# *Maximal subgroups of low rank of finite symmetric and alternating groups*

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## § 0. Introduction

The purpose of this note is to investigate maximal subgroups of low rank of finite symmetric groups $S_n$ and alternating groups $A_n$. Here the rank of a subgroup $H$ of a group $G$ is by definition the number of the $H$-double cosets $H\backslash G/H$. This equals the rank of the permutation group $G$ acting on the coset space $G/H$ via the natural action. Thus, classifying maximal subgroups of rank $l$ of $G$ up to inner automorphism of $G$ is equivalent to classifying primitive permutation representations of rank $l$ of $G$ up to equivalence as permutation representation.

In § 1 we shall present some general results on maximal subgroups of the groups $S_n$ and $A_n$. Our main results in § 1 are Theorems 1.3 and 1.4, which assert that the index of maximal subgroups of rank $l$ in $S_n$ (or $A_n$) is bounded by some relatively small value depending only on $n$ and $l$. In § 2 we will classify all maximal subgroups of the groups $S_n$ and $A_n$ which are of rank $\leqq 5$. (See also Tables 1 and 2 in Appendix). The Mathieu groups $M_{11}$ and $M_{12}$ are among them, with ranks 5 and 4 in $A_{11}$ and $A_{12}$ respectively. The results for those subgroups of rank 2 were formerly obtained by Ed. Maillet [2] in 1895. (Cf. E. Bannai [1][1).)

## § 1. General results on maximal subgroups of the groups $S_n$ and $A_n$

We state two elementary lemmas which are fundamental in our study.

LEMMA 1.1. *Let $G$ be a primitive permutation group of rank $l$ on a set $\Omega$ and let the lengths of the orbits of a stabilizer $G_a$ ($a\in\Omega$), ordered according to increasing magnitude, be $1=n_1\leqq n_2\leqq \cdots \leqq n_l$. Then $n_j\leqq n_2 n_{j-1}$ for every $j$, $2\leqq j\leqq l$.*

PROOF. See Theorem 17.4 in [6] page 47.

LEMMA 1.2. *Let the assumptions be the same as in Lemma 1.1. Then for*

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1) Permutation representations considered in [1] are all assumed to be faithful, although the author forgot to write down the assumption explicitly. The author takes this opportunity to correct some errata in [1]: page 296 on line 16, "$S_6/K$" should read "$S_6/K^\sigma$"; page 296 on line 20, "as permutation groups" should read "as permutation representations".

*every non-identity element $x$ of $G$, there exist at least $n_2$ elements of $G$ which are conjugate to $x$.*

(This lemma is a slight extension of Lemma 1 in Ed. Maillet [2] page 21, where the assertion was proved only for the case $l=2$.)

PROOF. Let $x=(c, d, \cdots)\cdots$, $c, d\in\Omega$, where the cycle containing $c$ is of length greater than 1. Let $d$ lie on the $i$-th orbit of the stabilizer $G_c$ of the length $n_i$. Then, for every element $d_j$ $(j=1, \cdots, n_i)$ in the $i$-th orbit, there exists an element $y_j\in G_c$ such that $d^{y_j}=d_j$. These $n_i$ elements $y_j^{-1}xy_j$ $(j=1, \cdots, n_i)$ are all distinct from each other and are of course conjugate to $x$. Hence the assertion is proved.

Now we state some properties about maximal subgroups of the groups $S_n$ and $A_n$. In the following, let $\Sigma_n$ be a fixed set of $n$ letters on which $S_n$ and $A_n$ act naturally.

I. Let $H$ be a maximal subgroup of $S_n$ (resp. $A_n$), and let $H$ be intransitive on $\Sigma_n$. Then $H$ consists of the elements of $S_n$ (resp. $A_n$) which fix a given subset $\Sigma_r$ (consisting of $r$ letters) of $\Sigma_n$, where $r<[n/2]$. Moreover, let $H'$ be such a subgroup fixing some subset $\Sigma'_{r'}$. Then $H$ and $H'$ are conjugate in $S_n$ (resp. $A_n$) if and only if $r=r'$. We denote by $H(\Sigma_r)$ (resp. $H(\Sigma_r)\cap A_n$) the subgroup of $S_n$ (resp. $A_n$) which consists of the elements fixing some given subset $\Sigma_r$ with $r<[n/2]$. Clearly $|H(\Sigma_r)|=r!(n-r)!$ (resp. $|H(\Sigma_r)\cap A_n|=1/2\cdot r!(n-r)!$), and in fact, $(S_n, S_n/H(\Sigma_r))$ (resp. $(A_n, A_n/(H(\Sigma_r)\cap A_n))$ is the primitive permutation group acting on the set of unordered $r$-tuples of elements in $\Sigma_n$, and we can easily show that the primitive permutation group is of rank $r+1$.

II. Let $H$ be a maximal subgroup of $S_n$ (resp. $A_n$), and let $H$ be transitive but imprimitive on $\Sigma_n$. Then $H$ consists of the elements of $S_n$ (resp. $A_n$) which permute a complete set of blocks $\{\pi_i\}$ $(i=1, \cdots, v)$ (where such $\pi_i$ consists of $u$ letters) of $\Sigma_n$ among themselves, where $|\pi_i|=u\neq 1$, $n=uv$ and $v\neq 1$. Let $H'$ also be such a subgroup which consists of the elements permuting a complete set of blocks $\{\pi'_i\}$ $(i=1, \cdots, v')$ with $|\pi'_i|=u'\neq 1$, $n=u'v'$ and $v'\neq 1$. Then $H$ and $H'$ are conjugate in $S_n$ (resp. $A_n$) if and only if $u=u'$ (hence of course $v=v'$). We denote by $H(\Pi_u)$ (resp. $H(\Pi_u)\cap A_n$) the subgroup of $S_n$ (resp. $A_n$) which consists of the elements permuting a fixed nontrivial complete set of blocks $\pi_i$ with $|\pi_i|=u$ among themselves. Clearly $|H(\Pi_u)|=n!/(u!)^v v!$ (resp. $|H(\Pi_u)\cap A_n|=1/2\cdot n!/(u!)^v v!$ where $uv=n$, and $H(\Pi_u)$ (resp. $H(\Pi_u)\cap A_n$) are maximal in $S_n$ (resp. $A_n$). Moreover it is immediately proved that

$$|S_n : H(\Pi_n)| \geqq \left[\frac{n+1}{2}\right]! \quad \left(\text{resp. } |A_n : H(\Pi_n)\cap A_n| \geqq \frac{1}{2}\left[\frac{n+1}{2}\right]!\right).$$

**III.** Let $H$ be a maximal subgroup of $S_n$ (resp. $A_n$), and let $H$ be primitive on $\Sigma_n$. Then we have for

$$H\neq A_n,\ |S_n : H| \geqq \left[\frac{n+1}{2}\right]! \quad \left(\text{resp. } |A_n : H| \geqq \frac{1}{2}\cdot\left[\frac{n+1}{2}\right]!\right)$$

by the theorem of Bochert (Theorem 14.2 in [6] page 41).

Now we state our main theorems.

THEOREM 1.3. *If*

$$\left[\frac{n+1}{2}\right]! > 2\cdot\left(\frac{n(n-1)}{2}\right)^{l-1},$$

*then the $H(\Sigma_{l-1})$ are the only maximal subgroups ($\neq A_n$) of rank $l$ of the group $S_n$.*

THEOREM 1.4. *If*

$$\frac{1}{2}\cdot\left[\frac{n+1}{2}\right]! > 2\cdot\left(\frac{n(n-1)(n-2)}{3}\right)^{l-1},$$

*then the $H(\Sigma_{l-1})\cap A_n$ are the only maximal subgroups of rank $l$ of the group $A_n$.*

PROOF OF THEOREM 1.3. Let $G=S_n$ be represented as a primitive permutation group of rank $l$ on a set $\Omega$. Let $x$ be a transposition (i.e., an element consisting of one 2-cycle) in $S_n$. Then there exist $\frac{1}{2}n(n-1)$ elements of $S_n$ which are conjugate to $x$. Thus by Lemma 1.2, for the permutation group $(G, \Omega)$, the length $n_2$ of the smallest orbit $\neq\{a\}$ of the stabilizer $H=G_a$ $(a\in\Omega)$ is not greater than $\frac{1}{2}n(n-1)$. Therefore by Lemma 1.1, we obtain

$$\begin{aligned}|\Omega| &= 1+n_2+n_3+n_4+\cdots+n_l\\ &\leqq 1+n_2+n_2^2+n_2n_3+\cdots+n_2n_{l-1}\\ &\leqq 1+n_2+n_2^2+n_2^3+\cdots+n_2^{l-1}\\ &\leqq 2\cdot n_2^{l-1}\\ &\leqq 2\cdot\left(\frac{1}{2}n(n-1)\right)^{l-1}.\end{aligned}$$

Let us assume that $H$ is transitive on $\Sigma_n$. Then from the preceding argument in subsections II and III we obtain $|\Omega| \geqq \left[\frac{n+1}{2}\right]!$ Therefore

$$\left[\frac{n+1}{2}\right]! \leqq 2\cdot\left(\frac{1}{2}n(n-1)\right)^{l-1}.$$

But this is a contradiction. Hence $H$ is intransitive on $\Sigma_n$, and by the argument in subsection I, the assertion is proved.

PROOF OF THEOREM 1.4. For an element $x$ of $A_n$ which consists of one 3-cycle, there exist $\frac{1}{3}n(n-1)(n-2)$ elements of $A_n$ which are conjugate to $x$. By the argument similar to the proof of Theorem 1, the assertion is proved.

REMARK 1.5. The inequality

$$\left[\frac{n+1}{2}\right]! > 2\cdot\left(\frac{1}{2}n(n-1)\right)^{l-1}$$

holds, if $n>8$ for $l=2$, if $n>14$ for $l=3$, if $n>20$ for $l=4$, and if $n>26$ for $l=5$. Generally, the above inequality holds, e.g., if $n>14+6(l-3)$.

REMARK 1.6. The inequality

$$\frac{1}{2}\cdot\left[\frac{n+1}{2}\right]! > 2\cdot\left(\frac{n(n-1)(n-2)}{3}\right)^{l-1}$$

holds, if $n>14$ for $l=2$, if $n>20$ for $l=3$, if $n>26$ for $l=4$, and if $n>32$ for $l=5$. Generally the above inequality holds, e.g., if $n>14+6(l-2)$.

## § 2. Maximal supgroups of rank $\leqq 5$

From now on we shall always assume that $H$ is a maximal subgroup of rank $\leqq 5$ of the group $S_n$ or $A_n$. We also assume that $H\neq A_n$ when $H$ is a maximal subgroup of $S_n$.

I. Let us assume that $H$ is a maximal subgroup of the group $S_n$ and that $H$ is primitive on the set $\Sigma_n$.

By Remark 1.5, we have $n\leqq 26$. The list of primitive permutation groups of degree up to 20 has been known for a long time. (Cf. the book of R. D. Carmichael, Introduction to the theory of groups of finite order, page 162, Exercise 10). Here the references will be made to Table I of C. C. Sims [5], where primitive permutation groups of degree $n\leqq 20$ are classified up to conjugacy in $S_n$. Clearly $|H|\cdot 4\geqq |S_n : H|-1$ must hold, since $H$ is maximal and of rank $\leqq 5$ in $S_n$, and $H$ must not be contained in $A_n$. Thus, we can easily see from Table I of C. C. Sims [5] that if $n\leqq 20$ then only the following primitive subgroup $H$ listed below as 1)~6) satisfy the property mentioned above.

(2.1)

| | | | |
|---|---|---|---|
| 1) | $n=5$, | No. 3, | order 20; |
| 2) | $n=6$, | No. 2, | order 120; |
| 3) | $n=7$, | No. 4, | order 42; |
| 4) | $n=8$, | No. 4, | order 336; |
| 5) | $n=9$, | No. 7, | order 432; |
| 6) | $n=10$, | No. 7, | order 1440; |

where the No. are that of Table I in [5].

A quick examination of the character tables of the groups $S_n$ ($n \leqq 10$) shows that the subgroups $H$ of cases 3), 5) and 6) are of rank $\geqq 6$, whereas the ranks of the $H$ of cases 1), 2) and 4) are 2, 2 and 5 respectively.

Now assume that $21 \leqq n \leqq 26$. Then the following lemma can be applied.

LEMMA 2.1. *Let $H$ be a maximal subgroup of $S_n$ and of rank $\leqq 5$ and let $H$ be primitive on $\Sigma_n$. If $21 \leqq n \leqq 26$, then the minimal degree*[2] *of the primitive permutation group $(H, \Sigma_n)$ is $\leqq 15$.*

PROOF. The proof will be by the method of contradiction, so let us assume that the minimal degree is $>15$. By the proof of Theorem 1.3, we have

$$|S_n : H| \leqq 2 \cdot \left(\frac{1}{2} n(n-1)\right)^4 .$$

On the other hand, from the theorem of W. A. Manning [4] (cf. Theorem 14.1 in [6] page 40), $|S_n : H|$ must be divisible by $\prod_{q=1}^{\infty} \tau_q$. Here $\tau_q$ denotes the product of all prime numbers $p$ in the following intervals:

$$(2.2) \qquad \begin{cases} q=1: & 2 \leqq p < n-2 \, ; \\ q=2,\ 3,\ 4: & q+1 < p < \dfrac{n}{q} - 1 \, ; \\ q=5: & 5 < p < \dfrac{n-6}{5} \, ; \\ q=6: & 5 < p < \dfrac{n-10}{6} \, ; \\ q=7: & 2q-2 < p < (n-4q+4)/q \, . \end{cases}$$

Vacuous products are set to be 1. Now let $K (\cong S_{15})$ be the subgroup of $S_n$ consisting of all the elements which fix pointwisely the letters $16, \cdots, n$ of $\Sigma_n$. Let $K^{(p)}$ be a Sylow $p$-subgroup of the group $K$. Tnen $|K^{(2)}| = 2^{11}$ and $|K^{(3)}| = 3^6$. From the assumption that the minimal degree is $>15$, $H$ must not contain any element of $K^{(p)}$; hence $|S_n : H|$ must be divisible by $|K^{(p)}|$ for any prime $p$. Since $\prod_{q=1}^{\infty} \tau_q$ is divisible by 2 and 3 only by the first power, $|S_n : H|$ must be divisible by $2^{10} \cdot 3^5 \cdot \prod_{q=1}^{\infty} \tau_q$. But we can easily check that if $21 \leqq n \leqq 26$, then

$$2^{10} \cdot 3^5 \cdot \prod_{q=1}^{\infty} \tau_q > 2 \left(\frac{n(n-1)}{2}\right)^4 ,$$

and this contradicts the inequality

[2] For the definition of the minimal degree, see [6]. In [3] this is called the class of a permutation group.

$$|S_n : H| \leqq 2\left(\frac{n(n-1)}{2}\right)^4 .$$

Hence the lemma is proved.

Incidentally, those primitive permutation groups whose minimal degrees are $\leqq 15$ were completely classified formerly by the series of papers of W. A. Manning [3] (cf. [6], chap. II, § 15). Exploiting those results of W. A. Manning [3], we can easily conclude that if $21 \leqq n \leqq 26$, then there exists no primitive permutation group $H$ of degree $n$ such that $|H| \cdot 4 \geqq |S_n : H| - 1$.

Summarizing the above argument, we obtain the following theorem.

THEOREM 2.2. *Let $H$ ($\neq A_n$) be a maximal subgroup of rank* $\leqq 5$ *of $S_n$ and let $H$ be primitive on the set $\Sigma_n$. Then $H$ is in one of the cases* 1), 2) *and* 4) *in* (2.1). *Moreover, the subgroups $H$ of the same case form a single class up to conjugacy in $S_n$.*

II. Let us assume that $H$ is a maximal subgroup of the group $A_n$ and that $H$ is primitive on the set $\Sigma_n$.

Noting the fact that $|H| \cdot 4 \geqq |A_n : H| - 1$ and that $H$ is contained in $A_n$, we can easily see from Table 1 of C. C. Sims that if $n \leqq 20$, then $H$ is in one of the cases 7)$\sim$14) given below. Here the cases $n=8$, No. 2; $n=8$, No. 3 and $n=9$, No. 8 are eliminated because these subgroups are not maximal in $A_n$.

(2.3)

| | | | |
|---|---|---|---|
| 7) | $n=5$, | No. 2, | order 10; |
| 8) | $n=6$, | No. 1, | order 60; |
| 9) | $n=7$, | No. 5, | order 168; |
| 10) | $n=8$, | No. 5, | order 1344; |
| 11) | $n=9$, | No. 9, | order 1512; |
| 12) | $n=10$, | No. 6, | order 720; |
| 13) | $n=11$, | No. 6, | order 7920; |
| 14) | $n=12$, | No. 4, | order 95040; |

By examining the character tables of the groups $S_n$ ($n \leqq 12$), we see easily that the subgroup $H$ in the case 12) is of rank $\geqq 6$, and that the ranks of $H$ in the cases 7), 8), 9), 10), 11), 13), 14) are 2, 2, 2, 2, 3, 5 and 4 respectively.

Now we may assume that $21 \leqq n \leqq 32$ by Remark 1.6. A similar argument as in the proof of Lemma 2.1 shows that if the minimal degree of $(H, \Sigma_n)$ is $>15$, then $|A_n : H|$ is divisible by $2^9 \cdot 3^5 \cdot \prod_{q=1}^{\infty} \tau_q$, and we have

$$2^9 \cdot 3^5 \cdot \prod_{q=1}^{\infty} \tau_q > 2\left(\frac{n(n-1)(n-2)}{3}\right)^4$$

for every $n$ with $21 \leqq n \leqq 32$. Hence we obtain the following lemma.

LEMMA 2.3. *Let $H$ be a maximal subgroup of $A_n$ and of rank $\leqq 5$, and let $H$ be primitive on $\Sigma_n$. If $21 \leqq n \leqq 32$, then the minimal degree of the primitive permutation group $(H, \Sigma_n)$ is $\leqq 15$.*

The results of W. A. Manning [3] show that if $21 \leqq n \leqq 32$, then $A_n$ contains no primitive permutation subgroup $H$ such that $|H| \cdot 4 \geqq |A_n : H| - 1$. Hence, summarizing the above argument together with some additional considerations, we obtain the following theorem.

THEOREM 2.4. *Let $H$ be a maximal subgroup of rank $\leqq 5$ of the group $A_n$, and let $H$ be primitive on the set $\Sigma_n$. Then $H$ is in one of the cases* 7), 8), 9), 10), 11), 13) *and* 14) *in* (2.3). *Moreover, the subgroups $H$ in case* 7) *or* 8) *form a single class up to conjugacy in $A_n$, and the groups $H$ in case* 9) *or* 10), 11), 13), 14) *form two classes up to conjugacy in $A_n$.*

**III.** Let us assume that $H$ is a maximal subgroup of the groups $S_n$ and that $H$ is imprimitive on the set $\Sigma_n$.

If $H$ is intransitive on $\Sigma_n$, then the argument in the subsection I of § 1 shows that $H$ is conjugate to some $H(\Sigma_r)$ in $S_n$ with $r < [n/2]$ and $r \leqq 4$ since the rank of $H$ is $\leqq 5$, and that $H$ is of rank $r+1$.

Let us assume that $H$ is transitive on $\Sigma_n$. The argument in the subsection II of § 1 shows that $H$ is conjugate to some $H(\Pi_u)$ in $S_n$, where $uv = n$, $u \neq 1$, $v \neq 1$. Now let us count the rank of the subgroups $H(\Pi_u)$ in $S_n$. Let $M_v(u)$ denote the set of all $v \times v$ matrices $(a_{ij})$ whose coefficients are nonnegative integers satisfying the relations

$$\sum_{i=1}^{v} a_{ij} = \sum_{j=1}^{v} a_{ij} = u \ . \tag{2.4}$$

Two elements $(a_{ij})$ and $(b_{ij})$ of $M_v(u)$ will be called equivalent if there exists a permutation matrix $(p_{ij})$ such that $(p_{ij})^{-1}(a_{ij})(p_{ij}) = (b_{ij})$. Let $m_v(u)$ be the number of equivalence classes in $M_v(u)$. Then it is easily verified that the rank of the permutation group $(S_n, S_n/H(\Sigma_n))$ is equal to $m_v(u)$. Although we do not know the explicit value of $m_v(u)$ for an arbitrary pair of $u$ and $v$, we can easily determine those pairs of $u$ and $v$ for which the inequalities $m_v(u) \leqq 5$ hold:

THEOREM 2.5. *Let $H (\neq A_n)$ be a maximal subgroup of rank $\leqq 5$ of the group $S_n$, and let $H$ be imprimitive on the set $\Sigma_n$. Then $H$ is conjugate in $S_n$ to one of the members listed below:*

*a)* $H(\Sigma_r)$ *with* $r < [n/2]$ *and* $r \leqq 4$, *and of rank* $r+1$;

*b)* $H(\Pi_u)$ *with the following pairs of* $u$ *and* $v (= n/u)$:

| | | | |
|---|---|---|---|
| 1) | $u=2,\ v=2$ | $(n=4)$; | *of rank* 2 |
| 2) | $u=3,\ v=2$ | $(n=6)$; | *of rank* 2 |
| 3) | $u=2,\ v=3$ | $(n=6)$; | *of rank* 3 |
| 4) | $u=4,\ v=2$ | $(n=8)$; | *of rank* 3 |
| 5) | $u=2,\ v=4$ | $(n=8)$; | *of rank* 3 |
| 6) | $u=3,\ v=3$ | $(n=9)$; | *of rank* 5 |
| 7) | $u=5,\ v=2$ | $(n=10)$; | *of rank* 3 |
| 8) | $u=6,\ v=2$ | $(n=12)$; | *of rank* 4 |
| 9) | $u=7,\ v=2$ | $(n=14)$; | *of rank* 4 |
| 10) | $u=8,\ v=2$ | $(n=16)$; | *of rank* 5 |
| 11) | $u=9,\ v=2$ | $(n=16)$; | *of rank* 5. |

**IV.** Let us assume that $H$ is a maximal subgroup of the group $A_n$, and that $H$ is imprimitive on the set $\Sigma_n$.

If $H$ is intransitive on $\Sigma_n$, then the argument in the subsection I of §1 shows that $H$ is conjugate to some $H(\Sigma_r)\cap A_n$ in $A_n$, where $r<[n/2]$ and $r\leqq 4$, since the rank of $H$ is $\leqq 5$.

Let us assume that $H$ is transitive on $\Sigma_n$. The argument in the subsection II of §1 shows that $H$ is conjugate to some $H(\Pi_u)\cap A_n$ in $A_n$, where $uv=n$, $u\neq 1$ and $v\neq 1$. Clearly the rank of $H(\Pi_u)\cap A_n$ in $A_n$ is no less than the rank of $H(\Pi_u)$ in $S_n$. And we can easily determine the rank of the subgroup $H(\Pi_u)\cap A_n$ in $A_n$ by consulting the list in Theorem 2.5 b). Namely, we have the following theorem.

THEOREM 2.6. *Let $H$ be a maximal subgroup of rank $\leqq 5$ of the group $A_n$, and let $H$ be imprimitive on the set $\Sigma_n$. Then $H$ is conjugate in $A_n$ to one of the members listed below:*

*a)* $H(\Sigma_r)\cap A_n$ *with* $r<[n/2]$ *and* $r\leqq 4$; *and of rank* $r+1$;

*b)* $H(\Pi_u)\cap A_n$ *with the following pairs of* $u$ *and* $v$:

| | | | |
|---|---|---|---|
| 1) | $u=2,\ v=2$ | $(n=4)$; | *of rank* 2 |
| 2) | $u=3,\ v=2$ | $(n=6)$; | *of rank* 2 |
| 3) | $u=2,\ v=3$ | $(n=6)$; | *of rank* 3 |
| 4) | $u=4,\ v=2$ | $(n=8)$; | *of rank* 3 |
| 5) | $u=2,\ v=4$ | $(n=8)$; | *of rank* 3 |
| 6) | $u=3,\ v=3$ | $(n=9)$; | *of rank* 5 |
| 7) | $u=5,\ v=2$ | $(n=10)$; | *of rank* 3 |
| 8) | $u=6,\ v=2$ | $(n=12)$; | *of rank* 4 |
| 9) | $u=7,\ v=2$ | $(n=14)$; | *of rank* 4 |

10) $u=8,\ v=2$ $(n=16)$; *of rank* 5

11) $u=9,\ v=2$ $(n=18)$; *of rank* 5.

## Appendix

Summing up the results obtained in § 2, we have the following Tables 1 and 2 which list all primitive permutation representations of rank $\leqq 5$ of the groups $S_n$ and $A_n$, together with the decompositions of the permutation characters. The decompositions of the permutation characters are easily obtained, and so we omit the proof.

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Table 1. The list of primitive permutation representations of rank $\leqq 5$ of the symmetric groups $S_n$.
(Here $H$ denotes the stabilizer of a point.)

| $n$ | Degree $= \lvert S_n : H \rvert$ | $\lvert H \rvert$ | Structure of $H$ | Rank | Decomposition of the permutation character $(1_H)^{S_n}$ |
|---|---|---|---|---|---|
| $n \geqq 2$*) | 2 | $(1/2)\cdot n!$ | Primitive on $\Sigma_n$, $(=A_n)$ | 2 | $[1]+[1^n]$ |
| $n \geqq 3$ | $n$ | $(n-1)!$ | $H(\Sigma_1)$ | 2 | $[n]+[n-1, 1]$ |
| $n \geqq 5$ | $(1/2)n(n-1)$ | $2\cdot(n-2)!$ | $H(\Sigma_2)$ | 3 | $[n]+[n-1, 1]+[n-2, 2]$ |
| $n \geqq 7$ | ${}_nC_3$ | $6\cdot(n-3)!$ | $H(\Sigma_3)$ | 4 | $[n]+[n-1, 1]+[n-2, 2]+[n-3, 3]$ |
| $n \geqq 9$ | ${}_nC_4$ | $24\cdot(n-4)!$ | $H(\Sigma_4)$ | 5 | $[n]+[n-1, 1]+[n-2, 2]+[n-3, 3]+[n-4, 4]$ |
| 4*) | 3 | 8 | $H(\Pi_2)$ | 2 | $[4]+[2, 2]$ |
| 5 | 6 | 20 | Primitive on $\Sigma_5$, (No. 3) | 2 | $[5]+[2^2, 1]$ |
| 6 | 6 | 120 | Primitive on $\Sigma_6$, (No. 2), $\cong S_5$ | 2 | $[6]+[2^3]$ |
| 〃 | 10 | 72 | $H(\Pi_3)$ | 2 | $[6]+[4, 2]$ |
| 〃 | 15 | 48 | $H(\Pi_2)$ | 3 | $[6]+[4, 2]+[2^3]$ |
| 8 | 35 | 1152 | $H(\Pi_4)$ | 3 | $[8]+[6, 2]+[4^2]$ |
| 〃 | 105 | 384 | $H(\Pi_2)$ | 5 | $[8]+[6, 2]+[4^2]+[4, 2^2]+[2^4]$ |
| 〃 | 120 | 336 | Primitive on $\Sigma_8$, (No. 4), $\cong PGL(2, 7)$ | 5 | $[8]+[4^2]+[4, 2^2]+[4, 1^4]+[2^4]$ |
| 9 | 280 | 1296 | $H(\Pi_3)$ | 5 | $[9]+[7, 2]+[6, 3]+[5, 2^2]+[4^2, 1]$ |
| 10 | 1260 | 2880 | $H(\Pi_5)$ | 3 | $[10]+[8, 2]+[6, 4]$ |
| 12 | $12!/2\cdot(6!)^2$ | $2\cdot(6!)^2$ | $H(\Pi_6)$ | 4 | $[12]+[10, 2]+[8, 4]+[6, 6]$ |
| 14 | $14!/2\cdot(7!)^2$ | $2\cdot(7!)^2$ | $H(\Pi_7)$ | 4 | $[14]+[12, 2]+[10, 4]+[8, 6]$ |
| 16 | $16!/2\cdot(8!)^2$ | $2\cdot(8!)^2$ | $H(\Pi_8)$ | 5 | $[16]+[14, 2]+[12, 4]+[10, 6]+[8, 8]$ |
| 18 | $18!/2\cdot(9!)^2$ | $2\cdot(9!)^2$ | $H(\Pi_9)$ | 5 | $[18]+[16, 2]+[14, 4]+[12, 6]+[10, 8]$ |

Remark. 1) *) denotes that the permutation representation is not faithful.
2) The No. when $H$ is primitive on $\Sigma_n$ denotes the number given in Table 1 in C. C. Sims [5].
3) [*] denotes the irreducible character of $S_n$ associated with a Young diagram of type [*].

Table 2. The list of primitive permutation representations of rank$\leqq 5$ of the alternating groups $A_n$.
(Here $H$ denotes the stabilizer of a point).

| $n$ | Degree$=\lvert A_n:H\rvert$ | $\lvert H\rvert$ | Structure of $H$ | Rank | Decomposition of the permutation character $(1_H)^{A_n}$ |
|---|---|---|---|---|---|
| 3 | 3 | 1 | 1 | 3 | $[3]+[2,1]_1+[2,1]_2$ |
| $n\geqq 4$ | $n$ | $(1/2)\cdot(n-1)!$ | $H(\Sigma_1)\cap A_n$ | 2 | $[n]+[n-1,1]$ |
| $n\geqq 5$ | $(1/2)n(n-1)$ | $(n-2)!$ | $H(\Sigma_2)\cap A_n$ | 3 | $[n]+[n-1,1]+[n-2,2]$ |
| $n\geqq 7$ | ${}_nC_3$ | $3\cdot(n-3)!$ | $H(\Sigma_3)\cap A_n$ | 4 | $[n]+[n-1,1]+[n-2,2]+[n-3,3]$ |
| $n\geqq 9$ | ${}_nC_4$ | $12\cdot(n-4)!$ | $H(\Sigma_4)\cap A_n$ | 5 | $[n]+[n-1,1]+[n-2,2]+[n-3,3]$ $+[n-4,4]$ |
| 4*) | 3 | 4 | $H(\Pi_2)\cap A_4$ | 3 | $[4]+[2,2]_1+[2,2]_2$ |
| 5 | 6 | 10 | Primitive on $\Sigma_5$, (No. 2) | 2 | $[5]+[2^2,1]$ |
| 6 | 6 | 60 | Primitive on $\Sigma_6$, (No. 1), $\cong A_5$ | 2 | $[6]+[2^3]$ |
| 〃 | 10 | 36 | $H(\Pi_3)\cap A_6$ | 2 | $[6]+[4,2]$ |
| 〃 | 15 | 24 | $H(\Pi_2)\cap A_6$ | 3 | $[6]+[4,2]+[4^2]$ |
| 7 | 15 | 168 | Primitive on $\Sigma_7$, (No. 5), $\cong PSL(3,2)$ | 2 | $[7]+[4,3]$ |
| 〃 | 〃 | 〃 | 〃 , 〃 | 〃 | 〃 |
| 8 | 15 | 1344 | Primitive on $\Sigma_8$, (No. 5) | 2 | $[8]+[4^2]$ |
| 〃 | 〃 | 〃 | 〃 , 〃 | 〃 | 〃 |
| 〃 | 35 | 576 | $H(\Pi_4)\cap A_8$ | 3 | $[8]+[6,2]+[4^2]$ |
| 〃 | 105 | 192 | $H(\Pi_2)\cap A_8$ | 5 | $[8]+[6,2]+[4^2]+[4,2^2]+[2^4]$ |
| 9 | 120 | 1512 | Primitive on $\Sigma_9$, (No. 8), $\cong P\Gamma L(2,8)$ | 3 | $[9]+[4^2,1]+[5,1^4]_1$ |
| 〃 | 〃 | 〃 | 〃 , 〃 | 〃 | $[9]+[4^2,1]+[5,1^4]_2$ |
| 〃 | 280 | 648 | $H(\Pi_3)\cap A_9$ | 5 | $[9]+[7,2]+[6,3]+[5,2^2]+[4^2,1]$ |
| 10 | 1260 | 1440 | $H(\Pi_5)\cap A_{10}$ | 3 | $[10]+[8,2]+[6,4]$ |
| 11 | 2520 | 7920 | Primitive on $\Sigma_{11}$, (No. 6), $\cong M_{11}$ | 5 | $[11]+[6,5]+[6,2^2,1]+[5,2^3]+[4^2,3]$ |
| 〃 | 〃 | 〃 | 〃 , 〃 | 〃 | 〃 |
| 12 | 2520 | 95040 | Primitive on $\Sigma_{12}$, (No. 4), $\cong M_{12}$ | 4 | $[12]+[6^2]+[6,2^3]+[4^3]$ |
| 〃 | 〃 | 〃 | 〃 , 〃 | 〃 | 〃 |

| $n$ | Degree $= \lvert A_n : H \rvert$ | $\lvert H \rvert$ | Structure of $H$ | Rank | Decomposition of the permutation character $(1_H)^{A_n}$ |
|---|---|---|---|---|---|
| 12 | $12!/2\cdot(6!)^2$ | $(6!)^2$ | $H(\Pi_6)\cap A_{12}$ | 4 | $[12]+[10,2]+[8,4]+[6,6]$ |
| 14 | $14!/2\cdot(7!)^2$ | $(7!)^2$ | $H(\Pi_7)\cap A_{14}$ | 4 | $[14]+[12,2]+[10,4]+[8,6]$ |
| 16 | $16!/2\cdot(8!)^2$ | $(8!)^2$ | $H(\Pi_8)\cap A_{16}$ | 5 | $[16]+[14,2]+[12,4]+[10,6]+[8,8]$ |
| 18 | $18!/2\cdot(9!)^2$ | $(9!)^2$ | $H(\Pi_9)\cap A_{18}$ | 5 | $[18]+[16,2]+[14,4]+[12,6]+[10,8]$ |

Remark. 1) *) denotes that the permutation representation is not faithful.

2) The No. when $H$ is primitive on $\Sigma_n$ denotes the number given in Table 1 in C. C. Sims [5].

3) $[*]|_{A_n}=[*]_1+[*]_2$, if $[*]$ is an associated character of $S_n$. (Note that $[*]_1$ and $[*]_2$ are irreducible character of $A_n$).

$[*]|_{A_n}=[*]$, if $[*]$ is not an associated character of $S_n$. (Note that $[*]$ are irreducible character of $A_n$.)

4) $\left\{ \begin{matrix} n \\ '' \end{matrix} \right.$ denotes that the both $H$'s are transformed to each other by an inner automorphism of $S_n$.