Primitive extensions of rank 4 of multiply transitive permutation groups, II.

(Part II. The case where there exist non-self-paired orbits)

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Introduction.

This note is a continuation of [1] and [2]. Here we consider the primitive extensions of rank 4 with non-self-paired orbits of multiply (4-ply) transitive permutation groups. Our main result (which is of negative nature) is as follows:

THEOREM 1. *There exists no primitive extension of rank 4 $(\mathfrak{G}, \Omega)$ of a 4-ply transitive permutation group (G, Δ) having non-self-paired orbits of $\mathfrak{G}_a$ $(a \in \Omega)$.*

Combining this theorem with Theorem 1 given in [2], we are able to determine the primitive extensions of rank 4 of 5-ply transitive permutation groups. That is, we have

THEOREM 2. *Let (G, Δ) be a 5-ply transitive permutation group. If (G, Δ) has a primitive extension of rank 4, then $|\Delta|=7$ and $G \cong S_7$ or A_7 (symmetric and alternating groups on 7 letters respectively.)*[1]

The special case of Theorem 2 where $(G, \Delta) \cong (A_n, \Delta)$, $|\Delta|=n$ has been proved by S. Iwasaki [7].

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§1. Notation and preliminary results.

We fix the following notation throughout this note. $(\mathfrak{G}, \Omega)$ is a primitive extension of rank 4 of a 4-ply transitive permutation group (G, Δ). That is, $\mathfrak{G}$ is primitive on Ω, and there exist 4 orbits $\{a\}$, $\Delta(a)$, $\Gamma(a)$ and $\Lambda(a)$ of $\mathfrak{G}_a (a \in \Omega)$ on Ω, and moreover $\mathfrak{G}_a$ is faithful on an orbit $\Delta(a)$ and $(\mathfrak{G}_a, \Delta(a))$ is identified with the permutation group (G, Δ). Henceforth we assume that there exist non-self-paired orbits of $\mathfrak{G}_a$. Here we take the orbits so that $\Delta(a)^g = \Delta(a^g)$, $\Gamma(a)^g = \Gamma(a^g)$ and $\Lambda(a)^g = \Lambda(a^g)$ for all $g \in \mathfrak{G}$ and $a \in \Omega$.

Now under the assumption of Theorem 1 that such $(\mathfrak{G}, \Omega)$ exists, we have

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1) In these cases each (G, Δ) has indeed primitive extensions of rank 4 $(\mathfrak{G}, \Omega)$ with a regular normal subgroup of order 64.

LEMMA 1. *$\Delta(a)$ is not a self-paired orbit.*

PROOF. Since $(\mathfrak{G}_a, \Delta(a))$ is doubly transitive, if $\Delta(a)$ is a self-paired orbit of $\mathfrak{G}_a$, then $(\Delta\circ\Delta)(a)=\Gamma(a)$ or $\Lambda(a)$ and moreover $\Gamma(a)$ or $\Lambda(a)$ is self-paired by Theorem 1 in P. J. Cameron [3], a contradiction. Here $(\Delta\circ\Delta)(a)=\{b \mid b\in\Delta(c)$ for some $c\in\Delta(a)$, $a\neq b\}$ by definition (cf. [3]).

Thus from now on we may assume that $\Delta(a)$ and $\Lambda(a)$ are paired orbits and $\Gamma(a)$ is a self-paired orbit.

Let us recall some fundamental properties about the intersection matrices due to D. G. Higman [4]. Set $\Gamma_0(a)=\{a\}$, $\Gamma_1(a)=\Delta(a)$, $\Gamma_2(a)=\Gamma(a)$ and $\Gamma_3(a)=\Lambda(a)$ and 1, k, l, m $(=k)$ be the length of the orbit $\Gamma_i(a)$ $(i=0,1,2,3)$ respectively. Let us define the intersection number $\mu_{ij}^{(\alpha)}$ by $\mu_{ij}^{(\alpha)}=|\Gamma_\alpha(b)\cap\Gamma_i(a)|$ for $b\in\Gamma_j(a)$. For the fundamental relations among the $\mu_{ij}^{(\alpha)}$ and the lengths of $\Gamma_i(a)$, see D. G. Higman [4], (4.1) and (4.2). Among them, the following relations will be used later in this note: (Here we use the conventional notation that $\mu_{ij}=\mu_{ij}^{(1)}$.)

$\mu_{11}+\mu_{21}+\mu_{31}=\mu_{12}+\mu_{22}+\mu_{32}=1+\mu_{13}+\mu_{23}+\mu_{33}=k$, $k\mu_{23}=l\mu_{12}$, $k\mu_{21}=l\mu_{32}$, and moreover (since $m=k$) $\mu_{11}=\mu_{13}=\mu_{33}$.

We use the following results proved in S. Iwasaki [6] and N. Ito [5].

LEMMA 2 ([6], Proposition 1.3). $l\neq k(k-1)$.

LEMMA 3 ([5], Satz 3). *Let (H, Σ) be a triply transitive permutation group, and let X be a subgroup of H of index $|\Sigma|$, then either X is transitive on Σ or $X=H_a$ for some $a\in\Sigma$.*

§ 2. Proof of Theorem 1.

We assume that there exists a $(\mathfrak{G}, \Omega)$ which satisfies the assumption of Theorem 1. Thus we may assume that $k>4$.

Since $\Delta(a)$ and $\Lambda(a)$ are paired orbits, there exists an element $x\in\mathfrak{G}$ such that $b^x=a$, $a^x=c$ for some $b\in\Delta(a)$ and $c\in\Lambda(a)$. We denote by σ the inner automorphism of $\mathfrak{G}$ induced by the element x. From the doubly (4-ply) transitivity of $\mathfrak{G}_a$ on $\Delta(a)$, we have $\mu_{11}=0$, therefore $\mu_{13}=\mu_{33}=0$ and $\mu_{23}=k-1$. Thus we have $\mu_{21}^{(3)}=k-1$ since $\mu_{11}^{(3)}=\mu_{31}^{(3)}=0$ and $1+\mu_{11}^{(3)}+\mu_{21}^{(3)}+\mu_{31}^{(3)}=k$. Since $\mu_{21}^{(3)}=|(\Delta(a)\cap\{b\})^x\cap\Gamma(a)|\neq 0$, there exist points $e\in\Delta(a)-\{b\}$ and $d\in\Gamma(a)$ such that $e^x=d$, hence $(\mathfrak{G}_{a,b,e})^\sigma\leqq\mathfrak{G}_{a,d}$. Since $(\mathfrak{G}_{a,b})^\sigma$ is a subgroup of index k of the 4-ply transitive permutation group $(\mathfrak{G}_a, \Delta(a))$, we have either

(I) $(\mathfrak{G}_{a,b})^\sigma$ is transitive on $\Delta(a)$, or

(II) $(\mathfrak{G}_{a,b})^\sigma=\mathfrak{G}_{a,b'}$ for some $b'\in\Delta(a)$,

by Lemma 3. While $(\mathfrak{G}_{a,b,e})^\sigma$ is a subgroup of index $k-1$ of the group $(\mathfrak{G}_{a,b})^\sigma$.

Therefore if Case (I) holds, then $(\mathfrak{G}_{a,b,c})^{\sigma}$ is transitive on $\Delta(a)$ by Theorem 17.3 in Wielandt [8]. If Case (II) holds, then either $(\mathfrak{G}_{a,b,c})^{\sigma}$ is transitive on $\Delta(a)-\{b'\}$ or $(\mathfrak{G}_{a,b,c})^{\sigma}=\mathfrak{G}_{a,b',e'}$ for some $e'\in\Delta(a)$, $e'\neq b'$, also by Lemma 3. Thus we have either $\mu_{12}=0$, 1, 2, $k-2$, $k-1$ or k, since $\mu_{12}=|\Delta(d)\cap\Delta(a)|$ and $\Delta(d)\cap\Delta(a)$ is a union of orbits of $\mathfrak{G}_{a,d}$ on $\Delta(a)$. The case $\mu_{12}=0$ is impossible since $\mu_{12}=\frac{k}{l}\mu_{23}=\frac{k(k-1)}{l}\neq 0$. If $\mu_{12}=k$ or $k-1$, then $l=k-1$ or k, respectively, and this contradicts a theorem of Manning (cf. [3], Theorem 1). Moreover $\mu_{12}\neq k-2$ for $k>4$, otherwise $l=\frac{k(k-1)}{k-2}$ is not an integer.

Thus we have proved that $\mu_{12}=1$ or $\mu_{12}=2$. If $\mu_{12}=1$ then $l=k(k-1)$ and this contradicts Lemma 2. Thus $\mu_{12}=2$. Now from the proof of Proposition 1 we immediately have $\mathfrak{G}_{a,d}=\mathfrak{G}_{a,\{b',e'\}}$. Thus the action of $\mathfrak{G}_a$ on $\Gamma(a)$ is isomorphic to that of $\mathfrak{G}_a$ on the set of unordered pairs of $\Delta(a)$. From the 4-ple transitivity of $(\mathfrak{G}_a, \Delta(a))$, we immediately conclude that $(\mathfrak{G}_{a,d}, \Gamma(a))$ is of rank 3 and the subdegrees are 1, $2(k-2)$ and $\frac{1}{2}(k-2)(k-3)$.

Now $\Gamma(a)\cap\Delta(d)$ is a union of orbits of $\mathfrak{G}_{a,d}$ on $\Gamma(a)$ whose total length is μ_{22}. While $|\Delta(a)\cap\Gamma(d)|=|\Gamma(a)\cap\Delta(d)|=\mu_{22}$, since there exists an element $y\in\mathfrak{G}$ which interchanges a and d (because $\Gamma(a)$ is self-paired). Now we have $\mu_{22}\geqq 1$, otherwise $\mu_{32}=k-2$ and $\mu_{21}=\frac{l}{m}\mu_{32}=(k-1)(k-2)>k$ (for $k\geqq 4$), a contradiction. While $\mu_{22}\leqq |\Delta(a)|-|\Delta(a)\cap\Delta(d)|=k-\mu_{12}=k-2$. Thus, if $k\geqq 6$ we have a contradiction, since $2(k-2)>k-2$ and $\frac{1}{2}(k-2)(k-3)>k-2$ and $\mathfrak{G}_{a,d}$ has no union of orbits $(\neq\{d\})$ whose total length μ_{22} is $1\leqq\mu_{22}\leqq k-2$. If $k=5$ (resp. $k=4$), then $(G,\Delta)\cong(S_k,\Delta)$, $|\Delta|=k$. Since S_5 (resp. S_4) has only one subgroup of index 5 (resp. 4) up to conjugacy, $(G,\Delta(a))\cong(G,\Lambda(a))$. Thus a transposition $\tau\in S_5$ $(=\mathfrak{G}_a)$ (resp. S_4) fixes $1+3+3+4=11$ (resp. $1+2+2+2=7$) points on Ω. While non-transpositional involutive elements of S_5 (resp. S_4) fix $1+1+1+2=5$ (resp. $1+0+0+2=3$) points. Thus the number of elements of $\mathfrak{G}$ which are conjugate to τ is given by $10\cdot\frac{21}{11}\left(\text{resp. } 6\cdot\frac{15}{7}\right)$[2], and this is a contradiction, since this number must be an integer.

Thus we have completed the proof of Theorem 1.

2) For this calculation, cf. [2 bis]. [2 bis] contains serious misprints: the words "not" in page 131, line 10; page 132, line 1 and line 17 should be omitted.

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