On continuation of regular solutions of partial differential equations to compact convex sets II

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This is a continuation of an earlier paper [3]. In [3], Theorem 3, we proved that any real analytic solution of a single equation $P(D)u=0$ defined on the complement $U\backslash K$ of a compact convex set K relative to a domain U can be extended to a real analytic solution on U if and only if no factor of the irreducible decomposition of $p(\zeta)$ is elliptic. Here we extend this result to the case of *systems of* equations (Theorem 2.3). As a tool for this, we prove the fundamental principle for the space $\widetilde{\mathscr{B}[K]}$ of entire functions of the Fourier transform of $\mathscr{B}[K]$ (Theorem 3.8). The same proof also applies to $\widetilde{\mathscr{O}(K)'}$ where K is any compact convex set in C^n. The readers are expected to have some knowledge of the definitions and the results in the book [7] of Palamodov and in the lecture notes [5] (or [6]) of Komatsu.

§1. General results on continuation of solutions of systems with constant coefficients.

Let $0\in K\subset U\subset R^n$, where K is a compact convex set containing the origin, U is one of its convex open neighborhoods in the n-dimensional real Euclidean space R^n. Let $p(\zeta): \mathscr{P}^s\rightarrow\mathscr{P}^t$ be a $t\times s$-matrix with elements in the ring of polynomials of n variables $\zeta_1, \cdots, \zeta_n$. We study the associated differential operator $p(D)$, where $D=(D_1, \cdots, D_n)$ and $D_j=\sqrt{-1}\dfrac{\partial}{\partial x_j}$. Put $M=\mathrm{Coker}\, p'=\mathscr{P}^s/p'\mathscr{P}^t$, where p' is the transposed matrix of p. From now on we often say "the system M" instead of "the system p". M is said to be *determined* when $\mathrm{Hom}(M, \mathscr{P})=0$, *over-determined* when $\mathrm{Hom}(M, \mathscr{P})=\mathrm{Ext}^1(M, \mathscr{P})=0$. (cf. [7] Chap. VIII, §14, 1°.) We employ the notations in [3]. Namely, $\mathscr{B}_M(U)$ denotes the space of hyperfunction solutions of the system M in U, $\mathscr{B}[K]$ denotes the space of hyperfunctions with support in K etc.

Let

$$0\longleftarrow M\longleftarrow\mathscr{P}^s\xleftarrow{p'}\mathscr{P}^t\xleftarrow{p_1'}\mathscr{P}^{t_2}\longleftarrow\cdots \tag{1.1}$$

be *a free resolution* of M. Applying the functor $\mathrm{Hom}(\cdot, \mathscr{P})$ we obtain a cochain complex:

$$0 \longrightarrow \mathscr{P}^s \xrightarrow{p} \mathscr{P}^t \xrightarrow{p_1} \mathscr{P}^{t_2} \longrightarrow \cdots .$$

Here the following is exact

$$0 \longrightarrow \mathrm{Hom}\,(M, \mathscr{P}) \longrightarrow \mathscr{P}^s \xrightarrow{p} \mathscr{P}^t .$$

Recalling that $\mathscr{B}[K]$ is a *faithfully flat* $\mathscr{P}(D)$*-module* for any compact convex set K ([6] p. 230), we have the exact sequence

$$0 \longrightarrow \mathrm{Hom}\,(M, \mathscr{P}) \otimes \mathscr{B}[K] \longrightarrow (\mathscr{B}[K])^s \xrightarrow{p(D)} (\mathscr{B}[K])^t , \tag{1.2}$$

from which we can see the well-known fact that M is determined if and only if the corresponding equation $p(D)u=0$ does not have any non-trivial solution with compact support.

Next we consider the sequence

$$0 \longrightarrow \mathrm{Ext}^1\,(M, \mathscr{P}) \longrightarrow \mathscr{P}^t/p\mathscr{P}^s \xrightarrow{p_1} \mathscr{P}^{t_2} , \tag{1.3}$$

which is exact by definition. Again using the fact that $\mathscr{B}[K]$ is $\mathscr{P}(D)$-flat, we obtain the exact sequence:

$$0 \longrightarrow \mathrm{Ext}^1\,(M, \mathscr{P}) \otimes \mathscr{B}[K] \longrightarrow (\mathscr{P}^t/p\mathscr{P}^s) \otimes \mathscr{B}[K] \xrightarrow{p_1(D)} (\mathscr{B}[K])^{t_2} .$$

Thus

$$\mathrm{Ext}^1\,(M, \mathscr{P}) \otimes \mathscr{B}[K] \cong \mathscr{B}_{p_1}[K]/p(D)(\mathscr{B}[K])^s . \tag{1.4}$$

On the other hand, we recall that $\mathscr{B}(U)$ is an *injective* $\mathscr{P}(D)$*-module* for any convex open set U ([6] p. 228). Therefore we obtain from (1.1) the exact sequence of sheaves

$$0 \longrightarrow \mathscr{B}_M \longrightarrow \mathscr{B}^s \xrightarrow{p(D)} \mathscr{B}^t \xrightarrow{p_1(D)} \mathscr{B}^{t_2} \longrightarrow \cdots , \tag{1.5}$$

from which we get the following cochain complexes of modules of sections

$$\left\{ \begin{array}{l} 0 \longrightarrow \mathscr{B}_M(U) \longrightarrow (\mathscr{B}(U))^s \xrightarrow{p(D)} (\mathscr{B}(U))^t \xrightarrow{p_1(D)} (\mathscr{B}(U))^{t_2} \longrightarrow \cdots \\ 0 \longrightarrow \mathscr{B}_M[K] \longrightarrow (\mathscr{B}[K])^s \xrightarrow{p(D)} (\mathscr{B}[K])^t \xrightarrow{p_1(D)} (\mathscr{B}[K])^{t_2} \longrightarrow \cdots . \end{array} \right. \tag{1.6}$$

Since $\mathscr{B}$ is a *flabby* sheaf, (1.5) is a flabby resolution of $\mathscr{B}_M$. Thus the cohomology groups of the complexes (1.6) agree with the cohomology groups of $\mathscr{B}_M$. Especially, we have

$$H^1(U, \mathscr{B}_M) \cong \mathscr{B}_{p_1}(U)/p(D)(\mathscr{B}(U))^s = 0 \tag{1.7}$$

$$H_K^1(U, \mathscr{B}_M) \simeq \mathscr{B}_{p_1}[K]/p(D)(\mathscr{B}[K])^s . \tag{1.8}$$

Finally we can write the exact sequence of *the relative cohomology* (*the local cohomology*) of the sheaf $\mathscr{B}_M$ with respect to the pair $K \subset U$:

$$0 \to \mathscr{B}_M[K] \to \mathscr{B}_M(U) \to \mathscr{B}_M(U \setminus K) \to H_K^1(U, \mathscr{B}_M) \to \underset{\substack{\| \\ 0}}{H^1(U, \mathscr{B}_M)} \to \cdots .$$

Thus if we define $\widehat{\mathscr{B}_M(U)} = \mathscr{B}_M(U)/\mathscr{B}_M[K]$ we have

$$\mathscr{B}_M(U \setminus K)/\widehat{\mathscr{B}_M(U)} \simeq H_K^1(U, \mathscr{B}_M) . \tag{1.9}$$

Combining (1.4), (1.8) and (1.9) we obtain the isomorphism

$$\mathscr{B}_M(U \setminus K)/\widehat{\mathscr{B}_M(U)} \simeq \mathrm{Ext}^1(M, \mathscr{P}) \otimes \mathscr{B}[K] , \tag{1.10}$$

from which we can see the well-known fact that M is overdetermined if and only if any hyperfunction solution can be continued uniquely to compact convex exceptional sets (cf. Komatsu [5]).

Now for $\mathscr{D}'$ (analogously for $\mathscr{E}^\infty$), we take some open convex neighborhood V of K which is relatively compact in U. Noticing that $\mathscr{D}'$ and $\mathscr{E}^\infty$ are soft sheaves, we obtain in a similar way using the corresponding results for $\mathscr{D}'$ or $\mathscr{E}^\infty$

$$\begin{aligned} H_V^1(U, \mathscr{D}'_M) &\simeq \frac{\mathscr{D}'_M(U \setminus V)}{\mathscr{D}'_M(U)/\mathscr{E}'_M(V)} \simeq \mathscr{E}'_{p_1}(V)/p(D)(\mathscr{E}'(V))^s \\ &\simeq \mathrm{Ext}^1(M, \mathscr{P}) \otimes \mathscr{E}'(V) . \end{aligned} \tag{1.11}$$

Here we used the symbol $\mathscr{E}'(V)$ in the usual sense of the distributions with compact supports contained in V. Passing to the limit we obtain (cf. Palamodov [7])

(1.12)

$$\mathscr{D}'_M(U \setminus K)/\widehat{\mathscr{D}'_M(U)} \cong \varprojlim_{V \supset K} \frac{\mathscr{D}'_M(U \setminus V)}{\mathscr{D}'_M(U)/\mathscr{E}'_M(V)} \cong \varprojlim_{V \supset K} (\mathrm{Ext}^1(M, \mathscr{P}) \otimes \mathscr{E}'(V)) ,$$

where the symbol $\widehat{\mathscr{D}'_M(U)} = \{u \in \mathscr{D}'_M(U \setminus K);\ U \supset^{\forall} V \supset K, {}^{\exists}u_1 \in \mathscr{D}'_M(U)$ such that $u = u_1$ on $U \setminus V\}$ is the one defined by Palamodov [7], Chap. VIII, § 14, 2°, and the first isomorphism is the one defined by the mapping which takes $u \in \mathscr{D}'_M(U \setminus K)$ to the element $\{u|_{U \setminus V} \bmod . (\mathscr{D}'_M(U)/\mathscr{E}'_M(V))\}_{V \supset K}$. Here $\varprojlim$ and $\otimes$ are, of course, non-commutative. We give a similar meaning to $\widehat{\mathscr{E}^\infty_M(U)}$, then similar isomorphisms hold for $\mathscr{E}^\infty_M$.

PROPOSITION 1.1. *If M is determined, then*

$$u \in \mathscr{B}_M(U) \cap \mathscr{D}'(U \setminus K)^s \textit{ implies } u \in \mathscr{D}'_M(U) ,$$

$$u \in \mathscr{B}_M(U) \cap \mathscr{E}^\infty(U \setminus K)^s \text{ implies } u \in \mathscr{E}^\infty_M(U),$$

$$u \in \mathscr{B}_M(U) \cap \mathscr{A}(U \setminus K)^s \text{ implies } u \in \mathscr{A}_M(U).$$

PROOF. Since p is determined by the assumption, $0 \longrightarrow \mathscr{P}^s \xrightarrow{p} \mathscr{P}^t$ is exact, so that $p'(D)u=f$ can be solved in $[\mathscr{D}'(R^n)]^t$ with no compatibility condition on $f \in [\mathscr{D}'(R^n)]^s$. Let δ^s be the diagonal matrix $\begin{bmatrix} \delta & & 0 \\ & \ddots & \\ 0 & & \delta \end{bmatrix}$ of size $s \times s$. Let $E \in [\mathscr{D}'(R^n)]^{t \times s}$ be a solution of $p'(D)E=\delta^s$. (Such E exists, e.g., by [7], Chap. VII, §8, 1°, Theorem 1.) Then, for a function $\alpha \in \mathscr{E}^\infty_0(U)$ which is equal to one in a neighborhood of K, we have on K

$$u=\alpha u=\delta^s*(\alpha u)=(E'*p(D)\delta)*(\alpha u)=E'*(p(D)(\alpha u)).$$

Here E' is the transposed matrix of E. Hereafter the proof goes just in the same way as in the proof of Theorem 1 in [3].

The following weaker result holds even in the underdetermined case.

PROPOSITION 1.2. *For any compact convex set $K' \subset U$ which contains K in its interior, the relations*

$$\mathscr{D}'_M(U \setminus K) \cap \mathscr{B}_M(U) \subset \mathscr{D}'_M(U) + \mathscr{B}_M[K'],$$

$$\mathscr{E}^\infty_M(U \setminus K) \cap \mathscr{B}_M(U) \subset \mathscr{E}^\infty_M(U) + \mathscr{B}_M[K']$$

take place.

PROOF. Let u be an element of $\mathscr{D}'_M(U \setminus K) \cap \mathscr{B}_M(U)$. Let α be a function in $\mathscr{E}^\infty_0(U)$ such that $\operatorname{supp} \alpha \subset K'$, and $\alpha(x) \equiv 1$ in a neighborhood of K. Then $p(D)(\alpha u) \in (\mathscr{D}'[K'])^t$, so that, by the inequality of division (see [7], Chap. III, §5, Theorem, and Chap. V, §3, Propositions 5 and 6), there is an element $v \in (\mathscr{D}'[K'])^s$ satisfying $p(D)v=p(D)(\alpha u)$. Thus,

$$u=(1-\alpha)u+\alpha u=\{(1-\alpha)u+v\}+(\alpha u-v).$$

Here $(1-\alpha)u+v \in \mathscr{D}'_M(U)$ and $\alpha u - v \in \mathscr{B}_M[K']$. Since K' is arbitrarily close to K, our assertion is proved in the case of $\mathscr{D}'$. Similarly in the case of $\mathscr{E}^\infty$.

PROPOSITION 1.3. *If M is underdetermined,*

$$\mathscr{A}_M(U)+\mathscr{B}_M[K] \subsetneqq \mathscr{A}_M(U \setminus K) \cap \mathscr{B}_M(U),$$

that is, the left side is properly contained in the right.

PROOF. For simplicity, we assume that K contains the origin. Since p is underdetermined, there is a non-zero matrix q with polynomial elements for which

$$\mathscr{P}^{s_0} \xrightarrow{q} \mathscr{P}^s \xrightarrow{p} \mathscr{P}^t$$

is exact. Let $q=(q_{ij})$. Then, some of the elements, say, $q_{ij}(\zeta)$ is not identically equal to zero. Choose an elliptic polynomial $r(\zeta)$ which is not a constant and relatively prime to $q_{ij}(\zeta)$, and choose a fundamental solution E of $r(D)E=\delta$. Then the function

$$u=q(D)\begin{pmatrix} 0 \\ \vdots \\ E \\ \vdots \\ 0 \end{pmatrix} j ,$$

belongs to $\mathscr{A}_M(U\backslash K)\cap\mathscr{B}_M(U)$, but does not belong to $\mathscr{A}(U)$ however we modify u on K. In fact, if we have $q_{ij}(D)E=f+w$ for some elements $f\in\mathscr{A}(U)$, $w\in\mathscr{B}[K]$, then

$$r(D)f+r(D)w=r(D)q_{ij}(D)E=q_{ij}(D)r(D)E=q_{ij}(D)\delta .$$

Hence,

$$r(D)f=q_{ij}(D)\delta-r(D)w .$$

Here, the left hand side belongs to $\mathscr{A}(U)$, and the right hand side belongs to $\mathscr{B}[K]$. Therefore by the uniqueness of analytic continuation we have

$$r(D)f=0 , \qquad q_{ij}(D)\delta=r(D)w .$$

Applying the Fourier transform to the second equality, we have

$$q_{ij}(\zeta)=r(\zeta)\cdot\tilde{w} .$$

Due to Hilbert's Nullstellensatz, this contradicts the assumption that $r(\zeta)$ is relatively prime to $q_{ij}(\zeta)$. q.e.d.

In a way similar to the proof of Proposition 1.2, we can prove the $\mathscr{P}(D)$-flatness of $\mathscr{B}[K]/\mathscr{D}'[K]$ etc. Now, from these inclusion relations we see easily:

COROLLARY 1.4. *The following natural mappings are all injective*

$$\mathscr{E}^{\infty}_M(U\backslash K)/\widehat{\mathscr{E}^{\infty}_M(U)}\subsetneq\mathscr{D}'_M(U\backslash K)/\widehat{\mathscr{D}'_M(U)}\subsetneq\mathscr{B}_M(U\backslash K)/\widehat{\mathscr{B}_M(U)} . \tag{1.16}$$

As for $\mathscr{A}_M(U\backslash K)/\mathscr{A}_M(U)\longrightarrow\mathscr{B}_M(U\backslash K)/\widehat{\mathscr{B}_M(U)}$, *it is injective when* M *is determined, and is not when* M *is underdetermined.*

Next, we derive an isomorphic expression of $\mathscr{B}_M(U\backslash K)/\widehat{\mathscr{B}_M(U)}$ by a space of holomorphic functions on $\{N_\lambda(\mathrm{Ext}^1(M,\mathscr{P}))\}$, the family of algebraic varieties associated to the module $\mathrm{Ext}^1(M,\mathscr{P})$ (cf. [7] Chap. IV, §1, 1° also cf. Chap. VIII, §14, 2° Theorem 1). First, applying the Fourier transform to (1.10), we have

$$\mathscr{B}_M(U\backslash K)/\widehat{\mathscr{B}_M(U)}\cong\mathrm{Ext}^1(M,\mathscr{P})\otimes\widetilde{\mathscr{B}[K]} . \tag{1.10$'$}$$

On the other hand, we have the exact commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
\mathscr{P}^s \arrow[r, "p(\zeta)"] \arrow[d, equal] & \operatorname{Ker} p_1(\zeta) \arrow[r] \arrow[d] & \operatorname{Ext}^1(M, \mathscr{P}) \arrow[r] \arrow[d] & 0 \\
\mathscr{P}^s \arrow[r, "p(\zeta)"] & \mathscr{P}^t \arrow[r] \arrow[d, "p_1(\zeta)"'] & \operatorname{Coker} p \arrow[r] \arrow[d] & 0 \\
 & \mathscr{P}^{t_2} \arrow[r, equal] & \mathscr{P}^{t_2} . &
\end{tikzcd}$$

Now consider the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
(\widetilde{\mathscr{B}[K]})^s \arrow[r, "p(\zeta)"] \arrow[d, equal] & (\widetilde{\mathscr{B}[K]})_{p_1} \arrow[r] \arrow[d] & \operatorname{Ext}^1(M, \mathscr{P}) \otimes \widetilde{\mathscr{B}[K]} \arrow[r] \arrow[d] & 0 \\
(\widetilde{\mathscr{B}[K]})^s \arrow[r, "p(\zeta)"] & (\widetilde{\mathscr{B}[K]})^t \arrow[r, "d_p"] \arrow[d, "p_1(\zeta)"'] & \widetilde{\mathscr{B}[K]}\{\operatorname{Coker} p, d_p\} \arrow[r] \arrow[d, dotted] & 0 \\
 & (\widetilde{\mathscr{B}[K]})^{t_2} \arrow[r, equal] & (\widetilde{\mathscr{B}[K]})^{t_2} . &
\end{tikzcd}$$

Here $\widetilde{\mathscr{B}[K]}\{\operatorname{Coker} p, d_p\}$ is the space of holomorphic functions on the family of varieties associated to the module $\operatorname{Coker} p$, which have the growth order of type $\widetilde{\mathscr{B}[K]}$, and which are locally the images of *the noetherian operator d_p associated to p* ([7] Chap. IV, §5, 1°). In this diagram, the second row is exact by *the fundamental principle* of Ehrenpreis-Palamodov (see §3 Theorem 3.8). The first row and the second column are exact by the $\mathscr{P}(\zeta)$-flatness of $\widetilde{\mathscr{B}[K]}$. The first column and the third row is clearly exact. The dotted arrow in the third column is the restriction of $\widetilde{\mathscr{B}[K]}\{\operatorname{Coker} p, d_p\}$ to the component of dimension n of the varieties associated with $\operatorname{Coker} p$. It is known that $p_1(\zeta)$ is the component of the noetherian operator d_p corresponding to this component of dimension n associated to $\operatorname{Coker} p$ ([7], Chap. IV, §4, Proposition 1). Thus the diagram is commutative and we see by a simple diagram chase that the third column is also exact.

Therefore we have

$$\operatorname{Ext}^1(M, \mathscr{P}) \otimes \widetilde{\mathscr{B}[K[} \simeq \left\{ \begin{array}{l} \text{The components of } \widetilde{\mathscr{B}[K]}\{\operatorname{Coker} p, d_p\} \text{ correspond-} \\ \text{ing to the subfamily of } \{N_\lambda(\operatorname{Coker} p)\} \text{ of dimension} \\ \leq n-1. \end{array} \right\}.$$

We introduce the notation $\widetilde{\mathscr{B}[K]}\{\mathrm{Ext}^1(M, \mathscr{P}), d'_p\}$ for the space in the right hand side of this isomorphism. Thus we have the exact sequence

$$(\mathscr{B}[K])^s \xrightarrow{p(D)} \mathscr{B}_{p_1}[K] \xrightarrow{\tilde{d}'_p} \widetilde{\mathscr{B}[K]}\{\mathrm{Ext}^1(M, \mathscr{P}), d'_p\} \longrightarrow 0 ,$$

and the isomorphism

$$\mathscr{B}_{p_1}[K]/p(D)(\mathscr{B}[K])^s \overset{\tilde{d}'_p}{\cong} \widetilde{\mathscr{B}[K]}\{\mathrm{Ext}^1(M, \mathscr{P}), d'_p\} , \tag{1.17}$$

where $\tilde{d}'_p$ is the composition of Fourier transform and the map d'_p, the components of the noetherian operator d_p corresponding to the varieties of dimension $\leqq n-1$. Combining (1.17) with (1.9) and (1.8) we have finally

$$\mathscr{B}_M(U\backslash K)/\widehat{\mathscr{B}_M(U)} \overset{\tilde{d}}{\cong} \widetilde{\mathscr{B}[K]}\{\mathrm{Ext}^1(M, \mathscr{P}), d'_p\} . \tag{1.18}$$

Here the symbol $\tilde{d}$ denotes the composition of the operator $\tilde{d}'_p$ with some cohomological maps.

Again we can give another expression of the map $\tilde{d}$, so as to derive the estimates analogous to Lemma 5 of [3]. Let u be an element of $\mathscr{B}_M(U\backslash K)$, and $[u]$ be an arbitrary extension of u to an element of $\mathscr{B}(U)$. Then, $p(D)[u]$ defines an element of $\mathscr{B}_{p_1}[K]$ modulo $p(D)(\mathscr{B}[K])^s$. Thus $\tilde{d}\cdot u$ is obviously equal to $d'_p\widetilde{p(D)[u]}$. Let α be an element of $\mathscr{E}_0^\infty(U)$ whose value is equal to 1 on a sufficiently small neighborhood of K. Then, for $u \in \mathscr{D}'_M(U\backslash K)$, $p(D)(\alpha u)$ is identically zero on some neighborhood of K. Extending this function by zero on K, we obtain an element of $\mathscr{E}'(U\backslash K)$, which we denote by $[p(D)(\alpha u)]_0$. Thus we have obviously

$$p(D)(\alpha[u]) = [p(D)(\alpha u)]_0 + p(D)[u] .$$

Here $\alpha[u]$ really has a compact support. Applying the Fourier transform to both sides and operating d'_p, we have

$$0 = d'_p p(\zeta)\widetilde{\alpha[u]} = d'_p\widetilde{[p(D)(\alpha u)]_0} + d'_p(\widetilde{p(D)[u]}) .$$

Hence

$$\tilde{d}\cdot u = d'_p\widetilde{p(D)[u]} = -d'_p\widetilde{[p(D)(\alpha u)]_0} .$$

The last term is the desired expression (cf. [7] Chap. VIII, § 14, 2° Theorem 1).

§ 2. Continuation of real analytic solutions

From the expression derived at the end of the preceding section we can get the following estimates similar to those in Lemma 5 in [3].

LEMMA 2.1. *On each of the varieties* $N_\lambda(\widetilde{\mathrm{Ext}^1(M,\mathscr{P})})$ *the corresponding component of the vector-function* $v=d'_\mu[\widetilde{p(D)(\alpha u)}]_0$ *satisfies the following*:

If $u\in\mathscr{D}'_M(U\setminus K)$, *then* $|v(\zeta)|\leqq C_\varepsilon(1+|\zeta|)^k\exp(H_K(\zeta)+\varepsilon|\mathrm{Im}\,\zeta|)$ *for any* $\varepsilon>0$ *and for some* $k=k(\varepsilon)$.

If $u\in\mathscr{E}^\infty_M(U\setminus K)$, *then* $|v(\zeta)|\leqq C_{k,\varepsilon}(1+|\zeta|)^k\exp(H_K(\zeta)+\varepsilon|\mathrm{Im}\,\zeta|)$ *for any* $\varepsilon>0$ *and for any* k.

If $u\in\mathscr{A}_M(U\setminus K)$, *then* $|v(\zeta)J(\zeta)|\leqq C_{J,\varepsilon}\exp(H_K(\zeta)+\varepsilon|\mathrm{Im}\,\zeta|)$ *for any* $\varepsilon>0$ *and for any entire infra-exponential function* $J(\zeta)$. (For the definitions of infra-exponential functions and associated differential operators of infinite order see [3], §2.)

PROOF. Since the operator d'_μ has polynomial coefficients, the first two estimates are immediate. Moreover, the third estimate follows as in the proof of Lemma 5 in [3] when we take into account the following lemma:

LEMMA 2.2. *Let* $\partial(z, D)$ *be a normal noetherian operator* (see [7] Chap. IV, §4, 1°, Definition 1) *associated with a primary component* $\mathfrak{p}$ *of* $p\mathscr{P}^s$ *in* $\mathscr{P}^t$. *Then, for any* $F\in\mathscr{O}^t$ *and for any* $f\in\mathscr{O}$ *which is invertible at* 0, *we have*[1]

$$\partial(z, D)(fF)=\partial'(z, D)f\cdot\partial(z, D)F\,,$$

where $\partial'(z, D)$ *is a matrix of differential operators with coefficients in the rational functions whose denominators do not vanish identically on* $N(\mathfrak{p})$. *Moreover, if* $\partial(z, D)$ *is well ordered in the lexicographic way with respect to the order of the normal derivatives, then* $\partial'(z, D)$ *has the form*

$$\partial'(z, D)=\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{bmatrix}.$$

$\partial'(z, D)$ *is determined independently of* f *or* F.

PROOF. Expanding $\partial(z, D)(fF)$ by the Leibniz formula, we get a differential operator $\partial'(z, f, D)$ which satisfies $\partial'(z, f, D)F=\partial(z, D)F$ and whose coefficients are linear combinations of the derivatives of f with polynomial coefficients. If the function f does not vanish at the considered point, the operator $\partial'(z, f, D)$, obviously, also serves as a "noetherian operator", except that it has not polynomial coefficients. Thus from the proof of [7] Chap. IV, §4, 1° Theorem, we see that $\partial'(z, f, D)=A(z)\partial(z, D)$, where $A(z)$ is a matrix whose elements are linear functions

[1] Of course we assume that the operator $\partial(z, D)$ does not have any term whose coefficient vanishes identically on $N(p)$.

of the derivatives of f with coefficients in the rational functions. Thus $\partial(z)=\partial'(z, D)f$, where $\partial'(z, D)$ is a differential operator with coefficients in the rational functions whose denominators do not vanish identically on $N(p)$. Since the process used there is independent of the noetherian operator $d(z, D)$ in the left hand side of the formula (1.4) in [7] Chap. IV, § 4, we see that $\partial'(z, D)$ is determined independently of f. Comparing the both sides of $\partial(z, D)(fF)=\partial'(z, D)f\cdot\partial(z, D)F$, we see that $\partial'(z, D)$ has the form stated above. q.e.d.

Now we can prove:

THEOREM 2.3. *In order that the image of the natural map*

$$\mathscr{A}_M(U\setminus K)/\mathscr{A}_M(U)\longrightarrow\mathscr{B}_M(U\setminus K)/\mathscr{B}_M(U)$$

is zero, it is necessary and sufficient that $\mathrm{Ext}^1(M, \mathscr{P})$ *has no elliptic component in its primary decomposition. In order that* $\mathscr{A}_M(U\setminus K)/\mathscr{A}_M(U)=0$, *it is necessary and sufficient that* M *is determined and satisfies the condition above.*

PROOF. The proof of the first statement can be carried out just as in the case of single equations (cf. [3] Theorem 3). In fact, the proof of the necessity is reduced to the case of single equations using Lech's theorem. For the details see the proof of the necessity in [7] Chap. VIII, § 14, 4°, Corollary 4. We only remark that at the last step of the proof there we have in our case $F=pG'$, where F is a polynomial vector and G' is a vector of (infra-exponential) entire functions. Since by Corollary 3 in [7] Chap. IV, § 4, 2° we have $F=pG''$ for some $G''\in\mathscr{P}^s$, the proof goes as well. The proof of the sufficiency is carried out in the same way as in the single case ([3] Theorem 3) since Lemma 2.1 holds.

For the second assertion of the theorem, we see that $\mathscr{A}_M(U\setminus K)/\mathscr{A}_M(U)=0$ is equivalent to the assertion: "The map $\mathscr{A}_M(U\setminus K)/\mathscr{A}_M(U)\longrightarrow\mathscr{B}_M(U\setminus K)/\mathscr{B}_M(U)$ is injective and has zero image". Thus the second assertion follows from Corollary 1.4. q.e.d.

§ 3. Cohomology with bounds and the fundamental principle for $\widetilde{\mathscr{B}[K]}$.

We prove here the fundamental principle of Ehrenpreis-Palamodov for the space $\widetilde{\mathscr{B}[K]}$. We mainly follow Palamodov [7] and show only the necessary modifications. However, we employ Hörmander [1], [2] to prove the vanishing of cohomology with bounds. The proof of Theorem 3.1 is suggested from the work of T. Kawai [4]. I am grateful to Mr. Kawai who has kindly shown me the manuscript of his master's thesis.

Let $\mathfrak{U}_\alpha=\{U_z^{(\alpha)}, z\in C^n\}$, $\alpha=1, 2, \cdots$ be a countable fundamental system of elementary coverings of parameter zero (i.e., coverings whose elements are the polydisks with a fixed diameter, see [7], Chap. III, §3, 3°). We assume that the diameter $r(\mathfrak{U}_\alpha)$ of the elements of the covering $\mathfrak{U}_\alpha$ is not greater than 1, and $r(\mathfrak{U}_{\alpha+1}) \leqq \frac{1}{2} r(\mathfrak{U}_\alpha)$. We employ another fundamental system of coverings $\mathfrak{U}'_\alpha$, $\alpha=1, 2, \cdots$ with the following properties: 1) $\mathfrak{U}'_\alpha$ is a subfamily of $\mathfrak{U}_\alpha$; 2) elements of a $\mathfrak{U}'_\alpha$ have a non-void intersection only if the number of the elements does not exceed some constant; and 3) $\mathfrak{U}'_\alpha$, $\alpha=1, 2, \cdots$, are cofinal with $\mathfrak{U}_\alpha$, $\alpha=1, 2, \cdots$. An example of such $\mathfrak{U}'_\alpha$ is given in Hörmander [2], 7.6.

Let $C^\nu\{\mathfrak{U}_\alpha, \widetilde{\mathscr{B}[K]}\}$ denote the space of holomorphic ν-cochains $\varphi=\{\varphi_{z_0, \cdots, z_\nu}(\zeta)\}$ with the topology defined by the countable seminorms:

$$\|\varphi\|_{m, \mathfrak{U}_\alpha} = \sup_{z_0, \cdots, z_\nu} \sup_{\zeta \in U^{(\alpha)}_{z_0, \cdots, z_\nu}} \left| \varphi_{z_0, \cdots, z_\nu}(\zeta) \exp\left(-\frac{1}{m}|\zeta| - H_K(\zeta)\right)\right|, \quad m=1, 2, \cdots, \tag{3.1}$$

where $U^{(\alpha)}_{z_0, \cdots, z_\nu} = U_{z_0} \cap \cdots \cap U_{z_\nu}$. Thus $C^\nu\{\mathfrak{U}_\alpha, \widetilde{\mathscr{B}[K]}\}$, $\alpha=1, 2, \cdots$ are Fréchet spaces. We denote by

$$ {}^\nu\widetilde{\mathscr{B}[K]} = [C^\nu\{\mathfrak{U}_\alpha, \widetilde{\mathscr{B}[K]}\}, \alpha=1, 2, \cdots] \tag{3.2}$$

the increasing family of topological modules $C^\nu\{\mathfrak{U}_\alpha, \widetilde{\mathscr{B}[K]}\}$, $\alpha=1, 2, \cdots$ in the sense of Palamodov [7], Chap. I, §1, 3°, Definition 2.

Our first purpose is to prove:

THEOREM 3.1. *The following is an exact sequence of homomorphisms of the increasing families* (in the sense of Palamodov [7], Chap. I, §1, 5°, Definition 8):

$$0 \longrightarrow \widetilde{\mathscr{B}[K]} \xrightarrow{\delta} {}^0\widetilde{\mathscr{B}[K]} \xrightarrow{\delta} \cdots \xrightarrow{\delta} {}^\nu\widetilde{\mathscr{B}[K]} \xrightarrow{\delta} \cdots .$$

Here δ *denotes the coboundary operator.*

PROOF. First, note that $\mathfrak{U}'_\alpha$ is cofinal with $\mathfrak{U}_\alpha$, so that,

$$ {}^\nu\widetilde{\mathscr{B}[K]} = \varinjlim_\alpha C^\nu(\mathfrak{U}'_\alpha, \widetilde{\mathscr{B}[K]}),$$

where $C^\nu(\mathfrak{U}'_\alpha, \widetilde{\mathscr{B}[K]})$ denotes a space of holomorphic ν-cochains on the covering $\mathfrak{U}'_\alpha$ with seminorms similar to (3.1). Further, for the space $C^\nu(\mathfrak{U}'_\alpha, \widetilde{\mathscr{B}[K]})$ we can employ another seminorms to define the topology.

$$\|\varphi\|_{(m), \mathfrak{U}'_\alpha} = \left[\sum_{(i_0, \cdots, i_\nu)} \int_{U_{i_0 \cdots i_\nu}} |\varphi_{i_0 \cdots i_\nu}(\zeta)|^2 \exp\left(-\frac{1}{m}|\zeta| - 2H_K(\zeta)\right) d\mu\right]^{1/2} \tag{3.3}$$

where we put $\mathfrak{U}'_\alpha = \{U_i, i \in I\}$, and $U_{i_0 \cdots i_\nu} = U_{i_0} \cap \cdots \cap U_{i_\nu}$. The symbol $d\mu$ denotes the Lebesgue measure on C^n. These seminorms are well defined since by the assumption the number of summands is finite in a neighborhood of every point

$z \in \boldsymbol{C}^n$. From the inequality

$$\exp\left(\frac{1}{m+1}|\zeta|+2H_K(\zeta)\right) \leqq C(1+|\zeta|)^{-n} \exp\left(\frac{1}{m}|\zeta|+2H_K(\zeta)\right)$$

and Cauchy's theorem, we see easily that these seminorms are as a whole equivalent to the seminorms (3.1) of supremum type for $\mathfrak{U}'_a$.

Now we have

LEMMA 3.2. *Let $U \subset \boldsymbol{C}^n$ be a domain of holomorphy which is either bounded or equal to $\boldsymbol{C}^n$ itself. We put*

$$X_m = L^2_{(p,q)}\left(U, \frac{1}{m}|\zeta|+2H_K(\zeta)+4\log(1+|\zeta|^2)\right),$$

$$Y_m = L^2_{(p,q+1)}\left(U, \frac{1}{m}|\zeta|+2H_K(\zeta)+2\log(1+|\zeta|^2)\right),$$

$$Z_m = L^2_{(p,q+2)}\left(U, \frac{1}{m}|\zeta|+2H_K(\zeta)\right).$$

Here we employ the notations of Hörmander [2]. *Namely, $L^2(U,\varphi)$ denotes the Hilbert space of measurable functions with the inner product $\int_U f\cdot\bar{g}e^{-\varphi}d\mu$. $L^2_{(p,q)}(U,\varphi)$ denotes the Hilbert space of (p,q)-forms with coefficients in $L^2(U,\varphi)$ defined above. Let $\bar{\partial}$ be the Cauchy-Riemann operator in the sense of the maximal operator. Then,*

$$X_m \xrightarrow{\bar{\partial}} Y_m \xrightarrow{\bar{\partial}} Z_m$$

is an exact sequence. The norm of the right inverse map of the second $\bar{\partial}$ depends only on the diameter of U and does not depend on the shape or the position of U.

For proof, see Hörmander [1] Theorem 2.2.3, and [2] Theorem 4.4.2. Note that our φ are obviously plurisubharmonic.

LEMMA 3.3. *Put*

$$X_m = C^\nu\left(\mathfrak{U}'_a, L^2_{(p,q)}\left(\frac{1}{m}|\zeta|+2H_K(\zeta)+4\log(1+|\zeta|^2)\right)\right),$$

$$Y_m = C^\nu\left(\mathfrak{U}'_a, L^2_{(p,q+1)}\left(\frac{1}{m}|\zeta|+2H_K(\zeta)+2\log(1+|\zeta|^2)\right)\right),$$

$$Z_m = C^\nu\left(\mathfrak{U}'_a, L^2_{(p,q+2)}\left(\frac{1}{m}|\zeta|+2H_K(\zeta)\right)\right),$$

where $\nu=0,1,2,\cdots$, or $\nu=-1$ (i.e. $X_m = L^2_{(p,q)}\left(\boldsymbol{C}^n, \frac{1}{m}|\zeta|+2H_K(\zeta)+4\log(1+|\zeta|^2)\right)$

etc. in this case). These are Hilbert spaces of L^2-cochains with the norms

$$\|\{u_{i_0\cdots i_\nu}\}\|_m=\left[\sum_{i_0\cdots i_\nu}\int_{U_{i_0\cdots i_\nu}}|u_{i_0\cdots i_\nu}|^2 e^{-\varphi}d\mu\right]^{1/2},$$

where φ denotes the weight function corresponding to each space. Then, the sequence

$$X_m \xrightarrow{\bar\partial} Y_m \xrightarrow{\bar\partial} Z_m\,,$$

with the maps naturally induced from the $\bar\partial$ operator, is exact.

PROOF. This follows easily from the preceding lemma, when we notice that the norm of the inverse correspondence of the second $\bar\partial$ in Lemma 3.2 is bounded by a constant independent of $U_{i_0\cdots i_\nu}$, and the number of different elements $U_i\in\mathfrak{U}'_a$ with a non-empty intersection does not exceed a constant, so that, we can construct the inverse correspondences of $\bar\partial$ in the above sequence by taking the direct product of all the inverse correspondences of $\bar\partial$ on each $U_{i_0\cdots i_\nu}$ constructed in Lemma 3.2.

Next, we must pass to the projective limit $\varprojlim_m$. For this purpose, we give:

LEMMA 3.4. *Let $0\longrightarrow X_m\longrightarrow Y_m\longrightarrow Z_m\longrightarrow 0$, $m=1,2,\cdots$ be exact sequences of Fréchet spaces. Put $X=\varprojlim_m X_m$, $Y=\varprojlim_m Y_m$, $Z=\varprojlim_m Z_m$. Assume that these maps are commutative with the maps in the projective systems. Assume that $X_m\longleftarrow X_{m+1}$ has a dense range for $m=1,2,\cdots$. Then $0\longrightarrow X\longrightarrow Y\longrightarrow Z\longrightarrow 0$ is also exact.*

For proof see Palamodov [7] Chap. V, §1, Proposition 11. Direct verification is not difficult.

LEMMA 3.4′. *Let $0\longrightarrow X_m\xrightarrow{f_m} Y_m\xrightarrow{g_m} Z_m\longrightarrow 0$, $m=1,2,\cdots$ be exact sequences of Fréchet spaces, where f_m are continuous linear operators, and g_m are closed linear operators. Assume that*

1) *in the projective system Y_m, the domain of g_{m+1} is mapped in the domain of g_m, and f_m, g_m are commutative with the maps in the projective systems and,*

2) *$X_m\longleftarrow X_{m+1}$ has dense range for $m=1,2,\cdots$.*

Then the limit sequence

$$0\longrightarrow \varprojlim_m X_m\xrightarrow{f}\varprojlim_m Y_m\xrightarrow{g}\varprojlim_m Z_m\longrightarrow 0$$

is also exact.

PROOF. Consider

$$0 \longrightarrow X_m \xrightarrow{(f_m, 0)} \Gamma(g_m) \xrightarrow{\pi_m} Z_m \longrightarrow 0\ , \quad m=1, 2, \cdots .$$

Here $\Gamma(g_m)$ is the graph of g_m, which is a Fréchet space, $(f_m, 0)$ is a map which brings $x \in X_m$ to $(f_m x, 0) \in \Gamma(g_m)$, and π_m is the projection operator of $\Gamma(g_m)$ to the second component Z_m. We see easily that these are exact sequences. Thus Lemma 3.4 is applied to these sequences and we obtain an exact sequence

$$0 \longrightarrow \varprojlim_m X_m \xrightarrow{(f, 0)} \varprojlim_m \Gamma(g_m) \xrightarrow{\pi} \varprojlim_m Z_m \longrightarrow 0\ .$$

Obviously, $\varprojlim_m \Gamma(g_m)$ is equal to $\Gamma(g)$, and π is equal to the projection of $\Gamma(g)$ to the second component $\varprojlim_m Z_m$. Thus we see that the limit sequence in the lemma is exact. q.e.d.

END OF PROOF OF THEOREM 3.1. First we show that the following are exact sequences:

$$\varprojlim_m X_m \xrightarrow{{}_m\bar{\partial}_q} \varprojlim_m Y_m \xrightarrow{{}_m\bar{\partial}_{q+1}} \varprojlim_m Z_m\ , \quad m=1, 2, \cdots .$$

Here, X_m etc. are the spaces in Lemma 3.3 or the spaces in Lemma 3.2 corresponding to the case of $U=C^n$. We decompose the sequences into two parts:

$$\text{(3.4)}\qquad \begin{array}{l} 0 \longrightarrow \operatorname{Ker}{}_m\bar{\partial}_q \xrightarrow{\iota_m} X_m \xrightarrow{\bar{\partial}} \operatorname{Image}{}_m\bar{\partial}_q \longrightarrow 0 \\ 0 \longrightarrow \operatorname{Ker}{}_m\bar{\partial}_{q+1} \xrightarrow{\iota_m} Y_m \xrightarrow{\bar{\partial}} \operatorname{Image}{}_m\bar{\partial}_{q+1} \longrightarrow 0 \\ \qquad\quad \| \\ \qquad \operatorname{Image}{}_m\bar{\partial}_q \end{array}$$

and verify the conditions of Lemma 3.4′.

X_m, $\operatorname{Ker}{}_m\bar{\partial}_q$ etc. are Hilbert spaces and the condition 1) of Lemma 3.4′ is clearly satisfied. We must verify that the projective system of the first terms consists of the maps with dense range. For the spaces X_m etc. of ν-cochains ($\nu \geqq 0$) or the corresponding subspaces $\operatorname{Ker}{}_m\bar{\partial}_q$ etc., we can easily see that $X_m \longleftarrow X_{m+1}$ or $\operatorname{Ker}{}_m\bar{\partial}_q \longleftarrow \operatorname{Ker}{}_{m+1}\bar{\partial}_q$ etc. have dense ranges, when we take into account that in these spaces the following cochains are dense: the cochains whose components, corresponding to those U_i sufficiently apart from the origin, are equal to zero. For the spaces X_m of global functions or forms ($\nu=-1$) or for the corresponding subspaces $\operatorname{Ker}{}_m\bar{\partial}_q = \operatorname{Image}{}_m\bar{\partial}_{q-1}$ with $q \geqq 1$, we can employ the functions in $C_0^\infty(C^n)$

and confirm also that $X_m \longleftarrow X_{m+1}$ or $\mathrm{Ker}\,{}_m\bar{\partial}_q \longleftarrow \mathrm{Ker}\,{}_{m+1}\bar{\partial}_q$ have dense ranges. Finally, for the spaces $\mathrm{Ker}\,{}_m\bar{\partial}_0$ with $q=0$, namely, for the spaces of entire holomorphic functions or forms, we need the following lemma.

LEMMA 3.5. *Let $K\subset C^n$ be a compact convex set which contains the origin in its interior. Let $H_K(\zeta)$ be its supporting function. Put $\varphi(\zeta)=H_K(\zeta)+\psi(\zeta)$, where $\psi(\zeta)\geqq 0$, $\psi(t\zeta)\leqq\psi(s\zeta)$ for $0<t<s$, and $\psi(\zeta)=o(|\zeta|)$ for large $|\zeta|$. Let $H(\varphi)$ be the Hilbert space of entire functions with the inner product:*

$$(f, g)_\varphi=\int_{C^n} f(\zeta)\overline{g}(\zeta)\, e^{-2\varphi(\zeta)} d\mu\,.$$

Then, the polynomials of ζ are dense in $H(\varphi)$.

PROOF. For any compact convex set $L\subset C^n$ $(O\in L)$, let $\mathcal{O}(L)$ denote the space of holomorphic functions in a neighborhood of L. Let $\mathcal{O}'(L)$ be its dual space. By Fourier transform, $\mathcal{O}'(L)$ is isomorphic to the Fréchet Schwartz space $H(L)$ of entire functions such that:

$$\|f\|_j=\sup_\zeta\left|f(\zeta)\cdot\exp\left(-H_L(\zeta)-\frac{1}{j}|\zeta|\right)\right|$$

are finite for $j=1, 2, \cdots$. Note that the polynomials are dense in $H(L)$. For, by the unique continuation property of holomorphic functions the linear combinations of the derivatives of the delta function with support at the origin are weakly dense in $\mathcal{O}'(L)$. Since $\mathcal{O}(L)$ is reflexive, these are also strongly dense.

Choose a sequence of compact convex sets $L_k\subset K^0$, $k=1, 2, \cdots$, such that $O\in L_k$, $L_k\Subset L_{k+1}$, $\bigcup_k L_k=K^0$. (K^0 is the interior of K. We assumed that $K^0\neq\emptyset$.) Then the natural injections $H(L_k)\longrightarrow H(\varphi)$ are clearly continuous. We first prove that $\bigcup_k H(L_k)$ is dense in $H(\varphi)$. Choose $f(\zeta)\in H(\varphi)$ arbitrary. Then for any real number t $(0<t<1)$, $g_t(\zeta)=f(t\zeta)$ belongs to $H(\varphi(t\zeta))$. Choose k so that $\varphi(t\zeta)=H_K(t\zeta)+\psi(t\zeta)\leqq H_{L_k}(\zeta)+c$ for a constant c independent of ζ and t. The following inequality can be obtained by the termwise integration of the Taylor series, and is well-known.

$$|g_t(\zeta)|^2\leqq\frac{1}{\pi}\int_{|z-\zeta|\leqq 1}|g_t(z)|^2 d\mu\,.$$

Thus for any ζ we have

$$\begin{aligned}|g_t(\zeta)\exp(-H_{L_k}(\zeta))|^2&\leqq\frac{C}{\pi}\int_{|z-\zeta|\leqq 1}|g_t(z)|^2\exp(-2H_{L_k}(z))d\mu\\ &\leqq\frac{C\cdot e^{2c}}{\pi}\int_{C^n}|g_t(z)|^2\exp(-2\varphi(tz))d\mu\,.\end{aligned}$$

Here $C = \sup_{|z-\zeta|\leq 1} \exp(2H_{L_k}(z) - 2H_{L_k}(\zeta)) \leqq \sup_{|z|\leq 1} 2H_K(z)$ is a constant. Thus $g_t(\zeta)$ belongs to some of the $H(L_k)$.

Since we have for $1/2 \leqq t < 1$,

$$\|g_t(\zeta)\|_\varphi^2 = \int |f(t\zeta)|^2 e^{-2\varphi(\zeta)} d\mu$$
$$\leqq 2^{2n} \int |f(t\zeta)|^2 e^{-2\varphi(t\zeta)} t^{2n} d\mu = 2^{2n} \|f(\zeta)\|_\varphi^2 ,$$

there is some sequence $\{t_j\}$, $t_j \nearrow 1$, for which the sequence $g_{t_j}(\zeta)$ converges weakly to some element $g(\zeta) \in H(\varphi)$. Especially on each bounded region $\Omega \subset C^n$, $g_{t_j}(\zeta)$ converges weakly to $g(\zeta)$ in the Hilbert space $H(\Omega, \varphi)$ with the norm

$$\left(\int_\Omega |\cdot|^2 \exp(-2\varphi(\zeta)) d\mu\right)^{1/2} .$$

(Recall that the natural restriction $H(\varphi) \longrightarrow H(\Omega, \varphi)$ is continuous, so that weakly continuous.) On the other hand, by the Lebesgue convergence theorem $g_{t_j}(\zeta)$ converges strongly to $f(\zeta)$ in $H(\Omega, \varphi)$ on each of the Ω. Thus $g(\zeta) \equiv f(\zeta)$, and $g_{t_j} \longrightarrow f$ weakly in $H(\varphi)$. We have proved that $\bigcup_k H(L_k)$ is weakly dense, hence strongly dense in $H(\varphi)$. Recalling that polynomials are dense in each of the $H(L_k)$, we have proved that polynomials are also dense in $H(\varphi)$. q.e.d.

Thus, in any case, by Lemma 3.4′ we can pass to the projective limit in the sequences (3.4) and get the following exact sequences:

$$\begin{gathered} 0 \longrightarrow \varprojlim_m \operatorname{Ker} {}_m\bar\partial_q \longrightarrow \varprojlim_m X_m \longrightarrow \varprojlim_m \operatorname{Image} {}_m\bar\partial_q \longrightarrow 0 \\ 0 \longrightarrow \varprojlim_m \operatorname{Ker} {}_m\bar\partial_{q+1} \longrightarrow \varprojlim_m Y_m \longrightarrow \varprojlim_m \operatorname{Image} {}_m\bar\partial_{q+1} \longrightarrow 0 . \\ \| \\ \varprojlim_m \operatorname{Image} {}_m\bar\partial_q \end{gathered}$$

Since the growth condition in the spaces $\varprojlim_m X_m$ etc. agrees with the growth condition of type $\widetilde{\mathscr{B}[K]}$, connecting these short exact sequences for $q = 0, 1, 2, \cdots$ we obtain the following exact sequence:

$$\begin{aligned} (3.5) \qquad & 0 \longrightarrow C^\nu(\mathfrak{U}'_\alpha, \widetilde{\mathscr{B}[K]}) \longrightarrow C^\nu(\mathfrak{U}'_\alpha, L^2_{(0,0)}(\widetilde{\mathscr{B}[K]})) \xrightarrow{\bar\partial} \\ & \xrightarrow{\bar\partial} C^\nu(\mathfrak{U}'_\alpha, L^2_{(0,1)}(\widetilde{\mathscr{B}[K]})) \xrightarrow{\bar\partial} \cdots . \end{aligned}$$

Here, $L^2_{(0,0)}(\widetilde{\mathscr{B}[K]})$ denotes the Fréchet space of locally summable functions whose seminorms are the same as those in the L^2 form of the Fréchet space $\widetilde{\mathscr{B}[K]}$.

$L^2_{(0,q)}(\widetilde{\mathscr{B}[K]})$ denotes the space of $(0,q)$-forms with coefficients in the Fréchet space $L^2_{(0,q)}(\widetilde{\mathscr{B}[K]})$. Lastly $C^\nu(\mathfrak{U}'_\alpha, L^2_{(0,q)}(\widetilde{\mathscr{B}[K]}))$ denotes the space of cochains with values in $L^2_{(0,q)}(\widetilde{\mathscr{B}[K]})$. This is a Fréchet space with the norms $\|\cdot\|_{(m),\mathfrak{U}'_\alpha}$, $m=1,2,\cdots$ given in (3.3).

Now we consider $C^\nu(\mathfrak{U}'_\alpha, L^2_{(0,q)}(\widetilde{\mathscr{B}[K]}))$, $\alpha=1,2,\cdots$, as an increasing family of topological modules in the sense of Palamodov [7] Chap. I, §1, 3° Definition 2 and denote it by ${}^\nu L^2_{(0,q)}(\widetilde{\mathscr{B}[K]})$. Then we obtain the sequence:

$$0 \longrightarrow {}^\nu\widetilde{\mathscr{B}[K]} \longrightarrow {}^\nu L^2_{(0,0)}(\widetilde{\mathscr{B}[K]}) \xrightarrow{\bar\partial} {}^\nu L^2_{(0,1)}(\widetilde{\mathscr{B}[K]}) \xrightarrow{\bar\partial} \cdots .$$

(Recall that $C^\nu(\mathfrak{U}'_\alpha, \widetilde{\mathscr{B}[K]})$, $\alpha=1,2,\cdots$ is a family equivalent to $C^\nu(\mathfrak{U}_\alpha, \widetilde{\mathscr{B}[K]})$, $\alpha=1,2,\cdots$, because $\mathfrak{U}'_\alpha$ is cofinal with $\mathfrak{U}_\alpha$.) It is evident that this is an exact sequence of "homomorphisms" of the increasing families in the sense similar to [7] Chap. I, §1, 5° Definition 8. (We use here the word "homomorphism" in the meaning that "there exists a continuous right inverse".) Though in our case the mappings $\bar\partial$ are not defined on the whole spaces, we meet no difficulties in the following.

Now in the diagram

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & {}^1\widetilde{\mathscr{B}[K]} \arrow[r, "\bar\partial"] \arrow[u, "\delta"] & {}^1L^2_{(0,0)}(\widetilde{\mathscr{B}[K]}) \arrow[r, "\bar\partial"] \arrow[u, "\delta"] & {}^1L^2_{(0,1)}(\widetilde{\mathscr{B}[K]}) \arrow[r, "\bar\partial"] \arrow[u, "\delta"] & \cdots \\
0 \arrow[r] & {}^0\widetilde{\mathscr{B}[K]} \arrow[r, "\bar\partial"] \arrow[u, "\delta"] & {}^0L^2_{(0,0)}(\widetilde{\mathscr{B}[K]}) \arrow[r, "\bar\partial"] \arrow[u, "\delta"] & {}^0L^2_{(0,1)}(\widetilde{\mathscr{B}[K]}) \arrow[r, "\bar\partial"] \arrow[u, "\delta"] & \cdots \\
0 \arrow[r] & \widetilde{\mathscr{B}[K]} \arrow[r, "\bar\partial"] \arrow[u, "\delta"] & L^2_{(0,0)}(\widetilde{\mathscr{B}[K]}) \arrow[r, "\bar\partial"] \arrow[u, "\delta"] & L^2_{(0,1)}(\widetilde{\mathscr{B}[K]}) \arrow[r, "\bar\partial"] \arrow[u, "\delta"] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

all the rows are exact in the sense stated above. On the other hand, all the columns except the first one are exact in the sense of Palamodov quoted above. In fact, this is shown in the following way. Let $\varphi=\{\varphi_{i_0\cdots i_\nu}\}$ be an element of $C^\nu(\mathfrak{U}'_\alpha, L^2_{(0,q)}(\widetilde{\mathscr{B}[K]}))$ which satisfy $\delta\varphi=0$. Let $\{\chi_i\}$ be a C^∞-partition of unity subordinate to the covering $\mathfrak{U}'_\alpha$. Without loss of generality we may assume that $\mathfrak{U}'_{\alpha+1}$ is a refinement of the covering consisting of the sets $\{\chi_i=1\}$. (For detailed study of the construction see Hörmander [2], 7.6). We define $\psi\in C^{\nu-1}(\mathfrak{U}'_\alpha, L^2_{(0,q)}(\widetilde{\mathscr{B}[K]}))$ by the following formula:

$$\psi=\{\psi_{i_0\cdots i_{\nu-1}}\}\,,\quad \psi_{i_0\cdots i_{\nu-1}}=\sum_i \chi_i\varphi_{ii_0\cdots i_{\nu-1}}\,.$$

The sum in the right hand side is well defined since there are only finitely many U_i which have non-void intersection with $U_{i_0\cdots i_{\nu-1}}$. We see easily that $\rho_\alpha\delta\psi=\rho_\alpha\varphi$, where ρ_α is the restriction map corresponding to a refinement $\mathfrak{U}'_{\alpha+1}\to\mathfrak{U}_\alpha$. Thus the columns are exact except the first one. Moreover, we see easily that if φ is in the domain of $\bar\partial$ operator, this ψ is also in the domain of $\bar\partial$. Noticing this fact, we can perform a diagram chase similar to that of Weil's lemma to the above diagram. Thus the first column is also exact.

Finally by a more precise diagram chase similar to that of [7] Chap. I, §2, Theorem 1, we see that all the δ in the first column are topological homomorphisms of the increasing families, since the δ in the other columns and the $\bar\partial$ in the rows are of similar nature. Thus the theorem 3.1 is proved.

From now on, we follow the proof in [7]. The part pp. 138-150 of Chap. III, §5 is a local theory and does not depend on the family of majorants, so that we can employ it without any modification.

Next we consider

LEMMA 3.6. (cf. [7] Chap. III, §5, 7°, Lemma 4.) *For any $\mathscr{P}$-matrix p and for every $\nu\geqq 0$,*

$$({}^\nu\widetilde{\mathscr{B}[K]})^s\xrightarrow{\ p\ }({}^\nu\widetilde{\mathscr{B}[K]})^t\cap\operatorname{Ker}\mathscr{D}\longrightarrow 0$$

is exact and p is a homomorphism. Here $\mathscr{D}$ is the p-operator of Palamodov [7] (Chap. II, §14, 1°, Theorem 1).

PROOF. Fix $\nu\geqq 0$ and α. Put $\mathfrak{U}_\alpha=\{U_z\}$ and $\mathfrak{U}_{\alpha+2}=\{V_z\}$. By assumption $r(\mathfrak{U}_{\alpha+2})\leqq\dfrac{1}{4}r(\mathfrak{U}_\alpha)$. Thus, employing the notations of Palamodov (cf. Lemma 4 quoted above), we have

$$V_{z_0}\cap\cdots\cap V_{z_\nu}\subset V_{z_0}\subset e_a^{z_0,r}\subset e_1^{z_0,r}=2V_{z_0}\subset U_{z_0}\cap\cdots\cap U_{z_\nu}\,,\qquad r=r(\mathfrak{U}_{\alpha+2})\,.$$

Let φ be an arbitrary cochain of $(C^\nu(\mathfrak{U}_\alpha,\widetilde{\mathscr{B}[K]}))^t\cap\operatorname{Ker}\mathscr{D}$, and let $\varphi_{z_0\cdots z_\nu}$ be its component corresponding to the domain $U_{z_0\cdots z_\nu}$. To the restriction of $\varphi_{z_0\cdots z_\nu}$ on $e_1^{z_0,r}$, we can apply the operator

$$B_{z_0,r}:{}^{-1}\Pi_1\cap\operatorname{Ker}\mathscr{D}\longrightarrow{}^{-1}\Pi_a/{}^{-1}\Pi_a\cap\operatorname{Ker}p\,,$$

constructed by the local theory ([7] Chap. III, §5, 5°). Let $\psi_{z_0\cdots z_\nu}$ be a representative of the class $B_{z_0,r}\varphi_{z_0\cdots z_\nu}$. By the inequality (19) of [7] Chap. III, §5, 5°, we have

$$\inf\{\sup_{e_a^{z_0,r}}|\psi_{z_0\cdots z_\nu}-\chi|,\ \chi\in E_a^{z_0,r}\cap\operatorname{Ker} p\}\leqq C(|z_0|+1)^{a-1}\sup_{e_1^{z_0,r}}|\varphi_{z_0\cdots z_\nu}|\ .$$

Thus we have with another constant C,

$$\inf_{\chi}\sup_{z\in V_{z_0\cdots z_\nu}}|\psi_{z_0\cdots z_\nu}-\chi|\exp\Big(-\frac{1}{m}|z|-H_K(z)\Big)$$
$$\leqq C\sup_{z\in V_{z_0\cdots z_\nu}}|\varphi_{z_0\cdots z_\nu}|\exp\Big(-\frac{1}{m+1}|z|-H_K(z)\Big)\ .$$

Here we used the fact that

$$\inf_{|z-\zeta|\leqq 1/4}(|z|+1)^{a-1}\exp\Big(\frac{1}{m}|z|+H_K(z)\Big)\geqq C\exp\Big(\frac{1}{m+1}|\zeta|+H_K(\zeta)\Big)\ .$$

If we define $\psi=\sum\psi_{z_0\cdots z_\nu}V_{z_0}\wedge\cdots\wedge V_{z_\nu}$, we obtain the inequality

$$\inf\|\psi-\chi\|_{m,\mathfrak{U}_{a+2}}\leqq C\|\varphi\|_{m+1,\mathfrak{U}_a}\ .$$

Here infimum is taken for all the cochains $\chi\in C^\nu(\mathfrak{U}_{a+2},\widetilde{\mathscr{B}[K]})$ such that $p\chi=0$. From the property of the operator $B_{z_0,r}$ we have $p\psi=\varphi$. Thus we have constructed a continuous inverse of the map p. q.e.d.

THEOREM 3.7. *For any $\mathscr{P}$-matrix p, the following is an exact sequence of homomorphisms.*

$$(\widetilde{\mathscr{B}[K]})^s\xrightarrow{p}(\widetilde{\mathscr{B}[K]})^t\cap\operatorname{Ker}\mathscr{D}\longrightarrow 0\ .$$

We can prove this theorem in the same way as in the proof of [7] Chap. III, §5, 1°, Theorem given in p. 152, using Theorem 3.1 and Lemma 3.6, so we do not repeat it.

THEOREM 3.8. *Let $\widetilde{\mathscr{B}[K]}\{p,d\}$ be the space of holomorphic p-functions $f=\{f^\lambda\}$ with the seminorms*

$$\|f\|_m=\max_{\lambda}\sup_{N^\lambda}|f^\lambda(z)|\exp\Big(-\frac{1}{m}|z|-H_K(z)\Big)$$

(cf. [7], Chap. IV, §5, 1°). *Here p is a $\mathscr{P}$-matrix and $\{N^\lambda\}$ is the associated family of algebraic varieties, d is the associated noetherian operator. Then,*

$$(\widetilde{\mathscr{B}[K]})^s\xrightarrow{p}(\widetilde{\mathscr{B}[K]})^t\xrightarrow{d}\widetilde{\mathscr{B}[K]}\{p,d\}\longrightarrow 0$$

is an exact sequence of homomorphisms.

The proof of this theorem is similar to that of [7] Chap. IV, §5, 2°, Theorem 2. The only difference to be noted is the use of seminorms of Fréchet type. The corresponding modification in the estimates in [7] Chap. IV, § 5, 2°, Theorem 1 is the same as the modification of [7] Chap. III, § 5, 7°, Lemma 4 to our Lemma 3.6, so that we do not repeat it here.

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