Classification of embeddings of a non-orientable manifold

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1. Introduction and announcement of results

In 1963, Haefliger and Hirsch [2] classified the embeddings of an orientable manifold M^n in the Euclidean $2n$-space R^{2n} when $n \geq 4$.

The purpose of this paper is to classify the embeddings of a non-orientable manifold M^n in R^{2n} when n is even ≥ 4.

When a non-orientable manifold M^n is embedded in R^{2n}, the Euler class of the normal bundle can be identified with an integer. We will call it the normal Euler class of the embedding.

A necessary condition for the normal Euler classes of a non-orientable manifold was given by Mahowald [4], and Malyi [5] determined all the possible values for the normal Euler classes when n is even ≥ 4. In the case $n=2$, Massey [6] also determined all the possible values for the normal Euler classes as a proof of Whitney's conjecture [7].

Our line of proof is to classify the embeddings by normal Euler classes. After I completed the first draft of this paper, I found out Malyi's paper. As we can recover both Mahowald's theorem [4] and Malyi's result in our line independently of their works, we will present the proofs briefly in Theorem 2.2 and in the appendix.

Our main result is as follows.

THEOREM 1.1. *Let M^n be a non-orientable manifold with n even ≥ 4. Then the isotopy classes of embeddings of M^n in R^{2n} correspond bijectively with $Z+H^{n-1}(M; Z)/K$ where K is a certain subgroup of $H^{n-1}(M; Z)$ and the first factor Z corresponds to the difference of normal Euler classes.*

COROLLARY 1.2. If in addition $H^{n-1}(M; Z)=0$, then the isotopy classes of embeddings are classified by normal Euler classes.

We can easily see that Corollary 1.2 holds for real projective spaces RP^n with n even ≥ 4 and for Dold manifolds $P(m, n)$ with both m and n even. Here $P(m, n)$ is obtained from $S^m \times CP^n$ by identifying (x, z) with $(-x, \bar{z})$ and is of dimension $m+2n$.

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2. Euler classes of normal bundles

In this paper we will assume that M^n is a non-orientable closed manifold with n even ≥ 4.

Let W^{n-1} be the submanifold of M^n expressing the Poincaré dual of the first Stiefel-Whitney class $w_1(M)$. W^{n-1} is orientable and moreover we may assume that it is connected. Let N be the closed tubular neighborhood of W^{n-1} in M^n and put $Q=M-\operatorname{int} N$.

Let f be an embedding of M^n in R^{2n}. Then the Euler class of the normal bundle $X(\nu_f)$ can be considered as the obstruction to constructing a non-singular normal vector field over M^n. Since N is a manifold with boundary, the embedding $f|N$ is unique up to isotopy ([2]). Therefore we may assume that $f(bN)\subset R^{2n-1}$ and that the intersection of M^n to R^{2n-1} is normal. We can find non-singular normal vector fields u and v over Q and N respectively because $bQ=bN\neq\phi$. Then the obstruction to constructing a non-singular normal vector field over M^n is given by

$$Lk(f(bQ), u(bQ))-Lk(f(bN), v(bN))$$

where Lk is the linking coefficient in R^{2n-1} and $u(bQ)$ denotes the $(n-1)$-chain determined as the image of bQ under the normal vector field u, similarly for $v(bN)$. Hence $X(\nu_f)=Lk(f(bQ), u(bQ))-Lk(f(bN), v(bN))$.

Now, let f_j $(j=0, 1)$ be any two embeddings of M^n in R^{2n}. As before we may assume that $f_0|N=f_1|N$ and that the intersection of M^n with R^{2n-1} is normal. Let u_0, u_1 and v be non-singular normal vector fields over $f_0(Q)$, $f_1(Q)$ and $f_0(N)=f_1(N)$ respectively. Then

$$X(\nu_{f_1})-X(\nu_{f_0})=Lk(f_1(bN), u_1(bQ))-Lk(f_0(bN), u_0(bQ)) .$$

Take an n-chain C in R^{2n-1} whose homological boundary dC is equal to the $(n-1)$-chain $f_j(bN)$. Decompose C as $C'+C''$ where C'' has no intersection with the tubular neighborhood of $f_j(bN)$ in R^{2n-1}. Let C_ε, C'_ε, and C''_ε be the n-chains in R^{2n} obtained from the n-chains C, C' and C'' respectively by parallel translation along the x_{2n}-axis by the small amount ε. Here we may assume that $f_j(N)$ is contained in the lower half-space of R^{2n} given by $x_{2n}\leq 0$. Define an n-chain T in R^{2n} by

$$T: bQ\times[0, \varepsilon]\rightarrow R^{2n-1}\times R=R^{2n}$$

$$T(x, t)=(f_0(x), t)=(f_1(x), t) .$$

Then

$$\begin{aligned} Lk(f_i(bN), u_j(bQ)) &= C\circ u_j(bQ) \\ &= (C'_\varepsilon+T)\circ u_j(Q) \quad \text{in} \quad R^{2n} \\ &= (C_\varepsilon+T-f_j(Q))\circ u_j(Q)-C''_\varepsilon\circ u_j(Q) \quad \text{in} \quad R^{2n} \end{aligned}$$

since $f_j(Q)\cap u_j(Q)=\emptyset$. $C_\varepsilon+T$ and $f_j(Q)$ coincide near the boundary, therefore there exists an $(n+1)$-chain K_j in R^{2n} whose homological boundary dK_j does not meet $f_i(N)$ and satisfies $dK_j=C_\varepsilon+T-f_j(Q)$. Then we have

$$(C_\varepsilon+T-f_j(Q))\circ u_j(Q)=dK_j\circ u_j(Q)=-K_j\circ bN \quad \text{and} \quad -C''_\varepsilon\circ u_j(Q)=-C''_\varepsilon\circ f_j(Q) ,$$

because $u_j(Q)$ is homologous to $f_j(Q)$ in $R^{2n}-C''_\varepsilon$. Hence

$$X(\nu_{f_1})-X(\nu_{f_0})=C_\varepsilon''\circ(f_0(Q)-f_1(Q))+(K_0-K_1)\circ f_i(bN) .$$

Let Q'' be the complement of a neighborhood of N, which is also diffeomorphic to Q. Then $f_0|Q''$ and $f_1|Q''$ define a map $F: DQ''\to R^{2n}$ (DQ'' is the double of Q'') with the orientation of $f_1(Q'')$ reversed. Then

$$\begin{aligned} X(\nu_{f_1})-X(\nu_{f_0}) &= C''_\varepsilon\circ F(DQ'')-(K_0-K_1)\circ f_i(bN) \\ &= Lk(dC''_\varepsilon, F(DQ''))+Lk(d(K_0-K_1), f_i(bN)) \\ &= Lk(dC, F(DQ''))+Lk(-F(DQ''), f_i(bN)) \\ &= 2Lk(f_i(bN), F(DQ'')) \\ &= 4Lk(f_i(W), F(DQ'')) . \end{aligned}$$

Thus, we have shown the following lemma.

LEMMA 2.1. *With the notations above,*

$$X(\nu_{f_1})-X(\nu_{f_0})=4Lk(f_i(W), F(DQ'')) \quad in \quad R^{2n} .$$

As a direct application of Lemma 2.1, we obtain the result of Malyi [5].

THEOREM 2.2. *Let f_0 be an embedding of M^n in R^{2n}. Then for any integer k, there exists an embedding f of M^n in R^{2n} with*

$$X(\nu_f)-X(\nu_{f_0})=4k .$$

PROOF. There exists a map

$$f': Q''\to R^{2n}-f_0(N)$$

such that $f'|bQ''=f_0|bQ''$ and that

$$F'=(-f'\cup f_0|Q''): DQ''\to R^{2n}-f_0(N)$$

represents kw in $H_n(R^{2n}-f(N))$ where w is the generator of $H_*(R^{2n}-f_0(N))$ corresponding to the Alexander dual of $f_0(W^{n-1})$ in R^{2n}. Without loss of generality, we may assume that f' is an immersion. Using the method of removing the double points of f' [8], we can make f' an embedding f''. Then clearly f'' also satisfies the same conditions as f'. Define an embedding f of M^n by $f|M-\text{int}\,Q''=f_0|M-\text{int}\,Q''$ and $f|Q''=f''$. It is easily verified that f is the required embedding by Lemma 2.1.

3. Proof of Theorem 1.1

First we will examine the relation between the Euler class of normal bundles and embeddings of a non-orientable manifold more closely.

Take two disks D_1^n and D_2^n in M^n such that $D_1^n\subset\text{int}\,D_2^n$, and put $M_j=M^n-\text{int}\,D_j^n$ $(j=1, 2)$. The embedding of M_1 in R^{2n} is unique up to isotopy ([2]). Therefore, if we fix an embedding f_0 of M^n in R^{2n}, then for any other embedding f of M^n, there exists and embedding f' which is isotopic to f and coincides with f_0 restricted on M_1. Then $f_0|D_1^n$ and $f'|D_1^n$ define a map

$$(-f_0\cup f'): D(D_1^n)\to R^{2n}-f_0(M_2)$$

where $D(D_1^n)$ is the double of D_1^n and $-f_0$ means that the orientation of $f_0(D_1^n)$ is reversed. By a general position argument and Alexander duality, $R^{2n}-f_0(M_2)$ is $(n-1)$-connected. Then the map $(-f_0\cup f')$ defines an element in

$$\pi_n(R^{2n}-f_0(M_2))\cong H_n(R^{2n}-f_0(M_2))\cong H^{n-1}(M_2)\ .$$

(In this section, all homology and cohomology groups have Z-coefficients.) This element will be denoted by $d(f', f_0)$.

LEMMA 3.1 *Let f_0 and f_1 be embeddings of M^n in R^{2n} with $f_0|M_1=f_1|M_1$. If h_t $(0\leq t\leq 1)$ is an isotopy of R^{2n} with $h_1|f_0(M_1)=$identity, then $d(h_1\circ f_1, h_1\circ f_0)=d(f_1, f_0)$.*

PROOF. Let L be an $(n+1)$-chain in R^{2n} satisfying $dL=d(f_1, f_0)$, then $d(f_1, f_0)\in H^{n-1}(M_2)$ is defined by $d(f_1, f_0)(e)=L\circ e$ for each $(n-1)$-cell e of M_2. On the other hand, since $d(h_1(L))=d(h_1\circ f_1, h_1\circ f_0)$, we have $d(h_1\circ f_1, h_1\circ f_0)(e)=h_1(L)\circ e$ which is equal to $L\circ e$.

Let $K(f_0)$ be the subset of $H^{n-1}(M_2)$ which is composed of elements of the form $d(f, f_0)$ where f is an embedding of M^n in R^{2n} isotopic to f_0 and $f|M_1=f_0|M_1$.

LEMMA 3.2 *$K(f_0)$ is a subgroup of $H^{n-1}(M_2)$.*

PROOF. Take an element of $K(f_0)$ expressed as $d(f_1, f_0)$. Let h_t be an isotopy of R^{2n} which covers an isotopy between f_0 and f_1, then $f_1=h_1\circ h_0$. Clearly, $d(h_1^{-1}\circ f_0, f_0)$ is the inverse element of $d(f_1, f_0)$. Take two elements $d(f', f_0)$ and $d(f'', f_0)$ in

$K(f_0)$ and let h'_t (resp. h''_t) be an isotopy of R^{2n} covering an isotopy between f_0 and f' (resp. f''). Then it is easy to see that the composition of $d(f', f_0)$ with $d(f'', f_0)$ is given by $d(h''_1 \circ h'_1 \circ f_0, f_0)$.

LEMMA 3.3. *Let f and g be two embeddings of M^n in R^{2n} with $f|M_1 = g|M_1$, then $K(f) = K(g)$.*

PROOF. Let h_t be an isotopy of R^{2n} with $h_1|f(M_1)$=identity. Then any element of $K(f)$ has the form $d(h_1 \circ f, f)$, hence it is enough to show that $d(h_1 \circ f, f) = d(h_1 \circ g, g)$.

$$d(h_1 \circ g, g) - d(h_1 \circ f, f) = d(h_1 \circ g, h_1 \circ f) + d(h_1 \circ f, g) - d(h_1 \circ f, f) .$$

But from Lemma 3.1,

$$d(h_1 \circ g, h_1 \circ f) = d(g, f) .$$

Hence

$$\begin{aligned} d(h_1 \circ g, g) &- d(h_1 \circ f, f) \\ &= d(g, f) - d(g, h_1 \circ f) - d(h_1 \circ f, f) \\ &= d(g, f) - d(g, f) = 0 . \end{aligned}$$

Let f be an embedding of M^n in R^{2n}. We can find an embedding f' which is isotopic to f and $f'|M_1 = f_0|M_1$. Then we define $\bar{d}(f, f_0) \in H^{n-1}(M_2)/K(f_0)$ to be the quotient class of $d(f', f_0)$. This class is independent of the choice of f' and is well-defined.

LEMMA 3.4. *Let* Emb(M^n) *denote the isotopy classes of embeddings of M^n in R^{2n}. Then there is a bijective correspondence between* Emb(M^n) *and* $H^{n-1}(M_2)/K(f_0)$ *where f_0 is a fixed embedding.*

PROOF. To each class of Emb(M^n) represented by an embedding f, we assign the class $\bar{d}(f, f_0)$. Surjectivity: Any element of $H^{n-1}(M_2)$ can be represented by a map $(-f_0 \cup f_1) : D(D_1^n) \to R^{2n} - f_0(M_2)$ where f_1 is a map of D_1^n into $R^{2n} - f_0(M_2)$ with $f_1|bD_1^n = f_0|bD_1^n$. Moreover, we may assume that f_1 is an embedding ([2]). $f_0|M_1$ and f_1 (by smoothing corners if necessary) give the required embedding. Injectivity: Let f and g be embeddings of M^n in R^{2n} with $\bar{d}(f, f_0) = \bar{d}(g, f_0)$. We can find an embedding f' (resp. g') which is isotopic to f (resp. g) and $f'|M_1 = f_0|M_1$ (resp. $g'|M_1 = f_0|M_1$). Then $d(f', f_0) - d(g', f_0)$ belongs to $K(f_0)$. From Lemma 3.3, $d(f', g')$ is an element of $K(g')$. Therefore there exists an embedding g'' which is isotopic to g', $g''|M_1 = g'|M_1$ and $d(f', g') = d(g'', g')$. Then $d(f', g'') = 0$ hence $f'|D_1^n$ and $g''|D_1^n$ are homotopic keeping the boundary fixed. Therefore $f'|D_1^n$ and $g''|D_1^n$ are isotopic keeping the boundary fixed. This shows that f' and g'' are isotopic. Consequently f and g are isotopic.

PROOF OF THEOREM 1.1. Fix an embedding f_0 of M^n in R^{2n}. Let f be an embedding of M^n and f' be an embedding isotopic to f with $f'|M_1=f_0|M_1$. As in section 2, we have

$$\begin{aligned} X(\nu_f)-X(\nu_{f_0}) &= 2Lk((-f_0\cup f')D(D_1^n), f_0(bN)) \\ &= 4Lk((-f_0\cup f')D(D_1^n), f_0(W^{n-1})) \\ &= 4(i''^*d(f', f_0))[W]. \end{aligned}$$

Here i'' is the inclusion map $W\to M_2$. Consider the cohomology exact sequences of the pairs (M, N) and (M_2, N).

$$\begin{array}{ccccccccc} H^{n-2}(N) & \xrightarrow{d^*} & H^{n-1}(M_2, N) & \xrightarrow{j'^*} & H^{n-1}(M_2) & \xrightarrow{i'^*} & H^{n-1}(N) & \longrightarrow & H^n(M_2, N) \\ \| & & \uparrow{\scriptstyle k^*} & & \uparrow{\scriptstyle k'^*} & & \| & & \uparrow \\ H^{n-2}(N) & \xrightarrow[d^*]{} & H^{n-1}(M, N) & \xrightarrow[j^*]{} & H^{n-1}(M) & \xrightarrow[i^*]{} & H^{n-1}(N) & \longrightarrow & H^n(M, N) \end{array}$$

Since the map $H^n(M, N)\to H^n(M)$ is onto and $H^{n-1}(N)$, $H^n(M, N)$ are infinite cyclic, the map $H^{n-1}(N)\to H^n(M, N)$ is injective and the map $H^{n-1}(M)\to H^{n-1}(N)$ is trivial. In addition, $H^n(M_2, N)$ must vanish because $H^n(M_2, N)=H^n(Q-\mathring{D}_2^n, bN)$ and $b(Q-\mathring{D}_2^n)=bN\cup bD_2^n$. If we identify $H^{n-1}(N)\cong H^{n-1}(W)$ with the group of integers Z, the map i'^* may be substituted by $\alpha: H^{n-1}(M_2)\to Z$ where α is defined by $\alpha(x)=(i''^*(x))[W]$ for $x\in H^{n-1}(M_2)$. k^* is an isomorphism from the exact sequence of the triple (M, M_2, N). By diagram chasing, k'^* is injective. Now it is clear that the following sequence is exact.

$$0\longrightarrow H^{n-1}(M)\xrightarrow{k'^*} H^{n-1}(M_2)\xrightarrow{\alpha} Z\longrightarrow 0$$

Therefore

$$\alpha(d(f', f_0))=(X(\nu_f)-X(\nu_{f_0}))/4\ .$$

Hence $K(f_0)\subset\operatorname{Ker}\alpha$ and $K(f_0)$ is contained in the image of $H^{n-1}(M)$ under k'^*. Thus, by Lemma 3.4, the proof is complete.

Appendix: Proof of Mahowald's theorem

We will give a geometric proof of the theorem of Mahowald [4] in our line.

MAHOWALD'S THEOREM. *Let $\sigma(M^n)$ be an integer defined by $\sigma(M^n)=0$ if $\bar{w}_1(M)\bar{w}_{n-1}(M)=0$ and $\sigma(M^n)=1$ if $\bar{w}_1(M)\bar{w}_{n-1}(M)\neq 0$. Then for any embedding f of M^n in R^{2n}, $X(\nu_f)\equiv 2\sigma(M^n)$ (mod. 4) holds.*

PROOF. By Lemma 2.1, the normal Euler class of M^n is an invariant mod. 4.

Therefore it is sufficient to show that $X(\nu_g)=2\sigma(M^n)$ for a special embedding g which we are going to construct.

Let e' be an embedding of W^{n-1} in R^{2n-2}. By the natural inclusion of R^{2n-2} in R^{2n-1}, we obtain an embedding e of W^{n-1} in R^{2n-1}. Let i denote the inclusion map of W^{n-1} in M^n and ν_i its normal bundle. The vector bundle $\mathrm{Hom}(\nu_i, \nu_e)$ admits a non-singular section since $\dim \mathrm{Hom}(\nu_i, \nu_e) > \dim W^{n-1}$. This non-singular section can be identified with an inclusion map of vector bundles because ν_i is a line bundle. Therefore, e can be extended to an embedding $\bar{e}$ of N^n in R^{2n-1}. Let R_+^{2n} denote the set of points of R^{2n} with $x_{2n}\geq 0$. Then $\bar{e}$ can be regarded as an embedding of N^n in bR_+^{2n}. Since R_+^{2n} is contractible, there exists an embedding g_+ of Q^n in R_+^{2n} with $g_+|bQ^n=\bar{e}|bN^n$ and $g_+^{-1}(bR_+^{2n})\cap Q^n=bQ^n$. $\bar{e}$ and g_+ yield an embedding g of M^n in R^{2n} by smoothing corners along $bN=bQ$.

Next we will compute $X(\nu_g)$. From §2, we have

$$X(\nu_g)=Lk(g(bQ), u(bQ))-Lk(\bar{e}(bN), v(bN)) \quad \text{in} \quad R^{2n-1}.$$

Let S be an n-chain in R^{2n-1} with $dC=g(bQ)$ and let S_ε be the n-chain in R^{2n} obtained by moving S parallel along the x_{2n}-axis in the positive direction by small amount ε. Define an n-chain V in R^{2n} by $V: bQ\times[0, \varepsilon]\to R^{2n}=R^{2n-1}\times R$, $V(x, t)=(g(x), t)$. Then from the definition of linking coefficients,

$$\begin{aligned} Lk(g(bQ), u(bQ)) &= S\circ u(bQ) \quad \text{in} \quad R^{2n-1} \\ &= (S_\varepsilon+V)\circ u(Q) \quad \text{in} \quad R^{2n}. \end{aligned}$$

The difference between $S_\varepsilon+V$ and $g(Q)$ is equal to the boundary of an $(n+1)$-chain in the interior of R_+^{2n}. Hence,

$$(S_\varepsilon+V)\circ u(Q)=g(Q)\circ u(Q)=0$$

since $g(Q)\cap u(Q)=\emptyset$. Therefore we have

$$X(\nu_g)=-Lk(\bar{e}(bN), v(bN)) \quad \text{in} \quad R^{2n-1}.$$

Since bN is a 2-fold covering of W^{n-1} and $\bar{e}(N)$ does not meet $v(bN)$, $\bar{e}(bN)$ is homologous to $2e(W)$ in $R^{2n-1}-v(bN)$. Therefore, we have

$$X(\nu_g)=-2Lk(e(W), v(bN)) .$$

Similarly, $v(bN)$ is homologous to $\bar{e}(bN)$ in $R^{2n-1}-e(W)$. Hence,

$$X(\nu_g)=-2Lk(e(W), \bar{e}(bN)) \quad \text{in} \quad R^{2n-1}.$$

Lastly we will show that

$$Lk(e(W), \bar{e}(bN))=\sigma(M^n) \quad \text{mod. } 2.$$

In what follows ($)_2$ and [$]_2$ mean reduction mod. 2.

$$(\sigma(M^n))_2=\bar{w}_1\bar{w}_{n-1}[M]_2=\bar{w}_{n-1}(\bar{w}_1\cap[M]_2)=i^*\bar{w}_{n-1}[W]_2=w_{n-1}(\nu_g|W)[W]_2$$
$$=(X(\nu_{\bar{e}}|W))_2[W]_2 .$$

Let W' be the image of a cross-section of $\nu_{\bar{e}}|W^{n-1}$ which is t-regular to W^{n-1}. Then since the Euler class can be identified with the self-intersection of the base space, we have

$$(X(\nu_{\bar{e}}|W))_2=(W'\circ\bar{e}(W))_2=(W'\circ\bar{e}(N))_2=(Lk(W',\bar{e}(bN)))_2=(Lk(e(W),\bar{e}(bN)))_2 .$$

This completes the proof.

References

[1] Haefliger, A., Plongements différentiables de variétés dans variétés, Comment. Math. Helv. **36** (1961), 47-82.
[2] Haefliger, A. and M. Hirsch, On the existence and classification of differentiable embeddings, Topology **2** (1963), 129-135.
[3] Hirsch, M., On imbedding differentiable manifolds in Euclidean space, Ann. of Math. **73** (1961), 566-571.
[4] Mahowald, M., On the normal bundle of a manifold, Pacific J. Math. **14** (1964), 1335-1341.
[5] Malyi, B. D., On the Whitney-Mahowald theorem concerning the normal numbers of smooth embeddings, Math. Notes Acad. Sci. USSR **5** (1969), 57-60. (Translated from Mat. Zametki **5** (1969), 91-97.)
[6] Massey, W. S., Proof of a conjecture of Whitney, Pacific J. Math. **31** (1969), 143-156.
[7] Whitney, H., On the topology of differentiable manifolds, Lectures in Topology, Michigan Press, 1940.
[8] Whitney, H., The self-intersections of a smooth n-manifolds, in $2n$-space, Ann. of Math. **45** (1944), 220-246.

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