Fourier ultra-hyperfunctions in the Euclidean n-space

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§1. Introduction

The tempered ultra-hyperfunctions and the Fourier ultra-hyperfunctions were introduced and studied, in one dimensional case, by J. Sebastião e Silva [6] under the name of tempered ultra-distribution and ultra-distribution of exponential type. Afterwards M. Hasumi [1] described explicitly the space of tempered ultra-hyperfunctions in the Euclidean n-space. Then the second named author [3] defined the space of ultra-hyperfunctions on $\boldsymbol{R}^n$ using the relative cohomology with coefficients in the sheaf of holomorphic functions, in connection with the hyperfunction of M. Sato and the analytic functional of A. Martineau. The ultra-hyperfunction was called in [3] the cohomological ultradistribution but we proposed to change its name into the ultra-hyperfunction in [4] in order to distinguish it from the continuous linear functional on the space of non-quasianalytic functions. The Fourier ultra-hyperfunction is named after the Fourier hyperfunction of M. Sato (see [2]).

As far as the authors know (see for example the references of [7]), the Fourier ultra-hyperfunctions have not been studied fully, especially in the Euclidean n-space. We study in this paper their basic properties.

Our results are roughly as follows. Let $\mathfrak{H}(\boldsymbol{C}^n)$ (resp. $Q(\boldsymbol{C}^n)$) be the space of all entire functions on $\boldsymbol{C}^n$ which decrease rapidly (resp. exponentially) on any tube with compact base. Then $\mathfrak{H}(\boldsymbol{C}^n)$ and $Q(\boldsymbol{C}^n)$ are Fréchet nuclear spaces. A tempered ultra-hyperfunction on $\boldsymbol{R}^n$ is, by definition, a continuous linear functional on $\mathfrak{H}(\boldsymbol{C}^n)$ and a Fourier ultra-hyperfunction on $\boldsymbol{R}^n$ is, by definition, a continuous linear functoinal on $Q(\boldsymbol{C}^n)$. As $Q(\boldsymbol{C}^n)$ is dense in $\mathfrak{H}(\boldsymbol{C}^n)$, the tempered ultra-hyperfunctions form a subspace of the space of Fourier ultra-hyperfunctions: $\mathfrak{H}'(\boldsymbol{C}^n)\subset Q'(\boldsymbol{C}^n)$. As the Fourier transformation $\mathscr{F}$ is a topological automorphism of $Q(\boldsymbol{C}^n)$, the space of Fourier ultra-hyperfunctions is also stable under the Fourier transformation. Most of these results can be proved using the results of Hasumi [1], for the space $Q(\boldsymbol{C}^n)$ is the projective limit of the space $\mathfrak{H}(\boldsymbol{C}^n)$ with respect to some system of linear mappings.

We shall show in the forthcoming paper that the space $Q'(\boldsymbol{C}^n)$ can be represented as a cohomology space of holomorphic functions with some bounds. An analogous representation of $\mathfrak{H}'(C^n)$ was obtained by Hasumi [1].

§ 2. Results of Hasumi [1].

We recall some results of Hasumi [1] which are necessary to us in the sequel. Let $\boldsymbol{R}^n$ (resp. $\boldsymbol{C}^n$) be the real (resp. complex) n-space whose generic points are denoted by $x=(x_1, \cdots, x_n)$ (resp. $z=(z_1, \cdots, z_n)$). We shall use the following notations: $x+y=(x_1+y_1, \cdots, x_n+y_n)$, $\alpha x=(\alpha x_1, \cdots, \alpha x_n)$ and $|x|=|x_1|+ \cdots +|x_n|$. By the inner product $\langle x, \xi\rangle=x_1\xi_1+ \cdots +x_n\xi_n$ the dual space of $\boldsymbol{R}^n$ (resp. $\boldsymbol{C}^n$) will be identified with $\boldsymbol{R}^n$ (resp. $\boldsymbol{C}^n$) but the generic point of the dual space $\boldsymbol{R}^n$ (resp. $\boldsymbol{C}^n$) will be denoted by the Greek letter ξ (resp. ζ). If necessary, we will distinguish $\boldsymbol{R}^n$ from its dual, writing $\boldsymbol{R}^n_x$ and $\boldsymbol{R}^n_\xi$. $\boldsymbol{C}^n_z$ and $\boldsymbol{C}^n_\zeta$ have the similar meanings. Let p be a system of non-negative integers $(p_1, \cdots, p_n)$. We shall write $|p|=p_1+ \cdots +p_n$ and $D^p=\partial^{|p|}/\partial x_1^{p_1} \cdots \partial x_n^{p_n}$ is the differential operator.

Let $H(\boldsymbol{R}^n)$ be the space of all C^∞-functions $\varphi(x)$ on $\boldsymbol{R}^n$ such that $\exp(k|x|)D^p\varphi(x)$ is bounded on $\boldsymbol{R}^n$ for any non-negative integer k and multi-index p. A fundamental system of seminorms in $H(\boldsymbol{R}^n)$ is defined by

$$|\varphi|_k = \sup\{\exp(k|x|)|D^p\varphi(x)|\ ;\ 0 \leqq |p| \leqq k,\ x \in \boldsymbol{R}^n\}$$

for $k=0, 1, 2, \cdots$.

PROPOSITION 2.1. *The space $H(\boldsymbol{R}^n)$ is Fréchet nuclear and therefore reflexive.*

PROPOSITION 2.2. *The space $\mathscr{D}(\boldsymbol{R}^n)$ of C^∞-functions with compact support* ([5]) *is dense in the space $H(\boldsymbol{R}^n)$.*

COROLLARY. *$H'(\boldsymbol{R}^n)$ is the subspace of the space $\mathscr{D}'(\boldsymbol{R}^n)$ of distributions on $\boldsymbol{R}^n$ whose elements are distributions with exponential growth* ([1]).

Let $\mathfrak{H}(\boldsymbol{C}^n)$ be the space of entire functions $f(z)$ on $\boldsymbol{C}^n$ such that, for any polynomial $P(z)$ of z and any compact set K of $\boldsymbol{R}^n$, $|P(z)f(z)|$ is bounded for $z \in T(K)=\boldsymbol{R}^n \times iK$. A fundamental system of seminorms in $\mathfrak{H}(\boldsymbol{C}^n)$ is defined by

$$p_k(f) = \sup\{|z^k f(z)|\ ;\ z \in T_k\}$$

for $k=0, 1, 2, \cdots$, where $z^k=z_1^k \cdots z_n^k$ and $T_k=\{z \in \boldsymbol{C}^n\ ;\ z=x+iy,\ |y_j| \leqq k$ for $j=1, 2, \cdots, n\}$. An element of the space $\mathfrak{H}(\boldsymbol{C}^n)$ is called entire function rapidly decreasing in any tube with compact base.

We define the Fourier transform $\mathscr{F}\varphi$ of $\varphi \in H(\boldsymbol{R}^n)$:

$$\mathscr{F}\varphi(\zeta) = \int \underset{\boldsymbol{R}^n}{\cdots} \int \exp(-i\langle x, \zeta\rangle)\varphi(x)dx_1 \cdots dx_n\ .$$

PROPOSITION 2.3. *The Fourier transformation* $\mathcal{F}$ *is a topological isomorphism of* $H(\boldsymbol{R}_x^n)$ *onto* $\mathfrak{H}(\boldsymbol{C}_\xi^n)$. *The inverse Fourier transformation is given by the following formula*:

$$\bar{\mathcal{F}} f(x)=\frac{1}{(2\pi)^n}\int\cdots\int_{R^n}\exp(i\langle x,\xi\rangle)f(\xi)d\xi_1\cdots d\xi_n\ .$$

COROLLARY 1. *The space* $\mathfrak{H}(\boldsymbol{C}^n)$ *is Fréchet nuclear and therefore reflexive.*

COROLLARY 2. *If we define the Fourier transformation* $\mathcal{F}$ *on* $\mathfrak{H}'(\boldsymbol{C}_\xi^n)$ *by the duality, the Fourier image of* $\mathfrak{H}'(\boldsymbol{C}_\xi^n)$ *coincides with the space* $H'(\boldsymbol{R}_x^n)$. *Therefore the space* $\mathfrak{H}'(\boldsymbol{C}_\xi^n)$ *of tempered ultra-hyperfunctions is not stable under the Fourier transformation.*

§ 3. The space $Q(\boldsymbol{C}^n)$

Let $Q(\boldsymbol{C}^n)$ be the space of entire functions of $f(z)$ such that

$$\sup\{\exp(k|x|)|f(z)|\ ;\ z\in T(K)\}<\infty$$

for any compact set K in $\boldsymbol{R}^n$ and any non-negative integer k, where we write $z=x+iy$. We define a fundamental system of seminorms in $Q(\boldsymbol{C}^n)$ by

$$\|f\|_k=\sup\{\exp(k|x|)|f(z)|\ ;\ z\in T_k\}\quad\text{for } k=0,1,2,\cdots.$$

Then $Q(\boldsymbol{C}^n)$ is clearly a Fréchet space and the canonical injection of $Q(\boldsymbol{C}^n)$ into $\mathfrak{H}(\boldsymbol{C}^n)$ is continuous. Hence we have the continuous injections:

$$Q(\boldsymbol{C}^n)\subset\mathfrak{H}(\boldsymbol{C}^n)\subset\mathcal{O}(\boldsymbol{C}^n)\ ,$$

where $\mathcal{O}(\boldsymbol{C}^n)$ denotes the space of entire functions endowed with the usual locally uniform convergence topology.

We are going to prove $Q(\boldsymbol{C}^n)$ is nuclear using the notion of projective limit (see [8]). Let X be a linear space, X_α be a locally convex space and $\phi_\alpha: X\to X_\alpha$ a linear mapping for any index $\alpha\in A$. The space X endowed with the weakest locally convex topology which makes all the mappings ϕ_α continuous is, by definition, the projective limit of X_α with respect to ϕ_α and will be denoted as $X=\varprojlim(X_\alpha,\phi_\alpha)$. We need the following property of the projective limit.

PROPOSITION 3.1 (Grothendieck, see [8]). *If the spaces* X_α *are nuclear and* $X=\varprojlim(X_\alpha,\phi_\alpha)$ *is Hausdorff, then* X *is nuclear.*

THEOREM 3.1. *The space* $Q(\boldsymbol{C}^n)$ *is Fréchet nuclear and therefore reflexive.*

PROOF. We have only to show that $Q(\boldsymbol{C}^n)$ is nuclear. Put

$$\Phi_{ks}(f)=\exp(\langle z,ks\rangle)f(z)$$

for $k=0, 1, 2, \cdots$ and $s=(s_1, \cdots, s_n)$, $s_j=\pm 1$. Then Φ_{ks} is a continuous linear mapping of $Q(\boldsymbol{C}^n)$ into $\mathfrak{H}(\boldsymbol{C}^n)$. In fact, if $f_\nu \to 0$ in $Q(\boldsymbol{C}^n)$, then $\exp(\langle z, (k+k')s\rangle)f_\nu(z)$ converges to 0 uniformly in the tube $T_{k+k'}$ for any non-negative integer k'. Hence $|z^{k'}\exp(\langle z, ks\rangle)f_\nu(z)| \to 0$ uniformly in $T_{k'}$, i.e. $p_{k'}(\Phi_{ks}(f_\nu)) \to 0$ as $\nu \to \infty$. As k' is arbitrary, $\Phi_{ks}(f) \to 0$ in $\mathfrak{H}(\boldsymbol{C}^n)$.

Now suppose that $\Phi_{ks}(f_\nu) \to 0$ in $\mathfrak{H}(\boldsymbol{C}^n)$ for any non-negative integer k and any s. Then $|z^k \exp(\langle z, ks\rangle)f_\nu(z)| \to 0$ uniformly in the tube T_k, which implies $|\exp(\langle z, ks\rangle)f_\nu(z)| \to 0$ uniformly in the tube T_k. Therefore $f_\nu \to 0$ in $Q(\boldsymbol{C}^n)$. We have thus established that $Q(\boldsymbol{C}^n)$ is the projective limit of the space $\mathfrak{H}(\boldsymbol{C}^n)$ with respect to the mappings Φ_{ks}:

$$Q(\boldsymbol{C}^n) = \varprojlim (\mathfrak{H}(\boldsymbol{C}^n), \Phi_{ks}) \,. \tag{1}$$

Since $\mathfrak{H}(\boldsymbol{C}^n)$ is nuclear and $Q(\boldsymbol{C}^n)$ is Hausdorff, $Q(\boldsymbol{C}^n)$ is also nuclear by Proposition 3.1. Q.E.D.

REMARK. Put

$$Q_k = \{f \in \mathcal{O}(\boldsymbol{C}^n)\,;\ \exp(\langle z, ks\rangle)f(z) \in \mathfrak{H}(\boldsymbol{C}^n) \text{ for any index } s=(s_1, \cdots, s_n), s_j=\pm 1\} \,.$$

We define a fundamental system of seminorms in Q_k by

$$p_{k's}(f) = p_{k'}(\exp(\langle z, ks\rangle)f(z))$$

for any k' and s. Then Q_k are Fréchet nuclear and the above proof shows

$$Q(\boldsymbol{C}^n) = \varprojlim Q_k \,. \tag{2}$$

As we have $Q_{k+1} \subset Q_k$ for $k=0, 1, 2, \cdots$, the projective limit (2) reduces to the intersection: $Q(\boldsymbol{C}^n) = \bigcap_{k=0}^{\infty} Q_k$.

LEMMA 3.1. *The differential operator* $D_z^p = \partial^{|p|}/\partial z_1^{p_1} \cdots \partial z_n^{p_n}$ *is a continuous linear operator of* $\mathfrak{H}(\boldsymbol{C}^n)$ *into* $\mathfrak{H}(\boldsymbol{C}^n)$.

PROOF. For simplicity we consider it in the case of single variable. No essential modification is necessary in the case of several variables. We have only to prove the continuity of d/dz in $\mathfrak{H}(\boldsymbol{C})$.

Let $f \in \mathfrak{H}(\boldsymbol{C})$. Put $f_1(z) = \dfrac{df}{dz}(z) = f'(z)$. Then we have

$$z^k f_1(z) = [z^k f(z)]' - kz^{k-1} f(z) \,.$$

By Cauchy's formula, we have

$$[z^k f(z)]' = \frac{1}{2\pi i} \int_{|w-z|=1} \frac{w^k f(w)}{(w-z)^2} dw \,.$$

Hence we have

$$\sup\{|[z^k f(z)]'|\,;\ z\in T_k\}\leqq \sup\{|z^k f(z)|\,;\ z\in T_{k+1}\}\leqq p_{k+1}(f)\ .$$

Therefore we get

$$p_k(f_1)\leqq p_{k+1}(f)+kp_k(f)\ . \qquad \text{Q.E.D.}$$

THEOREM 3.2. *The differential operator D^p is a continuous linear operator of $Q(C^n)$ into $Q(C^n)$.*

PROOF. Let $f(z)\in Q(C^n)$. Then, for any non-negative integer k and any s, we have

$$(3)\qquad \exp(\langle z, ks\rangle)\frac{\partial}{\partial z_j}f(z)=\frac{\partial}{\partial z_j}[\exp(\langle z, ks\rangle)f(z)]-ks_j\exp(\langle z, ks\rangle)f(z)\ .$$

By Lemma 3.1 and Remark, $\exp(\langle z, ks\rangle)\frac{\partial}{\partial z_j}f(z)\in \mathfrak{H}(C^n)$ for any k and s, which implies $\frac{\partial}{\partial z_j}f\in Q(C^n)$. Thanks to (1), the continuity of the operator $\frac{\partial}{\partial z_j}$ in $Q(C^n)$ results from Lemma 3.1 and the formula (3). Iterating this procedure, we can establish the continuity of D^p in the space $Q(C^n)$. Q.E.D.

Identify R^n_x with the real part of $C^n_z: R^n_x=R^n_x\times i0\subset C^n_z$, $z=x+iy$. The restriction mapping of $Q(C^n_z)$ into $H(R^n_x)$ being injective, we consider the space $Q(C^n_z)$ as a subspace of $H(R^n_x)$ by this injection.

THEOREM 3.3. *The space $Q(C^n_z)$ is dense in the space $H(R^n_x)$.*

PROOF. As the space $\mathscr{D}(R^n_x)$ is dense in $H(R^n_x)$ by Proposition 2.2., we have only to show that the elements of $\mathscr{D}(R^n_x)$ can be approximated by those of $Q(C^n_z)$ in the topology of $H(R^n_x)$. Put, for $\varepsilon>0$,

$$g_\varepsilon(x)=(2\pi)^{-n/2}\varepsilon^{-n}\exp(-(x_1^2+\cdots+x_n^2)/2\varepsilon^2)\ .$$

Then $g_\varepsilon(z)\in Q(C^n_z)$. Let $\varphi\in\mathscr{D}(R^n_x)$. As φ has compact support, the convolutions

$$g_\varepsilon*\varphi(z)=\int\cdots\int_{R^n} g_\varepsilon(z-u)\varphi(u)du_1\cdots du_n$$

are entire functions and

$$\sup\{\exp(k|x|)|g_\varepsilon*\varphi(z)|\,;\ z\in T_k\}<\infty$$

for any non-negative integer k. This implies $g_\varepsilon*\varphi(z)\in Q(C^n_z)$ for any $\varepsilon>0$.

On the other hand,

$$g_\varepsilon*\varphi(x)-\varphi(x)=\int\cdots\int_{R^n} g_\varepsilon(u)[\varphi(x-u)-\varphi(x)]du_1\cdots du_n\ .$$

Hence, by a routine argument we can show, for any k,

$$\sup\{\exp(k|x|)|g_\varepsilon*\varphi(x)-\varphi(x)|\,;\ x\in R^n\}\to 0$$

as $\varepsilon \to 0$. Since we have

$$\exp(k|x|)D_x^p(g_\varepsilon * \varphi(x) - \varphi(x)) = \exp(k|x|)(g_\varepsilon * (D^p\varphi)(x) - (D^p\varphi)(x)),$$

$g_\varepsilon * \varphi \to \varphi$ in the topology of $H(\boldsymbol{R}_x^n)$ as $\varepsilon \to 0$. Q.E.D.

Restricting the Fourier transformation $\mathscr{F}: H(\boldsymbol{R}_x^n) \to \mathfrak{H}(\boldsymbol{C}_\zeta^n)$ on $Q(\boldsymbol{C}_z^n)$, we define the Fourier transformation on $Q(\boldsymbol{C}_z^n)$.

THEOREM 3.4. *The Fourier transformation $\mathscr{F}$ is a topological isomorphism of $Q(\boldsymbol{C}_z^n)$ onto $Q(\boldsymbol{C}_\zeta^n)$.*

PROOF. Let $f(z) \in Q(\boldsymbol{C}_z^n)$. For any non-negative integer k and any index s, we put $f_{ks}(x) = f(x + iks)$. Then $f_{ks} \in H(\boldsymbol{R}_x^n)$ and $\mathscr{F} f_{ks} \in \mathfrak{H}(\boldsymbol{C}_\zeta^n)$ by Proposition 2.3. As we have

$$(4) \qquad \begin{aligned} \mathscr{F} f_{ks}(\zeta) &= \int \cdots \int_{\boldsymbol{R}^n} \exp(-i\langle x, \zeta\rangle) f(x + iks)\, dx_1 \cdots dx_n \\ &= \int \cdots \int_{\boldsymbol{R}^n + iks} \exp(-i\langle z - iks, \zeta\rangle) f(z)\, dz_1 \cdots dz_n \\ &= \exp(-\langle ks, \zeta\rangle) \int \cdots \int_{\boldsymbol{R}^n + iks} \exp(-i\langle z, \zeta\rangle) f(z)\, dz_1 \cdots dz_n \\ &= \exp(-\langle ks, \zeta\rangle) \mathscr{F} f(\zeta), \end{aligned}$$

the Fourier transform $\mathscr{F} f$ of f belongs to $Q(\boldsymbol{C}_\zeta^n)$ by Remark. Thanks to (1), the continuity of the mapping $\mathscr{F}: Q(\boldsymbol{C}_z^n) \to Q(\boldsymbol{C}_\zeta^n)$ results from the continuity of $\mathscr{F}: H(\boldsymbol{R}_x^n) \to \mathfrak{H}(\boldsymbol{C}_\zeta^n)$ and the formula (4).

As the space $Q(\boldsymbol{C}_z^n)$ is stable under the transformation $f(z) \to f(-z)$, the inverse Fourier transformation $\bar{\mathscr{F}}$ maps the space $Q(\boldsymbol{C}_\zeta^n)$ into $Q(\boldsymbol{C}_z^n)$. Therefore the inverse mapping of $\mathscr{F}: Q(\boldsymbol{C}_z^n) \to Q(\boldsymbol{C}_\zeta^n)$ is given by restricting the inverse Fourier transformation $\bar{\mathscr{F}}$ of $\mathfrak{H}(\boldsymbol{C}_\zeta^n)$ onto $H(\boldsymbol{R}_x^n)$. Q.E.D.

Theorems 3.3 and 3.4 give the following corollaries.

COROLLARY 1. *The space $Q(\boldsymbol{C}^n)$ is dense in the space $\mathfrak{H}(\boldsymbol{C}^n)$.*

COROLLARY 2. *The space of tempered ultra-hyperfunctions on $\boldsymbol{R}^n$ is a subspace of the space of Fourier ultra-hyperfunctions on $\boldsymbol{R}^n$, namely, $\mathfrak{H}'(\boldsymbol{C}^n) \subset Q'(\boldsymbol{C}^n)$.*

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