition, $\mathscr{C}(f)\in \boldsymbol{Z}$ is clearly a transversal regular homotopy invaraint. So by (9), we have the desired conclusion. Q.E.D.

To prove that $\mu(f)$ depends only on the homotopy class of f, it remains to show that $\mu(f)$ does not change if f is changed by the deformations of type (II) or (III) (Lemma 2.4).

LEMMA 2.7. *$\mu(f)$ is unchanged under the deformation of type (III).*

PROOF OF 2.7. By introducing a Whitney self-intersection at a point p in $\varpi f(S^n)$, we have that $\beta(f)$ remains unchanged and $\alpha(f)$ changes to $\alpha(f)+\varepsilon_p$, where ε_p is the sign at p with respect to an order θ of the two points of $f(S^n)$ over p (cf. Definition of $\alpha(f)$).

ASSERTION 2.7.1.*) $\mathscr{C}(f)$ changes to $\mathscr{C}(f)+(-1)^n\varepsilon_p$.

PPOOF OF 2.7.1. Since the problem is local, we may restrict our attention to the case of euclidean spaces.

We use the following notations:

$$\begin{cases} H^{2n+2}=\{(x_1, \cdots, x_{2n+2})\in \boldsymbol{R}^{2n+2}\,|\,x_{2n+2}\geqq 0\} \\ \cup \\ \boldsymbol{R}^{2n+1}=\{(x_1, \cdots, x_{2n+1})\} \\ \cup \\ \boldsymbol{R}^{2n}\ =\{(x_1, \cdots, x_{2n})\}\ . \end{cases}$$

*) The proof of this assertion is due to Professor A. Hattori.

$\varpi: \boldsymbol{R}^{2n+1} \longrightarrow \boldsymbol{R}^{2n}$ is the projection; $\varpi(x_1 \cdots, x_{2n}, x_{2n+1}) = (x_1, \cdots, x_{2n})$.
dx_i is the positive direction of the x_i-axis.

$\boldsymbol{R}^k$ has a canonical orientation $[\boldsymbol{R}^k] \equiv dx_1 \times \cdots \times dx_k$. Let $f: (D^{n+1}, S^n) \longrightarrow (H^{2n+2}, \boldsymbol{R}^{2n+1})$ be a nice immersion, i.e., f is a generic immersion such that $f|S^n$ is an embedding and such that $\varpi \circ (f|S^n): S^n \longrightarrow \boldsymbol{R}^{2n}$ is also a generic immersion. Then $\mathcal{O}(f) \in \boldsymbol{Z} = \pi_n(\boldsymbol{R}^{n+1} - \{0\})$ is defined in a similar way to $\mathcal{O}(f)$ in the definition of $\mu(f)$. Now suppose that $\varpi \circ (f|S^n): S^n \longrightarrow \boldsymbol{R}^{2n}$ is an immersion with a unique self-intersection p (an ordinary double point). Fix an order θ of the two points $p^{(1)}$ and $p^{(2)}$ of $f(S^n)$ over p such that (the x_{2n+1}-coordinate of $p^{(1)}$)<(the x_{2n+1}-coordinate of $p^{(2)}$) in the order $\theta = (p^{(1)}, p^{(2)})$. Let ε_p, the sign of the intersection p, be given by the equation

$$\varepsilon_p = \frac{\varpi_*[f(S^n)]_{p^{(1)}} \times \varpi_*[f(S^n)]_{p^{(2)}}}{[\boldsymbol{R}^{2n}]_p} . \tag{13}$$

We assert that $\mathcal{O}(f) = (-1)^n \varepsilon_p$. Let S_1^n and S_2^n be two embedded n-spheres in $\boldsymbol{R}^{2n+1}$ such that $S_1^n \cap S_2^n$ is empty. If S_2^n bounds a singular $n+1$-disk D_2^{n+1}, then the linking number $L(S_1^n, S_2^n) \in \boldsymbol{Z}$ is defined to be the intersection number of S_1^n and D_2^{n+1}.

By D_t^{n+1}, we denote a fibre of the normal $n+1$-disk bundle of $f|S^n: S^n \longrightarrow \boldsymbol{R}^{2n+1}$. Following the orientation convention (8), we define the orientation of D_t^{n+1} by $[\boldsymbol{R}^{2n+1}] = [f(S^n)] \times [D_t^{n+1}]$. Thus

$$L(f(S^n), \partial D_t^{n+1}) = 1 . \tag{14}$$

Let $w: f(S^n) \longrightarrow f(S^n) \times (D_t^{n+1} - \{0\})$ be a small normal cross-section which is pointing to the positive direction of the x_{2n+1}-axis. By definition, $\mathcal{O}(f) \in \pi_n(D_t^{n+1} - \{0\}) = \boldsymbol{Z}$ is represented by $f(S^n) \xrightarrow{w} f(S^n) \times (D_t^{n+1} - \{0\}) \xrightarrow{\text{proj.}} (D_t^{n+1} - \{0\})$. Thus $w(f(S^n))$ is homologous to the cycle $f(S^n) + \mathcal{O}(f) \cdot \partial D_t^{n+1}$ in $f(S^n) \times (D_t^{n+1} - \{0\})$. Hence we have

$$\begin{aligned} & L(f(S^n), w(f(S^n))) \\ &= L(f(S^n), f(S^n)) + L(f(S^n), \mathcal{O}(f) \cdot \partial D_t^{n+1}) \\ &= \mathcal{O}(f) \cdot L(f(S^n), \partial D_t^{n+1}) \\ &= \mathcal{O}(f) \qquad (14) . \end{aligned} \tag{15}$$

Let $CS^n = S^n \times [0, 1]/S^n \times \{1\}$ be a cone over S^n. Let $\pi: \boldsymbol{R}^{2n+1} \longrightarrow \boldsymbol{R}^1$ be the projection to the x_{2n+1}-axis. Define a singular $n+1$-disk $F: CS^n \longrightarrow \boldsymbol{R}^{2n+1}$ as follows:

$$\underset{(x,\tau) \in S^n \times [0,1]}{F(x, \tau)} = \begin{cases} (\varpi(wf(x)), \pi(wf(x)) + N\tau) \in \boldsymbol{R}^{2n} \times \boldsymbol{R} , & (0 \leqq \tau \leqq 1/2) \\ \text{a point} \in \boldsymbol{R}^{2n+1} - f(S^n) , & (\tau = 1) , \end{cases}$$

where N is a sufficiently large positive number. For $1/2<\tau<1$, define $F(x, \tau)$ appropriately so that F is continuous and $F(S^n\times[1/2, 1])\cap f(S^n)=\phi$. See Fig. 4.

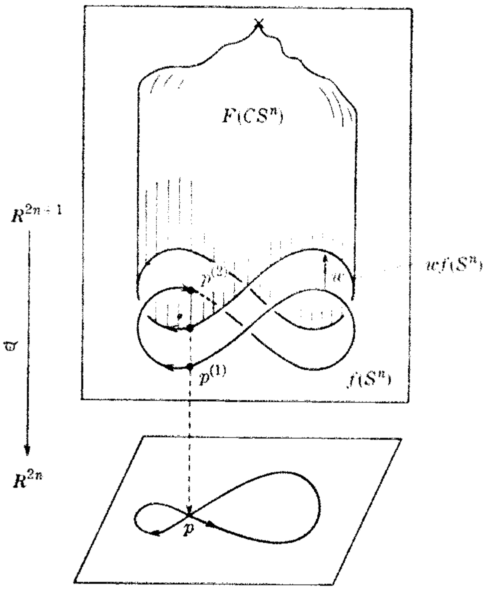


Fig. 4.

Clearly $F(CS^n)$ is a singular n+1-disk such that $\partial F(CS^n)=wf(S^n)$. One immediately verifies that $f(S^n)\cap F(CS^n)=\{p^{(2)}\}$. The sign $\varepsilon_{p^{(2)}}$ of the intersection at $p^{(2)}$ is

$$\varepsilon_{p^{(2)}}=\frac{[f(S^n)]_{p^{(2)}}\times[F(CS^n)]_{p^{(2)}}}{[\boldsymbol{R}^{2n+1}]}\ .$$

It is easy to see that $[F(CS^n)]_{p^{(2)}}=[f(S^n)]_{p^{(1)}}\times dx_{2n+1}$. (Recall the orientation convention (5).) Hence

$$\begin{aligned}\varepsilon_{p^{(2)}}&=\frac{[f(S^n)]_{p^{(2)}}\times[f(S^n)]_{p^{(1)}}\times dx_{2n+1}}{[\boldsymbol{R}^{2n}]\times dx_{2n+1}}\\&=\frac{\varpi_*[f(S^n)]_{p^{(2)}}\times\varpi_*[f(S^n)]_{p^{(1)}}}{[\boldsymbol{R}^{2n}]_p}\\&=(-1)^n\varepsilon_p\ ,\qquad\text{see (13)}\ .\end{aligned}$$

This means that $L(f(S^n), wf(S^n))=(-1)^n\varepsilon_p$, and by (15), $\mathcal{O}(f)=(-1)^n\varepsilon_p$. This completes the proof of Assertion 2.7.1.

PROOF OF 2.7 (continued). By introducing an intersection point p to $\varpi f(S^n)$, one sees that $\mu(f)$ changes to $\alpha(f)+\varepsilon_p+(-1)^{n+1}(1-t)\cdot\beta(f)+(-1)^{n+1}(\mathcal{O}(f)+(-1)^n\varepsilon_p)$, but this is equal to $\mu(f)$. Q.E.D.

LFMMA 2.8. *$\mu(f)$ is unchanged under the deformation of type (II).*

PROOF OF 2.8. In this case a Whitney self-intersection q is introduced to $f(D^{n+1})$. $\alpha(f)$ does not change. $\beta(f)$ changes to $\beta(f)+\varepsilon'_q$, where ε'_q is the sign at the new self-intersection q with respect to an order of the branches.

ASSERTION 2.8.1.

$$\mathcal{O}(f) \text{ changes to } \begin{cases} \mathcal{O}(f)-2\varepsilon'_q & (n+1\colon \text{even}) \\ \mathcal{O}(f) & (n+1\colon \text{odd}) . \end{cases}$$

PROOF OF 2.8.1. Since the problem is local, we consider the case of euclidean spaces. The notations are the same as in the proof of 2.7.1.

Let $k_\tau : S^n \longrightarrow \boldsymbol{R}^{2n+1}$ $(-1\leqq\tau\leqq1)$ be a homotopy such that (i) $k_{-1}(S^n)=k_1(S^n)=$a point, (ii) $k_0 : S^n \longrightarrow \boldsymbol{R}^{2n+1}$ is an immersion with only one self-intersection point q, (iii) if $|\tau|<2\delta$, where δ is a small positive number, $\varpi\circ k_\tau : S^n \longrightarrow \boldsymbol{R}^{2n}$ is an immersion with a unique self-intersection point p_τ which is an ordinary double point, (iv) a map $h : S^{n+1}=(S^n\times[-1, 1]/S^n\times\{-1\})/S^n\times\{1\} \longrightarrow \boldsymbol{R}^{2n+2}=\boldsymbol{R}^{2n+1}\times\boldsymbol{R}$ which is defined by $h(x, \tau)=(k_\tau(x), \tau)$ is an immersion with the unique self-intersection point q which is an ordinary double point (here one takes τ to be the x_{2n+2}-coordinate).

Let v_0 be a non-zero normal vector field over $h(S^n\times[-\delta, \delta])$ in the direction of dx_{2n+1}, and D_δ^{n+1} a small disk in $h(S^n\times[-\delta, \delta])-\{q\}$. We must compute the obstruction $\mathcal{O}\in\boldsymbol{Z}$ to extending $v_0|\partial D_0^{n+1}$ to the whole of $h(S^{n+1})$ as a non-zero normal vector field. We assume that an orientation of $h(S^{n+1})$ is already given. Following the orientation convention (5), $\partial D_0^{n+1}$is oriented by

$$\begin{aligned}[h(S^{n+1})]&=[\partial D_0^{n+1}]\times(\text{the inward direction of } h(S^{n+1})-D_0^{n+1})\\ &=[\partial D_0^{n+1}]\times(\text{the outward direction of } D_0^{n+1}) .\end{aligned}$$

Denote by $\mathcal{O}_+$ (or $\mathcal{O}_-$) the obstruction to extend $v_0|h(S^n\times\{\delta\})$ (or $v_0|h(S^n\times\{-\delta\})$ to the whole of $D_\delta^{n+1}=h(S^n\times[\delta, 1])$ (or $D_{-\delta}^{n+1}=h(S^n\times[-1, -\delta])$). We can identify the radial inward direction of D_δ^{n+1} (or $D_{-\delta}^{n+1}$) with dx_{2n+2} (or $-dx_{2n+2}$). Thus $k_\delta(S^n)=\partial D_\delta^{n+1}$ and $k_{-\delta}(S^n)=\partial D_{-\delta}^{n+1}$ are oriented as follows:

$$(16)\qquad \begin{cases} [D_\delta^{n+1}]=[k_\delta(S^n)]\times dx_{2n+2} , \\ [D_{-\delta}^{n+1}]=[k_{-\delta}(S^n)]\times(-dx_{2n+2}) . \end{cases}$$

In (16), $[D_\delta^{n+1}]$ and $[D_{-\delta}^{n+1}]$ are induced from $[h(S^{n+1})]$. Define

$$\begin{aligned} H_\delta^{2n+2}&=\{(x_1, \cdots, x_{2n+2})\in\boldsymbol{R}^{2n+2}|x_{2n+2}\geqq\delta\} , \\ H_{-\delta}^{2n+2}&=\{(x_1, \cdots, x_{2n+2})\in\boldsymbol{R}^{2n+2}|x_{2n+2}\leqq-\delta\} . \end{aligned}$$

We define the orientations of H_{δ}^{2n+2} and $H_{-\delta}^{2n+2}$ by the induced ones from $[\boldsymbol{R}^{2n+2}]$. Let $\boldsymbol{R}_{\delta}^{2n+1}=\partial H_{\delta}^{2n+2}$ and $\boldsymbol{R}_{-\delta}^{2n+1}=\partial H_{-\delta}^{2n+2}$. Following the convention (1) of §1, we define the orientations $[\boldsymbol{R}_{\delta}^{2n+1}]'$ and $[\boldsymbol{R}_{-\delta}^{2n+1}]'$ by

$$(17)\qquad \begin{cases} [H_{\delta}^{2n+2}]=[\boldsymbol{R}_{\delta}^{2n+1}]'\times dx_{2n+2}\,, \\ [H_{-\delta}^{2n+2}]=[\boldsymbol{R}_{-\delta}^{2n+1}]'\times(-dx_{2n+2})\,. \end{cases}$$

Note that $[\boldsymbol{R}_{\delta}^{2n+1}]'=[\boldsymbol{R}_{\delta}^{2n+1}]$, where $[\boldsymbol{R}_{\delta}^{2n+1}]$ is the canonical orientation, and $[\boldsymbol{R}_{-\delta}^{2n+1}]'=-[\boldsymbol{R}_{-\delta}^{2n+1}]$. The obstruction $\mathcal{O}_+$ and $\mathcal{O}_-$ are computed with respect to pairs of orientations $([k_{\delta}(S^n)], [\boldsymbol{R}_{\delta}^{2n+1}]')$ and $([k_{-\delta}(S^n)], [\boldsymbol{R}_{-\delta}^{2n+1}]')$.

There already exists a vector field v_0 over $h(S^n\times[-\delta, \delta])$ extending $v_0|\partial D_{\delta}^{n+1}$ and $v_0|k_{\delta}(S^n)\cup k_{-\delta}(S^n)$, so the obstruction $\mathcal{O}$ is related to $\mathcal{O}_+$ and $\mathcal{O}_-$;

$$(18)\qquad \mathcal{O}=\mathcal{O}_+ + \mathcal{O}_-\,.$$

The inverse image $(\varpi\circ k_{\tau})^{-1}(p_{\tau})\subset S^n$ consists of two points $\tilde{p}_{\tau}^{(1)}$ and $\tilde{p}_{\tau}^{(2)}$, where p_{τ} denotes the unique double point of $\varpi k_{\tau}(S^n)$, $-\delta\leqq\tau\leqq\delta$. If τ varies from $-\delta$ to δ, the points $p_{\tau}^{(1)}=h(\tilde{p}_{\tau}^{(1)}, \tau)$ and $p_{\tau}^{(2)}=h(\tilde{p}_{\tau}^{(2)}, \tau)$ traces two arcs A and B which intersect at q transversally. We may suppose that A and B lie in the (x_{2n+1}, x_{2n+2})-plane. By exchanging the names of the two points, if necessary, we may also assume that (the x_{2n+1}-coordinate of $p_{\delta}^{(1)}$)<(the x_{2n+1}-coordinate of $p_{\delta}^{(2)}$). The situation is as in Fig. 5.

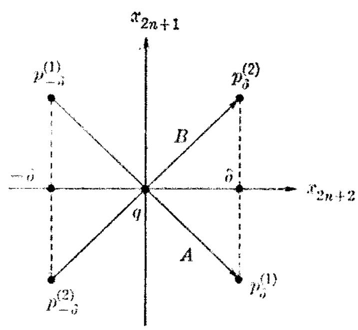


Fig. 5.

By Fig. 5, if the "positive directions" of the arcs A and B are defined to be $(q\longrightarrow p_{\delta}^{(1)})$ and $(q\longrightarrow p_{\delta}^{(2)})$, respectively, we have

$$(19)\qquad [A]\times[B]=-dx_{2n+1}\times dx_{2n+2}\,.$$

By (13), the signs $\varepsilon_{p_{\delta}}$ (or $\varepsilon_{p_{-\delta}}$) of the intersections p_{δ} (or $p_{-\delta}$) in $\boldsymbol{R}_{\delta}^{2n}$ (or $\boldsymbol{R}_{-\delta}^{2n}$) are defined as follows:

(20)
$$\varepsilon_{p_\delta}=\frac{\varpi_*[k_\delta(S^n)]_{p^{(1)}_\delta}\times\varpi_*[k_\delta(S^n)]_{p^{(2)}_\delta}}{[\boldsymbol{R}^{2n}_\delta]'_{p_\delta}}\ ,$$

and

(21)
$$\varepsilon_{p_{-\delta}}=\frac{\varpi_*[k_{-\delta}(S^n)]_{p^{(2)}_{-\delta}}\times\varpi_*[k_{-\delta}(S^n)]_{p^{(1)}_{-\delta}}}{[\boldsymbol{R}^{2n}_{-\delta}]'_{p_{-\delta}}}\ .$$

We notice that in (20) and (21) $p^{(1)}$ and $p^{(2)}$ are exchanged, because (the x_{2n+1}-coordinate of $p^{(1)}_{-\delta}$)>(the x_{2n+1}-coordinate of $p^{(2)}_{-\delta}$). See Fig. 5. Moreover, following the convention (2)′ in §1, one defines $[\boldsymbol{R}^{2n}_\delta]'$ and $[\boldsymbol{R}^{2n}_{-\delta}]'$ by

(22)
$$\begin{cases}[\boldsymbol{R}^{2n+1}_\delta]'=[\boldsymbol{R}^{2n}_\delta]'\times dx_{2n+1}\ , & \text{and}\\ [\boldsymbol{R}^{2n+1}_{-\delta}]'=[\boldsymbol{R}^{2n}_{-\delta}]'\times dx_{2n+1}\ . \end{cases}$$

By Assertion 2.7.1,

(23)
$$\begin{cases}\mathcal{O}_+=(-1)^n\varepsilon_{p_\delta}\ ,\\ \mathcal{O}_-=(-1)^n\varepsilon_{p_{-\delta}}\ .\end{cases}$$

If δ is sufficiently small, the image $\varpi k_\tau(S^n)$ scarcely moves for $-\delta\leqq\tau\leqq\delta$, and the branch of $\varpi k_\delta(S^n)$ at the point p_δ determined by $p^{(1)}_\delta$ corresponds to the branch at the point $p_{-\delta}$ determined by $p^{(1)}_{-\delta}$. This together with $[\boldsymbol{R}^{2n}_\delta]'_{p_\delta}=-[\boldsymbol{R}^{2n}_{-\delta}]'_{p_{-\delta}}$, which follows from (17) and (22), give

(24)
$$\varepsilon_{p_\delta}=(-1)^{n+1}\varepsilon_{p_{-\delta}}\qquad((20),\ (21))\ .$$

Finally ε'_q, the sign of q, is computed as follows:

$$\begin{aligned}\varepsilon'_q&=\frac{[h(S^{n+1})]_{p^{(1)}}\times[h(S^{n+1})]_{p^{(2)}}}{[\boldsymbol{R}^{2n+2}]_q}\\ &=\frac{[k_0(S^n)]_{p^{(1)}}\times[A]\times[k_0(S^n)]_{p^{(2)}}\times[B]}{[\boldsymbol{R}^{2n}]_p\times dx_{2n+1}\times dx_{2n+2}}\\ &=(-1)^n\frac{[k_0(S^n)]_{p^{(1)}}\times[k_0(S^n)]_{p^{(2)}}\times[A]\times[B]}{[\boldsymbol{R}^{2n}]_p\times dx_{2n+1}\times dx_{2n+1}}\\ &=(-1)^{n+1}\frac{\varpi_*[k_0(S^n)]_{p^{(1)}}\times\varpi_*[k_0(S^n)]_{p^{(2)}}}{[\boldsymbol{R}^{2n}]_p}\qquad(19)\\ &=(-1)^{n+1}\varepsilon_{p_\delta}\ .\end{aligned}$$

Therefore, from (18), (23), (24), we have

$$\begin{aligned}\mathcal{O}&=(-1)^n\varepsilon_{p_\delta}-\varepsilon_{p_\delta}\\ &=-\varepsilon'_q-(-1)^{n+1}\varepsilon'_q\end{aligned}$$

$$=\begin{cases}-2\varepsilon_q' & n+1:\ \text{even}\\ 0 & n+1:\ \text{odd}\ ,\end{cases}$$

completing the proof of 2.8.1. Q.E.D.

PROOF OF 2.8 (continued). Under the deformation of type (II) $\mu(f)$ changes to

$$\begin{aligned}&\alpha(f)+(-1)^{n+1}(1-t)\cdot\{\beta(f)+\varepsilon_q'\}+(-1)^{n+1}\{\mathscr{C}(f)+\mathscr{C}\}\\ &=\mu(f)+(-1)^{n+1}(1-t)\varepsilon_q'+(-1)^{n+1}\mathscr{C}\\ &=\mu(f)+\begin{cases}-(1+t)\varepsilon_q' & n+1:\ \text{even}\\ -(1-t)\varepsilon_q' & n+1:\ \text{odd}\ .\end{cases}\qquad(2.8.1)\\ &=\mu(f)\ \mathrm{mod}\ \{\nu-(-1)^n\bar{\nu}\cdot t\,|\,\nu\in \boldsymbol{Z}[\pi_1\mathscr{F}N]\}\ .\end{aligned}$$

Thus in $Q_n'(\pi_1\mathscr{F}N)$, $\mu(f)$ does not change. Q.E.D.

Summarizing 2.6, 2.7, 2.8 and 2.4, we can conclude

THEOREM 2.9. *$\mu(f)$ depends only on the homotopy class of a pathed nice immersion f. Approximating any pathed map $(D^{n+1}, S^n)\longrightarrow(E, \mathscr{F}N)$ by a nice immersion, we see that $\mu(f)$ defines a map*

$$\pi_{n+1}(E, \mathscr{F}N)\longrightarrow Q_n'(\pi_1\mathscr{F}N)\ ,$$

which is also denoted by μ.

REMARK. We have obtained an "intersection form" (λ, μ) associated with exterior 2-connected submanifolds in codimension 2. This is a higher dimensional analogy of a form which is defined by Seifert in his work of classical knot theory [22].

§ 3. Properties of (λ, μ)

We continue to use the same notations W^{2n+2}, L^{2n}, N, E, $\mathscr{F}N$ as in §§1~2; L^{2n} is an exterior 2-connected $2n$-submanifold of W^{2n+2} with $2n\geqq 4$.

$$1\longrightarrow C\longrightarrow \pi_1(\mathscr{F}N)\xrightarrow{\tilde{\omega}_\sharp}\pi_1(L)\longrightarrow 1$$

is the cyclic extension associated with W (1.6). Let t denote the special generator of C (1.4).

Let $*\in\mathscr{F}N$ be the base point of $\pi_1(\mathscr{F}N)$ and $\pi_{n+1}(E, \mathscr{F}N)$. Let $\{*\xrightarrow{l_i}*\}$ denote a loop l_i representing an element $g_i\in\pi_1(\mathscr{F}N)$. The composition $g_1\cdot g_2$ is defined by $\{*\xrightarrow{l_1}*\xrightarrow{l_2}*\}$. A left $\pi_1(\mathscr{F}N)$-module structure on $\pi_{n+1}(E, \mathscr{F}N)$ is defined as follows: Let $f:(D^{n+1}, S^n)\longrightarrow(E, \mathscr{F}N)$ be a pathed map representing $x\in\pi_{n+1}(E, \mathscr{F}N)$.

Let $\gamma(f)=\{*\xrightarrow{\gamma(f)}p\}$ $(p\in f(S^n))$ be the path assigned to f. Then gx $(g=\{*\xrightarrow{l}*\})$ is represented by the map f together with the new path $g\cdot\gamma(f)=\{*\xrightarrow{l}*\xrightarrow{\gamma(f)}p\}$.

We simply write π_1 instead of $\pi_1(\mathscr{F}N)$.

LEMMA 3.1. *Let* $x, y\in\pi_{n+1}(E, \mathscr{F}N)$.

(i) $\lambda(x, y)=(-1)^n\overline{\lambda(y, x)}\cdot t$.

(ii) *For any fixed* y, $\lambda(*, y):\pi_{n+1}(E, \mathscr{F}N)\longrightarrow \boldsymbol{Z}[\pi_1]$ *is a* $\boldsymbol{Z}[\pi_1]$*-homomorphism.*

(iii) $\mu(x+y)=\mu(x)+\mu(y)+\lambda(x, y)$.

(iv) $\lambda(x, x)=\mu(x)+(-1)^n\overline{\mu(x)}\cdot t$.

(v) $\mu(ax)=a\mu(x)\bar{a}$ *for* $\forall a\in \boldsymbol{Z}[\pi_1]$.

PROOF OF 3.1. (i) is proved in 1.9.

Proof of (ii): Let $f+f'$ denote the "connected sum along the path $\gamma(f)\cdot\gamma(f')$". Then clearly $\alpha(f+f', g)=\alpha(f, g)+\alpha(f', g)$ and similarly for β. Thus $\lambda(x+x', y)=\lambda(x, y)+\lambda(x', y)$. On the other hand, by the definitions of g_p and g'_q in the definition of λ, we have $\alpha(g\cdot f, g)=g\cdot\alpha(f, g)$ and $\beta(g\cdot f, g)=g\beta(f, g)$. So $\lambda(g\cdot x, y)=g\cdot\lambda(x, y)$ $(\forall g\in\pi_1)$.

Proof of (iii): $\alpha(f+g)=\alpha(f)+\alpha(g)+\alpha(f, g)$, $\beta(f+g)=\beta(f)+\beta(g)+\beta(f, g)$ and $\mathscr{O}(f+g)=\mathscr{O}(f)+\mathscr{O}(g)$. Therefore, $\mu(x+y)=\mu(x)+\mu(y)+\lambda(x, y)$.

Proof of (iv): Let x be represented by a pathed nice immersion f. First suppose $\mathscr{O}(f)=0$. Then f is framed so that a frame of the normal bundle of $\varpi f(S^n)$ in L^{2n} can be extended to D^{n+1}. So there is a nice immersion f' which is "parallel" to f, and such that its projected image $\varpi f'$ is parallel to ϖf.

Let $q_1, \cdots, q_r$ (or $p_1, \cdots, p_s$) be points of self-intersection of $f(D^{n+1})$ in E^{2n+2} (or of $\varpi f(S^n)$ in L^{2n}). Then for each point q_i, there are two intersection points q_i^1, q_i^2 of f and f'. Similarly for p_i, there are p_i^1 and p_i^2. Then by simple geometric considerations, $\varepsilon'_{q_i^1}(f,f')=(-1)^{n+1}\varepsilon'_{q_i^2}(f,f')$, $g'_{q_i^1}=(g'_{q_i^2})^{-1}$, $\varepsilon_{p_i^1}=(-1)^n\varepsilon_{p_i^2}$ and $g_{p_i^1}=(g_{p_i^2})^{-1}\cdot t$. Hence we have

$$\begin{aligned}\lambda(x, x)&=\lambda(f,f')\\&=\sum_i\{\varepsilon_{p_i^1}g_{p_i^1}+(-1)^n g_{p_i^1}^{-1}\cdot t\}\\&\quad+(-1)^{n+1}(1-t)\sum_i\{\varepsilon'_{q_i^1}g'_{q_i^1}+(-1)^{n+1}\varepsilon'_{q_i^1}(g'_{q_i^1})^{-1}\}\\&=\mu(f)+(-1)^n\overline{\mu(f)}\cdot t\ .\end{aligned}$$

Now we consider the general case when $\mathscr{O}(f)$ is not necessarily zero. However, by introducing a finite number of self-intersection points in $\varpi f(S^n)\subset L^{2n}$, we can make $\mathscr{O}(f)=0$ (2.7.1). Thus the problem is reduced to the first case.

Proof of (v):

$$\mu(g\cdot f)=\alpha(g\cdot f)+(-1)^{n+1}(1-t)\beta(g\cdot f)+(-1)^{n+1}\mathscr{C}(g\cdot f)$$
$$=g\cdot\alpha(f)\cdot g^{-1}+(-1)^{n+1}(1-t)\cdot g\cdot\beta(f)g^{-1}+(-1)^{n+1}\mathscr{C}(f)\ .$$

But $\mathscr{C}(f)$ can also be written as $g\cdot\mathscr{C}(f)\cdot g^{-1}$; thus we have $\mu(g\cdot x)=g\cdot\mu(x)\cdot g^{-1}$. The proof for $\mu(a\cdot x)$, $a=\sum m_g g$, is not difficult.

DEFINITION 3.2 (cf. [10]). For $r\geqq 0$, put $rD^k=\{x\in \boldsymbol{R}^k \mid \|x\|\leqq r\}$. A k-handle $H=(2D^k)\times D^{2n+1-k}$ (of total dimension $2n+1$) in W^{2n+2} is called a *normally embedded k-handle attached to* L^{2n} if it satisfies

(i) $H\cap L^{2n}=(\partial 2D^k)\times D^{2n+1-k}$,

(ii) $H\cap E=D^k\times D^{2n+1-k}$, where D^k is identified with $1D^k$ $(\subset 2D^k)$.

An element $x\in\pi_k(E,\mathscr{F}N)$ is said *to be represented by a normally embedded handle* if there exists a normally embedded handle $H=(2D^k)\times D^{2n+1-k}$ such that $(D^k\times 0, \partial(D^k\times 0))\subset(E,\mathscr{F}N)$ represents x.

LEMMA 3.3. *Suppose* $n\geqq 2$. *If* $x\in\pi_{n+1}(E,\mathscr{F}N)$ *is represented by a normally embedded handle, then* $\mu(x)=0$. *If* $n\geqq 3$, *the converse is true.*

PROOF OF 3.3. If x is represented by a normally embedded handle, we can take the core disk as a nice immersion f representing x. Clearly $f(D^{n+1})$ and $\varpi f(S^n)$ have no self-intersection points. Moreover, $w|f(S^n)$ is extended to $f(D^{n+1})$, where w is a "positive tangent vector field" over $\mathscr{F}N$ along the S^1-fibres; we have only to take the orthogonal complement of $D^{n+1}\times D^n$ in W^{2n+2}. Thus $\mathscr{C}(f)=0$, and we have $\mu(x)=0$. Conversely, suppose $n\geqq 3$ and $\mu(x)=0$. Let $f:(D^{n+1}, S^n)\longrightarrow (E,\mathscr{F}N)$ be a pathed nice immersion in the homotopy class of x. Let q be a self--intersection point of $f(D^{n+1})$. Draw two arcs from q to $\mathscr{F}N$ along the two branches. We then have an arc on $f(D^{n+1})$ which represents an element of $\pi_1(E,\mathscr{F}N)$. Since $\pi_1(E,\mathscr{F}N)=0$, this arc deforms into $\mathscr{F}N$ tracing a non-singular 2-disk. By deforming f by a regular homotopy along this 2-disk, we can reduce the number of self-intersection points of $f(D^{n+1})$. Proceeding inductively we can make f an embedding (i.e. $\beta(f)=0$). Furthermore, by introducing some self-intersection points to ϖf in L^{2n}, we can make $\mathscr{C}(f)=0$ (2.7.1). Now we have $\mu(f)=\alpha(f)$. Since $\mu(x)=0$ in $Q_n^t(\pi_1)$, $\alpha(f)$ is of the form $a-(-1)^n\bar{a}\cdot t$ $(a\in \boldsymbol{Z}[\pi_1])$ as an element of $\boldsymbol{Z}[\pi_1]$. Hence we know that the set of double points of $\varpi f(S^n)$ is divided into a set of pairs $(p_1,p_1'),\cdots,(p_r,p_r')$ such that $\varepsilon_{p_i'}=-(-1)^n\varepsilon_{p_i}$ and $g_{p_i'}=(g_{p_i})^{-1}\cdot t$. Let (p,p') be such a pair. Let $p_{(1)}, p_{(2)}$ (or $p'_{(1)}, p'_{(2)}$) be the two lifts of p (or p') to $f(S^n)$. We take the orders of these points so that $g_p(p_{(1)},p_{(2)})=g_p$, $\varepsilon_p(p_{(1)},p_{(2)})=\varepsilon_p$, $g_{p'}(p'_{(1)},p'_{(2)})=g_{p'}$ and $\varepsilon_{p'}(p'_{(1)},p'_{(2)})=\varepsilon_{p'}$. Then a loop $l=$

$$\{p_{(2)}\xrightarrow{\text{along }\varpi^{-1}(p)\text{ positively}}p_{(1)}\xrightarrow{\text{a path on }f(S^n)}p'_{(2)},\xrightarrow{\text{along }\varpi^{-1}(p')\text{ negatively}}p'_{(1)}$$

$\xrightarrow{\text{a path on } f(S^n)} p_{(2)}\}$ is contractible in $\mathscr{F}N$, for the composite loop $\gamma(f)\cdot l\cdot\gamma(f)^{-1}$ is homotopic to

$$\begin{aligned}
&\{* \xrightarrow{\gamma(f)} p_{(2)} \xrightarrow{\text{along } \varpi^{-1}(p),\ \text{posi.}} p_{(1)} \xrightarrow{\gamma(f)^{-1}} *\}\\
&\cdot\{* \xrightarrow{\gamma(f)} p_{(1)} \xrightarrow{\text{a path on } f(S^n)} p'_{(2)} \xrightarrow{\gamma(f)^{-1}} *\}\\
&\cdot\{* \xrightarrow{\gamma(f)} p'_{(2)} \xrightarrow{\text{along } \varpi^{-1}(p'),\ \text{nega.}} p'_{(1)} \xrightarrow{\gamma(f)^{-1}} *\}\\
&\cdot\{* \xrightarrow{\gamma(f)} p'_{(1)} \xrightarrow{\text{a path on } f(S^n)} p_{(2)} \xrightarrow{\gamma(f)^{-1}} *\}\\
&=(g_{p'}^{-1}\cdot t)\cdot(\text{null})\cdot(g_{p'}^{-1})\cdot(\text{null})=g_{p'}\cdot g_{p'}^{-1}=(\text{null})\,.
\end{aligned}$$

This loop l bounds in $\mathscr{F}N$ an embedded 2-disk $\varDelta^2$ such that $\varDelta^2\cap f(S^n)=l$. Note that $\varepsilon_p(p_{(2)}, p_{(1)})=(-1)^n\varepsilon_p(p_{(1)}, p_{(2)})=(-1)^n(-(-1)^n\varepsilon_{p'})=-\varepsilon_{p'}(p'_{(1)}, p'_{(2)})$. Hence by applying Whitney's process with the 2-disk $\varpi(\varDelta^2)\subset L^{2n}$, we can make p and p' pairwise vanish. (Here we use the assumption $n\geqq 3$.) Continuing this process, we finally obtain an embedding $\bar{f}: (D^{n+1}, S^n)\longrightarrow(E, \mathscr{F}N)$ such that the composite $\varpi\circ(\bar{f}|S^n): S^n\longrightarrow L^{2n}$ is also an embedding. On the other hand $\mathscr{O}(\bar{f})=0$. Thus we may find a cross section $s: \bar{f}(D^{n+1})\longrightarrow\bar{f}(D^{n+1})\times(\boldsymbol{R}^{n+1}-\{0\})$ of normal bundle of $\bar{f}$ extending the vector field $w|\bar{f}(S^n)$. Take a complementary sub-bundle ξ of s in $\bar{f}(D^{n+1})\times\boldsymbol{R}^{n+1}$, where $\bar{f}(D^{n+1})\times\boldsymbol{R}^{n+1}$ is the normal bundle of $\bar{f}$. Then since ξ is a bundle over $\bar{f}(D^{n+1})$, it is trivial and its disk bundle is $\bar{f}(D^{n+1})\times D^n$. Since s extends $w|\bar{f}(S^n)$, the projected image $\varpi(\bar{f}(S^n)\times D^n)$ is an embedding in L^{2n}. Therefore, if we extend $\bar{f}(S^n)\times D^n$ into N by attaching a mapping cylinder of $\varpi|\bar{f}(S^n)\times D^n$, we have constructed the desired normally embedded $(n+1)$-handle. This completes the proof. Q.E.D.

By a similar argument we have

LEMMA 3.4. *Suppose* $2n\geqq 4$. *If* x *and* $y\in\pi_{n+1}(E, \mathscr{F}N)$ *are represented by pathed nice immersions* f *and* g *respectively such that*

(i) $f(D^{n+1})\cap g(D^{n+1})=\phi$ *and*

(ii) $\varpi f(S^n)\cap\varpi g(S^n)=\phi$,

then $\lambda(x, y)=0$. *If* $2n\geqq 6$, *the converse is also true.*

In concluding this section, we give an invariance property of (λ, μ) which is stronger than the homopy invariance. Let $2n\geqq 4$. Suppose that the $2n$-Poincaré thickening W^{2n+2} is a regular submanifold in the boundary ∂Z of a $2n+3$-manifold Z^{2n+3}, and that there is another $2n$-Poincaré thickening W' in ∂Z which is disjoint from W. Let W (resp. W') have an exterior n-connected $2n$-submanifold L (resp. L'), and let E, $\mathscr{F}N$, E' and $\mathscr{F}N'$, be as usual. Moreover, suppose that there is a locally flat exterior 2-connected $2n+1$-submanifold Y^{2n+1} in Z such that $\partial Y\cap W=L$ and $\partial Y\cap W'=L'$. Let T, F and $\mathscr{F}T$ be a regular (or tubular) neighbourhood of Y, the exterior of T in Z, the frontier of T, respectively. We may take them so that $E=W\cap F$, $E'=W'\cap F$, $\mathscr{F}N=W\cap\mathscr{F}T$, and $\mathscr{F}N'=W'\cap\mathscr{F}T$. We assume that the inclusions induce the following isomorphisms

$$\begin{array}{ccccc}\pi_1(E) & \xrightarrow{\cong} & \pi_1(F) & \xleftarrow{\cong} & \pi_1(E') \\ \uparrow\cong & & \uparrow\cong & & \uparrow\cong \\ \pi_1(\mathscr{F}N) & \xrightarrow{\cong} & \pi_1(\mathscr{F}T) & \xleftarrow{\cong} & \pi_1(\mathscr{F}N')\,.\end{array}$$

Finally, the incidence relations of orientations are assumed to be positive (plus 1) between $[W]$ (or $[L]$) and $\partial[Z]$ (or $\partial[Y]$), and negative (minus 1) between $[W']$ (or $[L']$) and $\partial[Z]$ (or $\partial[Y]$).

THEOREM 3.5. *Let $i_\sharp:\pi_{n+1}(E,\mathscr{F}N)\longrightarrow\pi_{n+1}(F,\mathscr{F}T)$ and $i'_\sharp:\pi_{n+1}(E',\mathscr{F}N')\longrightarrow\pi_{n+1}(F,\mathscr{F}T)$ be induced by the inclusions. If $i_\sharp(x)=i'_\sharp(x')$ and $i_\sharp(y)=i'_\sharp(y')$, we have*

$$\lambda(x, y)=\lambda'(x', y')\,,$$

$$\mu(x)=\mu'(x')\,,$$

and

$$\mu(y)=\mu'(y')\,.$$

Here (λ', μ') is the Seifert form associated with L'.

PROOF OF 3.5. Let f and $g:(D^{n+1}, S^n)\longrightarrow(E,\mathscr{F}N)$ (or f' and $g':(D^{n+1}, S^n)\longrightarrow(E',\mathscr{F}N')$) be a pair of pathed nice immersions corresponding to x and y (or x' and y'). By the hypothesis, there are maps

$\tilde{f}$, $\tilde{g}:(D^{n+1}\times I, S^n\times I; D^{n+1}\times 0, D^{n+1}\times 1)\longrightarrow(F,\mathscr{F}T; E, E')$ such that $\tilde{f}|D^{n+1}\times 0=f$, $\tilde{f}|D^{n+1}\times 1=f'$, $\tilde{g}|D^{n+1}\times 0=g$ and $\tilde{g}|D^{n+1}\times 1=g'$. Let $f_S=\tilde{f}|S^n\times I$, $g_S=\tilde{g}|S^n\times I:S^n\times I\longrightarrow\mathscr{F}T$. Make f_S, g_S nice in the following sense:

(A-1°) f_S, g_S are generic immersions which intersect in general position (at finite set of points).

We now show that in addition f_S, g_S can be made to satisfy

(A-2°) the compositions $\tilde{\varpi}\circ f_S$, $\tilde{\varpi}\circ g_S:S^n\times I\longrightarrow Y^{2n+1}$ are generic immersions which intersect in general position, where $\tilde{\varpi}$ is the projection of the S^1-fibration $\mathscr{F}T\longrightarrow Y$.

For this, approximate $\tilde{\varpi}\circ f_S$ and $\tilde{\varpi}\circ g_S$ by generic maps. Their singularities are isolated points [30]. Let s_1 be a singular point of $\tilde{\varpi}\circ f_S(S^n\times I)$. s_1 is a terminal point of a double curve Δ_1 of $\tilde{\varpi}\circ f_S(S^n\times I)$. Introduce a Whitney double point d_2 in $\tilde{\varpi}\circ f_S(S^n\times 0)$ $(\subset L^{2n})$ by a non-transverse movement of $f(S^n)$ $(=f_S(S^n\times 0))$, cf. (2.4), and consider a double curve Δ_2 in $\tilde{\varpi}\circ f_S(S^n\times I)$ starting at d_2 and ending at a new singular points s_2. Taking s_2 sufficiently close to s_1, and choosing the sign of the double point d_2 appropriately, we obtain a double curve containing $\Delta_1\cup\Delta_2$ by cancelling s_1 and s_2 in pairs. In this way, all of the singularities of $\tilde{\varpi}\circ f_S(S^n\times I)$ can be removed.

The same argument is applied to $\tilde{\varpi}\circ g_S$. Now make $\tilde{\varpi}\circ f_S$ and $\tilde{\varpi}\circ g_S$ intersect in general position, then they will satisfy (A-2°). Note that introducing Whitney self-intersection points in $\tilde{\varpi}\circ f_S(S^n\times 0)$ does not change the value of $\lambda(f,g)$, $\mu(f)$, and $\mu(g)$ (2.5, 2.7).

In a similar manner we can get rid of isolated singular points of (generic) maps $\tilde{f}$ and $\tilde{g}$; we introduce double points in $f(D^{n+1})$ (or $g(D^{n+1})$) and double curves in $\tilde{f}(D^{n+1}\times I)$ (or $\tilde{g}(D^{n+1}\times I)$) without altering the values of λ and μ. Therefore, we may assume

(A-3°) $\tilde{f}$, $\tilde{g}$ are generic immersions which intersect in general position.

Let f_D and $g_D: (S^n\times I\cup D^{n+1}\times 1, S^n\times 0)\longrightarrow(\mathscr{F}T\cup E', \mathscr{F}N)$ be pathed maps defined by $f_S\cup f'$ and $g_S\cup g'$ respectively. In the same way as in §1, we define the intersection pairing $\lambda(f_D,g_D)\in \boldsymbol{Z}[\pi]$ and the self-intersection numbers $\mu(f_D)$, $\mu(g_D)\in Q'_n(\pi)$ by replacing $(E,\mathscr{F}N)$ in §1 with $(\mathscr{F}T\cup E',\mathscr{F}N)$. Here π denotes $\pi_1(\mathscr{F}N)$ $(\cong\pi_1(E)\cong\pi_1(F)\cong\pi_1(\mathscr{F}T))$. Precisely speaking, we define $\beta(f_D,g_D)$ by counting intersecting points of $f_S(S^n\times I)$ and $g_S(S^n\times I)$ in $\mathscr{F}T$ and those of $f'(D^n\times 1)$ and $g'(D^n\times 1)$ in E'. $\alpha(f_D,g_D)$ is defined by counting the intersection $\varpi f(S^n)\cap\varpi g(S^n)\subset L^{2n}$. $\lambda(f_D,g_D)$ is the pairing defined by

$$\alpha(f_D,g_D)+(-1)^{n+1}(1-t)\beta(f_D,g_D)\ .$$

The definitions of $\mu(f_D)$ and $\mu(g_D)$ are similar;

$$\mu(f_D)=\alpha(f_D)+(-1)^{n+1}(1-t)\beta(f_D)+(-1)^{n+1}\mathscr{C}(f_D)\ ,$$

and the formula for $\mu(g_D)$ is the same.

Step (I). $\lambda(f,g)=\lambda(f_D,g_D)$, $\mu(f)=\mu(f_D)$, $\mu(g)=\mu(g_D)$.

PROOF OF STEP (I). By (A-3°), the intersection $\tilde{f}(D^{n+1}\times I)\cap\tilde{g}(D^{n+1}\times I)$ consists of some arcs and circles. Let A be such an arc with terminal points q_0 and q_1. Two cases are possible:

Case (i): $\{q_0,q_1\}\subset E$ (or $\{q_0,q_1\}\subset\mathscr{F}T\cup E'$).

Case (ii): $q_0\in E$, $q_1\in\mathscr{F}T\cup E'$ (or $q_1\in E$, $q_0\in\mathscr{F}T\cup E'$).

In Case (i), it is easily seen that $\varepsilon'_{q_0}=-\varepsilon'_{q_1}$ and that $g'_{q_0}=g'_{q_1}$, so the contribution of $\{q_0,q_1\}$ to $\beta(f,g)$ (or to $\beta(f_D,g_D)$) is zero. In Case (ii), we have $\varepsilon'_{q_0}=\varepsilon'_{q_1}$, $g'_{q_0}=g'_{q_1}$, and the contribution of q_0 (or q_1) to $\beta(f, g)$ is equal to that of q_1 (or q_0) to $\beta(f_D, g_D)$. Since any intersection point of $f(D^{n+1})$ and $g(D^{n+1})$ (or of $f_D(S^n\times I\cup D^{n+1}\times 1)$ and $g_D(S^n\times I\cup D^{n+1}\times 1)$) is a terminal point of an intersecting arc, we can conclude that $\beta(f, g)=\beta(f_D, g_D)$. On the other hand, $\alpha(f, g)=\alpha(f_D, g_D)$ is obvious, (both are computed by the same $f(S^n)$ and $g(S^n)$). Therefore, $\lambda(f, g)=\lambda(f_D, g_D)$.

A similar argument using double curves of $\tilde{f}(D^{n+1}\times I)$ instead of intersecting arcs shows that $\alpha(f)=\alpha(f_D)$ and $\beta(f)=\beta(f_D)$. In order to see $\mu(f)=\mu(f_D)$, let ν be the normal bundle of $\tilde{f}: D^{n+1}\times I\longrightarrow F$ (which is an immersion by A-3°), then $\nu\cong(D^{n+1}\times I)\times \boldsymbol{R}^{n+1}$. Let w be the positive tangent vector field over $\mathscr{F}N$ along the S^1-fibres (§1. Definition of μ). Clearly the obstruction to extending the non-zero cross section $w|f(S^n)$ to a non-zero section of $\nu|f(D^{n+1})$ is equal to that of extending $w|f(S^n)$ to a non-zero section of $\nu|f_S(S^n\times I)\cup f'(D^{n+1}\times 1)$. So $\mathscr{O}(f)=\mathscr{O}(f_D)$; hence we have $\mu(f)=\mu(f_D)$. In a similar manner we show $\mu(g)=\mu(g_D)$. This completes the proof of step (I).

Step (II). $\lambda(f_D, g_D)=\lambda'(f', g')$, $\mu(f_D)=\mu'(f')$ and $\mu(g_D)=\mu'(g')$.

PROOF OF STEP (II). By (*A*-2°), the intersection $\tilde{\omega}f_S(S^n\times I)\cap\tilde{\omega}g_S(S^n\times I)$ consists of some arcs and circles. Let C be such a circle, C_f and C_g its liftings to $f_S(S^n\times I)$ and $g_S(S^n\times I)$ respectively, i.e., $C_f=\tilde{\omega}^{-1}(C)\cap f_S(S^n\times I)$ and $C_g=\tilde{\omega}^{-1}(C)\cap g_S(S^n\times I)$. Since $C_f\subset f_S(S^n\times I)$ and $C_g\subset g_S(S^n\times I)$, they are contractible in $\mathscr{F}T$ (recall $n\geqq 2$). From this, we see that the contribution of points in $C_f\cap C_g$ to $\lambda(f_D, g_D)$ is zero*). Similarly the contributions to $\mu(f_D)$ and $\mu(g_D)$ are zero. So in comparing $\lambda(f_D, g_D)$, etc. with $\lambda'(f', g')$, etc., only the intersecting *arcs* are relevant.

Next let A be an intersecting arc $(\subset\tilde{\omega}f_S(S^n\times I)\cap\tilde{\omega}g_S(S^n\times I)\subset Y^{2n+1})$ with terminal points p_0, p_1. Clearly $\{p_0, p_1\}\subset L\cup L'$. Let A_f and A_g be its liftings to $f_S(S^n\times I)$ and $g_S(S^n\times I)$. For the sake of convenience, let us introduce the following notations:

$\alpha_D(A)=$the contribution of points $\{p_0, p_1\}\cap L$ to $\alpha(f_D, g_D)$ $(=\alpha(f, g))$,

$\beta_D(A)=$the contribution of points in $A_f\cap A_g$ to $\beta(f_D, g_D)$,

$\alpha'(A)=$the contribution of points $\{p_0, p_1\}\cap L'$ to $\alpha'(f', g')$.

We want to show

$$(*)\qquad \alpha_D(A)+(-1)^{n+1}(1-t)\beta_D(A)=\alpha'(A)\ .$$

Let p_0^f and p_0^g (or p_1^f and p_1^g) be the liftings of p_0 (or p_1) to $f_S(S^n\times 0\cup S^n\times 1)$

*) See *Added in proof* at the end of this paper.

$(=f(S^n)\cup f'(S^n))$ and to $g_S(S^n\times 0\cup S^n\times 1)$ $(=g(S^n)\cup g'(S^n))$ respectively.

There are three cases (a)~(c). We consider them separately.

Case (a); $\{p_0, p_1\}\subset L$.

Subcase (a-0) *when* $A_f\cap A_g=\phi$. Then $g_{p_0}(f, g)=g_{p_1}(f, g)$ and $\varepsilon_{p_0}(f, g)=-\varepsilon_{p_1}(f, g)$. Thus $\alpha_D(A)=0$. On the other hand, $\beta_D(A)=0$ and $\alpha'(A)=0$ are trivial. Hence (*) follows.

Subcase (a-1) *when* $A_f\cap A_g=\{q\}$. By exchanging the notations p_0 and p_1, if necessary, we can suppose that the arc $\{p_0^f \xrightarrow{\text{along } \varpi^{-1}(p_0) \text{ positively}} p_0^g\}$ is homotopic in $\mathscr{F}T$ to the arc $\{p_0^f \xrightarrow{A_f} q \xrightarrow{A_g} p_0^g\}$ fixing p_0^f and p_0^g. Then

$$\begin{aligned}
g_{p_0}(f, g) &= \{* \xrightarrow{\gamma(f)} p_0^f \xrightarrow{+} p_0^g \xrightarrow{\gamma(g)^{-1}} *\} \\
&= \{* \xrightarrow{\gamma(f)} p_0^f \xrightarrow{A_f} q \xrightarrow{A_g} p_0^g \xrightarrow{\gamma(g)^{-1}} *\} \\
&= \{* \xrightarrow{\gamma(f)} p_0^f \xrightarrow{A_f} p_1^f \xrightarrow{-} p_1^g \xrightarrow{A_g} p_0^g \xrightarrow{\gamma(g)^{-1}} *\} \\
&= \{* \xrightarrow{\gamma(f)} p_1^f \xrightarrow{-} p_1^g \xrightarrow{\gamma(g)^{-1}} *\} \\
&= g_{p_1}(g, f)^{-1} \\
&= g_{p_1}(f, g)\cdot t^{-1},
\end{aligned}$$

by the proof of 1.9. Therefore, $g_{p_1}(f, g)=g_{p_0}(f, g)\cdot t$. $\varepsilon_{p_1}=-\varepsilon_{p_0}$ is obvious. So $\alpha_D(A)=\varepsilon_{p_0}g_{p_0}-\varepsilon_{p_0}g_{p_0}\cdot t=\varepsilon_{p_0}g_{p_0}(1-t)$.

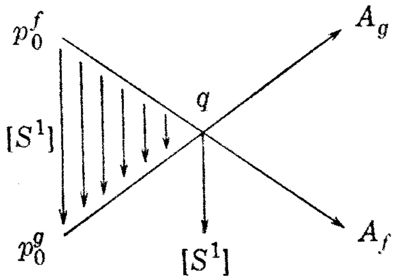
Now the sign ε_q' of the intersection $f_D(S^n\times I)\cap g_D(S^n\times I)$ in $\mathscr{F}T$ is computed as follows: First orient A_f and A_g by the directions $[p_0^f\longrightarrow p_1^f]$ and $[p_0^g\longrightarrow p_1^g]$. Then

$$\begin{aligned}
\varepsilon_q' &= \frac{[f_D(S^n\times I)]_q\times[g_D(S^n\times I)]_q}{[\mathscr{F}T]} \\
&= \frac{[f(S^n)]_{p_0}\times[A_f]\times[g(S^n)]_{p_0}\times[A_g]}{[L^{2n}]\times[S^1]\times[A_f]}
\end{aligned}$$

(Local orientations are those at p_0^f or p_0^g, but they are brought to q along A_f or A_g.)

$$= (-1)^n\frac{[f(S^n)]_{p_0}\times[g(S^n)]_{p_0}\times[A_f]\times[A_g]}{[L^{2n}]\times[S^1]\times[A_f]}\ .$$

In a neighbourhood of q, one may consider that the directions $[S^1]$, $[A_f]$, $[A_g]$ are in the same 2-plane, then clearly $[S^1]\times[A_f]=[A_f]\times[A_g]$ (see the figure below).



Therefore $\varepsilon'_q=(-1)^n\dfrac{[f(S^n)]_{p_0}\times[g(S^n)]_{p_0}}{[L^{2n}]}=(-1)^n\varepsilon_{p_0}$. As is easily seen, $g'_q=g_{p_0}$. So we have

$$\beta_D(A)=(-1)^n\varepsilon_{p_0}\cdot g_{p_0}\ .$$

Hence $\alpha_D(A)+(-1)^{n+1}(1-t)\beta_D(A)=\varepsilon_{p_0}g_{p_0}(1-t)+(-1)^{n+1}(1-t)\{(-1)^n\varepsilon_{p_0}g_{p_0}\}=0$. On the other hand $\alpha'(A)=0$, for $\{p_0, p_1\}\cap L'=\phi$, and (*) follows.

Subcase (a–r) *where* $A_f\cap A_g=\{q_1, \cdots, q_r\}$. This case is treated by essentially the same method as in (a–1). One can verify that

$$\alpha_D(A)=\varepsilon_{p_0}g_{p_0}(1-t^\chi)\ ,$$

and

$$\beta_D(A)=\begin{cases}(-1)^n\varepsilon_{p_0}g_{p_0}(1+t+\cdots+t^{\chi-1}) & \text{for } \chi>0\ ,\\ 0 & \text{for } \chi=0\ ,\\ (-1)^{n+1}\varepsilon_{p_0}g_{p_0}(t^{-1}+t^{-2}+\cdots+t^{\chi}) & \text{for } \chi<0\ ,\end{cases}$$

wherh χ=(the number of q_i's at which A_f crosses A_g positively)−(the number of q_i's at which A_f crosses A_g negatively). From this, (*) follows. ($\alpha'(A)=0$).

Case (b); $p_0\in L$ *and* $p_1\in L'$ *(by exchanging the notations* p_0, p_1, *if necessary.)*

Subcase (b–0) *when* $A_f\cap A_g=\phi$. Then we have $g_{p_0}(f, g)=g_{p_1}(f, g)$ and $\varepsilon_{p_0}(f, g)=\varepsilon_{p_1}(f, g)$. Therefore, $\alpha_D(A)=\alpha'(A)$, $\beta_D(A)=0$, and (*) follows.

Subcase (b–1) *when* $A_f\cap A_g=\{q\}$. In a similar way to (a–1), we have

$$g_{p_1}(f', g')=g_{p_0}(f, g)\cdot t\ ,$$

$$\varepsilon_{p_1}=\varepsilon_{p_0}\ ,$$

$$\varepsilon'_q=(-1)^n\varepsilon_{p_0}$$

and

$$g'_q=g_{p_0}\ .$$

Therefore $\alpha_D(A)=\varepsilon_{p_0}g_{p_0}$, $\beta_D(A)=(-1)^n\varepsilon_{p_0}g_{p_0}$, and $\alpha'(A)=\varepsilon_{p_0}g_{p_0}\cdot t$. From this, (*) is easily verified.

Subcase (b-r) *when* $A_f\cap A_g=\{q_1,\cdots,q_r\}$ is treated similarly.

Case (c); $\{p_0,p_1\}\subset L'$.

Subcase (c-0) *when* $A_f\cap A_g=\phi$. The verification of (*) is the same as in (a-0).

Subcase (c-1) *when* $A_f\cap A_g=\{q\}$. We may suppose that $\{p_0^f\xrightarrow{\tau}p_0^g\}\simeq\{p_0^f\xrightarrow{A_f}q\xrightarrow{A_g}p_0^g\}$ (fixing p_0^f, p_0^g). We have $g_{p_1}(f',g')=g_{p_0}(f',g')t$, $\varepsilon_{p_0}=-\varepsilon_{p_1}$, $g'_q=g_{p_0}$, and $\varepsilon'_q=(-1)^{n+1}\varepsilon_{p_0}$. Hence $\beta_D(A)=(-1)^{n+1}\varepsilon_{p_0}g_{p_0}$, $\alpha'(A)=\varepsilon_{p_0}g_{p_0}(1-t)$, and clearly $\alpha_D(A)=0$. (*) is now easily verified.

Subcase (c-r) *when* $A_f\cap A_g=\{q_1,\cdots,q_r\}$. The verification is the same as $(a-r)$, and we omit it.

By (a-1)$\sim$(c-r), the formula (*) is established. $\lambda(f_D,g_D)=\lambda'(f',g')$ is now proved as follows:

$$\begin{aligned}\lambda(f_D,g_D)&=\alpha(f_D,g_D)+(-1)^{n+1}(1-t)\{\beta(f_S,g_S)+\beta(f',g')\}\\&=\sum_A\alpha_D(A)+(-1)^{n+1}(1-t)\sum_A\beta_D(A)+(-1)^{n+1}(1-t)\beta(f',g')\\&\quad\text{(the summation runs over all the intersecting arcs}\\&\quad\text{of }\tilde{\tilde{\omega}}f_S(S^n\times I)\cap\tilde{\tilde{\omega}}g_S(S^n\times I).\\&=\sum_A\alpha'(A)+(-1)^{n+1}(1-t)\beta(f',g')\quad\text{(by (*))}\\&=\alpha(f',g')+(-1)^{n+1}(1-t)\beta(f',g')\\&=\lambda'(f',g')\,.\end{aligned}$$

In a similar way (by replacing the intersecting arcs with double curves), we can prove that $\alpha(f_D)+(-1)^{n+1}(1-t)\beta(f_D)=\alpha(f')+(-1)^{n+1}(1-t)\beta(f')$. In order to prove $\mu(f_D)=\mu'(f')$, it remains to show $\mathcal{O}(f_D)=\mathcal{O}(f')$. This follows from the facts that $f_S(S^n\times I)$ is transverse to the S^1-fibres (by A-2°) and $w|f(S^n)$ is extended over $f_S(S^n\times I)$. $\mu(g_D)=\mu'(g')$ is shown similarly. This completes the proof of Step (II).

Lemma 3.5 is now clear by Steps (I) and (II). Q.E.D.

COROLLARY 3.5.1. *Let W^{2n+2} be a 2n-Poincaré thickening embedded as a regular submanifold in ∂Z, the boundary of a $2n+3$-manifold Z^{2n+3}. Let L^{2n}, Y^{2n+1}, E, $\mathcal{F}N$, F, $\mathcal{F}T$ be as in 3.5. Suppose that the orientation of W is induced from that of ∂Z and that the number of connected components of W is at most two. Then if $x,y\in\pi_{n+1}(E,\mathcal{F}N)$ is in the kernel of $i_\sharp:\pi_{n+1}(E,\mathcal{F}N)\longrightarrow\pi_{n+1}(F,\mathcal{F}T)$, we have*

$$\lambda(x,y)=0\,,$$

and

$$\mu(x)=\mu(y)=0\,.$$

PROOF OF 3.5.1. In the case where W is connected, the corollary is a special case of 3.5 with $x'=y'=0$. (In this case even the existence of another W' is not necessary.) (In the case where W has two components W_1, W_2, some care is needed because an element x in $\pi_{n+1}(E, \mathscr{F}N)$ $(\cong\pi_{n+1}(E_1, \mathscr{F}N_1)\oplus\pi_{n+1}(E_2, \mathscr{F}N_2))$ is not in general represented by a pathed map of (D^{n+1}, S^n) to $(E, \mathscr{F}N)$, but by a pair of two pathed maps $f_1:(D^{n+1}, S^n)\longrightarrow(E_1, \mathscr{F}N_1)$ and $f_2:(D^{n+1}, S^n)\longrightarrow(E_2, \mathscr{F}N_2)$. (We are considering that $\mathscr{F}N_1$ and $\mathscr{F}N_2$ have separate base points.) Let a pair of pathed maps (g_1, g_2) represent y. Then clearly

$$\lambda(x, y)=\lambda_1(f_1, g_1)+\lambda_2(f_2, g_2) ,$$
$$\mu(x)=\mu_1(f_1)+\mu_2(f_2) .$$

By the hypothesis, $i_\sharp(x)=i_\sharp(f_1\oplus f_2)=i_\sharp(f_1)+i_\sharp(f_2)=0$, thus $i_\sharp(f_1)=-i_\sharp(f_2)$. Similarly $i_\sharp(g_1)=-i_\sharp(g_2)$. Therefore, by 3.5, if the orientation of W_2 were defined by the *inverse* of $[\partial Z]$, then we would have $\lambda_1(f_1, g_1)=\lambda_2(-f_2, -g_2)=\lambda_2(f_2, g_2)$ and $\mu_1(f_1)=\mu_2(-f_2)=\mu_2(f_2)$. However, in 3.5.1 we are assuming that the orientation $[W_2]$ is induced from $[\partial Z]$, so in fact we have $\lambda_1(f_1, g_1)=-\lambda_2(f_2, g_2)$ and $\mu_1(f_1)=-\mu_2(f_2)$. Therefore, $\lambda(x, y)=\lambda_1(f_1, g_1)-\lambda_1(f_1, g_1)=0$, and $\mu(x)=\mu_1(f_1)-\mu_1(f_1)=0$.
The proof for $\mu(y)$ is the same. Q.E.D.

CHAPTER II. THE OBSTRUCTION TO FINDING A LOCALLY FLAT SPINE.

§ 4. Necessary and sufficient conditions.

In § 1.1, we have defined m-Poincaré thickenings. Here we introduce the relative notions:

A pair (W^{m+2}, U^{m+1}) consisting of a compact $m+2$-manifold W^{m+2} and a regular $m+1$-submanifold U^{m+1} of ∂W is called an *m-Poincaré thickening pair* if it is a simple Poincaré pair of formal dimension m. Then, of course, U^{m+1} is an $m-1$-Poincaré thickening.

Moreover, a triad $(W^{m+2}; U^{m+1}, V^{m+1})$ is called an *m-Poincaré thickening triad* if it satisfies the following:

(i) U^{m+1} and V^{m+1} are regular $m+1$-submanifolds in ∂W^{m+2} such that $U\cap V=\partial U\cap\partial V$. Denote the intersection by X^m.

(ii) (U^{m+1}, X^m) and (V^{m+1}, X^m) are $m-1$-Poincaré thickening pairs.

(iii) $(W^{m+2}, U^{m+1}\cup V^{m+1})$ is an m-Poincaré thickening pair.

DEFINITION 4.1. A *spine* of an m-Poincaré thickening pair (W^{m+2}, U^{m+1}) is a proper m-submanifold $(L^m, \partial L)\subset(W, U)$ such that the inclusion $i:(L^m, \partial L)\longrightarrow$

(W, U) is a simple homotopy equivalence of pairs. Then clearly ∂L is a spine of U^{m+1}.

Suppose we are given a connected special m-Poincaré thickening pair (W^{m+2}, U^{m+1}) together with a *locally flat* spine K^{m-1} of U^{m+1}. (For the meaning of "special", see §1.1.) The main purpose of this section is to give necessary and sufficient conditions for (W, U) to have a locally flat spine L^m such that $\partial L^m = K^{m-1}$, (4.12). For this purpose, the Seifert form $(\pi_{n+1}(E, \mathscr{F}N), \lambda, \mu)$ plays an essential role.

In chapter I, the Seifert form $(\pi_{n+1}(E, \mathscr{F}N), \lambda, \mu)$ is defined for an exterior 2-connected *closed* submanifold L^{2n} of a $2n$-Poincaré thickening W^{2n+2}. However, there we used essentially neither of the hypotheses that W is a Poincaré thickening and that L is a closed submanifold. In fact, the Seifert form $(\pi_{n+1}(E, \mathscr{F}N), \lambda, \mu)$ is defined in the same way as in §1 for any oriented exterior 2-connected submanifold (with or without boundary) of an oriented compact manifold W^{2n+2}.

In our case of a Poincaré thickening pair (W, U), the tubular neighbourhood N of an exterior 2-connected submanifold $(L, \partial L) \subset (W, U)$ should be taken so that $N \cap U$ is a tubular neighbourhood of ∂L in U. Denote by E, $\mathscr{F}N$ the exterior $\overline{W-N}$ and the frontier $E \cap N$ respectively, then we have the Seifert form $(\pi_{n+1}(E, \mathscr{F}N), \lambda, \mu)$. All the results in §§1~3 are also valid for this form.

LEMMA 4.2. *There a exists locally flat submanifold L^m with $\partial L = K$ which represents the fundamental class of (W, U) as a Poincaré pair. Moreover we can perform codimension 2 surgeries on Int L to obtain an exterior $[m/2]$-connected submanifold.*

This is proved in [10], Lemma 3.4.

Hereafter we shall denote by L^m a locally flat exterior $[m/2]$-connected submanifold with $\partial L = K$ which represents the fundamental class of (W, U). Simply we write $\pi_1 = \pi_1(\mathscr{F}N)$, $\pi_1' = \pi_1(L)$, $\Lambda = \boldsymbol{Z}[\pi_1]$, $\Lambda' = \boldsymbol{Z}[\pi_1']$. Note that if $m \geqq 4$, $\pi_1(L) \cong \pi_1(W)$ (1.3. (ii)).

THE ODD-DIMENSIONAL CASE

This case is dealt with in [10], where the following theorem is proved:

THEOREM 4.3. *Suppose $m = 2n+1 \geqq 5$. There is an obstruction $\eta(W, U, K)$ in the oriented surgery obstruction group $L_m(\pi_1')$ such that $\eta = 0$ if and only if (W, U) has a locally flat spine extending K^{m-1}.*

THE EVEN-DIMENSIONAL CASE

This case occupies the rest of this section. Suppose $m = 2n \geqq 4$. As the inclusion $i: L^m \longrightarrow W^{m+2}$ is a degree 1 map between Poincaré pairs, $i_*: H_k(L^m; \Lambda') \longrightarrow$

$H_k(W; \Lambda')$ is onto for $\forall k$. Its kernel is denoted by $K_k(L; \Lambda')$. (Here $H_k(L^m; \Lambda')$, etc. indicate the k-th integral homology of the universal covering of L^m, etc.) The condition that L^m is exterior $[m/2]$-connected implies (1.3. (ii)) that $i: L^m \longrightarrow W^{m+2}$ is $[m/2]$-connected and that $K_k(L^m; \Lambda')=0$ except for $k=[m/2]=n$. Moreover, by Wall [29], $K_n(L^m; \Lambda')$ is a stably free, stably based Λ'-module.

Let $1 \longrightarrow C \longrightarrow \pi_1 \longrightarrow \pi_1' \longrightarrow 1$ be the cyclic extension associated with W (1.6), t the special generator of C.

LEMMA 4.4. *The sequence*

$$\pi_{n+1}(E, \mathscr{F}N) \xrightarrow{1-t} \pi_{n+1}(E, \mathscr{F}N) \xrightarrow{\partial} K_n(L; \Lambda') \longrightarrow 0$$

is exact. ∂ is defined by the composition

$$\pi_{n+1}(E, \mathscr{F}N) \xrightarrow{\partial} \pi_n(\mathscr{F}N) \xrightarrow{\varpi} \pi_n(L) \xrightarrow{\text{Hurewicz}} H_n(L; \Lambda') .$$

PROOF OF 4.4. Let $\tilde{W} \xrightarrow{p} W$ denote the universal covering of W. Let $\tilde{E}=p^{-1}(E)$, $\widetilde{\mathscr{F}N}=p^{-1}(\mathscr{F}N)$, $\tilde{N}=p^{-1}(N)$, $\tilde{L}=p^{-1}(L)$. $\tilde{N}$ and $\tilde{L}$ are the universal coverings of N and L, but, in general $\tilde{E}$ and $\widetilde{\mathscr{F}N}$ are not so. In, fact, $\pi_1(\tilde{E})\cong \pi_1(\widetilde{\mathscr{F}N})\cong C$. By the Hurewicz theorem,

$$\pi_{n+1}(\tilde{E}, \widetilde{\mathscr{F}N}) \xrightarrow{1-t} \pi_{n+1}(\tilde{E}, \widetilde{\mathscr{F}N}) \xrightarrow{\text{Hurewicz}} H_{n+1}(\tilde{E}, \widetilde{\mathscr{F}N}) \longrightarrow 0$$

is exact. On the other hand, $\pi_{n+1}(\tilde{E}, \widetilde{\mathscr{F}N})\cong\pi_{n+1}(E, \mathscr{F}N)$ and $H_{n+1}(\tilde{E}, \widetilde{\mathscr{F}N}) \overset{\text{excision}}{\cong} H_{n+1}(\tilde{W}, \tilde{N})\cong K_n(L; \Lambda')$. This completes the proof. Q.E.D.

REMARK 4.4.1. Lemma 4.4 is reformulated as $\Lambda'\otimes_{\Lambda}\pi_{n+1}(E, \mathscr{F}N)=K_n(L; \Lambda')$, where $\Lambda \longrightarrow \Lambda'$ is induced from $\pi_1 \xrightarrow{\varpi_\sharp} \pi_1'$. Then $\partial: \pi_{n+1}(E, \mathscr{F}N) \longrightarrow K_n(L; \Lambda')$ in 4.4 is identified with the mapping $x \mapsto 1\otimes x$.

Following Wall [29], $K_n(L; \Lambda')$ has the structure of a special $(-1)^n$-Hermitian form (λ_0, μ_0);

$$\lambda_0 : K_n(L; \Lambda')\times K_n(L; \Lambda') \longrightarrow \Lambda' ,$$

$$\mu_0 : K_n(L; \Lambda') \longrightarrow Q_n(\pi_1') ,$$

where $Q_n(\pi_1')=\Lambda'/\{\nu-(-1)^n\bar{\nu} | \nu\in\Lambda'\}$. Let $\varpi_* : \Lambda \longrightarrow \Lambda'$, $\bar{\varpi}_* : Q_n'(\pi_1) \longrightarrow Q_n(\pi_1')$ be induced by $\varpi_\sharp : \pi_1 \longrightarrow \pi_1'$.

LEMMA 4.5. Let (λ, μ) *be the Seifert form difined in* §1. *The following diagrams are commutative*:

(*i*)

$$\begin{array}{ccc} \pi_{n+1}(E, \mathscr{F}N)\times \pi_{n+1}(E, \mathscr{F}N) & \xrightarrow{\lambda} & \Lambda \\ \downarrow{\scriptstyle \partial\times\partial} & & \downarrow{\scriptstyle \varpi_*} \\ K_n(L; \Lambda')\times K_n(L; \Lambda') & \xrightarrow{\lambda_0} & \Lambda' , \end{array}$$

(ii)
$$\begin{array}{ccc} \pi_{n+1}(E, \mathscr{F}N) & \xrightarrow{\mu} & Q'_n(\pi_1) \\ \downarrow \partial & & \downarrow \bar{\varpi}_* \\ K_n(L; \Lambda') & \xrightarrow{\mu_0} & Q_n(\pi'_1) \end{array} .$$

PROOF OF 4.5. $\varpi_*\lambda(f, g) = \varpi_*(\alpha(f, g) + (-1)^{n+1}(1-t)\beta(f, g)) = \varpi_*\alpha(f, g) = \lambda_0(\partial f, \partial g)$. The proof for μ is similar. Q.E.D.

Next we study the effect on $(\pi_{n+1}(E, \mathscr{F}N), \lambda, \mu)$ of surgery along a trivial n-handle, i.e., an n-handle which represents the zero element of $\pi_n(E, \mathscr{F}N)$.

For the calculation we need a precise description [10]. Let $H=(2D^n)\times D_t^{n+1}$ be a normally embedded trivial n-handle attached to L^{2n} (3.2), where the suffix t indicates a transverse disk. We assume $H\cap L=2S^{n-1}\times D_t^{n+1}$ and $H\cap E=D^n\times D_t^{n+1}$. We want to study the surgery on L^{2n} along H, but for later convenience we shall perform a surgery along a slightly thinner handle $H'=(2D^n)\times \frac{2}{3}D_t^{n+1}$. Then we have the resulting submanifold $L'=\left(L^{2n}-2S^{n-1}\times \frac{2}{3}D_t^{n+1}\right)\cup 2D^n\times \frac{2}{3}S_t^n$. Let $\varpi: N \longrightarrow L^{2n}$ be the normal 2-disk bundle. It is proved in [10] that the restricted bundle $N|(2S^{n-1}\times D_t^{n+1})$ is trivial;

$$(*) \qquad N|(2S^{n-1}\times D_t^{n+1})=(2S^{n-1}\times D_t^{n+1})\times(D^1\times D^1)\ ,$$

with $N|(2S^{n-1}\times D_t^{n+1})\cap L^{2n}=(2S^{n-1}\times D_t^{n+1})\times 0\times 0$. Write $D^1=D_+^1\cup D_-^1$ where $D_+^1=[0,1]$ and $D_-^1=[-1,0]$. We may assume that $H\cap N=(2S^{n-1}\times D_t^{n+1})\times D_+^1\times 0$ by identifying $\overline{2D^n-D^n}\times D_t^{n+1}$ with $(2S^{n-1}\times D_t^{n+1})\times D_+^1\times 0$. Consider the normal D^1-bundle of H in W^{2n+2}. This is a trivial bundle $H\times D_f^1$, suffix f indicating the fibre D^1-disk. We assume $H\times D_f^1\cap N=(2S^{n-1}\times D_t^{n+1})\times(D_+^1\times D^1)$, where D_f^1 is identified with the fourth D^1 in $(*)$.

For the sake of simplicity, we put $V=H\times D_f^1\cap E(=D^n\times D_t^{n+1}\times D_f^1)$ and $V'=2D^n\times\frac{1}{3}D_t^{n+1}\times D_f^1\cup 2S^{n-1}\times\frac{1}{3}D_t^{n+1}\times D_-^1\times D^1$. We specify the regular neighbourhood N' of L' as follows:

$$N'=\overline{(N\cup V)-V'}\ .$$

Let $E'=\overline{W-N'}$ be the exterior of N', and $\mathscr{F}N'=E'\cap N'$ the frontier of N'. We also use the notations $N^*=N\cup V=N'\cup V'$, $E^*=\overline{W-N^*}$, $\mathscr{F}N^*=E^*\cap N^*$, $Y=\mathscr{F}N\cup V$, and $Y'=\mathscr{F}N'\cup V'$. Consider the following diagram (Λ-coefficients):

$$
(**)\qquad
\begin{array}{ccccccc}
H_i(Y, \mathscr{F}N) & \longrightarrow & H_i(E, \mathscr{F}N) & \longrightarrow & H_i(E, Y) & \longrightarrow & H_{i-1}(Y, \mathscr{F}N) \\
 & & & & \Vert\wr \text{ excision} & & \\
 & & & & H_i(E^*, \mathscr{F}N^*) & & \\
 & & & & \Vert\wr \text{ excision} & & \\
H_i(Y', \mathscr{F}N') & \longrightarrow & H_i(E', \mathscr{F}N') & \longrightarrow & H_i(E', Y') & \longrightarrow & H_{i-1}(Y', \mathscr{F}N') .
\end{array}
$$

Note that $(Y, \mathscr{F}N) \simeq (\mathscr{F}N \cup D^n, \mathscr{F}N)$ and $(Y', \mathscr{F}N') \simeq (\mathscr{F}N' \cup D^{n+1}, \mathscr{F}N')$, then by (**) we have

$$H_i(E, \mathscr{F}N) \longrightarrow H_i(E', \mathscr{F}N') \longrightarrow 0 .$$

for $i \leqq n$. Therefore, if L^{2n} is exterior n-conneted, L' is exterior n-connected, too. Also the second row of (**) with $i=n+1$ yields

$$
(***)\qquad
\begin{array}{c}
H_{n+1}(Y', \mathscr{F}N') \xrightarrow[\phi]{} H_{n+1}(E', \mathscr{F}N') \longrightarrow H_{n+1}(E', Y') \longrightarrow 0 . \\
\Vert\wr \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \\
\Lambda \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad
\end{array}
$$

$H_{n+1}(Y', \mathscr{F}N')$ has a canonical generator e^{n+1} which is represented by D^{n+1} of $(\mathscr{F}N' \cup D^{n+1}, \mathscr{F}N')$, and $\phi(e^{n+1})$ is represented by $0 \times \frac{1}{3} D_t^{n+1}$ of $2D^n \times \frac{1}{3} D_t^{n+1}$. Since $H(=2D^n \times D_t^{n+1})$ is a trivial n-handle, $S^{n-1} \times p$ (where $S^{n-1} = \partial(1D^n)$, $p \in S_t^n = \partial D_t^{n+1}$) bounds an n-disk D_0^n in $\overline{\mathscr{F}N - \{(2S^{n-1} \times D_t^{n+1}) \times \partial(D^1 \times D^1)\}}$ (see (*)), and the n-sphere $D_0^n \cup D^n \times p$ $(\subset \mathscr{F}N')$ is null-homotopic in E^*. Thus it bounds an $n+1$-disk D_1^{n+1} in E^*. As an element of $\pi_{n+1}(E', \mathscr{F}N')$, D_1^{n+1} satisfies $\lambda'(\phi(e^{n+1}), D_1^{n+1}) = 1$, where λ' is the Seifert form associated with L' (Chap. I). (Of course, the orientations of $\phi(e^{n+1})$ and D_1^{n+1} and their pathes have to be chosen appropriately.) Hence Ker $\phi = 0$. Moreover, if we define $\phi' : H_{n+1}(E', \mathscr{F}N') \longrightarrow H_{n+1}(Y', \mathscr{F}N')$ by $\phi'(z) = \lambda'(z, D_1^{n+1}) \cdot e^{n+1}$, we have $\phi' \circ \phi = id$. Therefore, the short exact sequence (***) splits; $H_{n+1}(E', \mathscr{F}N') \cong H_{n+1}(E', Y') \oplus \Lambda x$, x standing for the generator $\phi(e^{n+1}) = 0 \times \frac{1}{3} D_t^{n+1}$. The first row in (**) for $i=n+1$ also splits, for $H_n(Y, \mathscr{F}N) \cong \Lambda$ is generated by the boundary ∂D_1^{n+1}. If the right inverse of $H_{n+1}(E, Y) \longrightarrow H_n(Y, \mathscr{F}N) \longrightarrow 0$ is defined by $\partial D_1^{n+1} \mapsto D_1^{n+1}$, we have the splitting

$$H_{n+1}(E, Y) \cong H_{n+1}(E, \mathscr{F}N) \oplus \Lambda y ,$$

where $y = [D_1^{n+1}]$. Hence by (**) and the Hurewicz theorem, we have

$$\pi_{n+1}(E', \mathscr{F}N') \cong \pi_{n+1}(E, \mathscr{F}N) \oplus \Lambda x \oplus \Lambda y .$$

Note that the union $H \cup 0 \times \frac{1}{3} D_t^{n+1} \cup D_1^{n+1}$ is engulfed in a $2n+2$-ball B^{2n+2} such that $B^{2n+2} \cap L^{2n} =$ (a $2n$-ball), because H is a trivial handle. So a map $f : (D^{n+1}, S^n)$

$\longrightarrow(E, \mathscr{T}N)$ representing an element of $\pi_{n+1}(E, \mathscr{T}N)$ also represents the corresponding element of $\pi_{n+1}(E', \mathscr{T}N')$, if f is carefully chosen so that $f(D^{n+1})\cap B^{2n+2}=\phi$.

By the above, the Seifert form (λ', μ') on $\pi_{n+1}(E', \mathscr{T}N')$ is calculated as follows:

$$(25)\qquad \begin{cases} (\lambda', \mu')|\pi_{n+1}(E, \mathscr{T}N)=(\lambda, \mu)\ , \\ \lambda'(z, x)=\lambda'(z, y)=0 \quad \text{for all} \quad z\in\pi_{n+1}(E, \mathscr{T}N)\ , \end{cases}$$

$$(26)\qquad \begin{cases} \lambda'(x, y)=1\ , \quad \lambda'(y, x)=(-1)^n t\ , \\ \mu'(x)=\mu'(y)=0\ . \end{cases}$$

We shall call the form (λ', μ') over $\Lambda x\oplus\Lambda y$ which is given by (26) a *standard plane*.

Let us summarize the above result.

LEMMA 4.6. *Let L^{2n} be a locally flat exterior n-connected submanifold ($n\geqq 2$). If we perform a codimension 2 surgery on L^{2n} along a trivial n-handle, the resulting submanifold L' is also exterior n-connected, and the effect on the associated Seifert form $(\pi_{n+1}(E, \mathscr{T}N), \lambda, \mu)$ is to add a standard plane.*

This is the analogy of [29, Lemma 5.5] in our context.

SOME ALGEBRAIC DEFINITIONS

Here we clarify some algebraic formulations we will need. Let $\mathscr{E}: 1\longrightarrow C\longrightarrow \pi\xrightarrow{\varpi}\pi'\longrightarrow 1$ be an extension of finitely presented groups such that C is a (multiplicative) cyclic group with a specified generator t which is contained in the centre of π. Let $\Lambda=\boldsymbol{Z}[\pi]$, $\Lambda'=\boldsymbol{Z}[\pi']$, $Q'_n(\pi)=\Lambda/\{\nu-(-1)^n\bar{\nu}\cdot t|\nu\in\Lambda\}$, $Q_n(\pi')=\Lambda'/\{\nu'-(-1)^n\bar{\nu}'|\nu'\in\Lambda'\}$. $\varpi_*:\Lambda\longrightarrow\Lambda'$ and $\bar{\varpi}_*: Q'_n(\pi)\longrightarrow Q_n(\pi')$ are induced by ϖ.

DEFINITION 4.7. A triple $X=(G, \lambda, \mu)$ consisting of a finitely generated left Λ-module G and maps $\lambda: G\times G\longrightarrow\Lambda$ and $\mu: G\longrightarrow Q'_n(\pi)$ is called a $((-1)^n$-$)$ *Seifert form over* $\mathscr{E}$ if it satisfies the following:

(i) $\lambda(x, y)=(-1)^n\overline{\lambda(y, x)}\cdot t$ for $\forall x,\ y$,

(ii) for any fixed y, $\lambda(*, y): G\longrightarrow\Lambda$ is a Λ-homomorphism,

(iii) $\mu(x+y)=\mu(x)+\mu(y)+\lambda(x, y)$,

(iv) $\lambda(x, x)=\mu(x)+(-1)^n\overline{\mu(x)}\cdot t$,

(v) $\mu(ax)=a\mu(x)\bar{a}$ for $a\in\Lambda$,

(vi) the tensor product $(\Lambda'\otimes_\Lambda G, \varpi_*\circ\lambda, \bar{\varpi}_*\circ\mu)$ is a special $(-1)^n$-Hermitian form over Λ' in the sense of Wall [29].

Although $\Lambda'\otimes_\Lambda G$ is a stably free Λ'-module by (vi), G is not assumed to be a stably free Λ-module. This is because $\pi_{n+1}(E, \mathscr{F}N)$ is not necessarily a stably free Λ-module. If G is a free Λ-module, X is called a *free Seifert form*. This notion is needed in later sections (§5~). Neither do we assume that λ is nonsingular. This corresponds to the non-triviality of the "Alexander polynomial" $\Delta(t)=$"$\det\lambda$". However $\varpi_*\circ\lambda$ is nonsingular by (vi), and this corresponds to the fact that $\Delta(1)=\pm 1$ in the classical theory. Cf. Seifert [22].

By 3.1, 4.4 and 4.5, the Seifert form $(\pi_{n+1}(E, \mathscr{F}N), \lambda, \mu)$ defined in Chapter I is a $(-1)^n$-Seifert form in the sense of 4.7.

The direct sum $X\oplus Y$ of two $(-1)^n$-Seifert forms $X=(G, \lambda, \mu)$ and $Y=(G', \lambda', \mu')$ is defined by $(G\oplus G', \lambda\oplus\lambda', \mu+\mu')$.

DEFINITION 4.8. Two Seifert forms X and Y are *stably equivalent* if the direct sum of X and some number of standard planes is isomorphic to the direct sum of Y and standard planes

Now we study the necessary conditions for the group $K_n(L^{2n}; \Lambda')$ to be killed by the surgery in codimension 2. For this, we need

DEFINITION 4.9. A Seifert form $X=(G, \lambda, \mu)$ is said to be *null-cobordant* if

(i) there exists a sub Λ-module $H\subset G$ such that $\lambda(H\times H)=0$ and $\mu(H)=0$.

(ii) H is mapped under the canonical map $G\longrightarrow\Lambda'\otimes G$ onto a sub-kernel H' (in the sense of Wall) of the special Hermitian form $\Lambda'\otimes_\Lambda G$. (Hence $\Lambda'\otimes_\Lambda G$ is a kernel. Wall [29] Lemma 5.3.)

This definition is a generalization of cobordism of Seifert matrices due to Levine [13]. We refer to the sub Λ-module H as a *Seifert sub-kernel* of G. Note that we do not assume that a Seifert sub-kernel is a free direct summand. A Seifert form is *stably null-cobordant* if it is stably equivalent to a null-cobordant form.

DEFINITION 4.10.

(i) A triple $\theta=(W^{m+2}, U^{m+1}, K^{m-1})$ consisting of a special m-Poincaré thickening pair (W^{m+2}, U^{m+1}) and a locally flat spine K^{m-1} of U^{m+1} is called an *m-object*.

(ii) An m-object $\theta=(W^{m+2}, U^{m+1}, K^{m-1})$ is *null-cobordant* if there exists a special $m+1$-Poincaré thickening triad $(Z^{m+3}; W^{m+2}, X^{m+2})$ such that

(ii)-1°. $W\cap X=U^{m+1}$,

(ii)-2°. the m-Poincaré thickening pair (X, U) admits a locally flat spine M^m such that $\partial M=K^{m-1}$,

(ii)-3°. the associated cyclic extension of the pair $(Z, W\cup X)$ is isomorphic to that of (W, U). (cf. 1.6.)

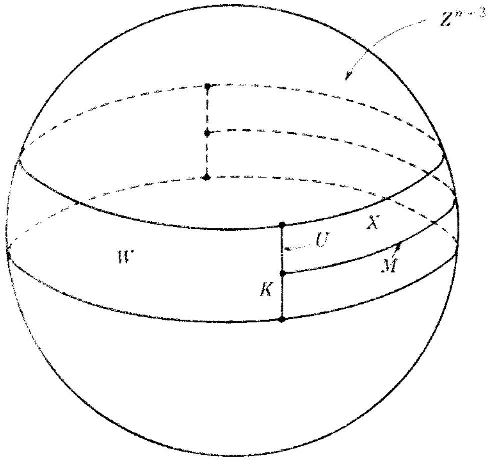


Fig. 6.

PROPOSITION 4.11. *Let $0=(W^{2n+2}, U^{2n+1}, K^{2n-1})$ be a $2n$-object with $2n\geqq 4$. Suppose the number of the connected components of W is at most two. Let L^{2n} be a locally flat exterior n-connected submanifold of W such that $\partial L=K$. If the object 0 is null-cobordant, the Seifert form $(\pi_{n+1}(E, \mathcal{F}N), \lambda, \mu)$ associated with L^{2n} is stably null-cobordant.*

PROOF OF 4.11. The argument is an extension of those in Levine [13] and Wall [29]. We use the notations of 4.10. M^{2n} and L^{2n} are L-equivalent [27], so there exists a locally flat submanifold Y^{2n+1} of Z^{2n+3} such that $\partial Y=L^{2n}\cup M^{2n}$. Applying 4.2 to $(Z, W\cup X)$, we can make Y^{2n+1} exterior n-connected. (In 4.2, ∂Y is assumed to be a spine, but to prove 4.2 this assumption is in fact not necessary. See Lemma 3.4 in [10].) As in §3.5, we denote by T, F and $\mathcal{F}T$ a regular (or tubular) neighbourhood of Y, the exterior of T in Z and the frontier of T $(=T\cap F)$, respectively. We assume that $F\cap W=E$, $\mathcal{F}T\cap W=\mathcal{F}N$, and that $T\cap X$ is a regular (or tubular) neighbourhood of M^{2n} in X^{2n+2}.

Let Φ denote $\left(\begin{array}{ccc} E & \longrightarrow & F \\ \uparrow & & \uparrow \\ \mathcal{F}N & \longrightarrow & \mathcal{F}T \end{array}\right)$, then we have

$$\begin{array}{ccccccc} H_{n+1}(E, \mathcal{F}N; \Lambda) & \longrightarrow & H_{n+1}(F, \mathcal{F}T; \Lambda) & \longrightarrow & H_{n+1}(\Phi; \Lambda) & \longrightarrow & 0 \\ \| \wr & & \| \wr & & \| \wr & & \\ \pi_{n+1}(E, \mathcal{F}N) & \longrightarrow & \pi_{n+1}(F, \mathcal{F}T) & \longrightarrow & \pi_{n+1}(\Phi) & \longrightarrow & 0\,. \end{array}$$

We want to make $H_{n+1}(\Phi; \Lambda)$ zero. The method is an analogy of [16]. Let $f:\left(\begin{array}{ccc} D^n_+ & \longrightarrow & D^{n+1} \\ \uparrow & & \uparrow \\ S^{n-1} & \longrightarrow & D^n_- \end{array}\right)\longrightarrow\Phi$ represent an element $x\in H_{n+1}(\Phi; \Lambda)$, where $\partial D^{n+1}=D^n_+\cup D^n_-$

is the standard decomposition. $\tilde{f}$ can be approximated by an embedding, and is extended to a "normally embedded knob $\tilde{J}$" attached to Y^{2n+1}. However, as in the observation preceding 4.6, we need more detailed descriptions.

Let $J=(D^{n+1}, D^n_-, D^n_-)\times D^{n+1}_t \subset (F^{2n+3}, E^{2n+2}, \mathscr{F}T)$ be a knob of dimension $2n+2$ such that the "core disk" $(D^{n+1}, D^n_-, D^n_-)\times 0$ coincides with the image of $\tilde{f}$, (here D^{n+1}_t is the transverse $n+1$-disk). We can suppose that $\tilde{\varpi}|(D^n_-, S^{n-1})\times D^{n+1}_t : (D^n_-, S^{n-1})\times D^{n+1}_t \longrightarrow (Y^{2n+1}, L^{2n})$ is an embedding, where $\tilde{\varpi} : \mathscr{F}T \longrightarrow Y$ is the projection. Clearly the restricted 2-disk bundle $T|\tilde{\varpi}(D^n_-\times D^{n+1}_t)$ is trivial;

$$(*) \qquad T|\tilde{\varpi}(D^n_-\times D^{n+1}_t)=\tilde{\varpi}(D^n_-\times D^{n+1}_t)\times(D^1\times D^1)\,.$$

D^1 denotes $[-1, 1]$. We can make the identification $J\cap\mathscr{F}T=D^n_-\times D^{n+1}_t=\tilde{\varpi}(D^n_-\times D^{n+1}_t)\times 1\times 0$, and let $\hat{D}^{n+1}=D^{n+1}\cup\{\tilde{\varpi}(D^n_-\times D^{n+1}_t)\times[0, 1]\times 0\}$ (D^{n+1} is the core disk of J).

For the later convenience, the modification is done along a thinner knob $J'=\hat{D}^{n+1}\times\frac{2}{3}D^{n+1}_t$ (as in 4.6). We obtain the resulting submanifold $Y'=\left\{Y-\tilde{\varpi}\left(D^n_-\times\frac{2}{3}D^{n+1}_t\right)\right\}\cup\left(\hat{D}^{n+1}\times\frac{2}{3}S^n_t\right)$. It is easily seen that we obtain $L'^{2n}=Y'\cap W$ from L^{2n} by performing a surgery in W^{2n+2} along a trivial n-handle, so by 4.6, $\pi_{n+1}(E, \mathscr{F}N)$ is changed to $\pi_{n+1}(E, \mathscr{F}N)\oplus$(a standard plane). However, this does not affect the stable equivalence class.

A normal D^1-bundle of J in Z^{2n+3} is clearly trivial; $J\times D^1_f$, where D^1_f denotes the D^1-fibre. We may asume $J\times D^1_f\cap\mathscr{F}T=\tilde{\varpi}(D^n_-\times D^{n+1}_t)\times\{1\}\times D^1$ (cf. (*)), and we write

$$V=J\times D^1_f\,,$$

and

$$V'=\left(\hat{D}^{n+1}\times\frac{1}{3}D^{n+1}_t\right)\times D^1_f\cup\tilde{\varpi}(D^n_-\times D^{n+1}_t)\times([-1, 0]\times D^1)\ \text{(cf. (*))}\,.$$

Let T' be the regular (or tubular) neighbourhood of Y' defined by $T'=\overline{(T\cup V)-V'}$, F' its exterior, and $\mathscr{F}T'$ its frontier. Let $N'=W\cap T'$, $E'=W\cap F'$, $\mathscr{F}N'=W\cap\mathscr{F}T'$.

Some more notations are needed;

$$T^*=T\cup V=T'\cup V',\quad F^*=\overline{Z-T^*},$$

$$\mathscr{F}T^*=T^*\cap F^*,$$

$$N^*=W\cap T^*,\quad E^*=W\cap F^*,\quad \mathscr{F}N^*=W\cap\mathscr{F}T^*.$$

Now we wish to compare $\Phi^*=\begin{pmatrix} E^* & \longrightarrow & F^* \\ \uparrow & & \uparrow \\ \mathscr{F}N^* & \longrightarrow & \mathscr{F}T^* \end{pmatrix}$ with $\Phi'=\begin{pmatrix} E' & \longrightarrow & F' \\ \uparrow & & \uparrow \\ \mathscr{F}N' & \longrightarrow & \mathscr{F}T' \end{pmatrix}$.
In what follows, coefficients are understood to be Λ. First, note that by excision, $H_*(E^*, \mathscr{F}N^*)\cong H_*(E^*\cup A', \mathscr{F}N^*\cup A')\cong H_*(E', \mathscr{F}N'\cup A')$, where $A'=W\cap V'$. Similarly, $H_*(F^*, \mathscr{F}T^*)\cong H_*(F', \mathscr{F}T'\cup V')$. However $(E', \mathscr{F}N'\cup A')\simeq\left(E', \mathscr{F}N' \cup \frac{1}{3}D_i^{n+1}\right)$ and $(F', \mathscr{F}T'\cup V')\simeq\left(F', \mathscr{F}T'\cup\frac{1}{3}D_i^{n+1}\right)$. Therefore, $H_i(E^*, \mathscr{F}N^*)\cong H_i(E', \mathscr{F}N')$ for $i\leqq n$ and $H_i(F^*, \mathscr{F}T^*)\cong H_i(F', \mathscr{F}T')$ for $i\leqq n$, so by the Five Lemma, we obtain

$$H_i(\Phi')\cong H_i(\Phi^*) \quad i\leqq n .$$

For $i=n+1$, we have the diagram

$$\begin{array}{ccccccc}
H_{n+1}\left(\mathscr{F}N'\cup\frac{1}{3}D_i^{n+1}, \mathscr{F}N'\right) & \xrightarrow{\cong} & H_{n+1}\left(\mathscr{F}T'\cup\frac{1}{3}D_i^{n+1}, \mathscr{F}T'\right)\cong\Lambda & & & & \\
\downarrow & & \downarrow & & & & \\
H_{n+1}(E', \mathscr{F}N') & \longrightarrow & H_{n+1}(F', \mathscr{F}T') & \longrightarrow & H_{n+1}(\Phi') & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H_{n+1}\left(E', \mathscr{F}N'\cup\frac{1}{3}D_i^{n+1}\right) & \longrightarrow & H_{n+1}(F', \mathscr{F}T'\cup D_i^{n+1}) & \longrightarrow & H_{n+1}(\Phi^*) & \longrightarrow & 0 . \\
\downarrow & & \downarrow & & & & \\
0 & & 0 & & & &
\end{array}$$

This shows that $H_{n+1}(\Phi')\cong H_{n+1}(\Phi^*)$.

Next we compare $H_*(\Phi^*)$ with $H_*(\Phi)$. Let $A=W\cap V$. By excision, $H_*(E^*, \mathscr{F}N^*)\cong H_*(E^*\cup A, \mathscr{F}N^*\cup A)\cong H_*(E, \mathscr{F}N\cup D_+^n)$. On the other hand, $H_*(F^*, \mathscr{F}T^*)\cong H_*(F^*\cup V, \mathscr{F}T^*\cup V)\cong H_*(F, \mathscr{F}T\cup J)$. Here note that $(F, \mathscr{F}T\cup J)\simeq (F, \mathscr{F}T)$, because J is a knob! So $H_*(F^*, \mathscr{F}T^*)\cong H_*(F, \mathscr{F}T)$. By these isomorphisms and the assumption $H_i(\Phi)=0$ for $i\leqq n$, we have $H_i(\Phi^*)=0$ for $i\leqq n$. For $i=n+1$, consider the following diagram

$$\begin{array}{ccccccc}
0 & & & & & & \\
\downarrow & & & & & & \\
H_{n+1}(E, \mathscr{F}N) & \longrightarrow & H_{n+1}(F, \mathscr{F}T) & \longrightarrow & H_{n+1}(\Phi) & \longrightarrow & 0 \\
\downarrow & & \downarrow\cong & & \downarrow \text{onto} & & \\
H_{n+1}(E, \mathscr{F}N\cup D_+^n) & \xrightarrow{\psi} & H_{n+1}(F, \mathscr{F}T\cup J) & \longrightarrow & H_{n+1}(\Phi^*) & \longrightarrow & 0 \\
\downarrow & & & & & & \\
H_n(\mathscr{F}N\cup D_+^n, \mathscr{F}N)\cong\Lambda . & & & & & & \\
\downarrow & & & & & & \\
0 & & & & & &
\end{array}$$

We assert thatt he element $x \in H_{n+1}(\Phi)$, which is represented by the core disk $(D^{n+1}, D^n_+, D^n_-)\times 0 \subset (F, E, \mathscr{T}T)$, is in the kernel of $H_{n+1}(\Phi) \longrightarrow H_{n+1}(\Phi^*)$. Since $(E, \mathscr{T}N)$ is n-connected, $(D^n_+, \partial D^n_+) \subsetneq (E, \mathscr{T}N)$ is null-homotopic, so there is an $n+1$-disk Δ^{n+1} in E such that $\Delta^n_+ = D^n_+$, $\Delta^n_- \subset \mathscr{T}N$, where $\partial\Delta^{n+1} = \Delta^n_+ \cup \Delta^n_-$ is the standard decomposition. By gluing $n+1$-disks Δ^{n+1} and D^{n+1} (the core of J) along D^n_+, we have an $n+1$-disk $(D^{n+1}_0, \partial D^{n+1}_0) \subset (F, \mathscr{T}T)$. This is a lift of $x \in H_{n+1}(\Phi)$ to $H_{n+1}(F, \mathscr{T}T)$. Considered as an element of $H_{n+1}(F, \mathscr{T}T \cup J)$, it coincides with the image under ψ of the element which is represented by $(\Delta^{n+1}, \partial\Delta^{n+1}) \subset (E, \mathscr{T}N \cup D^n_+)$. Thus by the exactness of the second row, we have $H_{n+1}(\Phi^*) \cong H_{n+1}(\Phi)/\Lambda x$. Combining this and $H_{n+1}(\Phi') \cong H_{n+1}(\Phi^*)$, we obtain $H_{n+1}(\Phi') \cong H_{n+1}(\Phi)/\Lambda x$. If L^{2n} and Y^{2n+1} are exterior n-connected, so are L' and Y' and the induction argument can be applied. Since $H_{n+1}(\Phi)$ is a finitely generated Λ-module, $H_{n+1}(\Phi)$ can be reduced to zero after a finite number of processes of modifications as above.

The proof of 4.11 proceeds as follows: Make $\pi_{n+1}(\Phi)(\cong H_{n+1}(\Phi))$ zero. Then $\pi_{n+1}(E, \mathscr{T}N)$ is changed to $\pi_{n+1}(E, \mathscr{T}N) \oplus$ (some copies of standard planes), but this does not change the stable equivalence class. Hereafter suppose $\pi_{n+1}(\Phi) = 0$. Let H be the kernel of $\pi_{n+1}(E, \mathscr{T}N) \longrightarrow \pi_{n+1}(F, \mathscr{T}T)$. This map is surjective because $\pi_{n+1}(\Phi) = 0$. Let $\Lambda' = \boldsymbol{Z}[\pi_1 L] = \boldsymbol{Z}[\pi_1 Y]$. Since the tensor product functor $\Lambda' \underset{\Lambda}{\otimes}$ is right exact, we have

$$\begin{array}{ccccccc}
\Lambda'\underset{\Lambda}{\otimes} H & \longrightarrow & \Lambda'\underset{\Lambda}{\otimes}\pi_{n+1}(E, \mathscr{T}N) & \longrightarrow & \Lambda'\underset{\Lambda}{\otimes}\pi_{n+1}(F, \mathscr{T}T) & \longrightarrow & 0 \\
\Big\downarrow \text{onto} & & \Big\| \wr \; 4.4.1 & & \Big\| \wr \; 4.4.1 & & \\
0 \longrightarrow K_{n+1}(Y, L; \Lambda') & \longrightarrow & K_n(L^{2n}; \Lambda') & \longrightarrow & K_n(Y; \Lambda') & \longrightarrow & 0\,.
\end{array}$$

Wall [29, Lemma 5.7] proved that $K_{n+1}(Y, L: \Lambda')$ is a subkernel in $K_n(L; \Lambda')$. Hence $\Lambda' \underset{\Lambda}{\otimes} H$ is mapped onto a subkernel of $\Lambda' \underset{\Lambda}{\otimes} \pi_{n+1}(E, \mathscr{T}N)$. Finally by 3.5.1, $\lambda(H \times H) = 0$ and $\mu(H) = 0$; thus by Definition 4.9, the Seifert form $(\pi_{n+1}(E, \mathscr{T}N), \lambda, \mu)$ is (stably) null-cobordant. Q.E.D.

The next theorem is the main result in this section.

THEOREM 4.12. *Given a connected 2n-object $\theta = (W^{2n+2}, U^{2n+1}, K^{2n-1})$ with $2n \geqq 6$, the following four conditions on θ are equivalent*:

(*i*) *W^{2n+2} admits a locally flat spine M^{2n} such that $\partial M = K$.*

(*ii*) *θ is null-cobordant.*

(*iii*) *For any locally flat exterior n-connected submanifold L^{2n} such that $\partial L = K$, the Seifert form $(\pi_{n+1}(E, \mathscr{T}N), \lambda, \mu)$ associated with L is stably null-cobordant.*

(*iv*) *There exists a locally flat exterior n-connected submanifold L^{2n} such that*

$\partial L=K$ and the Seifert form associated with L is stably null-cobordant. If $2n=4$, we have the implcations: (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii)$\Longrightarrow$(iv).

REMARK. In (i), we do not assume any exterior connectivity on M^{2n}.

PROOF OF 4.12. Suppose $2n\geqq 4$. If (i) holds, let $Z^{2n+3}=W\times I$ and $X=U\times I\cup W\times\{1\}$, $I=[0,1]$. Then X has a locally flat spine $K\times I\cup M\times\{1\}$ extending $K\times\{0\}$. So by Definition 4.10, the $2n$-object $\theta=(W\times\{0\}, U\times\{0\}, K\times\{0\})$ is null cobordant. (ii)$\Longrightarrow$(iii) is proved in 4.11. By 4.2, there always exists a locally flat exterior n-connected submanifold L^{2n} in W such that $\partial L=K$. So (iii)$\Longrightarrow$(iv) is trivial. We now prove (iv)$\Longrightarrow$(i) assuming $2n\geqq 6$. By performing surgery on L^{2n} along trivial n-handles, we can add sufficiently many copies of standard planes to $\pi_{n+1}(E, \mathscr{F}N)$, (4.6). Therefore, we can assume that $\pi_{n+1}(E, \mathscr{F}N)$ is actually unll-cobordant. Then there is a Λ-submodule $H\subset\pi_{n+1}(E, \mathscr{F}N)$ such that $\lambda(H\times H)=0$, $\mu(H)=0$, and under the canonical map $\pi_{n+1}(E, \mathscr{F}N)\longrightarrow\Lambda'\otimes_{\Lambda}\pi_{n+1}(E, \mathscr{F}N)=K_n(L;\Lambda')$, H is mapped onto a subkernel H'. Let $e_1, \cdots, e_r\in H'$ form a preferred base of H'. Let $\tilde{e}_1, \cdots, \tilde{e}_r$ be their lifts in H, i.e., $\tilde{e}_i$ is an element which is mapped to e_i under $H\longrightarrow H'$. Since $\lambda(\tilde{e}_i, \tilde{e}_j)=0$, $\mu(\tilde{e}_i)=0$ $(\forall i, j)$, $\tilde{e}_i$'s are represented by normally embedded $n+1$-handles attached to L which are disjoint to each other (see 3.3 and 3.4). So we can perform the surgery on L^{2n} along these $n+1$ handles to obtain a new locally flat submanifold $M^{2n}\subset W^{2n+2}$. Since the attaching n-spheres of these handles represent $e_1, \cdots, e_r$, $K_n(M;\Lambda')$ vanisnes, and the resulting submanifold M^{2n} must be simple homotopy equivalent to W^{2n+2} (Wall [29], §5.6). This completes the proof. Q.E.D.

§5. The obstruction theory.

In this section we shall introduce certain abelian groups and reformulate the results in §4 in terms of these groups.

According to 4.3, we know that there is nothing new to do in the odd-dimensional case (except 5.11 below), so in this section we treat mainly the even-dimensional case.

With Theorem 4.12 it seems quite natural to define an even-dimensional obstruction group as the quotient of the Grothendieck group of *all* (fre or not) Seifert forms over a cyclic extension $\mathscr{E}$ modulo stably null-cobordant forms. However, this definition has two difficulties:

1° *Are all Seifert forms geometric?*; Given a Seifert form $X=(G, \lambda, \mu)$, is X isomorphic to a geometric Seifert form $(\pi_{n+1}(E, \mathscr{F}N), \lambda, \mu)$?

We do not know any complete answer. However, if X is a free Seifert form

(c.f. 5.1 below), we have an affirmative answer to this question (cf. 5.2).
2° *Is there a cancellation theorem for Seifert forms?*; If $X\oplus Y$ and Y are stably null-cobordant, is X also stably null-cobordant? Although we have no general solution, we can prove it for free Seifert forms (cf. 5.3).

To avoid these difficulties we must define the obstruction group by using only free Seifert forms. Unfortunately a geometric form $(\pi_{n+1}(E, \mathscr{T}N), \lambda, \mu)$ is not always free, so we need a certain device to modify the situation. This will be done in sections 5.6~5.9.

Let us begin a more detailed discussion, throughout which we fix a cyclic extension of finitely presented groups $\mathscr{E}$; $1\longrightarrow C\longrightarrow\pi\xrightarrow{\ \varpi\ }\pi'\longrightarrow 1$ such that $C\subset$centre of π, where C is a (multiplicative) cyclic group with a specified generator t called the *special generator*. A $2n$-object $\theta=(W, U, K)$ (or a Poincaré thickening triad $(W; U, V)$) is said to be *over* $\mathscr{E}$ if the cyclic extension associated with W (1.6) is isomorphic (preserving the special generators) to $\mathscr{E}$.

All $2n$-objects and Seifert forms considered below are understood to be over $\mathscr{E}$. Let $\Lambda=\boldsymbol{Z}[\pi]$, $\Lambda'=\boldsymbol{Z}[\pi']$.

DEFINITION 5.1. A $(-1)^n$-Seifert form $X=(G, \lambda, \mu)$ is *free* if
(i) G is a free Λ-module with a specified basis $\{\tilde{e}_1, \cdots, \tilde{e}_k\}$.
(ii) The special $(-1)^n$-Hermitian form $\Lambda'\underset{\Lambda}{\otimes}X$ is actually Λ'-free.
(iii) $\{1\otimes\tilde{e}_1, \cdots, 1\otimes\tilde{e}_k\}$ is a preferred basis of $\Lambda'\underset{\Lambda}{\otimes}X$.

The next lemma tells us that all free Seifert forms are geometric. Let U^{2n+1} be the total space of an oriented D^2-bundle over a compact connected oriented $2n-1$-manifold K^{2n-1}, such that, considered as a $2n-1$-Poincaré thickening pair, the cyclic extension associated with $(U, U|\partial K)$ is isomorphic to $\mathscr{E}$ (preserving t). The existence of such U^{2n+1} is stated in §1.5. K^{2n-1} is considered as a submanifold of U^{2n+1} by the zero-section.

LEMMA 5.2. *Suppose $2n\geqq 6$. Given a free $(-1)^n$-Seifert form $X=(G, \lambda^0, \mu^0)$, we can find a $2n$-Poincaré thickening triad $(W^{2n+2}; U^{2n+1}, U_+^{2n+1})$ over $\mathscr{E}$ such that*
(i) *U_+^{2n+1} has the structure of the total space of an oriented D^2-bundle over a compact manifold K_+^{2n-1} which is simple homotopy equivalent to K. We think of K_+^{2n-1} as a submanifold of U_+^{2n+1} by the zero-section,*
(ii) *$U\cap U_+=U|\partial K=U_+|\partial K_+$ and $K\cap K_+=\partial K=\partial K_+$,*
(iii) *there exists an exterior n-connected locally flat submanifold L^{2n} of W such that $\partial L=K\cup K_+$ and the Seifert form $(\pi_{n+1}(E, \mathscr{T}N), \lambda, \mu)$ associated with L is isomorphic to X,*

(iv) $(W; U, U_+)$ *is simple homotopy equivalent to* $(K\times I; K\times 0, K\times 1\cup\partial K\times I)$.

PROOF OF 5.2. Note that $\Lambda'\otimes_\Lambda X$ is a special Hermitian form. Theorem 5.8 of Wall [29] asserts that there exists a compact manifold triad $(L^{2n}; K^{2n-1}, K_+^{2n-1})$ and an n-connected map $\phi:(L; K, K_+)\longrightarrow(K\times I; K\times 0, K\times 1\cup\partial K\times I)$ of degree 1, such that

(1°) $\phi|K: K\longrightarrow K\times 0$ is the identity map,

(2°) $\phi|K_+$ is a simple homotopy equivalence,

(3°) there are pathed framed immersions $g_i: S^n\longrightarrow L^{2n}$, $1\leqq i\leqq k$, with each g_i representing $1\otimes\tilde{e}_i$ in $K_n(L^{2n}; \Lambda')$ $(=H_{n+1}(\phi; \Lambda'))$ and

(4°) (self-intersection of g_i)$=\bar{\varpi}_*\circ\mu^0(\tilde{e}_i)$, (intersection of g_i and g_j)$=\varpi_*\circ\lambda^0(\tilde{e}_i, \tilde{e}_j)$.

Let rN^{2n+2} denote the total space of rD^2-bundle (rD^2 denoting a 2-disk of radius $r\geqq 0$) which is associated with the induced bundle $(p\circ\phi)^*(U)$ over L^{2n}, where $p: K\times I\longrightarrow K\times 0$ is the projection. We simply write N^{2n+2} instead of $1N^{2n+2}$. Clearly $N|K\cong U$. By U_+ we denote $N|K_+$.

We can find disjoint pathed framed embeddings $\tilde{g}_i: S^n\longrightarrow\mathscr{T}N$ such that $\varpi\circ\tilde{g}_i=g_i$, where $\varpi:\mathscr{T}N\longrightarrow L^{2n}$ denotes the associated S^1-bundle of N. Consider $\alpha(\tilde{g}_i, \tilde{g}_j)$ $(i<j)$ and $\alpha(\tilde{g}_i)$ (§1). $\alpha(\tilde{g}_i, \tilde{g}_j)$'s are elements of Λ, and $\alpha(\tilde{g}_i)$'s belong to $Q_n'(\pi)$. Note that $\pi_1(\mathscr{T}N)\xrightarrow{\varpi_*}\pi_1(L)$ is identified with $\pi\xrightarrow{\varpi}\pi'$, and that the following sequences are exact:

$$\Lambda\xrightarrow{\times(1-t)}\Lambda\xrightarrow{\varpi_*}\Lambda', \tag{27}$$

$$Q_{n+1}(\pi)\xrightarrow{\times(1-t)}Q_n'(\pi)\xrightarrow{\bar{\varpi}_*}Q_n(\pi'). \tag{28}$$

(Proof of (28): Suppose $a=\sum_\kappa m_\kappa g\in\Lambda$ is such that $\varpi_*(a)\equiv 0 \bmod \{\nu'-(-1)^n\bar{\nu}'|\nu'\in\Lambda'\}$. For the sake of simplicity, we assume that $\varpi_*(a)=mg'-(-1)^n mg'^{-1}$, where $g'\in\pi'$. Suppose g' is not involutive; $g'\neq g'^{-1}$, and fix $g\in\pi$ such that $\varpi_*(g)=g'$. Then since $x\equiv xt^{\pm1}\equiv xt^{\pm2}\equiv\cdots \bmod (1-t)\Lambda$ for $\forall x\in\pi$, we have $a\equiv mg-(-1)^n mg^{-1}t$ mod $(1-t)\Lambda$. So as an element of $Q_n'(\pi)$, a is in the image of $\times(1-t)$; this completes the proof in the non-involutive case. In the case when g' is involutive, the proof is similar.)

We continue the proof of 5.2. Since $\varpi_*\{\alpha(\tilde{g}_i, \tilde{g}_j)\}=$intersection of g_i with g_j, we have $\varpi_*\{\alpha(\tilde{g}_i, \tilde{g}_j)\}=\varpi_*\{\lambda^0(\tilde{e}_i, \tilde{e}_j)\}$ by (4°). Similarly, $\bar{\varpi}_*\{\alpha(\tilde{g}_i)\}=\bar{\varpi}_*(\mu^0(\tilde{e}_i))$. Thus by (27) and (28) we can find certain elements $b_{ij}\in\Lambda$ and $b_i\in Q_{n+1}(\pi)$ such that

$$\begin{cases}\lambda^0(\tilde{e}_i, \tilde{e}_j)-\alpha(\tilde{g}_i\,\tilde{g}_j)=(-1)^{n+1}(1-t)b_{ij}, \\ \mu^0(\tilde{e}_i)-\alpha(\tilde{g}_i)=(-1)^{n+1}(1-t)b_i.\end{cases} \tag{29}$$

Now we apply again Wall's argument in [29], Th. 5.8, and we construct framed regular homotopies of $\tilde{g}_i$'s, $F_i: S^n\times I \longrightarrow \mathscr{F}N\times I$ such that

$$(30)\qquad \begin{cases} \text{self-intersection of } F_i = b_i, \text{ and} \\ \text{intersection of } F_i \text{ and } F_j = b_{ij}\,. \end{cases}$$

Now identify $\mathscr{F}N\times I$ with $\overline{2N-N}$, and attach $n+1$-handles to $2N$ by attaching maps $F_i|S^n\times 1$. Let W be the resulting manifold. Also let E be the exterior $\overline{W-N}$. By the construction, $\pi_{n+1}(E, \mathscr{F}N)$ is a free Λ-module and its basis is given by pathed framed immersions $f_i: (D^{n+1}, S^n) \longrightarrow (E, \mathscr{F}N)$ which are defined by adjoining the images of F_i in $\overline{2N-N}$ and the core disks of attached handles. From (29) and (30), we know $\lambda(f_i, f_j) = \lambda^0(\tilde{e}_i, \tilde{e}_j)$. To show that $\mu(f_i) = \mu^0(\tilde{e}_i)$, we must prove $\mathscr{O}(f_i) = 0$. However, this is clear, for the frame of f_i is obtained by extending the frame of $\tilde{g}_i$; the latter is a direct sum ϖ^*(frame of g_i)$\oplus$(1-frame tangential to S^1-fibres). So we have $\mu(f_i) = \mu^0(\tilde{e}_i)$. Hence by the correspondence $\tilde{e}_i \mapsto \{f_i\}$, two Seifert forms (G, λ^0, μ^0) and $(\pi_{n+1}(E, \mathscr{F}N), \lambda, \mu)$ are isomorphic. Now (i), (ii) and (iii) having been established it remains to prove (iv). By the hypothesis, the compositions $\partial D^{n+1} \xrightarrow{f_i|\partial D^{n+1}} \mathscr{F}N \xrightarrow{\varpi} L \xrightarrow{\phi} K\times I$ are null-homotopic, so ϕ is extended to $\bar{\phi}: W \longrightarrow K\times I$. We wish to prove that $\bar{\phi}$ is a simple homotopy equivalence. Let $\psi: L^{2n} \longrightarrow W$ denote the inclusion, then we have the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_*(L) & \xrightarrow{\psi} & C_*(W) & \longrightarrow & C_*(\psi) & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\bar{\phi}} & & \downarrow{\scriptstyle\theta} & & \\ 0 & \longrightarrow & C_*(L) & \xrightarrow{\phi} & C_*(K\times I) & \longrightarrow & C_*(\phi) & \longrightarrow & 0\,. \end{array}$$

According to Lemma 2.5 in [29], in order to prove that $\bar{\phi}$ is a simple equivalence, we have only to show it for θ. Let $\mathscr{H}$ denote the homology exact sequence associated with $C_*(\psi) \xrightarrow{\theta} C_*(\phi)$, then we have $\tau(C_*(\phi)) = \tau(C_*(\psi)) + \tau(\theta) + \tau(\mathscr{H})$ ([19] Th. 3.2). However $\mathscr{H}$ has only two non-zero terms $H_{n+1}(\psi)$ and $H_{n+1}(\phi)$ both of which are based isomorphic to $K_n(L; \Lambda')$. By the construction, θ_* induces the identity of $K_n(L; \Lambda')$, so $\tau(\mathscr{H}) = 0$. On the other hand, the basis of $H_{n+1}(\psi)$ and that of $H_{n+1}(\phi)$ were chosen so that $\tau(C_*(\phi)) = \tau(C_*(\psi)) = 0$ ([29] p. 27). Hence we have $\tau(\theta) = 0$ as desired. Q.E.D.

As an application of 5.2, we can prove a cancellation theorem for free Seifert forms. Although the statement of the following lemma is purely algebraic, its proof is geometrical. It would be interesting to find algebraic proof.

LEMMA 5.3. *Let X and Y be free $(-1)^n$-Seifert forms over $\mathscr{E}$. If $X\oplus Y$*

and Y are both stably null-cobordant, then so is X.

PROOF OF 5.3. Let U^{2n+1} be as in 5.2, with a sufficiently large n. Then applying 5.2, we can find $2n$-Poincaré thickening triads $(W_1^{2n+2}; U, V_1)$ and $(W_2^{2n+2}; -U, V_2)$ "representing" X and Y, respectively, where $-U$ denotes U with the inverse orientation. Glue W_1 to W_2 along U, and denote the resulting manifold by W_3. Then clearly, W_3 represents $X\oplus Y$. Apply 4.12 to W_2 and W_3. Then since Y and $X\oplus Y$ are stably null-cobordant, we can find $2n+1$-Poincaré thickening triads $(Z_2^{2n+3}; W_2, X_2)$ and $(Z_3^{2n+3}; -W_3, X_3)$ over $\mathscr{E}$ with X_2, X_3 admitting locally flat spines M_2, M_3. Gluing Z_2 to Z_3 along W_2, we obtain a $2n+1$-Poincaré thickening triad $(Z_2\cup Z_3; W_1, X_2\cup X_3)$ over $\mathscr{E}$. See Fig. 7. Note that $X_2\cup X_3$ admits a locally flat spine $M_2\cup M_3$. (If we denote the zero-section manifold of D^2-bundles U, V_1, V_2 by K, K_1, K_2, respectively, we have $\partial M_2=K\cup K_2$ and $\partial M_3=-K_2\cup K_1$. So $M_2\cup M_3$ is a submanifold with $\partial(M_2\cup M_3)=K\cup K_1$.) Therefore, by Definition 4.10 the $2n$-object $(W_1^{2n+2}, U^{2n+1}\cup V_1^{2n+1}, K^{2n-1}\cup K_1^{2n-1})$ is null-cobordant. Hence by applying 4.11, the Seifert form X, which is associated with this $2n$-object, is stably null-cobordnt. Q.E.D.

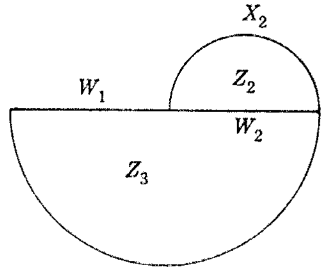


Fig. 7.

For a Seifert form $X=(G, \lambda, \mu)$, we denote by $-X$ the inverse Seifert form $(G, -\lambda, -\mu)$.

LEMMA 5.4. *$X\oplus(-X)$ is stably null-cobordant.*

PROOF OF 5.4. Adding some copies of standard planes, we can assume that $\Lambda'\otimes_\Lambda X$ is actually a free special Hermitian form. Let $\{e_1, \cdots, e_r\}$ denote a preferred basis of $\Lambda'\otimes_\Lambda X$, $\{\tilde{e}_1, \cdots, \tilde{e}_r\}$ its lift to X. By $\{\tilde{e}_1', \cdots, \tilde{e}_r'\}$, $\{\tilde{e}_1'', \cdots, \tilde{e}_r''\}$, we denote the corresponding elements of the two copies. Then the sub Λ-module H of $X\oplus(-X)$ generated by the elements $\tilde{e}_i'+\tilde{e}_i''$, $1\leqq i\leqq r$, satisfies $\lambda(H\times H)=0$ and $\mu(H)=0$, and the image in $\Lambda'\otimes_\Lambda(X\oplus-X)$ is a subkernel ([29] Lemma 5.4). By Definition 4.9, $X\oplus(-X)$ is null-cobordant. Q.E.D.

We define a relation $X\sim Y$ between $(-1)^n$-Seifert forms over $\mathscr{E}$ as follows: $X\sim Y$, if and only if $X\oplus(-Y)$ is stably null-cobordant.

LEMMA 5.5. *Being restricted to the category of free Seifert forms, the relation $\sim$ becomes an equivalence relation.*

PROOF OF 5.5. The relation is obviously reflexive (by 5.4) and symmetric. Suppose $X\sim Y$ and $Y\sim Z$, then $X\oplus(-Y)$ and $Y\oplus(-Z)$ are stably null-cobordant, thus so is $(X\oplus-Y)\oplus(Y\oplus-Z)$, but this is isomorphic to $(X\oplus-Z)\oplus(Y\oplus-Y)$. Therefore, if X, Y, Z are free Seifert forms, $X\oplus(-Z)$ is stably null-cobordant by 5.3, for $Y\oplus(-Y)$ is stably null-cobordant by 5.4. Hence the relation is transitive. Q.E.D.

DEFINITION OF GROUPS $P_m(\mathscr{E})$.

The odd-dimensional case; We define $P_{2n+1}(\mathscr{E})$ to be the Wall group $L_{2n+1}(\pi', 1)$, where $1:\pi'\longrightarrow \boldsymbol{Z}_2$ stands for a trivial homomorphism. This is motivated by 4.3. Hereafter we shall write $L_{2n+1}(\pi')$ instead of $L_{2n+1}(\pi', 1)$ for the sake of simplicity.

The even-dimensional case; Let $\mathscr{F}_{2n}(\mathscr{E})$ denote the semi-group under $\oplus$ of *free* $(-1)^n$-Seifert forms over $\mathscr{E}$. Since the equivalence relation $\sim$ is compatible with $\oplus$, we may consider the quotient semi-group $\mathscr{F}_{2n}(\mathscr{E})/\sim$, which is an (abelian) group by 5.4. $P_{2n}(\mathscr{E})$ is defined to be $\mathscr{F}_{2n}(\mathscr{E})/\sim$. Note that a free Seifert form X represents 0 in $P_{2n}(\mathscr{E})$ if and only if it is stably null-cobordant.

ALGEBRAIC PERIODICITY

As is immediately seen from the definition, we have an isomorphism $\rho: P_m(\mathscr{E}) \xrightarrow{\cong} P_{m+4}(\mathscr{E})$ called the algebraic periodicity. For m odd, this is shown in [29]. For m even, this is obvious since $\mathscr{F}_{2n}(\mathscr{E})$ and $\mathscr{F}_{2(n+2)}(\mathscr{E})$ coincide as sets.

FUNCTORIAL PROPERTIES

By a *morphism* $\mathscr{E}_1\longrightarrow\mathscr{E}_2$ we mean a triple (θ, φ, ψ) of homomorphisms such that

(i) the diagram

$$\begin{array}{ccccccccc} \mathscr{E}_1: 1 & \longrightarrow & C_1 & \longrightarrow & \pi_1 & \longrightarrow & \pi_1' & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} & & \\ \mathscr{E}_2: 1 & \longrightarrow & C_2 & \longrightarrow & \pi_2 & \longrightarrow & \pi_2' & \longrightarrow & 1 \end{array}$$

is commutative, and

(ii) θ preserves the special generators.

A morphism $\mathscr{E}_1 \longrightarrow \mathscr{E}_2$ induces a homomorphism $P_m(\mathscr{E}_1) \longrightarrow P_m(\mathscr{E}_2)$ as follows: For m odd $\varphi: \pi_1' \longrightarrow \pi_2'$ induces a homomorphism $L_m(\pi_1') \longrightarrow L_m(\pi_2')$ [29]. For m even, let $X=(G, \lambda, \mu)$ be a free $(-1)^n$-Seifert form over $\mathscr{E}_1$. Then $(\Lambda_2 \otimes_{\Lambda_1} G, \varphi_* \lambda, \bar{\varphi}_* \mu)$ is a free $(-1)^n$-Seifert form over $\mathscr{E}_2$, defining the desired homomorphism.

We want to reformulate the result in §4.12 in terms of the groups $P_m(\mathscr{E})$. As we remarked at the beginning of this section, geometric Seifert forms $(\pi_{n+1}(E, \mathscr{T}N), \lambda, \mu)$ are not always free, and we have to proceed as follows. Let $X=(G, \lambda, \mu)$ be a (not necessarily free) $(-1)^n$-Seifert from over $\mathscr{E}$. By adding some copies of standard planes, we may assume that $\Lambda' \otimes_{\Lambda} G$ is an actually free special Hermitian form.

DEFINITION 5.6. A *free core* of X is a free $(-1)^n$-Seifert form $X^*=(G^*, \lambda^*, \mu^*)$ over $\mathscr{E}$ together with a Λ-homomorphism $c: G^* \longrightarrow G$ such that
(i) c preserves the structure of $(-1)^n$-Seifert forms. (Precisely, $\lambda^*=\lambda \circ (c \times c)$, $\mu^*=\mu \circ c$.)
(ii) c induces a simple isomorphism $1 \otimes c: \Lambda' \otimes_{\Lambda} G^* \longrightarrow \Lambda' \otimes_{\Lambda} G$.

LEMMA 5.7. *Any $(-1)^n$-Seifert form $X=(G, \lambda, \mu)$ such that $\Lambda' \otimes_{\Lambda} G$ is actually Λ'-free has a free core.*

PROOF OF 5.7. Let $\{e_1, \cdots, e_r\}$ be a preferred Λ'-basis of $\Lambda' \otimes_{\Lambda} G$, $\{\tilde{e}_1, \cdots, \tilde{e}_r\}$ its lifting to G, i.e., $\tilde{e}_i$ is such that $1 \otimes \tilde{e}_i = e_i$. Let G^* be a free Λ-module of rank r with the indeterminant basis $\{x_1, \cdots, x_r\}$. We define a structure of $(-1)^n$-Seifert form (λ^*, μ^*) on G^* by setting

$$\lambda^*(x_i, x_j) = \lambda(\tilde{e}_i, \tilde{e}_j) ,$$

$$\mu^*(x_i) = \mu(\tilde{e}_i) .$$

$X^*=(G^*, \lambda^*, \mu^*)$ is a free Seifert form over $\mathscr{E}$, and by defining $c(x_i)=\tilde{e}_i$, we can easily prove that it is a free core of X. Q.E.D.

The following lemma is an essential step in our formulation of the obstruction theory.

LEMMA 5.8. *Let $\theta=(W^{2n+2}, U^{2n+1}, K^{2n-1})$ be a connected $2n$-object over $\mathscr{E}$, L^{2n} a locally flat exterior n-connected submanifold such that $\partial L=K$. By X we denote the associated Seifert form $(\pi_{n+1}(E, \mathscr{T}N), \lambda, \mu)$ with L^{2n}, and let $X^*=(G^*, \lambda^*, \mu^*)$ be a free core of X (we may assume that $\Lambda' \otimes_{\Lambda} X$ is Λ'-free).*

If $2n \geqq 6$, the following two statements are equivalent:
(i) θ is null-cobordant.
(ii) X^* is stably null-cobordant.

If $2n=4$, we have the implication (i) $\Longrightarrow$ (ii).

PROOF OF 5.8. Let $\{x_1, \cdots, x_r\}$ be the preferred basis of G^*, and consider elements $c(x_i)$, $1 \leqq i \leqq r$, where $c: G^* \longrightarrow \pi_{n+1}(E, \mathscr{F}N)$ is the Λ-homomorphism in the definition of the free core (5.6). Let $f_i: (D^{n+1}, S^n) \longrightarrow (E, \mathscr{F}N)$ be pathed immersions which represent $c(x_i)$. Since $\pi_1(\mathscr{F}N) \cong \pi_1(E)$, these f_i's are regular homotopic to mutually disjoint embeddings f_i' (for a precise discussion, see the proof of 3.3). (One should remark here that this does not imply $\lambda(f_i', f_j')=0$, unless $\varpi f_i'(S^n) \cap \varpi f_j'(S^n) = \phi$ (3.4). This does not imply $\mu(f_i')=0$, either (3.3).)

Let W^* be a (smooth) regular neighbourhood of $N \cup \bigcup_{i=1}^{r} f_i'(D^{n+1})$ in W, E^* the exterior of N in W^*, i.e., $\bar{E}^* = \overline{W^* - N}$. Obviously $E^* \simeq \mathscr{F}N \cup \bigcup_{i=1}^{r} f_i'(D^{n+1})$, and $\pi_{n+1}(E^*, \mathscr{F}N)$ is a free Λ-module with a preferred basis $\{f_i'\}$, $1 \leqq i \leqq r$. Since $\lambda(f_i', f_j') = \lambda(c(x_i), c(x_j)) = \lambda^*(x_i, x_j)$ and $\mu(f_i') = \mu(c(x_i)) = \mu^*(x_i)$, two Seifert forms (G^*, λ^*, μ^*) and $(\pi_{n+1}(E^*, \mathscr{F}N), \lambda, \mu)$ are isomorphic to each other. In this sense, we shall call W^* a *geometric realization* of X^* or a *geometric free core of* W.

REMARK. In the case of $(2n-1, 2n+1)$-knots, the concept of geometric free core corresponds to that of simple knots [13].

PROOF OF 5.8 (continued). In the same way as we did at the end of the proof of 5.2, it is proved that W^* is simple homotopy equivalent to W by the inclusion and that a triple $\theta^* = (W^*, U^*, K^{2n-1})$ is a $2n$-object over $\mathscr{E}$, where $U^* = N \cap U^{2n+1}$. Glue $W^* \times [0, 1]$ to $W \times [1, 2]$ along $W^* \times 1$, and we have a $2n+3$-manifold denoted by Z^{2n+3}.

We now prove that θ^* is null-cobordant if and only if θ is null-cobordant. This does not need any requirement on the dimension $2n$.

First suppose θ is null-cobordant, then by Definition 4.10, we can find a $2n+1$-Poincaré thickening triad $(Z'; W^{2n+2}, X^{2n+2})$ over $\mathscr{E}$ such that X^{2n+2} admits a locally flat spine M^{2n} with $\partial M = K^{2n-1}$. Glue Z' to Z by identifying W (of Z') with $W \times 2$ (of Z). Then we obtain a $2n+1$-Poincaré thickening triad $(Z \cup Z'; W^* \times 0, U^* \times [0, 1] \cup U \times [1, 2] \cup X^{2n+2})$. Since a $2n$-object $(U^* \times [0, 1] \cup U \times [1, 2] \cup X^{2n+2}, U^* \times 0, K^{2n-1} \times 0)$ admits a locally flat spine $K \times [0, 2] \cup M^{2n}$, the $2n$-object $\theta^* = (W^* \times 0, U^* \times 0, K \times 0)$ is null-cobordant by Definition 4.10. The proof of the converse is the same.

To complete the proof of 5.8, we have only to make the following observations
(i) If $2n \geqq 6$, θ^* is null-cobordant if and only if X^*, the associated Seifert form with θ^*, is stably null-cobordant.
(ii) For $2n=4$ if θ^* is null-cobordant then X^* is stably null-cobordant (4.12).
This completes the proof of 5.8. Q.E.D.

LEMMA 5.9. *Let $X = (\pi_{n+1}(E, \mathscr{F}N), \lambda, \mu)$ be the Seifert form associated with* *an exterior n-connected submanifold* L^{2n} *of* $\theta=(W^{2n+2}, U^{2n+1}, K^{2n-1})$ *with* W *connected. Suppose* $2n\geqq 4$. *Then the equivalence class (under* $\sim$*) of a free core of* X *depends only on* θ *and not on* L^{2n}.

PROOF OF 5.9. Let X^* be a free core of X, and W^* the geometric realization of X^*. Take another exterior n-connected submanifold L' and the associated Seifert form Y. Let Y^* be a free core of Y, and V^* the realization of Y^*.

Consider a $2n+3$-manifold Z^{2n+3} defined by

$$W^*\times[0,1]\underset{W^*\times 1}{\cup} W\times[1,2]\underset{V^*\times 2}{\cup} V^*\times[2,3]\,.$$

A triple $(W^*\times 0\cup -V^*\times 3, (W^*\cap U)\times 0\cup(V^*\cap U)\times 3, K\times 0\cup K\times 3)$ is a (non-connected) $2n$-object representing $X^*\oplus(-Y^*)$, and this $2n$-object is null-cobordant via a $2n+1$-Poincaré thickening triad $(Z^{2n+3}; W^*\times 0\cup -V^*\times 3, (W^*\cap U)\times[0,1]\cup U\times[1,2]\cup(V^*\cap U)\times[2,3])$, for a $2n$-object $((W^*\cap U)\times[0,1]\cup U\times[1,2]\cup(V^*\cap U)\times[2,3], (W^*\cap U)\times 0\cup(V^*\cap U)\times 3, K\times 0\cup K\times 3)$ admits a locally flat spine $K^{2n-1}\times[0,3]$. So by 4.11, $X^*\oplus(-Y^*)$ is stably null-cobordant, i.e., $X^*\sim Y^*$. Q.E.D.

According to 4.12, 5.8 and 5.9, we can now establish our obstruction theory for the even-dimensional case, and combining this with the result (4.3) for the odd-dimensional case, we obtain our main theorem.

THEOREM 5.10. *Let* $\theta=(W^{m+2}, U^{m+1}, K^{m-1})$ *be a connected m-object over* $\mathscr{E}$ *with* $m\geqq 4$. *There is a unique obstruction element* $\eta(\theta)\in P_m(\mathscr{E})$ *such that the following three statements are equivalent for* $m\geqq 5$.

(*i*) $\eta(\theta)=0$,

(*ii*) θ *is null-cobordant,*

(*iii*) θ *admits a locally flat spine* M^m *with* $\partial M=K$.

If $m=4$, *we have the implications* (*iii*)$\Longrightarrow$(*ii*)$\Longrightarrow$(*i*).

PROOF OF 5.10. We have now only to define $\eta(\theta)$.

The odd-dimensional case [10]; Let L^m be a locally flat exterior $[m/2]$-connected submanifold with $\partial L=K$, $\rho:(W,\partial W-U)\longrightarrow(CP_s, CP_{s-1})$ (s: large) the Pontrjagin-Thom map for L. Then the normal bundle N is induced from that of CP_{s-1}, but the latter is extended to CP_s. Thus the 2-disk bundle N is extended to W; denote the bundle by ξ. Let ν be the stable normal bundle of W. Then $i^*(\xi\oplus\nu)\oplus\tau_L$ has a canonical trivialization F. ($i: L^m\longrightarrow W$ denotes the inclusion, which is degree 1.) Therefore we have a normal map ([1])

$$\begin{array}{ccc}(\text{normal bundle of } L) & \longrightarrow & \xi\oplus\nu\\ \downarrow & & \downarrow\\ L & \xrightarrow{\ i\ } & W\,,\end{array}$$

and by the usual surgery theory an obstruction element $\sigma(L, i, F) \in L_m(\pi')$ ($=P_m(\mathscr{E})$, for m odd) is defined. Our $\eta(\theta)$ is defined to be this element σ.

The even-dimensional case $m=2n$; L^m denotes again a locally flat exterior n-connected submanifold with $\partial L=K$. We take a free core of the Seifert form associated with L^{2n}. Its cobordism class (i.e., the equivalence class under $\sim$) depends only on (the cobordism class of) θ. We define $\eta(\theta)$ to be the element ($\in P_m(\mathscr{E})$) represented by the free core. Q.E.D.

COMPLEMENT 0. *As a matter of fact, $\eta(\theta)$ does not depend on U^{m+1} in θ, but depends only on W^{m+2} and K^{m-1}.*

PROOF. Let T^{m+1} be a tubular neighbourhood of K^{m-1} in ∂W which is contained in U^{m+1}. Then a triad $(W\times[0,2]; W\times 0\cup -W\times 2, U\times[0,1]\cup T\times[1,2])$ provides a cobordism between θ and $\theta_T=(W^{m+2}, T^{m+1}, K^{m-1})$. In fact, $U\times[0,1]\cup T\times[1,2]$ admits a locally flat spine $K^{m-1}\times[0,2]$. Thus $\eta(\theta)=\eta(\theta_T)$.

Q.E.D.

COMPLEMENT 1. (Additivity and naturality)

Let U^{m+1} be a D^2-bundle over a compact m-1-manifold Q^{m-1}. Suppose that there are two Poincaré thickening triads $(W_1^{m+2}; U^{m+1}, X_1^{m+1})$ and $(W_2^{m+2}; -U^{m+1}, X_2^{m+1})$ containing $\pm U^{m+1}$, and that X_i^{m+1} has a locally flat spine K_i^{m-1} such that $K_i\cap\partial X_i=\partial Q^{m-1}$ $(i=1,2)$. Then, $\eta(\theta)=j_1(\eta(\theta_1))+j_2(\eta(\theta_2))$. Here $\theta=(W_1\cup W_2, X_1\cup X_2, K_1\cup K_2)$, $\theta_1=(W_1^{m+2}, U^{m+1}\cup X_1^{m+1}, Q^{m-1}\cup K_1^{m-1})$, $\theta_2=(W_2^{m+2}, -U^{m+1}\cup X_2^{m+1}, -Q^{m-1}\cup K_2^{m-1})$, and

$$j_i: P_m(\mathscr{E}_i)\longrightarrow P_m(\mathscr{E}) \quad (i=1,2)$$

is the homomorphism corresponding to the morphism $\mathscr{E}_i\longrightarrow\mathscr{E}$ which is induced by the inclusion $W_i\longrightarrow W_1\cup W_2$. $\mathscr{E}$ and $\mathscr{E}_i$, $i=1,2$, are the associated extensions with θ and θ_i, $i=1,2$.

In particular, if θ_2 admits a locally flat spine extending $-Q\cup K_2$, then we have $\eta(\theta)=j_1(\eta(\theta_1))$ which reveals the naturality of the obstruction elements.

The proof is not difficult.

As is shown later (§6.6), our groups $P_m(\mathscr{E})$ are in general very large, but they are not too large;

COMPLEMENT 2. (Realization theorem)

Suppose $m\geqq 6$. Given any element $\eta\in P_m(\mathscr{E})$, we can find an m-object θ such that $\eta(\theta)=\eta$.

PROOF. For the even-dimensional case, this follows from 5.2. The corresponding fact for the odd-dimensional case is stated in the next lemma.

Q.E.D.

LEMMA 5.11. *Given any element* η *in* $P_{2n+1}(\mathscr{E})$ *with* $2n+1\geqq 7$, *we can find a* $2n+1$*-object* θ *such that* $\eta(\theta)=\eta$.

PROOF OF 5.11. We use the notations of Wall [29]. Suppose η is represented by a matrix $A\in SU_r(\Lambda')$. By 1.5, the extension $\mathscr{E}$ is realized by an S^1-bundle over an oriented manifold X^m with the fundamental group π'. Here we assume $m(=\dim X)=2n\geqq 6$. Let U^{2n+2} be the mapping cylinder. U^{2n+2} is a 2-disk bundle over X^{2n}. Fisrt perform codimension 2 surgery on X^{2n} (within U^{2n+2}) to "kill" r trivial $(n-1)$-spheres. The surgery trace M_1^{2n+1} is isomorphic to

$$X\times I \natural \underbrace{S^n\times D^{n+1}\natural S^n\times D^{n+1}\natural \cdots}_{r \text{ copies}},$$

where $\natural$ denotes the boundary connected sum. M_1^{2n+1} is also realized as a locally flat submanifold in $U^{2n+2}\times I$ such that $M_1\cap U\times 0=X$ and $M_1\cap U\times 1=X$ # (r-copies of $S^n\times S^n$). the latter is denoted by X'. Note that X' is an exterior n-connected submanifold in $U\times 1$ and the associated Seifert form is isomorphic to the direct sum of r copies of standard planes $\bigoplus_{i=1}^{r}(\Lambda x_i\oplus\Lambda y_i)$, see 4.6. Let $\tilde{A}$ be a Λ-matrix such that $\varpi_*(\tilde{A})=A$, where $\varpi_*:\Lambda\longrightarrow\Lambda'$ is induced by $\varpi:\pi\longrightarrow\pi'$. By (26) and 3.1, we have $\lambda(\tilde{A}y_i, \tilde{A}y_j)=0$ $(\forall i, j)$, $\mu(\tilde{A}y_i)=0$ $(\forall i)$. Moreover $\{\partial\tilde{A}y_i\}_{i=1,\dots,r}$ $(\partial:\pi_{n+1}(E, \mathscr{F}N)\longrightarrow K_n(X';\Lambda'))$ generate a subkernel (for $\partial\tilde{A}y_i=A\partial y_i$, and A is a simple isomorphism). So we can perform codimension 2 surgery on X' (within $U\times 1$) to kill n+1-disks $\tilde{A}y_1, \cdots, \tilde{A}y_r$ (cf. the proof of 4.12). Denote the resulting submanifold by X''. Then X'' is simple homotopy equivalent to X, and the surgery trace from X' to X'' is realized as a submanifold M_2^{2n+1} in $U\times[1,2]$. Glue M_1 to M_2 along X' to obtain a $2n+1$-submanifold M^{2n+1} in $U\times[0,2]$. Wall [29, p. 66] proves that the manifold M represents the surgery obstruction η which is determined by A. So if we define a $2n+1$-object θ as $(U^{2n+2}\times[0,2], U\times 0\cup(U|\partial X)\times[0,2]\cup U\times 2, X\times 0\cup\partial X\times I\cup X''\times 2)$, we have $\eta(\theta)=\eta$. Q.E.D.

"AN IMPORTANT SPECIAL CASE" IN OUR CONTEXT

Let $\theta=(W^{m+2}, U^{m+1})$ be a connected m-Poincaré thickening pair with $m\geqq 6$.

THEOREM 5.10. A. *Suppose the extensions* $\mathscr{E}_W$ *and* $\mathscr{E}_U$ *which are associated with W and U are isomorphic via the inclusion. Then we can find a locally flat spine* $(L^m, \partial L^m)$ *of* θ.

PROOF. U^{m+1} is itself an $m-1$-Poincaré thickening whose obstruction $\eta(U^{m+1})$ equals zero. (For U^{m+1} is null-cobordant in the sense of 4.10. Here we take ϕ as X of 4.10.) So there is a locally flat spine K^{m-1} of U^{m+1}. Let $\bar{\eta}$ be the obstruction of the m-object $(W^{m+2}, U^{m+1}, K^{m-1})$. By 5.2 and 5.11, one can find an

m-Poincaré thickening triad $(V^{m+2}; T', T^{m+1})$ with T^{m+1} a tubular neighbourhood of K^{m-1} in U^{m+1}, such that

(i) T' has a locally flat spine K'.

(ii) $\eta(V^{m+2}; T' \cup -T, K' \cup -K) = -\bar{\eta}$.

Gluing V to W along T, we obtain a new object (W', U', K'). However, V is isomorphic to a product

$$T^{m+1} \times [0, 1]\ .$$

This follows from the construction in the case with m odd; in the case with m even, it follows from the construction and the relative s-cobordism theorem. Therefore, $(W', U') \cong (W, U)$, and by the construction of W' and the additivity of η, we have $\eta(W', U', K') = 0$. So (W', U') (and hence (W, U)) admits a locally flat spine L^m extending K'. Q.E.D.

GEOMETRIC PERIODICITY

It is well-known ([24], [29]) that there is a geometric Periodicity of period 4 in the usual surgery obstruction groups which is induced by multiplying a complex projective plane CP_2. Here we will give an analogous theorem for our groups $P_m(\mathscr{C})$. As a special case, this gives a geometric interpretation of the periodicity of the knot cobordism groups [13] which is known only algebraically. This point will be clarified in §6.5.

THEOREM 5.12. *Suppose $m \geqq 5$. Denote by $\theta \times CP_2$ the $m+4$-object $(W^{m+2} \times CP_2, U^{m+1} \times CP_2, K^{m-1} \times CP_2)$. Then we have $\eta(\theta \times CP_2) = \rho(\eta(\theta))$, where ρ is the algebraic periodicity isomorphism given earlier.*

PROOF OF 5.12. The proof is similar to that of the usual case [29], so we give only an outline of it. We have only to consider the even-dimensional case; $m = 2n$. Also by considering a geometric free core (§5.8), we may suppose that

$$H_i(E, \mathscr{F}N; \Lambda) = \begin{cases} 0 & (i \neq n+1) \\ \text{a free } \Lambda\text{-module} & (i = n+1)\ , \end{cases} \tag{31}$$

where E, $\mathscr{F}N$ are the exterior or the frontier of an exterior n-connected submanifold L^{2n}.

Let $(e_1, \cdots, e_r)$ be a Λ-basis of $H_{n+1}(E, \mathscr{F}N; \Lambda) \cong \pi_{n+1}(E, \mathscr{F}N)$ which are represented by pathed nice immersions $f_i : (D^{n+1}, S^n) \longrightarrow (E, \mathscr{F}N)$. According to 2.7.1, by introducing some self-intersection points to $\varpi \circ f_i$, we can make $\mathscr{O}(f_i) = 0 \forall i$. Let S^2 denote a complex projective line in CP_2 with the canonical orientation. Consider immersions $g_i : (D^{n+1} \times S^2, S^n \times S^2) \longrightarrow (E \times CP_2, \mathscr{F}N \times CP_2)$ defined by $g_i = f_i \times \text{inclusion}$. Since the intersection number $S^2 \cdot S^2 = 1$ in CP_2, we can put g_i's into general position so that the numbers of the (self-) intersection points of $\varpi g_i(S^n\times S^2)$ (or $g_i(D^{n+1}\times S^2)$) are equal to those of $\varpi f_i(S^n)$ (or $f_i(D^{n+1})$). We may assume that $g_i(D^{n+1}\times 1, S^n\times 1)$ ($1\in S^2$) are mutually disjoint embeddings. Perform surgeries on $L\times CP_2$ in codimension 2 along these $n+1$-disks to obtain a new pair $(E', \mathscr{T}N')$. Here note that $\dim E'=2n+6$. It is shown that

$$(32)\qquad H_i(E', \mathscr{T}N'; \Lambda)\cong\begin{cases} H_i(E\times CP_2, \mathscr{T}N\times CP_2; \Lambda) & i\neq n+1,\quad i\leqq n+3,\\ 0 & i=n+1.\end{cases}$$

By (31), (32) and the Künneth formula, we have

$$(33)\qquad H_i(E', \mathscr{T}N'; \Lambda)\cong\begin{cases} 0, & i\leqq n+2\\ H_{n+1}(E, \mathscr{T}N; \Lambda), & i=n+3.\end{cases}$$

We have performed surgery along the $n+1$-disks $g_i(D^{n+1}\times 1, S^n\times 1)$, but by this, $g_i(D^{n+1}\times S^2, S^n\times S^2)$ are changed to immersions $g'_i: (D^{n+3}, S^{n+2})\longrightarrow(E', \mathscr{T}N')$. Recall here that $g_i(D^{n+1}\times S^2, S^n\times S^2)$'s have the same intersection as $f_i(D^{n+1}, S^n)$. So the immersions $g'_i(D^{n+3}, S^{n+2})$ have the same intersection numbers as $f_i(D^{n+1}, S^n)$, too. It is proved that $\{g'_i\}$ represent a Λ-basis of $H_{n+3}(E', \mathscr{T}N'; \Lambda)$ and that the isomorphism $H_{n+1}(E, \mathscr{T}N: \Lambda)\longrightarrow H_{n+3}(E', \mathscr{T}N'; \Lambda)$ is given by $f_i\mapsto g'_i$. This together with the above remark tells us that the Seifert form defined on $H_{n+1}(E, \mathscr{T}N; \Lambda)$ is isomorphic to the form defined on $H_{n+3}(E', \mathscr{T}N'; \Lambda)$. From this we have the desired conclusion. Q.E.D.

CONCLUDING REMARKS

REMARK A. If the cyclic group C in $\mathscr{E}$ is a trivial group, i.e., $\mathscr{E}=\{1\longrightarrow 1\longrightarrow\pi'\xrightarrow{id}\pi'\longrightarrow 1\}$, then $P_m(\mathscr{E})$ coincides with the Wall group $L_m(\pi')$.

Indeed a free $(-1)^n$-Seifert form over $\mathscr{E}$ is nothing other than a special $(-1)^n$-Hermitian form over $\boldsymbol{Z}[\pi']$.

In particular, if $\mathbf{1}$ denotes the trivial extension

$$1\longrightarrow 1\longrightarrow 1\longrightarrow 1\longrightarrow 1\,,$$

then $P_m(\mathbf{1})\cong$the Kervaire-Milnor group $L_m(1)$ [12]. This fact is essentially contained in the work of Montgomery and Yang [20].

REMARK B. With an extension $\mathscr{E}=\{1\longrightarrow C\longrightarrow\pi\xrightarrow{\varpi}\pi'\longrightarrow 1\}$, we associate an "almost trivial" extension $\mathscr{E}'=\{1\longrightarrow 1\longrightarrow\pi'\xrightarrow{id}\pi'\longrightarrow 1\}$ and a morphism $b:\mathscr{E}\longrightarrow\mathscr{E}'$ defined by the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C & \longrightarrow & \pi & \longrightarrow & \pi' & \longrightarrow & 1\\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle\varpi} & & \downarrow{\scriptstyle id} & & \\ 1 & \longrightarrow & 1 & \longrightarrow & \pi' & \xrightarrow{id} & \pi' & \longrightarrow & 1\,. \end{array}$$

The morphism b induces a homomorphism $P_m(\mathscr{E}) \xrightarrow{b_*} P_m(\mathscr{E}')$. By Remark A, $P_m(\mathscr{E}') \cong L_m(\pi')$. Thus we have a canonical homomorphism

$$b_* : P_m(\mathscr{E}) \longrightarrow L_m(\pi') .$$

ASSERTION. b_* is surjective.

The proof is not difficult and is omitted.

More generally, it can be proved that if $\bar{b}$ is a morphism of the form

$$\begin{array}{ccccccccc} \mathscr{E} : 1 & \longrightarrow & C & \longrightarrow & \pi & \longrightarrow & \pi' & \longrightarrow & 1 \\ \bar{b}\downarrow & & \downarrow & & \downarrow & & \downarrow id & & \\ \mathscr{E}_1 : 1 & \longrightarrow & C_1 & \longrightarrow & \pi_1 & \longrightarrow & \pi' & \longrightarrow & 1 , \end{array}$$

then $\bar{b}_* : P_m(\mathscr{E}) \longrightarrow P_m(\mathscr{E}_1)$ is surjective.

§ 6. Simply connected cases.

In this section we will give some remarks about the simply connected cases where $\pi'=1$. The extension appears as

$$1 \longrightarrow C \xrightarrow{id} C \longrightarrow 1 \longrightarrow 1 .$$

Here we use the notation $P_m(C)$ instead of $P_m(1 \longrightarrow C \longrightarrow C \longrightarrow 1 \longrightarrow 1)$. This notation was used in [17]. An object over $\{1 \longrightarrow C \longrightarrow C \longrightarrow 1 \longrightarrow 1\}$ will be simply called an object over C.

Let $\theta=(W^{m+2}, U^{m+1}, K^{m-1})$ be a connected m-object over C with $m \geqq 4$.

PROPOSITION 6.1. *There is a simple relation*

$$C \cong H_1(\partial W - U ; \boldsymbol{Z}) .$$

PROOF OF 6.1. Let L^m be an exterior 2-connected submanifold extending K^{m-1}. The proposition follows from the definition of C; $C=\pi_1(E) \cong H_1(E)$, and the following diagram

$$\begin{array}{ccccccc} & & H^m(W, U) \cong \boldsymbol{Z} & & & & \\ & & \wr\Vert & & & & \\ H_2(W) & \longrightarrow & H_2(W, \partial W - U) & \longrightarrow & H_1(\partial W - U) & \longrightarrow & 0 \\ \Vert & & \downarrow\cong & & \downarrow & & \\ H_2(W) & \longrightarrow & H_2(W, E) & \longrightarrow & H_1(E) & \longrightarrow & 0 . \\ & & \wr\Vert & & & & \\ & & H^m(L, \partial L) \cong \boldsymbol{Z} & & & & \end{array}$$

Q.E.D.

Next we show that the $P_m(Z)$ are isomorphic to the Kervaire-Levine knot cobordism groups C_{m-1} (of $(m-1, m+1)$-knots) with $m\geqq 5$.

Let $\Sigma^{m-1}\subset S^{m+1}$ be an $(m-1, m+1)$-knot with $m\geqq 5$. Let U^{m+1} be a tubular neighbourhood of Σ^{m-1} in S^{m+1}, and D^{m+2} an $m+2$-disk bounded by S^{m+1}. Since $H_1(S^{m+1}-U^{m+1})\cong Z$ (cf. [9]), the m-object $\theta=(D^{m+2}, U^{m+1}, \Sigma^{m-1})$ is *over* Z by 6.1.

The obstruction $\eta(\theta)$ of 5.10 belongs to $P_m(Z)$. One easily verifies that $\eta(\theta)$ depends only on the cobrdism class (in Levine's sense [13]) of the knot $\Sigma^{m-1}\subset S^{m+1}$. Thus $\eta(\theta)$ provides a map $\eta: C_{m-1}\longrightarrow P_m(Z)$ which is a homomorphism by the additivity of η (Complement 1 to 5.10).

PROPOSITION 6.2. *$\eta: C_{m-1}\longrightarrow P_m(Z)$ is an isomorphism ($m\geqq 5$).*

PROOF OF 6.2. Suppose $\eta(\theta)=0$, then by 5.10, θ admits a locally flat spine M^m such that $\partial M^m=\Sigma^{m-1}$. Since $M^m\simeq D^{m+2}$, M^m is contractible, so the knot is null-cobordant (in the sense of Levine). Therefore, η is injective. The surjectivity of η follows from 5.2 and 5.11. (Although 5.11 is not applied to the case $m=5$, the surjectivity is trivial, for $P_5(Z)\cong L_5(1)\cong 0$.) Q.E.D.

REMARK. We can show *geometrically* that the above isomorphism $C_{m-1}\longrightarrow P_m(Z)$ is induced by sending a Seifert matrix A to the Seifert form$(-1)^n A+tA'$ (up to the sign ± 1). A' denotes the transposed matrix of A. However, we have no *algebraic* proof that the map is an isomorphism.

ISOLATED SINGULARITIES OF SPINES

As a corollary of the fact that $P_{odd}(C)\cong L_{odd}(1)\cong 0$, one sees that any 1-connected m-object with $m=$odd$\geqq 5$ admits a locally flat spine (cf. [10]).

An even-dimensional ($2n$-) object θ does not generally admit a locally flat spine, but it is known [10] that if $2n\geqq 6$, *a* 1-connected $2n$-object $\theta=(W^{2n+2}, U, K)$ admits a spine L^{2n} which is locally flat except (possibly) at a finite number of points. We define the *singularity at a point* $p\in L^{2n}$ by the $(2n-1, 2n+1)$-knot $\sigma_p(L)=(Lk(p, L), Lk(p, W))$, where Lk denotes the link of the point p in L (or in W).

The *total singularity of* L is defined by $\sum_{p\in L}\sigma_p(L)$ (in C_{2n-1}). $\sigma_p(L)=0$ in C_{2n-1} if L is locally flat at p, so the summation is in fact a finite sum.

PROPOSITION 6.3. *The total singularity of L^{2n} is related to $\eta(\theta)$ as follows:*

$$\eta(\theta)=j(\sum_{p\in L}\sigma_p(L)) .$$

Here j denotes the composition $C_{2n-1}\xrightarrow{\cong} P_{2n}(Z)\xrightarrow{\bar{b}*} P_{2n}(H_1(\partial W-U))$. *$\bar{b}_*$ is given in Concluding remark B in* §5. (See also 6.1.)

COROLLARY 6.3.1. *As an element of $P_{2n}(H_1(\partial W-U))$, the total singularity of a spine L^{2n} depends only on the object θ and not on the choice of the spine.*

In particular, if $H_1(\partial W-U)\cong \boldsymbol{Z}$, the knot cobordism class of the total singularity of a spine is uniquely determined by θ.

PROOF OF 6.3. (Outline) Let W' be a regular neighbourhood of L in W. Then W' and W are "cobordant", i.e., $W'\cup(-W)$ is null-cobordant in the sense of 4.10. (The cobordism is $W'\times[0,1]\cup W\times[1,2]$.) Therefore, $\eta(\theta)=\eta(W)=\eta(W')$, but the obstruction $\eta(W')$ is "concentrated" to the knots at singular points. Hence we have the desired relation by the naturality of $\eta(\theta)$. Q.E.D.

REMARK. The relation 6.3, is stated in [17] with the extra assumption $\pi_1(W-L)\cong\pi_1(\partial W-U)$ (there $U=\phi$). However, this assumption is not necessary.

ON THE REDUCTION OF CODIMENSION 2 SINGULARITIES

Let M^m be an arbitrary PL m-manifold ($m\geqq 5$) which is PL embedded in another PL $m+2$-manifold T^{m+2}. The singular set $\Sigma(M^m)$ of M^m is, by definition, the set of points of M^m at which M^m is not locally flat in T^{m+2}. $\Sigma(M^m)$ is a subcomplex of M^m [21].

Suppose that $\Sigma(M)$ is connected and 1-connected, and $\Sigma(M^m)\cap\partial M=\phi$. A regular neighbourhood of $\Sigma(M^m)$ in M^m (or in T^{m+2}) is denoted by N_M (or N_T). We may assume that

$$N_T\cap M=N_M\ .$$

The pair $(N_T,\partial N_M)$ is homotopically equivalent to $(N_M,\partial N_M)$, and N_T is 1-connected. As remarked above, there is a submanifold L^m of N_T extending ∂N_M which is locally flat except at a finite number of points. (Without loss of generality, we may assume that the number of the singular points is at most one [10].) Therefore, we have

PROPOSITION 6.4. *If $\Sigma(M)$ is 1-connected, there is a submanifold M' such that*

(i) $\Sigma(M')$ consists of at most one point,

(ii) $M'\subset$ a regular neighbourhood of M in T, and

(iii) the inclusion in (ii) is a homotopy equivalence.

PROOF. We have only to define M' by the following:

$$M'=\overline{(M-N_M)}\cup L^m\ . \qquad \text{Q.E.D.}$$

REMARK. Of course, if $m=\text{odd}\geqq 5$, then we can take M' such that $\Sigma(M')=\phi$, for $P_{\text{odd}}(C)\cong L_{\text{odd}}(1)\cong 0$. We will call the singularity at the point of $\Sigma(M')$ a *reduced form* of $\Sigma(M)$. In general, a reduced form has some ambiguity, but in the following situation, it is uniquely determined as an element of C_{m-1}.

PROPOSITION 6.5. *Let $\kappa=(S_1^{m-1}\subset S_2^{m+1})$ be a PL $(m-1,m+1)$-knot ($m\geqq 5$). The* *reduced form of the singularity*

$$CP_2\times \text{cone}\,\kappa=(CP_2\times \text{cone}\,S_1^{m-1}\subset CP_2\times \text{cone}\,S_2^{m+1})$$

is $\rho(\kappa)$. *Here* $\rho: C_{m-1}\longrightarrow C_{m+3}$ *is the algebraic periodicity.*

This is a corollary of 6.4, 6.3.1, and 5.12.

INFINITENESS OF $P_{4k+2}(\boldsymbol{Z}_2)$

The calculation of our groups $P_m(\mathscr{C})$ seems to be very difficult. In the cases when $C=\{1\}$, the calculation is reduced to that of the Wall groups (Remark A at the end of §5). The only case of non-trivial C for which we have full information is the case where $\mathscr{C}=\{1\longrightarrow \boldsymbol{Z}\longrightarrow \boldsymbol{Z}\longrightarrow 1\longrightarrow 1\}$; the group $P_m(1\longrightarrow \boldsymbol{Z}\longrightarrow \boldsymbol{Z}\longrightarrow 1\longrightarrow 1)$ is isomorphic to the knot cobordism group C_{m-1} (§6.2) [13]. It is known that this group is not finitely generated if m=even [11]. It would be natural to conjecture that $P_{2n}(C)$ $(=P_{2n}(1\longrightarrow C\longrightarrow C\longrightarrow 1\longrightarrow 1))$ is also infinitely generated if $C\neq\{1\}$.

In this paragraph we will show that if C is a cyclic group of even order, then $P_{4k+2}(C)$ is infinitely generated. Since there is a surjection $P_m(C)\longrightarrow P_m(\boldsymbol{Z}_2)$ which is induced by the quotient map $C\longrightarrow \boldsymbol{Z}_2$ (Remark B at the end of §5), it will be sufficient to show.

PROPOSITION 6.6. $P_{4k+2}(\boldsymbol{Z}_2)$ *is infinitely generated.*

We need a lemma.

Consider $\boldsymbol{Z}_2$ as a multiplicative group with the generator t of order 2. Let $\Lambda=\boldsymbol{Z}[\boldsymbol{Z}_2]$.

Let (G, λ, μ) be a free $(-1)^n$-Seifert form over $\boldsymbol{Z}_2$ (i.e., over $1\longrightarrow \boldsymbol{Z}_2\longrightarrow \boldsymbol{Z}_2\longrightarrow 1\longrightarrow 1$) which is stably null-cobordant.

LEMMA 6.7. *G is stably equivalent* (§4.8) *to a null-cobordant form G' of which a Seifert sub-kernel H'* (§4.9) *is a free Λ-direct summand of G'.*

PROOF OF 6.7. We may suppose that G is itself a null-cobordant form with a Seifert sub-kernel $H\subset G$. Put $H_0=\{x\in G\,|\,\exists m\in \boldsymbol{Z}-\{0\},\ mx\in H\}$. Then it is easy to see that H_0 is also a Seifert sub-kernel of G and that G/H_0 is a free $\boldsymbol{Z}$-module. Thus we may regard G/H_0 as an integral representation of a group $\boldsymbol{Z}_2$.

Any integral representation of $\boldsymbol{Z}_2$ is a direct sum of some copies of the following Λ-modules (see for example [2]):

$$\begin{cases} \Lambda \\ \boldsymbol{Z}^-=\boldsymbol{Z} \text{ with a } \boldsymbol{Z}_2\text{-action defined by } t(1)=-1\,, \\ \boldsymbol{Z}^+=\boldsymbol{Z} \text{ with a trivial } \boldsymbol{Z}_2\text{-action.} \end{cases}$$

By the definition of a null-cobordant form (4.9), H_0 is mapped by $G\longrightarrow G\otimes_\Lambda \boldsymbol{Z}$ onto a $\boldsymbol{Z}$-free direct summand of $G\underset{\Lambda}{\otimes}\boldsymbol{Z}$ of rank $r=1/2\ \mathrm{rank}_Z(G\underset{\Lambda}{\otimes}\boldsymbol{Z})$, so $(G/H_0)\underset{\Lambda}{\otimes}\boldsymbol{Z}$ is a free $\boldsymbol{Z}$-module of rank r. Since $\boldsymbol{Z}^-\underset{\Lambda}{\otimes}\boldsymbol{Z}=\boldsymbol{Z}_2$, no copies of $\boldsymbol{Z}^-$ appear in the decomposition of G/H_0. Therefore, the decomposition is of the form

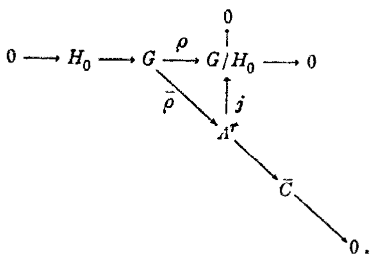
$$\underbrace{\Lambda\oplus\cdots\oplus\Lambda}_{a\text{ summands}}\oplus\underbrace{\boldsymbol{Z}^+\oplus\cdots\oplus\boldsymbol{Z}^+}_{b\text{ summands}},$$

where $a+b=r$.

Hence there exists a Λ-homomorphism

$$j:\Lambda^r\longrightarrow G/H_0$$

which is surjective. Here Λ^r denotes a free Λ-module of rank r. Let $\rho:G\longrightarrow G/H_0$ be the quotient map. Since G is a free Λ-module, there is a Λ-homomorphism $\bar\rho:G\longrightarrow\Lambda^r$ such that $\rho=j\circ\bar\rho$. Let $\bar C$ denote the cokernel of $\bar\rho$. (See the diagram below.)



We choose a set of Λ-generators $\{e_1,\cdots,e_s\}$ of $\bar C$, and their lifts $\{\tilde e_1,\cdots,\tilde e_s\}$ to Λ^r. Define G' by the direct sum

$$G\oplus\sum_{i=1}^{s}\{\Lambda x_i\oplus\Lambda y_i\},$$

where x_i and y_i are indeterminates. We give to each $\Lambda x_i\oplus\Lambda y_i$ the structure of a standard plane (see §4.6). Choose $f_1,\cdots,f_s\in G$ so that $\rho(f_i)=j(\tilde e_i)$ $(i=1,\cdots,s)$, and define $\rho':G'\longrightarrow\Lambda^{r+s}=\Lambda^r\oplus\sum_{i=1}^{s}\Lambda y_i$ as follows:

$$\begin{cases}\rho'|G=\bar\rho\,,\\ \rho'(x_i)=\tilde e_i-\bar\rho(f_i)\,,\quad\text{and}\\ \rho'(y_i)=y_i\,.\end{cases}$$

To complete the proof of 6.7, it will be sufficient to verify the following:

(1°) *(G,λ,μ) is stably equivalent to (G',λ',μ'), where $(\lambda',\mu')=(\lambda,\mu)\oplus\sum_{i=1}^{s}(\lambda_i,\mu_i)$*

and (λ_i, μ_i) *are the structures of standard planes over* $\Lambda x_i \oplus \Lambda y_i$.
(2°) $\rho' : G' \longrightarrow \Lambda^{r+s}$ *is surjective. Therefore,* H'=*kernel* (ρ') *is a* Λ*-free direct summand.*
(3°) H' *is a Seifert sub-kernel of* G'.
(1°) is obvious by the construction of G' and the definition of the stable equivalence. The elements y_i clearly belong to the image of ρ'. We will show that any element y in Λ^r is in the image of ρ'. Let $\pi : \Lambda^r \longrightarrow \tilde{C}$ be the quotient map. We can write $\pi(y) = \sum_{i=1}^{s} a_i e_i$, $a_i \in \Lambda$. Let $y' \in G'$ be $\sum_{i=1}^{s} a_i f_i \oplus \sum_{i=1}^{s} a_i x_i$. Then $\pi\rho'(y') = \pi(\sum_{i=1}^{s} a_i \tilde{\rho}(f_i) + \sum_{i=1}^{s} a_i(\tilde{e}_i - \tilde{\rho}(f_i))) = \pi(\sum_{i=1}^{s} a_i \tilde{e}_i) = \pi(y)$. Hence $y - \rho'(y')$ belongs to Kernel (π) = Image $(\tilde{\rho}) \subset$ Image (ρ'), so $y \in$ Image (ρ').

To verify (3°), first we prove that $H' \subset H_0 \oplus \sum_{i=1}^{s} \Lambda x_i$. In fact, let z be any element in Kernel $(\rho') = H'$. Write $z = u + v + w$, $u \in G$, $v \in \sum_{i=1}^{s} \Lambda x_i$, and $w \in \sum_{i=1}^{s} \Lambda y_i$. By the definition of ρ', ρ' is injective on $\sum_{i=1}^{s} \Lambda y_i$, so $w=0$. Then we have

$$0 = j\rho'(z) = j\rho'(u) + j\rho'(v) = \rho(u) + j\rho'(v) \ . \tag{$*$}$$

Since $j\rho'(x_i) = j(\tilde{e}_i - \tilde{\rho}(f_i)) = j(\tilde{e}_i) - \rho(f_i) = 0$ $(i=1, \cdots, s)$, we have $j\rho'(v)=0$ for $\forall v \in \sum_{i=1}^{s} \Lambda x_i$.

Substituting in $(*)$, we have $\rho(u)=0$, i.e., $u \in$ Kernel $(\rho) = H_0$. Therefore, $H' \subset H_0 \oplus \sum_{i=1}^{s} \Lambda x_i$ as asserted. From this, it is easy to see that $\lambda'(H' \times H')=0$ and $\mu'(H')=0$. Since H' is a Λ-free direct summand of G' of rank $r+s = 1/2\, \mathrm{rank}_\Lambda(G')$, H' is clearly a Seifert sub-kernel of G'. This completes the proof of (3°) and hence the proof of 6.7. Q.E.D.

PROOF OF 6.6. (cf. Kervaire [11]) Let (G, λ, μ) be a free $(-1)^n$-Seifert form over $\boldsymbol{Z}_2$. Define $\Delta_G \in \Lambda$ by $\det \lambda$. If G is null-cobordaht with a Seifert sub-kernel H which is a Λ-free direct summand of G, then λ is of the form

$$\left[\begin{array}{c|c} 0 & A \\ \hline (-1)^n \bar{A}' \cdot t & * \end{array}\right].$$

The involution $- : \Lambda \longrightarrow \Lambda$ is the identity for $\Lambda = \boldsymbol{Z}[\boldsymbol{Z}_2]$, and we have $\Delta_G = \pm t^l (\det A)^2$, $l=0$ or 1. If two Seifert forms G_1, G_2 are stably equivalent, we have $\Delta_{G_1} = \pm t^l \Delta_{G_2}$, $l=0$ or 1. So by Lemma 6.7, if G is stably null-cobordant, $\Delta_G = \pm t^l a^2$ for some $a \in \Lambda$. Consider a multiplicative abelian group $\mathcal{C}$;

$$\mathcal{O}=\boldsymbol{Z}-\{0\}/\pm(\boldsymbol{Z}-\{0\})^2 .$$

By $\Delta_G(\varepsilon)$ ($\varepsilon=\pm1$), we denote the value obtained by substituting $t=\varepsilon$. We get a homomorphism

$$d: P_{2n}(\boldsymbol{Z}_2)\longrightarrow\mathcal{O} ,$$

which is defined by $d(G, \lambda, \mu)=\Delta_G(-1)$, ($\Delta_G(-1)\neq0$, because $\Delta_G(-1)\equiv\Delta_G(1) \bmod 2$, and $\Delta_G(1)$ is the determinant of $\lambda\otimes1$ which is non-singular. cf. §4.7, (vi).) If $2n=4k$, $\Delta_G(-1)$ is always a square integer (cf. Levine [13] §14). Thus in the case $2n=4k$ d is a trivial homomorphism, and we cannot deduce any information.

(Problem: Is $P_{4k}(\boldsymbol{Z}_2)$ infinitely generated?)

Now let $2n=4k+2$, and m an arbitrary integer. Consider the following (-1)-Seifert form (G_m, λ_m, μ_m) over $\boldsymbol{Z}_2$ of rank 2:

$$\begin{cases} G_m=\Lambda x\oplus\Lambda y , \\ \lambda_m(x, y)=-\lambda_m(y, x)\cdot t=1 , \\ \mu_m(x)=mt , \quad \mu_m(y)=t . \end{cases}$$

(This is constructed by Kervaire [11].) Note that

$$\lambda_m=\begin{pmatrix} m(t-1), & 1 \\ -t, & t-1 \end{pmatrix} .$$

Therefore, $d(G_m, \lambda_m, \mu_m)=m(t-1)^2+t|_{t=-1}=4m-1$. The set $\{4m-1|m\in\boldsymbol{Z}\}$ generates an infinitely generated subgroup of $\mathcal{O}$, for there are infinitely many prime numbers in $\{4m-1|m\in\boldsymbol{Z}\}$ (Dirichlet's theorem!).

This completes the proof of 6.6.

Added in proof:

The following result was used in Step (II) of §3.5 without proof. Here we will give an indication of the proof. (For the notations, see §3.5.)

ASSERTION. *The contribution of $C_f\cap C_g$ to $\lambda(f_D, g_D)$ is equal to zero.*

OUTLINE OF PROOF. Consider the restriction of the S^1-bundle $\mathscr{T}T\longrightarrow Y^{2n+1}$ to the circle C. The total space is a torus T^2. We give it a trivialization $T^2\simeq S^1\times S^1$ with which C_g is identified with $S^1\times pt$. The homotopy class of the composite map $C_f\xrightarrow{incl.}T^2\xrightarrow{proj.}pt\times S^1$ defines an integer with the absolute value ρ. Since C_f and C_g are contractible in $\mathscr{T}T$, we have $t^\rho=1$. If $\rho\neq0$, the contribution of $C_f\cap C_g$ to $\beta(f_D, g_D)$ is proved to be of the form $\varepsilon g(1+t+\cdots+t^{\rho-1})$. Therefore, the contribution to $\lambda(f_D, g_D)$ is equal to $(-1)^{n+1}(1-t)\varepsilon g(1+t+\cdots+t^{\rho-1})=(-1)^{n+1}\varepsilon g(1-t^\rho)=0$. On the other hand, if $\rho=0$, the contribution to $\beta(f_D, g_D)$ is already equal to zero. Q.E.D.

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