A note on nonlinear dispersive operators

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1. Introduction. Let $\mathfrak{B}$ be a Banach lattice imbedded in a vector lattice $\tilde{\mathfrak{B}}$ and let e be an element of $\tilde{\mathfrak{B}}$ such that

$$e\wedge f\in\mathfrak{B} \quad \text{if } f\in\mathfrak{B}\,. \tag{1.1}$$

We have introduced in [4] a functional

$$\varphi_e(f, g)=\lim_{\varepsilon\to 0+}\varepsilon^{-1}(\|(f-e+\varepsilon g)^+\|-\|(f-e)^+\|) \tag{1.2}$$

for $f,\ g\in\mathfrak{B}$, and called an operator A (φ_e, γ)-*dispersive* if

$$\varphi_e(u, Au)\leqq\gamma\|(u-e)^+\| \quad \text{whenever} \quad u\in\mathfrak{D}(A) \quad \text{and} \quad (u-e)^+\neq 0\,, \tag{1.3}$$

γ being a real number. Under some additional conditions on e, we have proved that if A is the infinitesimal generator of a strongly continuous semigroup $\{T_t; t\geqq 0\}$ of linear operators, then (φ_e, γ)-dispersiveness is a necessary and sufficient condition in order that T_t be e-majoration preserving and have norm $\leqq e^{\gamma t}$. In this note we will prove similar results for nonlinear operators and show examples. A typical result is that if an operator A (nonlinear in general) is (φ_e, γ)-dispersive and u_t is a strongly continuous mapping of $[0, T]$ into $\mathfrak{B}$ and satisfies

$$(d/dt)u_t=Au_t\,, \qquad 0<t<T\,, \tag{1.4}$$

then

$$e^{-\gamma t_1}\|(u_{t_1}-e)^+\|\geqq e^{-\gamma t_2}\|(u_{t_2}-e)^+\| \quad \text{whenever} \quad 0\leqq t_1\leqq t_2\leqq T\,. \tag{1.5}$$

We will discuss various definitions of dispersiveness which are equivalent to (1.3) if the operator is linear and $e=0$. There is a freedom of choice of other functionals than φ_e, as is investigated in [4]. One of them is the functional φ'_e defined by

$$\varphi'_e(f, g)=-\varphi_e(f, -g)\,. \tag{1.6}$$

In concrete Banach lattices such as C and L_p, we can find explicit expression of the functional φ_e. Hence we can find typical examples of operators which are dispersive in these spaces.

In case $e=0$, closely related results are found in Calvert [1] and Konishi [2].

2. A lemma. The following lemma reveals that the introduction of the functional φ_e is quite natural.

LEMMA. *Let u_t be a strongly continuous mapping of* $[0, T]$ *into* $\mathfrak{B}$ *which possesses a strong right derivative*

$$(d/dt)_r u_t = \operatorname*{s\text{-}lim}_{h\to 0+} h^{-1}(u_{t+h}-u_t)$$

for $t\in(0, T)$, *and let γ be a real number. Then, the following three conditions are equivalent*:

(a) (1.5) *holds.*

(b) *For all* $t\in(0, T)$,

$$\varphi_e(u_t, (d/dt)_r u_t) \leqq \gamma \|(u_t-e)^+\| \tag{2.1}$$

holds.

(c) *There is a positive number α satisfying*

$$\|(u_0-e)^+\| < \alpha \tag{2.2}$$

and having the property that (2.1) *holds for any* $t\in(0, T)$ *that satisfies*

$$0 < e^{-\gamma t}\|(u_t-e)^+\| < \alpha\ . \tag{2.3}$$

PROOF. Let $(d/dt)_r u_t = w_t$. Noting that

$$\|(u_{t+h}-e)^+\| \leqq \|(u_{t+h}-u_t-hw_t)^+\| + \|(u_t-e+hw_t)^+\|\ ,$$

$$\|(u_t-e+hw_t)^+\| \leqq \|(u_{t+h}-u_t-hw_t)^-\| + \|(u_{t+h}-e)^+\|\ ,$$

we have

$$\|(u_{t+h}-e)^+\| = \|(u_t-e+hw_t)^+\| + o(h)\ , \qquad h\to 0+\ ,$$

since $\|u_{t+h}-u_t-hw_t\| = o(h)$. Hence we have

$$\lim_{h\to 0+} h^{-1}(\|(u_{t+h}-e)^+\| - \|(u_t-e)^+\|) = \varphi_e(u_t, w_t) \tag{2.4}$$

by the definition (1.2) of φ_e. It follows that

$$(d/dt)_r(e^{-\gamma t}\|(u_t-e)^+\|) = e^{-\gamma t}\{\varphi_e(u_t, w_t) - \gamma\|(u_t-e)^+\|\}\ , \tag{2.5}$$

where $(d/dt)_r$ denotes right derivative. Hence, using a generalization of the mean value theorem to right derivatives, we see the equivalence of (a) and (b). (b) implies (c) trivially. Suppose that (c) holds, and let us prove (a). First we claim

$$e^{-\gamma t}\|(u_t-e)^+\| < \alpha \tag{2.6}$$

for all $t\in[0, T]$. In fact, suppose that (2.6) does not hold for some $t\in(0, T]$.

Then we can find $0 \leqq t_1 < t_2 \leqq T$ such that

$$e^{-\gamma t_2} \|(u_{t_2} - e)^+\| = \alpha$$

and (2.3) holds for any $t \in [t_1, t_2)$. Hence $(d/dt)_r (e^{-\gamma t} \|(u_t - e)^+\|)$ is positive at some $t \in (t_1, t_2)$, which contradicts the assumption. Thus we have (2.6). Let $0 \leqq t_1 < t_2 \leqq T$. In case $(u_t - e)^+ \neq 0$ for all $t \in (t_1, t_2)$, then (2.1) holds for all $t \in (t_1, t_2)$ by (c), and hence the inequality in (1.5) holds. In case $(u_t - e)^+ = 0$ for some $t \in (t_1, t_2)$, then we have $(u_{t_2} - e)^+ = 0$. For, if $(u_{t_2} - e)^+ \neq 0$, then we can find $t_0 \in (t_1, t_2)$ such that $(u_{t_0} - e)^+ = 0$ and $\|(u_t - e)^+\| > 0$ for all $t \in (t_0, t_2]$, and it follows from (c) that

$$(d/dt)_r (e^{-\gamma t} \|(u_t - e)^+\|) \leqq 0$$

for all $t \in (t_0, t_2)$, which is absurd. Hence the proof is complete.

REMARK 2.1. Let us denote the strong left derivative of u_t by $(d/dt)_l u_t$. *The above lemma remains valid if we replace "right", $(d/dt)_r$, and φ_e by "left", $(d/dt)_l$, and φ'_e, respectively.* The proof is quite similar.

3. Various definitions of nonlinear dispersiveness. Let A be an operator (nonlinear in general) with domain and range in $\mathfrak{B}$. Let α be a positive number or $+\infty$, and γ be a real number. We introduce the following three conditions.

$D_1(e, \gamma, \alpha)$: $\varphi_e(u, Au) \leqq \gamma \|(u - e)^+\|$ for every $u \in \mathfrak{D}(A)$ such that $0 < \|(u - e)^+\| < \alpha$.

$D_2(e, \gamma, \alpha)$: $\varphi_e(-u, -Au) \leqq \gamma \|(u + e)^-\|$ for every $u \in \mathfrak{D}(A)$ such that $0 < \|(u + e)^-\| < \alpha$.

$D_3(e, \gamma, \alpha)$: $\varphi_e(u - v, Au - Av) \leqq \gamma \|(u - v - e)^+\|$ for every $u, v \in \mathfrak{D}(A)$ such that $0 < \|(u - v - e)^+\| < \alpha$.

If $0 \in \mathfrak{D}(A)$ and $A0 = 0$, then $D_3(e, \gamma, \alpha)$ implies $D_1(e, \gamma, \alpha)$ and $D_2(e, \gamma, \alpha)$. If we define B by $Bu = -A(-u)$, then the condition $D_2(e, \gamma, \alpha)$ for A is equivalent to $D_1(e, \gamma, \alpha)$ for B. If A is linear and $e = 0$, then each of $D_i(e, \gamma, \alpha)$, $i = 1, 2, 3$, is equivalent to (1.3), as is seen from the properties

$$\varphi_e(f, \alpha g) = \alpha \varphi_e(f, g) \quad \text{for} \quad \alpha \geqq 0 , \tag{3.1}$$

$$\varphi_0(\alpha f, g) = \varphi_0(f, g) \quad \text{for} \quad \alpha > 0 . \tag{3.2}$$

The following proposition shows the implication of the properties $D_i(e, \gamma, \alpha)$ of A for the mapping u_t which satisfies

$$(d/dt)_r u_t = A u_t . \tag{3.3}$$

PROPOSITION. *Let u_t be a strongly continuous mapping of $[0, T]$ into $\mathfrak{D}(A)$ which has a strong right derivative for $t \in (0, T)$ and satisfies (3.3). Let $\beta = \min\{\alpha e^{-\gamma T}, \alpha\}$.*

(i) *If A satisfies $D_1(e, \gamma, \alpha)$ and $\|(u_0 - e)^+\| < \beta$, then (1.5) holds.*

(ii) *If A satisfies $D_2(e, \gamma, \alpha)$ and $\|(u_0+e)^-\|<\beta$, then*

$$e^{-\gamma t_1}\|(u_{t_1}+e)^-\| \geqq e^{-\gamma t_2}\|(u_{t_2}+e)^-\| \quad \textit{whenever} \quad 0 \leqq t_1 \leqq t_2 \leqq T. \tag{3.4}$$

(iii) *If v_t, $t\in[0, T]$, satisfies the same assumption as is stated for u_t and if A satisfies $D_3(e, \gamma, \alpha)$ and $\|(u_0-v_0-e)^+\|<\beta$, then*

$$e^{-\gamma t_1}\|(u_{t_1}-v_{t_1}-e)^+\| \geqq e^{-\gamma t_2}\|(u_{t_2}-v_{t_2}-e)^+\| \quad \textit{whenever} \quad 0 \leqq t_1 \leqq t_2 \leqq T\,. \tag{3.5}$$

(iv) *If A satisfies $D_3(0, \gamma, \alpha)$, then*

$$e^{-\gamma t_1}\|(Au_{t_1})^+\| \geqq e^{-\gamma t_2}\|(Au_{t_2})^+\|\,, \tag{3.6}$$

$$e^{-\gamma t_1}\|(Au_{t_1})^-\| \geqq e^{-\gamma t_2}\|(Au_{t_2})^-\| \quad \textit{for} \quad 0<t_1 \leqq t_2 < T\,. \tag{3.7}$$

PROOF. Suppose that A satisfies $D_1(e, \gamma, \alpha)$ and $\|(u_0-e)^+\|<\beta$. By (3.3) the inequality in the condition $D_1(e, \gamma, \alpha)$ is no other than (2.1). If $0<e^{-\gamma t}\|(u_t-e)^+\|<\beta$, then we have $0<\|(u_t-e)^+\|<\alpha$. Thus the condition (c) follows with β in place of α, and (1.5) follows by the lemma. Hence we have (i). The assertions (ii) and (iii) are proved likewise. In order to prove (iv), choose $\varepsilon>0$ so small that $\|(u_\varepsilon-u_0)^+\|<\beta$. Noting (3.3) and $(d/dt)_r u_{t+\varepsilon}=Au_{t+\varepsilon}$ and using the assertion (iii) we have

$$e^{-\gamma t_1}\|(u_{t_1+\varepsilon}-u_{t_1})^+\| \geqq e^{-\gamma t_2}\|(u_{t_2+\varepsilon}-u_{t_2})^+\| \quad \text{for} \quad 0<t_1 \leqq t_2 < T-\varepsilon\,.$$

Divide the both sides by ε and let $\varepsilon\to 0+$. Then we get

$$e^{-\gamma t_1}\|w_{t_1}^+\| \geqq e^{-\gamma t_2}\|w_{t_2}^+\| \quad \text{for} \quad 0<t_1 \leqq t_2 < T$$

where $w_t=(d/dt)_r u_t$, and hence (3.6). Similarly we get (3.7), completing the proof.

REMARK 3.1. We define the conditions $D_i'(e, \gamma, \alpha)$, $i=1, 2, 3$, replacing φ_e by φ_e' in $D_i(e, \gamma, \alpha)$, $i=1, 2, 3$, respectively. *The above proposition remains true if we replace "right", $(d/dt)_r$, D_1, D_2 and D_3 by "left", $(d/dt)_l$, D_1', D_2' and D_3', respectively.* It is easy to rewrite the proof using Remark 2.1. The conditions D_1', D_2' and D_3' are weaker than the conditions D_1, D_2 and D_3, respectively, since

$$\varphi_e'(f, g) \leqq \varphi_e(f, g)\,. \tag{3.8}$$

REMARK 3.2. Let $i=1, 2$ or 3 and let c_1 and c_2 be nonnegative numbers. If A_1 and A_2 satisfy $D_i(e, \gamma_1, \alpha)$ and $D_i(e, \gamma_2, \alpha)$, respectively, then $c_1A_1+c_2A_2$ satisfies $D_i(e, c_1\gamma_1+c_2\gamma_2, \alpha)$. The same assertion does not hold for D_i' in place of D_i. But, if A_1 satisfies $D_i(e, \gamma_1, \alpha)$ and A_2 satisfies $D_i'(e, \gamma_2, \alpha)$, then $c_1A_1+c_2A_2$ satisfies $D_i'(e, c_1\gamma_1+c_2\gamma_2, \alpha)$. These are seen from (3.1) and

$$\varphi_e(f, g+h) \leqq \varphi_e(f, g)+\varphi_e(f, h)\,. \tag{3.9}$$

This property of φ_e is found in [4] pp. 434 and 436 together with the properties (3.1), (3.2) and (3.8).

REMARK 3.3. By the equivalence of (a), (b) and (c) proved in the Lemma, the conditions $D_i(e, \gamma, \alpha)$ of A are close to necessary for the solution of (3.3) to have the properties stated in the proposition.

4. Examples in lattices like C.

4.1. If $\mathfrak{B}$ is a Banach lattice of real-valued bounded functions on a set X with $\|f\|$ being the supremum of $|f(x)|$ and $f \wedge g$ being the pointwise minimum of f and g, and if $e(x)$ is a function (possibly unbounded) on X satisfying (1.1), then we have

$$\varphi_e(f, g) = \lim_{\varepsilon \to 0+} \sup_{x \in X(f-e, \varepsilon)} g(x) \quad \text{if} \quad (f-e)^+ \neq 0\,, \tag{4.1}$$

where $X(f-e, \varepsilon)$ is the set of points x such that $f(x)-e(x) > \|(f-e)^+\| - \varepsilon$. The proof is similar to [3] p. 433.

4.2. Consider the Banach lattice $C(R^d)$ of bounded continuous functions on R^d, and let Δ be the Laplacian restricted to C^2 functions u such that $\Delta u(x) \to 0$ as $|x| \to \infty$. Δ is then $(\varphi_e, 0)$-dispersive for any constant function e. Suppose that a given operator A is expressed as

$$(Au)(x) = \Delta u(x) + \Phi(u(x))\,, \tag{4.2}$$

where Φ is a mapping of R^1 into R^1. The following assertions are easily proved from (4.1). If $-\infty < a < b \leqq +\infty$ and if $\Phi(\xi) \leqq 0$ for $a < \xi < b$, then A satisfies $D_1(e, 0, b-a)$ where $e(x) = a$ (constant function). It follows that if $-\infty < b \leqq +\infty$ and $\Phi(\xi) \leqq 0$ for $\xi < b$, then $\max_x u_0(x) < b$ implies

$$\max_x u_{t_1}(x) \geqq \max_x u_{t_2}(x) \qquad \text{for} \quad 0 \leqq t_1 \leqq t_2 \leqq T\,,$$

provided that u_t is strongly continuous on $[0, T]$, has a strong right derivative on $(0, T)$ and satisfies (3.3). If $-\infty \leqq a < b < +\infty$ and $\Phi(\xi) \geqq 0$ for $a < \xi < b$ then A satisfies $D_2(e, 0, b-a)$ where $e(x) = -b$ (constant function). It similarly follows that if $-\infty \leqq a < +\infty$ and $\Phi(\xi) \geqq 0$ for $\xi > a$, then $\min_x u_0(x) > a$ implies

$$\min_x u_{t_1}(x) \leqq \min_x u_{t_2}(x) \qquad \text{for} \quad 0 \leqq t_1 \leqq t_2 \leqq T\,.$$

4.3. If Φ is a monotone nonincreasing mapping of R^1 into R^1 and if A is an operator in $C(R^d)$ such that

$$(Au)(x) = \Phi(u(x))\,, \tag{4.3}$$

then A satisfies $D_3(e, 0, +\infty)$ for any nonnegative continuous function e. In fact, we have $(Au-Av)(x)\leqq 0$ for any x such that $(u-v-e)(x)\geqq 0$.

4.4. In 4.4–4.6, we consider the Banach lattice $C_0(R^1)$ of continuous functions on R^1 vanishing at $\pm\infty$. For any continuous function e satisfying (1.1), we have

$$\varphi_e(f, g)=\max_{x\in X(f-e)} g(x) \qquad \text{if} \quad (f-e)^+\neq 0\,, \tag{4.4}$$

where $X(f-e)$ is the set of points x such that $(f-e)(x)=\|(f-e)^+\|$. (See [4] p. 437. This is also a consequence of (4.1).) Let $\Phi(\xi, \eta, \zeta)$ be a mapping of R^3 into R^1 which is monotone nonincreasing in ξ and monotone nondecreasing in ζ, and let

$$(Au)(x)=\Phi(u(x), u'(x), u''(x))\,. \tag{4.5}$$

Then, A satisfies $D_3(e, 0, +\infty)$ for any nonnegative constant function e. In fact, if $u-v-e$ attains a positive maximum at a point x_0, then $u\geqq v$, $u'=v'$, and $u''\leqq v''$ at x_0, and hence

$$Au-Av=\Phi(u, u', u'')-\Phi(v, v', v'')\leqq 0 \qquad \text{at} \quad x_0\,.$$

Simple examples are

$$Au=-u^{2n+1}(u')^{2m} \quad \text{and} \quad Au=(u'')^{2n+1}(u')^{2m}\,,$$

n and m being nonnegative integers.

4.5. Let Φ be a mapping of R^3 into R^1 such that $\Phi(\xi, 0, \zeta)\leqq 0$ if $\xi\geqq 0$ and $\zeta\leqq 0$, let B be an operator such that $Bu\geqq 0$ for any $u\in\mathfrak{D}(B)$, and let

$$(Au)(x)=\Phi(u(x),\ u'(x),\ u''(x))\cdot(Bu)(x)\,. \tag{4.6}$$

Then A satisfies $D_1(e, 0, +\infty)$ for any nonnegative constant function e, since $u\geqq 0$, $u'=0$, and $u''\leqq 0$ at the point x_0 where $u-e$ attains a positive maximum.

4.6. Let Φ be a mapping of R^1 into R^1 such that $\Phi(0)=0$, let B be any operator, and let

$$(Au)(x)=\Phi(u'(x))\cdot(Bu)(x)\,. \tag{4.7}$$

Then A satisfies $D_1(e, 0, +\infty)$ and $D_2(e, 0, +\infty)$ for any nonnegative constant function e, since $(Au)(x_0)=0$ at the point x_0 where u attains a maximum or minimum.

In the above examples the Laplacian (or the second order derivative in one-dimensional case) can be replaced by infinitesimal generators of Markov processes. Infinitesimal generators of spatially homogeneous Markov processes are $(\varphi_e, 0)$-dispersive for any nonnegative constant function e also in $L_p(R^d)$, $1\leqq p\leqq +\infty$. Hence, using Remark 3.2 on sums of operators, we can make many examples in L_1 and L_2 from operators considered below.

5. Examples in L_1.

5.1. Let $\mathfrak{B}$ be the real L_1 space on a measure space $(X, \mathscr{B}, m)$, and let e be a nonnegative $\mathscr{B}$-measurable function. Then we have, [4] p. 437,

$$\varphi_e(f, g)=\int_{\{x: f(x)>e(x)\}} g(x)m(dx)+\int_{\{x: f(x)=e(x)\}} g^+(x)m(dx) . \tag{5.1}$$

It follows that if A is an operator in this space represented as (4.3) by a monotone nonincreasing mapping Φ of R^1 into R^1, then A satisfies $D_3(e, 0, +\infty)$.

5.2. Let $\Phi(\xi, \eta)$ be a C^2 mapping of R^2 into R^1, which is monotone nondecreasing in η, and let A be an operator in $L_1(R^1)$ represented as

$$(Au)(x)=\frac{d}{dx}\Phi(u(x), u'(x)) \tag{5.2}$$

with the domain $\mathfrak{D}(A)$ being a class of C^2 functions u such that u and u' belong to $C_0(R^1)$. Let e be a nonnegative constant function. Then A satisfies $D_1(e, 0, +\infty)$, $D_2(e, 0, +\infty)$, and $D_3(0, 0, +\infty)$. In case Φ is a function only of η, A satisfies also $D_3(e, 0, +\infty)$. Proof is as follows. In order to show $D_3(0, 0, +\infty)$, it suffices to prove

$$\int_E \frac{d}{dx}\{\Phi(u, u')-\Phi(v, v')\}dx \leqq 0 , \tag{5.3}$$

$$\int_F \left(\frac{d}{dx}\{\Phi(u, u')-\Phi(v, v')\}\right)^+ dx=0 \tag{5.4}$$

for every pair of $u, v \in \mathfrak{D}(A)$, where $E=\{x: u(x)>v(x)\}$ and $F=\{x: u(x)=v(x)\}$. Let (a, b) be a connected component of E. If (a, b) is a finite interval, then

$$\int_a^b \frac{d}{dx}\{\Phi(u, u')-\Phi(v, v')\}dx=[\Phi(u, u')-\Phi(v, v')]_a^b \leqq 0 ,$$

since $u(a)=v(a)$, $u'(a)\geqq v'(a)$, $u(b)=v(b)$, and $u'(b)\leqq v'(b)$. If (a, b) is an infinite interval, we can get the same conclusion, noting that u, v, and their derivatives vanish at $\pm\infty$. Hence we get (5.3). Decompose $F=F_1\cup F_2$ where F_1 is the set of accumulation points of F and F_2 is the set of isolated points of F, and further $F_1=F_{11}\cup F_{12}$ where F_{11} is the set of accumulation points of F_1 and F_{12} is the set of isolated points of F_1. Since F_2 and F_{12} are countable, they do not contribute to the integral in (5.4). We see $u=v$ and $u'=v'$ everywhere on F_1, and hence $(d/dx)\{\Phi(u, u')-\Phi(v, v')\}$ vanishes on F_{11}. Thus (5.4) follows. The other assertions are proved quite similarly. This example includes an operator

$$(Au)(x)=\frac{d}{dx}\left(a(u)\frac{du}{dx}\right)+b(u)\frac{du}{dx} \qquad \text{with} \quad a(u)\geqq 0 .$$

6. Examples in L_2.

6.1. In case $\mathfrak{B}$ is the real L_2 space on a measure space $(X, \mathscr{B}, m)$ and e is a nonnegative $\mathscr{B}$-measurable function, we have, [4] p. 437,

$$\varphi_e(f, g)=\int_X (f-e)^+ g m(dx) \Big/ \|(f-e)^+\| \qquad \text{if} \quad (f-e)^+\neq 0 . \tag{6.1}$$

Hence an operator A represented as (4.3) by a monotone nonincreasing mapping $\Phi(\xi)$ satisfies $D_3(e, 0, +\infty)$ in this space.

6.2. If $\Phi(\xi)$ is a monotone nondecreasing C^1 mapping of R^1 into R^1 and A is an operator in $L_2(R^1)$ with the form

$$(Au)(x)=\frac{d}{dx}\Phi(u'(x)) , \tag{6.2}$$

the domain $\mathfrak{D}(A)$ being a class of bounded C^2 functions u such that u' belongs to $C_0(R^1)$, then A satisfies $D_3(e, 0, +\infty)$ for any nonnegative constant function e. In fact, let (a, b) be a connected component of the set $\{x : u(x)-v(x)-e>0\}$. If (a, b) is a finite interval, we have

$$\int_a^b (u-v-e)\frac{d}{dx}(\Phi(u')-\Phi(v'))dx$$
$$=[(u-v-e)(\Phi(u')-\Phi(v'))]_a^b-\int_a^b (u'-v')(\Phi(u')-\Phi(v'))dx ,$$

and in the right-hand side the first term is zero by the vanishing of $u-v-e$ at a and b and the integral is nonnegative by the monotonicity of Φ. If (a, b) is an infinite interval, the same conclusion holds by the assumption on the domain of A. Thus we have

$$\int_{-\infty}^{+\infty} (u-v-e)^+\frac{d}{dx}(\Phi(u')-\Phi(v'))dx\leqq 0 ,$$

and the assertion follows immediately.

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