Some remarks on the Kirby-Siebenmann class

By Shigeyuki MORITA

§ 1. Statement of results.

Let $k \in H^4(B\,\mathrm{Top}\,; \boldsymbol{Z}_2)$ be the Kirby-Siebenmann class, *i.e.* the unique obstruction to stable PL reducibility of Top bundles. In this note, we show some elementary properties of k and using them we show the existence of certain non-triangulable manifolds.

First we show

PROPOSITION 1. *k is primitive, i.e. let $\mu : B\,\mathrm{Top} \times B\,\mathrm{Top} \to B\,\mathrm{Top}$ be the natural H-space structure on $B\,\mathrm{Top}$ defined by the Whitney sum. Then*

$$\mu^*(k) = k \times 1 + 1 \times k\,.$$

For a topological manifold M, we define the Kirby-Siebenmann class of M, $k(M)$, to be that of the tangent micro-bundle of M. Then we have

COROLLARY 2.

(i) *$k(M) = k(\nu(M))$, where $\nu(M)$ is the stable normal bundle of M.*

(ii) *$k(M \times N) = k(M) \times 1 + 1 \times k(N)$.*

Thus, if M and N are closed topological manifolds of dimension greater than 4, then $M \times N$ is triangulable as a PL manifold if and only if so are M and N. (Of course, this is a well known fact.)

Next we consider the following commutative diagram

$$\begin{array}{ccccccc}
 & \nearrow^{i'} & G/\mathrm{PL} & \xrightarrow{p'} & G/\mathrm{Top} & \searrow^{m} & \\
\mathrm{Top}/\mathrm{PL} & & \downarrow j & & \downarrow j' & & B(\mathrm{Top}/\mathrm{PL}) = K(\boldsymbol{Z}_2, 4) \\
 & \searrow_{i} & B\,\mathrm{PL} & \xrightarrow[p]{} & B\,\mathrm{Top} & \nearrow_{k} &
\end{array}$$

and we show

PROPOSITION 3. $m = k_2^2 + x \bmod 2 \in H^4(G/\mathrm{Top}\,; \boldsymbol{Z}_2)$, *where k_2 is the first Kervaire obstruction and $x \bmod 2$ is the reduction mod 2 of the fundamental class of* $K(\boldsymbol{Z}_{(2)}, 4)$. (Recall that G/Top localized at $2 = \prod_{i \geqq 0} K(\boldsymbol{Z}_2, 4i+2) \times \prod_{i \geqq 1} K(\boldsymbol{Z}_{(2)}, 4i)$, cf. Sullivan [5] and Kirby-Siebenmann [4].

REMARK. Let $P_1 \in H^4(B\,\mathrm{Top}\,; \boldsymbol{Z})$ be the first Pontrjagin class. Then

$$24x = (j')^* P_1\,.$$

As a corollary, we obtain

COROLLARY 4. *Let $I_j=(i^j_{n_j}, \cdots, i^j_1)$ be admissible $(j=1, \cdots, m)$ n_j-ple such that,* $e(I_j)<4$ and $i^j_1 \neq 1$, *then*

$$P(\mathrm{Sq}^{I_1}(k), \cdots, \mathrm{Sq}^{I_m}(k)) \neq 0$$

for any polynomial $P(x_1, \cdots, x_m) \neq 0$.

On the other hand, it is easy to show

PROPOSITION 5. $\mathrm{Sq}^1(k) \neq 0$.

Now we make the following definition.

DEFINITION. Let M^n be a closed topological manifold. An element $\eta \in H^4(M; \boldsymbol{Z}_2)$ is said to be "k-realizable" if there is a homotopy equivalence of closed topological manifolds

$$f : N^n \to M^n$$

such that

$$k(N) = f^*(\eta) .$$

By using Proposition 3 and the surgery theory in the Top category, we obtain

PROPOSITION 6. *Let M^n $(n \geqq 5)$ be a closed topological manifold and let $\eta \in H^4(M; \boldsymbol{Z}_2)$ be an element. Then η is k-realizable if and only if there is a map*

$$g : M \to G/\mathrm{Top}$$

such that

(i) $\eta = g^*(k_2^2 + x \bmod 2) + k(M)$

(ii) $\mathscr{S}(g : M \to G/\mathrm{Top}) = 0$ *in* $L_n(\pi_1(M), w_1(M))$,

where $\mathscr{S}$ is the surgery obstruction and L_n is the Wall group.

COROLLARY 7. *Let M^n $(n \geqq 5)$ be a closed topological manifold. Assume $k(M)$ is not of the form $\alpha^2 + \beta \bmod 2$ for any $\alpha \in H^2(M; \boldsymbol{Z}_2)$ and $\beta \in H^4(M; \boldsymbol{Z})$. Then M is not homotopy equivalent to any closed PL manifold.*

REMARK. By Kirby's t-regularity theorem [5], the result of Browder, Liulevicious and Peterson on the unoriented cobordism group [1] is valid in the Top category except in dimension 4. Thus there is a closed topological manifold M^5 such that the number $\langle w_1(M)k(M), [M] \rangle$ is non zero. Since $w_1(M)k(M) = \mathrm{Sq}^1 k(M)$ by the Wu formula, M is not homotopy equivalent to any closed PL manifold by Corollary 7.

Now if we wish to apply Proposition 6 to a given manifold M, we must calculate the set (topological normal invariants) $[M, G/\mathrm{Top}]$, and the surgery obstruction $\mathscr{S}: [M, G/\mathrm{Top}] \to$ Wall group [7]. Since we know the homotopy type of G/Top, it is rather easy to calculate $[M, G/\mathrm{Top}]$ in some cases. However, since the known results of the Wall groups and the surgery obstruction formula are limited, we must assume that $\pi_1(M)$ is rather simple, e.g. 0, $\boldsymbol{Z}_2$ or free abelian.

By the argument of Sullivan [6] and Kirby-Siebenmann [4], we have

$$(G/\mathrm{Top})_7 = K(\boldsymbol{Z}_2, 2) \times K(\boldsymbol{Z}, 4) \times K(\boldsymbol{Z}_2, 6) ,$$

where $(G/\mathrm{Top})_7$ is the 7-coskeleton (the 7-th stage in the Postnikov system) of G/Top. Let $p: G/\mathrm{Top} \to (G/\mathrm{Top})_7$ be the natural map, and let X be a space homotopy equivalent to a finite CW-complex. Then we have

LEMMA 8. *If the cohomology (with any coefficient group) of X vanishes for degree $\geqq 7$, then the natural map*

$$[X, G/\mathrm{Top}] \xrightarrow{p_*} [X, (G/\mathrm{Top})_7]$$

is bijective.

Recall that, by Kirby and Siebenmann [4], any closed topological manifold M^n has the homotopy type of a finite CW-complex. Therefore we can apply Lemma 8 and obtain

$$[M, G/\mathrm{Top}] \approx H^2(M; \boldsymbol{Z}_2) \oplus H^4(M; \boldsymbol{Z}) \oplus H^6(M; \boldsymbol{Z}_2) .$$

Now we give an application of Proposition 6.

COROLLARY 9.

(i) *Let M^5 be an oriented closed PL manifold with $\pi_1(M) = \boldsymbol{Z}_2$, Then there is a non-triangulable topological manifold N^5 having the same homotopy type as M.*

(ii) *Let M^5 be a non-orientable closed topological manifold with $\pi_1(M) = \boldsymbol{Z}_2$. Then for any homotopy equivalence*

$$f: N \to M$$

we have

$$k(N) = f^*(k(M)) .$$

(iii) *Let M^6 be a closed PL orientable manifold with $\pi_1(M) = 0$ or $\boldsymbol{Z}$. Then any $\eta \in H^4(M; Z_2)$ is k-realizable.*

(iv) *Let M^6 be a closed PL orientable manifold with $\pi_1(M) = \boldsymbol{Z}_2$. Then $\eta \in H^4(M; \boldsymbol{Z}_2)$ is k-realizable if and only if η is the reduction mod 2 of an integral class.*

The k-realizability problem was first considered by S. Ichiraku [3]. Here I

wish to express my sincere thanks to him for introducing me to this subject. Thanks are also due to Y. Matsumoto for several discussions.

After I had finished the work in this note, I was informed that Siebenmann [5] and Hollingsworth and Morgan [2] have also obtained the main results of this note independently.

§2. Proofs.

PROOF OF PROPOSITION 1. Consider the following commutative diagram

$$\begin{array}{ccc} B\,\mathrm{PL}\times B\,\mathrm{PL} & \xrightarrow{\mu} & B\,\mathrm{PL} \\ \downarrow{\scriptstyle p\times p} & & \downarrow{\scriptstyle p} \\ B\,\mathrm{Top}\times B\,\mathrm{Top} & \xrightarrow[\mu]{} & B\,\mathrm{Top}\,. \end{array}$$

Clearly $p^*(k)=0$. Hence $(p\times p)^*\mu^*(k)=0$. But since

$$H^i(B\,\mathrm{Top}\,;\boldsymbol{Z}_2)\simeq H^i(B\,\mathrm{PL}\,;\boldsymbol{Z}_2)$$

for $i\leqq 3$, we have the result. Q.E.D.

PROOF OF PROPOSITION 3. By Sullivan [6] and Kirby-Siebenmann [4]

G/PL localized at 2

$$=K(\boldsymbol{Z}_2,2)\underset{\delta Sq^2}{\times}K(\boldsymbol{Z}_{(2)},4)\times\prod_{i\geqq 1}K(\boldsymbol{Z}_2,4i+2)\times\prod_{i\geqq 2}K(\boldsymbol{Z}_{(2)},4i)\,,$$

$$G/\mathrm{Top}\text{ localized at }2=\prod_{i\geqq 0}K(\boldsymbol{Z}_2,4i+2)\times\prod_{i\geqq 1}K(\boldsymbol{Z}_{(2)},4i)\,.$$

Therefore $H^4(G/\mathrm{Top}:\boldsymbol{Z}_2)\simeq Z_2\oplus Z_2$ generated by k_2^2 and $x \bmod 2$. The Serre exact sequence of the fibering

$$\mathrm{Top}/\mathrm{PL}\longrightarrow G/\mathrm{PL}\longrightarrow G/\mathrm{Top}$$

yields

$$0\longrightarrow H^3(\mathrm{Top}/\mathrm{PL}\,;\boldsymbol{Z}_2)\xrightarrow{\tau}H^4(G/\mathrm{Top}\,;\boldsymbol{Z}_2)\xrightarrow{p'^*}H^4(G/\mathrm{PL}\,;\boldsymbol{Z}_2)\longrightarrow\cdots.$$

Thus $m=\tau(u)$ is the non-zero element of $\operatorname{Ker} p'^*$. ($u\in H^3(\mathrm{Top}/\mathrm{PL}\,;\boldsymbol{Z}_2)$ is the fundamental class.)

Now clearly $p'^*(k_2^2)\neq 0$ and $p'^*(x \bmod 2)\neq 0$. Hence we have

$$m=k_2^2+x \bmod 2. \qquad \text{Q.E.D.}$$

PROOF OF COROLLARY 4. Consider the following natural projection map.

$$q:G/\mathrm{Top}\text{ localized at }2\to K(\boldsymbol{Z}_2,2)\times K(\boldsymbol{Z}_{(2)},4)\,.$$

Then

$$m=q^*(v^2+w)\ ,$$

where v and w are the generators of $H^2(K(\boldsymbol{Z}_2, 2)\,;\boldsymbol{Z}_2)$ and $H^4(K(\boldsymbol{Z}_{(2)}, 4)\,;\boldsymbol{Z}_2)$ respectively.

Now clearly

$$P(\mathrm{Sq}^{I_1}(v^2+w),\ \cdots,\ \mathrm{Sq}^{I_m}(v^2+w))\neq 0\ .$$

Since $q^*: H^*(K(\boldsymbol{Z}_2, 2)\times K(\boldsymbol{Z}_{(2)}, 4)\to H^*(G/\mathrm{Top}\,;\boldsymbol{Z}_2)$ is a monomorphism, we have the result. Q.E.D.

PROOF OF PROPOSITION 5. This follows from the Serre exact sequence of the fibering

$$\mathrm{Top/PL}\to B\,\mathrm{Spin\ PL}\to B\,\mathrm{Spin\ Top}\ . \qquad \text{Q.E.D.}$$

PROOF OF PROPOSITION 6. Let M^n $(n\geqq 5)$ be a closed topological manifold and assume that $\eta\in H^4(M\,;\boldsymbol{Z}_2)$ is k-realizable. Thus we have a homotopy equivalence

$$f: N^n\to M^n$$

such that

$$k(N)=f^*(\eta)\ .$$

Let $g: M\to G/\mathrm{Top}$ be the map for the associated topological normal invariant. Then clearly $\mathscr{S}(g: M\to G/\mathrm{Top})=0$ and

$$g^*(m)=k(\tau(M)\oplus\bar{f}^*(\nu(N)))\ ,$$

where $\bar{f}$ is a homotopy inverse to f. Now by Proposition 1, we have

$$k(\tau(M)\oplus\bar{f}^*(\nu(N)))=k(M)+\bar{f}^*(k(N))=k(M)+\eta\ .$$

Hence,

$$\eta=g^*(m)+k(M)\ .$$

Conversely assume that there is a map $g: M\to G/\mathrm{Top}$ satisfying the conditions (i) and (ii). Then by (ii), there is a homotopy equivalence

$$f: N\to M$$

whose normal invariant is the given one. Moreover we have

$$k(N)=f^*g^*(m)+f^*k(M)$$

by the above argument. Hence

$$k(N)=f^*(\eta)\ . \qquad \text{Q.E.D.}$$

PROOF OF COROLLARY 9.

(i) We first recall that $L_5(\boldsymbol{Z}_2, +)=0$ ([7]). Now there is a fibering sequence

$$\cdots \longrightarrow \mathrm{Top/PL} \longrightarrow G/\mathrm{PL} \longrightarrow G/\mathrm{Top} \xrightarrow{m} B(\mathrm{Top/PL}) \longrightarrow \cdots .$$

Thus we have an exact sequence

$$\cdots \longrightarrow H^3(M;\boldsymbol{Z}_2) \longrightarrow [M, G/\mathrm{PL}] \longrightarrow [M, G/\mathrm{Top}] \xrightarrow{m_*} H^4(M;\boldsymbol{Z}_2) \longrightarrow \cdots .$$

By Lemma 8, $[M, G/\mathrm{Top}]\simeq H^2(M;\boldsymbol{Z}_2)\oplus H^4(M;\boldsymbol{Z})$ and by Proposition 3, m_* is given by

$$m_*(\alpha\oplus\beta)=\alpha^2+\beta \bmod 2 ,$$

where $\alpha\in H^2(M;\boldsymbol{Z}_2)$ and $\beta\in H^4(M;\boldsymbol{Z})$. Since M is orientable and $\pi_1(M)=\boldsymbol{Z}_2$, we have

$$H^4(M;\boldsymbol{Z}) \xrightarrow[\bmod 2]{\sim} H^4(M;\boldsymbol{Z}_2) .$$

Therefore there is a map $g: M\to G/\mathrm{Top}$ such that $g^*(m)$ is the non-zero element of $H^4(M;\boldsymbol{Z}_2)$. Since the surgery obstruction is zero, by Proposition 6 we have the result.

(ii) Let $f: N\to M$ be a homotopy equivalence. It suffices to show that if M is triangulable, then the unique non-zero element $\eta\in H^4(M;\boldsymbol{Z}_2)$ is not k-realizable. Since M is non-orientable, we have

$$\mathrm{Sq}^1(\eta)\neq 0 .$$

Hence η is not of the form $\alpha^2+\beta \bmod 2$ for any $\alpha\in H^2(M;\boldsymbol{Z}_2)$ and $\beta\in H^4(M;\boldsymbol{Z})$. This proves (ii). (cf. Corollary 7 and the Remark after it.)

(iii) Recall that $L_6(0)\simeq L_6(\boldsymbol{Z}, +)\simeq \boldsymbol{Z}_2$ detected by the Kervaire invariant [7]. Consider the following exact sequence

$$\cdots \longrightarrow H^4(M;\boldsymbol{Z}) \xrightarrow{\bmod 2} H^4(M;\boldsymbol{Z}_2) \xrightarrow{\delta}$$
$$H^5(M;\boldsymbol{Z}) \xrightarrow{\times 2} H^5(M;\boldsymbol{Z}) \longrightarrow \cdots .$$

Since $H^5(M;\boldsymbol{Z})\simeq 0$ or $\boldsymbol{Z}$, the map δ is trivial. Hence any $\eta\in H^4(M;\boldsymbol{Z}_2)$ is the reduction mod 2 of an integral class. Thus choose an element $\beta\in H^4(M;\boldsymbol{Z})$ such that

$$\eta=\beta \bmod 2 .$$

Let

$$g: M\to G/\mathrm{Top}$$

be the map corresponding to $(0, \beta, 0)$ under the bijection $[M, G/\mathrm{Top}] \approx H^2(M; Z_2) \oplus H^4(M; Z) \oplus H^6(M; Z_2)$. Then clearly $g^*(m) = \eta$ and $\mathscr{S}(g: M \to G/\mathrm{Top}) = 0$. Hence, by Proposition 6, η is k-realizable.

(iv) Recall that $L_6(Z_2, +) = Z_2$ detected by the Kervaire obstruction [7]. By the proof of (iii), it is clear that if η is integral, then it is k-realizable. Conversely assume that η is k-realizable. Then by Proposition 6, we have

$$\mathrm{Sq}^1(\eta) = 0 .$$

Consider the following exact sequence

$$\cdots \longrightarrow H^4(M; Z) \longrightarrow H^4(M; Z_2) \xrightarrow{\delta}$$
$$H^5(M; Z) \xrightarrow{\times 2} H^5(M; Z) \xrightarrow{\mathrm{mod}\, 2} H^5(M; Z_2) \longrightarrow \cdots .$$

Since M is orientable and $\pi_1(M) = Z_2$, we have

$$H^5(M; Z) \xrightarrow[\mathrm{mod}\, 2]{\sim} H^5(M; Z_2) .$$

Hence if $\mathrm{Sq}^1(\eta) = 0$, then $\delta(\eta) = 0$. Therefore η is integral. Q.E.D.

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Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Tokyo
113 Japan