Characterization of families of finite permutation groups by the subdegrees. I

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1. Introduction.

Various families of finite permutation groups of rank 3 have been characterized by the subdegrees in [5], [6] and [3]. In a series of papers, we shall consider similar problems for some families of finite permutation groups not necessarily of rank 3.

THEOREM 1. *Let* T_r^t *be the set of all unordered* r*-tuples on* t *symbols with the following graph structure: two* r*-tuples are adjacent if and only if they have* $r-1$ *symbols in common. Suppose* $t>2r$. *Then the full automorphism group of* T_r^t *is isomorphic to* Σ_t, *the symmetric group of degree* t, *and the action is primitive of rank* $r+1$ *with subdegrees* 1 *and* $\binom{r}{i}\binom{t-r}{i}$, $1\leqq i\leqq r$.

This theorem is probably well-known, but we give the proof in section 3 for the sake of convenience. We prove the following main result in section 4.

THEOREM 2. *Let* (G, Ω) *be a primitive permutation group of rank* r_0+1 $(r_0\geqq 3)$ *with subdegrees* l_i, $0\leqq i\leqq r_0$. *Suppose there exist some integers* r *and* t *such that* $l_0=1$, $l_1=r(t-r)$, $l_2=\binom{r}{2}\binom{t-r}{2}$ *and* $l_i\geqq\binom{r}{3}\binom{t-r}{3}$ *for* $3\leqq i\leqq r_0$. *Then the graph structure of* Ω *defined by the orbital of length* l_1 *is isomorphic to* T_r^t *for* $r\geqq 6$ *and* $t>2r(r-1)+4$. *Consequently,* $|\Omega|=\binom{t}{r}$ *and* G *is isomorphic to some* $(r+1)$*-fold transitive subgroup of* Σ_t. *Furthermore, the followings are equivalent:*

(i) $r_0=r$,

(ii) (G, Ω) *is of maximal diameter,*

(iii) G *is isomorphic to some* $2r$*-fold transitive subgroup of* Σ_t.

The proof rests on T. A. Dowling's characterization of the graph T_r^t (see Lemma 4 below).

REMARK: Theorem II of Higman [5] corresponds to the case $r=2$. We have not completed the characterization for $3\leqq r\leqq 5$.

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2. Preliminaries.

The terminology and notation of Higman [4] for finite permutation groups are used throughout.

Graphs considered in this paper are finite undirected graphs without loops. For a vertex x of a graph Ω, $\Delta(x)$ denotes the set of vertices adjacent to x, and put $\Delta^*(x)=\{x\}\cup\Delta(x)$. A subset K of Ω is called a clique (resp. an internally stable set) if any two vertices of K are adjacent (resp. non-adjacent), and $|K|$, the number of vertices in K, is called the order of K. A graph Ω is regular of valence k if $|\Delta(x)|=k$ for every vertex x of Ω, and a regular graph Ω is called edge-regular of edge-degree λ if $|\Delta(x)\cap\Delta(y)|=\lambda$ for $y\in\Delta(x)$. The distance $d(x, y)$ of two vertices x and y is defined naturally.

LEMMA 1. *Suppose $\Delta(x)\cap\Delta(y)$ is internally stable when $d(x, y)=2$, $x, y\in\Omega$. Then $\Delta(x)$ can be expressed as a disjoint union of cliques such that there exists no edge joining different cliques. Furthermore, if Ω is regular of valence k and edge-regular of edge-degree λ, then $\Delta(x)$ is the disjoint union of cliques of order $\lambda+1$. Therefore, $\lambda+1$ divides k.*

PROOF. We show that $\Delta(x)\cap\Delta(y)$ is a clique when $y\in\Delta(x)$. By way of contradiction, suppose two vertices z and w in $\Delta(x)\cap\Delta(y)$ are not adjacent. Then we have $d(z, w)=2$ and $\Delta(z)\cap\Delta(w)$ must be internally stable by assumption. On the other hand, two adjacent vertices x and y are contained in $\Delta(z)\cap\Delta(w)$, which is a contradiction.

LEMMA 2. *Let Ω be regular of valence k and edge-regular of edge-degree λ. Suppose $\Delta(x)\cap\Delta(y)$ consists of two adjacent vertices when $d(x, y)=2$. Then $\lambda+1-\dfrac{k-1}{\lambda}$ is a positive integer. Therefore λ divides $k-1$ and $k\leqq\lambda^2+1$.*

PROOF. First, apply Lemma 1 to the subgraph $\Delta(x)$ and we know that $\Delta(x)\cap\Delta(y)$ is the disjoint union of cliques for $y\in\Delta(x)$. Suppose $\Delta(x)\cap\Delta(y)=\bigcup_{i=1}^{m}K_i$ and there exists no edge joining the cliques K_i and K_j for $i\neq j$. Put $|K_i|=s_i$, then we have $\lambda=\sum_{i=1}^{m}s_i$. Next, put $T=\{w\in\Delta(x)\,|\,d_0(y, w)=2\}$, where d_0 means the distance in $\Delta(x)$. Then $\Delta(x)\cap\Delta(y)\cap\Delta(w)$ consists of a single vertex for $w\in T$. On the other hand, $|\Delta(z)\cap T|=\lambda-s_i$ for $z\in K_i$. Therefore, $|T|=k-\lambda-1=\sum_{i=1}^{m}s_i(\lambda-s_i)=\lambda^2-\sum_{i=1}^{m}s_i^2$.

Choose the representatives $z_i\in K_i$, and put $T_i=\Delta(z_i)\cap T$. For $w\in T_i$, $\Delta(x)\cap\Delta(w)\cap\Delta(z_j)$ consists of single vertex for $i\neq j$ by assumption, and it must be in T_j. It is immediate that this correspondence is a bijection between T_i and T_j, and therefore

$$|T_i|=\lambda-s_i=|T_j|=\lambda-s_j\ .$$

This implies $s_i = s_j$ for all i, j, and we shall write s for this common value. Then we have $\lambda = ms$, $k = 1+\lambda+\lambda^2 - ms^2 = 1+\lambda+\lambda^2-\lambda s$, and therefore $s = \lambda+1-\dfrac{k-1}{\lambda}$.

LEMMA 3. *Let Ω be regular of valence k and edge-regular of edge-degree λ.*

(i) *If there exists an internally stable set of order s in $\Delta(x)$, then*

$$k \geqq s(\lambda+1) - \frac{s(s-1)}{2}(\mu-1),$$

where μ is the maximum number of $|\Delta(y)\cap\Delta(z)|$ for y, $z\in\Delta(x)$ and $d(y, z)=2$.

(ii) *If $k > s(\lambda+1)$, then any internally stable set of order smaller than $s+1$ in $\Delta(x)$ can be extended to an internally stable set of order $s+1$ in $\Delta(x)$.*

(iii) *Suppose $\{y_1, \cdots, y_s\}$ is an internally stable set in $\Delta(x)$. Suppose furthermore that there exists no internally stable set of order $s+1$ in $\Delta(x)$. Then $\Delta(x) - \bigcup_{i=1}^{s-1}\Delta^*(y_i)$ is a clique containing y_s. Especially, there exists a clique of order $k-(s-1)(\lambda+1)$ in $\Delta(x)$.*

PROOF. (i) This is equivalent to Theorem (2.2.1) of [1].

(ii) This is equivalent to Theorem (2.2.2) of [1].

(iii) This is equivalent to Lemma (2.3.1) of [1].

3. Proof of Theorem 1.

The action of Σ_t on T_r^t is defined naturally:

$$\{a_1, \cdots, a_r\}^\sigma = \{a_1^\sigma, \cdots, a_r^\sigma\} \text{ for } \sigma\in\Sigma_t .$$

Then the transitivity of the action is obvious and the primitivity follows from the maximality of the stabilizer of an r-tuple, which is isomorphic to $\Sigma_r\times\Sigma_{t-r}$, when $t>2r$. The distance $d(\boldsymbol{a}, \boldsymbol{b})$ of two r-tuples $\boldsymbol{a}=\{a_1, \cdots, a_r\}$ and $\boldsymbol{b}=\{b_1, \cdots, b_r\}$ is equal to $r-|\{a_1, \cdots, a_r\}\cap\{b_1, \cdots, b_r\}|$. Hence the diameter of T_r^t is equal to r. Now set

$$\Gamma_i(\boldsymbol{a}) = \{\boldsymbol{b}\in T_r^t \mid d(\boldsymbol{a}, \boldsymbol{b}) = i\} \text{ for } 0\leqq i\leqq r .$$

It is easily seen that the stabilizer of $\boldsymbol{a}$ in Σ_t is transitive on $\Gamma_i(\boldsymbol{a})$. Hence the subdegrees are $|\Gamma_i(\boldsymbol{a})| = \binom{r}{i}\binom{t-r}{i}$.

For any $(r-1)$-tuple $\boldsymbol{x}=\{x_1, \cdots, x_{r-1}\}$, set

$$K_{\boldsymbol{x}} = \{\boldsymbol{a}\in T_r^t \mid \boldsymbol{a}\supseteq\boldsymbol{x}\}, \text{ and } S=\{K_{\boldsymbol{x}} \mid \boldsymbol{x}\in T_{r-1}^t\} .$$

Two maximal cliques $K_{\boldsymbol{x}}$ and $K_{\boldsymbol{y}}$ are defined to be adjacent if and only if $K_{\boldsymbol{x}}\cap K_{\boldsymbol{y}} \neq\emptyset$, or $\boldsymbol{x}$ and $\boldsymbol{y}$ are adjacent in T_{r-1}^t. Hence the graph S is isomorphic to T_{r-1}^t. S coincides with the set of all maximal cliques of order $t-r+1$ in T_r^t (see [2]).

Hence any automorphism of T_r^t induces an automorphism of S, and therefore it belongs to Σ_t by induction. This completes the proof of Theorem 1.

4. Proof of Theorem 2.

Let Γ_i $(0 \leq i \leq r_0)$ be the orbital of (G, Ω) of length l_i. We shall only consider the intersection numbers relative to Γ_1, and put

$$\mu_{ij} = |\Gamma_1(y) \cap \Gamma_i(x)| \text{ for } y \in \Gamma_j(x),$$

and define the graph structure in Ω using $\Delta = \Gamma_1$.

LEMMA 4. *The graph Ω is isomorphic to T_r^t if the intersection matrix M takes the following form:*

$$M = (\mu_{ij}) = \begin{bmatrix} 0 & 1 & 0 & 0 \cdots 0 \\ r(t-r) & t-2 & 4 & 0 \cdots 0 \\ 0 & * & & \\ 0 & 0 & & \\ \vdots & \vdots & & * \\ 0 & 0 & & \end{bmatrix}.$$

PROOF. We verify the conditions (a_1)-(a_3) of [2], replacing n, m and G_m^n with t, r and T_r^t respectively. First, Ω is a regular graph of valence $l_1 = r(t-r)$, thus (a_1) is satisfied. Similarly, (a_2) is satisfied since Ω is edge-regular of edge-degree $\mu_{11} = t-2$. Finally, $\mu_{1i} \leq 4$ $(i \geq 2)$ implies (a_3). Furthermore, Ω is connected by the primitivity of (G, Ω). Hence Ω is isomorphic to T_r^t by Theorem 1 of [2].

We remark that $2r(r-1)+4 \geq r+58$ when $r \geq 6$. Therefore the assumption implies $t > r+58$. Now, we know that $\mu_{12} \leq 4$, since

$$\mu_{12} = \frac{\mu_{21} l_1}{l_2} < \frac{l_1^2}{l_2} = \frac{4r(t-r)}{(r-1)(t-r-1)} = 4\left(1+\frac{1}{r-1}\right)\left(1+\frac{1}{t-r-1}\right)$$
$$\leq 4 \cdot \frac{6}{5} \cdot \frac{59}{58} < 5\,.$$

Similarly, we have $\mu_{1i} = 0 = \mu_{i1}$ for $3 \leq i \leq r_0$, since

$$\mu_{1i} = \frac{\mu_{i1} l_1}{l_i} < \frac{l_1^2}{\binom{r}{3}\binom{t-r}{3}} = \frac{36\left(1+\frac{1}{r-1}\right)\left(1+\frac{1}{t-r-1}\right)}{(r-2)(t-r-2)} < \frac{36 \cdot 2 \cdot 2}{4 \cdot 56} < 1\,.$$

$\mu_{12} = 0$ contradicts the primitivity of (G, Ω).

If $\mu_{12} = 1$, then $\mu_{21} = \mu_{12} l_2 / l_1 = (r-1)(t-r-1)/4$, and $\mu_{11} = l_1 - 1 - \mu_{21} = (3rt - 3r^2 + t$

$-5)/4$. Then $\mu_{11}+1$ divides l_1 by Lemma 1, and therefore $3r(t-r)+t-1$ divides $r(t-r)-(t-1)$, but this is impossible, since

$$3r(t-r)+t-1>r(t-r)-(t-1)>0 .$$

If $\mu_{12}=2$, then $\mu_{21}=(r-1)(t-r-1)/2$ and $\mu_{11}=(rt-r^2+t-3)/2$. By Lemmas 1 and 2, $\mu_{11}+1$ divides l_1 or μ_{11} divides l_1-1. If $\mu_{11}+1$ divides l_1, then $r(t-r)+t-1$ divides $2(t-1)$, but this is impossible, since

$$r(t-r)+t-1>2(t-1)>0 .$$

Similarly, if μ_{11} divides l_1-1, then $rt-r^2+t-3$ divides $2t-4$, but this is impossible, since

$$rt-r^2+t-3>2t-4>0 .$$

If $\mu_{12}=3$, then $\mu_{21}=3(r-1)(t-r-1)/4$ and $\mu_{11}=(rt-r^2+3t-7)/4$. Now, we shall show that 3 is the maximum order of internally stable sets in $\Delta(x)=\Gamma_1(x)$. If there exists an internally stable set of order 4 in $\Delta(x)$, then $l_1\geqq 4(\mu_{11}+1)-6(\mu_{12}-1)$ by Lemma 3 (i). This implies $r(t-r)\geqq(rt-r^2+3t-3)-12$, and therefore $3t-15\leqq 0$, which contradicts the assumption. If there exists no internally stable set of order 3 in $\Delta(x)$, then $l_1\leqq 2(\mu_{11}+1)$ by Lemma 3 (ii). This implies

$$r(t-r)\leqq\frac{rt-r^2+3t-3}{2} ,$$

and therefore

$$t\leqq r+3+\frac{6}{r-3}\leqq r+5 ,$$

which contradicts the assumption.

Hence there exists an internally stable set of order 3, but no internally stable set of order 4 in $\Delta(x)$. Therefore $l_1\leqq 3(\mu_{11}+1)$ by Lemma 3 (ii), and then $(r-9)t\leqq r^2-9$. If $r\geqq 11$, then

$$t\leqq\frac{r^2-9}{r-9}=r+9+\frac{72}{r-9}\leqq r+45 ,$$

which contradicts the assumption. If $r=10$, then $t\leqq\frac{r^2-9}{r-9}=91$, but this is not the case, since $t>2r(r-1)+4=184$ when $r=10$. We shall consider the case $r\leqq 9$ in the following.

By Lemma 3 (iii), there exists a clique of order u in $\Delta(x)$, where $u=l_1-2(\mu_{11}+1)=(rt-r^2-3t+3)/2$. Now, we shall show that for any pair $\{x, y\}$ of adjacent vertices there exists a unique maximal clique of order greater than u containing x

and y. We have already proved the existence, and we shall show the uniqueness. By way of contradiction, suppose x and y are contained in two maximal cliques K, L of order greater than u. If $|K\cap L|\geqq 4$, then $|\Delta(z)\cap\Delta(w)|\geqq 4$ for $z\in K-K\cap L$ and $w\in L-K\cap L$. Therefore z and w are adjacent, since $|\Delta(z)\cap\Delta(w)|\leqq\mu_{12}=3$ if z and w are not adjacent. Then $K\cup L$ is a clique, and this contradicts the maximality of K and L. Hence $|K\cap L|\leqq 3$, and therefore

$$\mu_{11}=|\Delta(x)\cap\Delta(y)|\geqq|K\cup L|-2=|K|+|L|-|K\cap L|-2\geqq 2u-3 .$$

This implies

$$\frac{rt-r^2+3t-7}{4}\geqq rt-r^2-3t ,$$

and therefore

$$3(r-5)t\leqq 3r^2-7 .$$

Since we have assumed $r\geqq 6$, this implies

$$t\leqq\frac{3r^2-7}{3(r-5)}=r+5+\frac{68}{3(r-5)}<r+28 ,$$

which contradicts the assumption.

Let v be the order of the maximal cliques of order greater than u. (v is well-defined by the edge-transitivity of G.) Then $\Delta(x)$ is divided into m cliques $K_1,\cdots,K_m$ of order $v-1$. Clearly, $u\leqq v-1=l_1/m$. If $m\geqq 4$, then $l_1\geqq 4u$, and therefore

$$r(t-r)\geqq 2(rt-r^2-3t+3) ,$$

thus

$$(r-6)t\leqq r^2-6 .$$

This is impossible when $r>6$. Similarly, it is easily shown that $m\leqq 4$ when $r=6$. On the other hand, there exists an internally stable set of order 3 in $\Delta(x)$, hence $m\geqq 3$. Suppose $y\in K_i$ and $|\Delta(y)\cap K_j|\geqq 3$ for some $j\neq i$. Then for any vertex z in $K_j-\Delta(y)\cap K_j$, we have $|\Delta(y)\cap\Delta(z)|\geqq 4$, since

$$\Delta(y)\cap\Delta(z)\supset(\Delta(y)\cap K_j)\cup\{x\} .$$

Hence y and z are adjacent, and therefore $\{x, y\}\cup K_j$ is a clique, which contradicts the maximality of $\{x\}\cup K_j$. Therefore

$$|\Delta(y)\cap K_j|\leqq 2 \text{ for } y\in K_i,\ i\neq j.$$

Then

$$\mu_{11}=|\Delta(x)\cap\Delta(y)|\leqq v-2+2(m-1) .$$

If $m=3$, then $v-1=l_1/3$. Therefore

$$\frac{rt-r^2+3t-7}{4} \leqq \frac{r(t-r)}{3}-1+4 ,$$

and then

$$(r-9)t \geqq r^2-57 .$$

This is impossible, since we have assumed $r \leqq 9$. If $m=4$, then $v-1=l_1/4$. Therefore

$$\frac{rt-r^2+3t-7}{4} \leqq \frac{r(t-r)}{4}-1+6 ,$$

and then $3t \leqq 27$, which contradicts the assumption.

Therefore the only possibility is $\mu_{12}=4$, and then $\mu_{21}=(r-1)(t-r-1)$ and $\mu_{11}=t-2$. Hence the graph Ω is isomorphic to T_r^t by Lemma 4. The rest of the theorem follows immediately from [7, Theorem 2 (b)].

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