On some degenerate system of parabolic semi-linear equations

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§ 1. Introduction and summary

This paper is concerned with the following initial value problem for the degenerate system of parabolic semi-linear equations,

$$
(1.1) \qquad \left\{
\begin{aligned}
\frac{\partial u_1}{\partial t} &= \Delta u_1 - d_1 u_1 u_4 - d_2 u_1 u_3 , \\
\frac{\partial u_2}{\partial t} &= \Delta u_2 - d_3 u_2 u_4 + d_2 u_1 u_3 , \\
\frac{\partial u_3}{\partial t} &= \qquad d_3 u_2 u_4 - d_2 u_1 u_3 , \\
\frac{\partial u_4}{\partial t} &= \quad -d_1 u_1 u_4 - d_3 u_2 u_4 ,
\end{aligned}
\right.
$$

in $R^n \times [0, \infty)$ with the initial data

$$
(1.2) \qquad u_i(x, 0) = \varphi_i(x) \qquad i = 1, 2, 3, 4 ,
$$

where Δ denotes the n-dimentional Laplacian, and d_1, d_2 and d_3 are all nonnegative constants.

From now on, we shall denote this initial value problem by (I.V.P).

Such a system arises in describing the diffusion accompanied by an immobilizing reaction of second order. For this (I.V.P) M. Mimura [2] proved global existence of the nonnegative solution by difference scheme method. In this paper we shall prove similar results by the iteration method which is different from Mimura's. Our method is applicable to the mixed problem, that is, the initial value problem with the boundary condition (the Dirichlet, or the Neumann condition) considered in $\Omega \times [0, \infty)$, where Ω is an open set in R^n.

In construction of the nonnegative global solution, the parabolicity of the equation is essential. An *a priori* bound, important for global existence of the solution of (I.V.P), is easily obtained using elementary properties of parabolic equations.

Now we give some definitions.

DEFINITION 1.1. $\mathscr{B}(R^n)$ is the set of all $\Phi(x)=\{\varphi_i(x)\}$ $(i=1, 2, 3, 4)$ such that each component $\varphi_i(x)$ of $\Phi(x)$ is boundedly continuous in R^n. $\mathscr{B}(R^n)$ is a Banach space normed by

$$\|\Phi\|_{\mathscr{B}}=\sum_{i=1}^{4}\sup_{x\in R^n}|\varphi_i(x)| .$$

DEFINITION 1.2. $\mathscr{B}_T$ is the set of all $U(x, t)=\{u_i(x, t)\}$ $(i=1, 2, 3, 4)$ such that each component $u_i(x, t)$ of $U(x, t)$ is boundedly continuous in $(x, t)\in R^n\times[0, T)$. $\mathscr{B}_T$ is a Banach space, its norm being

$$\|U\|_{\mathscr{B}_T}=\sum_{i=1}^{4}\sup_{\substack{x\in R^n\\ 0\leq t<T}}|u_i(x, t)| .$$

$W(t, x, y)$ means the fundamental solution of the heat equation, that is,

$$W(t, x, y)=\frac{1}{(4\pi t)^{n/2}}\exp\left(-\frac{|x-y|^2}{4t}\right) \qquad (x, y\in R^n,\ t>0) .$$

Next we shall define solutions of (I.V.P). We treat only bounded classical solutions which belong to $\mathscr{B}_T$.

DEFINITION 1.3. $U(x, t)=\{u_i(x, t)\}$ is said to be a solution of (I.V.P) in $R^n\times[0, T)$, if $u_i(x, t)$ $(i=1, 2, 3, 4)$ satisfy the equations (1.1) in $R^n\times(0, T)$ in the classical sense, and as for initial values the relations

$$\lim_{t\downarrow 0}u_i(x, t)=\varphi_i(x) \qquad (i=1, 2, 3, 4)$$

hold.

Our main theorem is the following.

MAIN THEOREM. *Let the initial value $\Phi(x)=\{\varphi_i(x)\}$ be in $\mathscr{B}(R^n)$, and suppose that all $\varphi_i(x)$ are nonnegative and $\varphi_3(x)$ and $\varphi_4(x)$ are locally Hölder continuous. Then the solution $U(x, t)=\{u_i(x, t)\}$ of* (I.V.P), *whose components $u_i(x, t)$ are all nonnegative, exists globally in time. It belongs to $\mathscr{B}_T$ and is unique there for an arbitrary $T>0$.*

§ 2. Elementary properties of parabolic equations

In this section we shall summarize elementary properties of parabolic equations (heat equation) in lemmas. (See, Friedman [1]). We shall need these lemmas in later sections.

Set

$$L=\frac{\partial}{\partial t}-\Delta-q(x, t) , \tag{2.1}$$

where $q(x, t)$ is bounded and continuous in $R^n \times [0, T) \equiv R_T$.

In the following two lemmas, we shall assume that $f(x, t)$, a continuous function defined in $R^n \times [0, T)$, is twice continuously differentiable in x and once in t in $R^n \times (0, T) \equiv \mathring{R}_T$.

LEMMA 2.1. *Suppose that a bounded function $f(x, t)$ satisfies $Lf \geqq 0$ and $f(x, +0) \geqq 0$. Then $f(x, t) \geqq 0$ in R_T.*

Since it is well-known, the proof is omitted.

LEMMA 2.2. *For a nonnegative bounded function $f(x, t)$, Lf is non-positive ($Lf \leqq 0$) and boundedly continuous in $\mathring{R}_T$. Then*

$$f(x, t) \leqq e^{kt} \sup_{x \in R^n} |f(x, 0)| , \tag{2.2}$$

where

$$k = \sup_{(x,t) \in R_T} q(x, t) .$$

PROOF. According to the representation formula for the non-homogeneous initial value problem, we have, noticing nonnegativity of $W(t, x, y)$ and $f(x, t)$,

$$0 \leqq f(x, t) \leqq \int_{R^n} W(t, x, y) f(y, 0) dy + \int_0^t ds \int_{R^n} W(t-s, x, y) q(y, s) f(y, s) dy . \tag{2.3}$$

Putting $F(t) = \sup_{x \in R^n} f(x, t)$, we can easily derive the inequality

$$0 \leqq F(t) \leqq F(0) + \int_0^t kF(s) ds . \tag{2.4}$$

In getting (2.4), we have made use of $\int_{R^n} W(t, x, y) dy = 1$. From the integral inequality (2.4),

$$F(t) \leqq F(0) e^{kt} ,$$

can be shown.

This implies (2.2). Q.E.D.

LEMMA 2.3. *Let $g(x, t)$ and $h(x, t)$ be in $\mathscr{B}_T$. Let $u(x, t)$ be the solution, uniquely determined in $\mathscr{B}_T$, of the integral equation,*

$$u(x, t) = \int_0^t ds \int_{R^n} W(t-s, x, y) g(y, s) u(y, s) dy + \int_0^t ds \int_{R^n} W(t-s, x, y) h(y, s) dy .$$

Then $u(x, t)$ is continuously differentiable in x in $R^n \times (0, T)$. In addition, if $g(x, t)$ and $h(x, t)$ are locally Hölder continuous in x uniformly with respect to t ($t > 0$), then $u(x, t)$ satisfies the differential equation,

$$\frac{\partial u}{\partial t} - \Delta u - g(x, t) u - h(x, t) = 0 ,$$

and $\lim_{t\downarrow 0} u(x, t)=0$ *holds.*

These facts are based on the potential theory, that is, estimates of the fundamental solution $W(t, x, y)$. We refer the reader to Friedman [1] for the details.

§ 3. Uniqueness

PROPOSITION 3.1. *Let* $U(x, t)$ *and* $V(x, t)$ *be two solutions of* (I.V.P) *in* $R^n\times[0, T)$ *whose initial values are* $\Phi(x)$ *and* $\Psi(x)$ *respectively. Assume that* $\|U\|_{\mathscr{B}_T}\|V\|_{\mathscr{B}_T}<M$.

Then we have the following estimate

$$\|U-V\|_{\mathscr{B}_T}\leqq\|\Phi-\Psi\|_{\mathscr{B}}\exp(5kMT)\,,$$

where

$$k=\max(d_1, d_2, d_3)\,.$$

PROOF. It is obvious that (I.V.P) can be transformed to the following system of the integral equations for $U(x, t)=\{u_i(x, t)\}$ with the aid of the fundamental solution $W(t, x, y)$:

$$(3.1)\quad\begin{cases} u_1(x, t)=\displaystyle\int_{R^n} W(t, x, y)\varphi_1(y)dy \\ \qquad -\displaystyle\int_0^t ds\int_{R^n} W(t-s, x, y)\{d_1u_1(y, s)u_4(y, s)+d_2u_1(y, s)u_3(y, s)\}dy\,, \\ u_2(x, t)=\displaystyle\int_{R^n} W(t, x, y)\varphi_2(y)dy \\ \qquad -\displaystyle\int_0^t ds\int_{R^n} W(t-s, x, y)\{d_3u_2(y, s)u_4(y, s)-d_2u_1(y, s)u_3(y, s)\}dy\,, \\ u_3(x, t)=\varphi_3(x)+\displaystyle\int_0^t d_3u_2(x, s)u_4(x, s)ds-\int_0^t d_2u_1(x, s)u_3(x, s)ds\,, \\ u_4(x, t)=\varphi_4(x)-\displaystyle\int_0^t d_1u_1(x, s)u_4(x, s)ds-\int_0^t d_3u_2(x, s)u_4(x, s)ds\,. \end{cases}$$

Also, for $V(x, t)=\{v_i(x, t)\}$ similar equations are obtained.

Introducing

$$(3.2)\qquad |u_i-v_i|_{\mathscr{B}_t}=\sup_{x\in R^n}|u_i(x, t)-v_i(x, t)|$$

and

$$(3.3)\qquad |\varphi_i-\psi_i|_{\mathscr{B}}=\sup_{x\in R^n}|\varphi_i(x)-\psi_i(x)|\,,$$

and using $\int W(t, x, y)dy=1$, we have for (u_i-v_i),

$$
(3.4)\quad \begin{cases}
|u_1-v_1|_{\mathscr{B}_t} \leqq |\varphi_1-\psi_1|_{\mathscr{B}} + kM\int_0^t \{2|u_1-v_1|_{\mathscr{B}_s} + |u_3-v_3|_{\mathscr{B}_s} + |u_4-v_4|_{\mathscr{B}_s}\}ds\,, \\
|u_2-v_2|_{\mathscr{B}_t} \leqq |\varphi_2-\psi_2|_{\mathscr{B}} + kM\int_0^t \{|u_1-v_1|_{\mathscr{B}_s} + |u_2-v_2|_{\mathscr{B}_s} + |u_3-v_3|_{\mathscr{B}_s} \\
\qquad\qquad + |u_4-v_4|_{\mathscr{B}_s}\}ds\,, \\
|u_3-v_3|_{\mathscr{B}_t} \leqq |\varphi_3-\psi_3|_{\mathscr{B}} + kM\int_0^t \{|u_1-v_1|_{\mathscr{B}_s} + |u_2-v_2|_{\mathscr{B}_s} + |u_3-v_3|_{\mathscr{B}_s} \\
\qquad\qquad + |u_4-v_4|_{\mathscr{B}_s}\}ds\,, \\
|u_4-v_4|_{\mathscr{B}_t} \leqq |\varphi_4-\psi_4|_{\mathscr{B}} + kM\int_0^t \{|u_1-v_1|_{\mathscr{B}_s} + |u_2-v_2|_{\mathscr{B}_s} + 2|u_4-v_4|_{\mathscr{B}_s}\}ds\,.
\end{cases}
$$

Adding these four inequalities, we obtain the integral inequality

$$
(3.5)\qquad \sum_{i=1}^{4} |u_i-v_i|_{\mathscr{B}_s} \leqq \|\Phi-\Psi\|_{\mathscr{B}} + 5kM\int_0^t \sum_{i=1}^{4} |u_i-v_i|_{\mathscr{B}_s}ds\,.
$$

After an elementary calculation, (3.5) gives

$$
(3.6)\qquad \sum_{i=1}^{4} |u_i-v_i|_{\mathscr{B}_t} \leqq \|\Phi-\Psi\|_{\mathscr{B}} \exp(5kMt)\,.
$$

Finally, (3.6) implies

$$
\|U-V\|_{\mathscr{B}_T} \leqq \|\Phi-\Psi\|_{\mathscr{B}} \exp(5kMT)\,. \qquad \text{Q.E.D.}
$$

COROLLARY 3.2. (uniqueness).
The solution of (I.V.P) is unique in $\mathscr{B}_T$.

§ 4. Construction of a local (in time) nonnegative solution

In this section we construct a nonnegative solution of (I.V.P) for a nonnegative initial value $\Phi(x)=\{\varphi_i(x)\}$ in $\mathscr{B}(R^n)$ locally in time by the iteration method. We assume that $\varphi_3(x)$ and $\varphi_4(x)$ are locally Hölder continuous.

Define the sequence $U^n(x, t)=\{u_i^n(x, t)\}$ in the following way. $U^0(x, t)$ is the solution of the initial value problem:

$$
(4.1)\quad \begin{cases}
\dfrac{\partial u_1^0}{\partial t} = \Delta u_1^0\,, \\
\dfrac{\partial u_2^0}{\partial t} = \Delta u_2^0\,, \\
\dfrac{\partial u_3^0}{\partial t} = 0\,,
\end{cases}
$$

$$\left\{ \frac{\partial u_4^0}{\partial t} = 0 \right. ,$$

with the initial data $u_i^0(x, 0) = \varphi_i(x)$ $(i=1, 2, 3, 4)$.

For $n \geqq 1$ we define $U^n(x, t)$ by the solution of the following initial value problem:

$$(4.2) \quad \left\{ \begin{aligned} \frac{\partial u_1^n}{\partial t} &= \Delta u_1^n - d_1 u_1^n u_4^{n-1} - d_2 u_1^n u_3^{n-1} , \\ \frac{\partial u_2^n}{\partial t} &= \Delta u_2^n - d_3 u_2^n u_4^{n-1} + d_2 u_1^n u_3^{n-1} , \\ \frac{\partial u_3^n}{\partial t} &= \qquad d_3 u_2^{n-1} u_4^n - d_2 u_1^{n-1} u_3^n , \\ \frac{\partial u_4^n}{\partial t} &= \quad -d_1 u_1^{n-1} u_4^n - d_3 u_2^{n-1} u_4^n , \end{aligned} \right.$$

with the initial data $u_i^n(x, 0) = \varphi_i(x)$ $(i=1, 2, 3, 4)$.

The sequence $U^n(x, t)$ is well-defined. From (4.1) $U^0(x, t)$ is defined and obviously $u_3^0(x, t)$ and $u_4^0(x, t)$ are locally Hölder continuous in x, uniformly with respect to t $(t>0)$. Suppose that $U^{n-1}(x, t)$ is known and $u_3^{n-1}(x, t)$ and $u_4^{n-1}(x, t)$ are locally Hölder continuous in x, uniformly with respect to t $(t>0)$. Then $u_3^n(x, t)$ and $u_4^n(x, t)$ are easily obtained, and it is clear that they satisfy the same condition of Hölder continuity as $u_3^{n-1}(x, t)$ and $u_4^{n-1}(x, t)$ satisfy. To get $u_1^n(x, t)$ we solve the integral equation,

$$(4.3) \quad \begin{aligned} u_1^n(x, t) &= \int_{R^n} W(t, x, y) \varphi_1(y) dy \\ &\quad - \int_0^t ds \int_{R^n} W(t-s, x, y) \{ d_1 u_1^n(y, s) u_4^{n-1}(y, s) + d_2 u_1^n(y, s) u_3^{n-1}(y, s) \} dy . \end{aligned}$$

(4.3) can be solved by means of successive approximation. Next we get $u_2^n(x, t)$ in the same way. By the assumption of Hölder continuity of $u_3^{n-1}(x, t)$ and $u_4^{n-1}(x, t)$, $U^n(x, t)$ satisfies the system (4.2) with the aid of Lemma 2.3.

LEMMA 4.1. *The sequence $U^n(x, t)$ defined by* (4.1) *and* (4.2) *is nonnegative, that is,* $u_i^n(x, t) \geqq 0$ $(i=1, 2, 3, 4)$.

PROOF. The proof is by induction. Since all $\varphi_i(x)$ are nonnegative, $u_i^0(x, t)$ are nonnegative. Assume that $u_i^{n-1}(x, t) \geqq 0$ $(i=1, 2, 3, 4)$. By Lemma 2.1 $u_1^n(x, t) \geqq 0$ and obviously $u_4^n(x, t) \geqq 0$. Hence

$$\frac{\partial u_2^n(x, t)}{\partial t} - \Delta u_2^n + d_3 u_2^n u_4^{n-1} \geqq 0 ,$$

$$\frac{\partial u_3^n}{\partial t}+d_2 u_1^{n-1} u_3^n \geqq 0 .$$

Thus it follows from Lemma 2.1 that $u_2^n(x, t) \geqq 0$, and $u_3^n(x, t) \geqq 0$.

LEMMA 4.2. *For the sequence $U^n(x, t)$ $(n=0, 1, 2, \cdots)$ we have an a priori estimate, for $0<T<\infty$,*

$$\|U^n\|_{\mathscr{B}_T} \leqq 2\|\Phi\|_{\mathscr{B}} . \tag{4.4}$$

PROOF. For $n=0$ it is obvious that $\|U^0\|_{\mathscr{B}_T} \leqq \|\Phi\|_{\mathscr{B}}$. For $n \geqq 1$, from equations (4.2) and Lemma 4.1, we have

$$\frac{\partial(u_1^n+u_2^n)}{\partial t}-\Delta(u_1^n+u_2^n)=-d_1 u_1^n u_4^{n-1}-d_3 u_2^n u_4^{n-1} \leqq 0 ,$$

$$\frac{\partial(u_3^n+u_4^n)}{\partial t}=-d_2 u_1^{n-1} u_3^n-d_1 u_1^{n-1} u_4^n \leqq 0 .$$

Hence, by Lemma 2.2, we conclude that

$$0 \leqq u_1^n(x, t)+u_2^n(x, t) \leqq \sup_{x \in R^n}|\varphi_1(x)|+\sup_{x \in R^n}|\varphi_2(x)| ,$$

$$0 \leqq u_3^n(x, t)+u_4^n(x, t) \leqq \sup_{x \in R^n}|\varphi_3(x)|+\sup_{x \in R^n}|\varphi_4(x)| .$$

Then $\|U^n(x, t)\|_{\mathscr{B}_T} \leqq 2\|\varphi\|_{\mathscr{B}}$. Q.E.D.

Existence of a local (in time) nonnegative solution is now established.

PROPOSITION 4.3. *The sequence $U^n(x, t)$ defined by* (4.1) *and* (4.2) *converges to a nonnegative solution $U(x, t)$ of* (I.V.P). *More precisely, $U^n(x, t)$ converges to $U(x, t)$ in $\mathscr{B}_{T_0}$, where $T_0=\dfrac{1}{18kM}$, $k=\max(d_1, d_2, d_3)$ and $M=\|\Phi\|_{\mathscr{B}}$.*

PROOF. The initial value problem for (4.2) may be rewritten in the form of the following system of integral equations:

$$\left\{\begin{aligned}
u_1^n(x, t)= & \int_{R^n} W(t, x, y)\varphi_1(y)dy \\
& -\int_0^t ds \int_{R^n} W(t-s, x, y) d_1 u_1^n(y, s) u_4^{n-1}(y, s) dy \\
& -\int_0^t ds \int_{R^n} W(t-s, x, y) d_2 u_1^n(y, s) u_3^{n-1}(y, s) dy , \\
u_2^n(x, t)= & \int_{R^n} W(t, x, y)\varphi_2(y)dy \\
& -\int_0^t ds \int_{R^n} W(t-s, x, y) d_3 u_2^n(y, s) u_4^{n-1}(y, s) dy \\
& +\int_0^t ds \int_{R^n} W(t-s, x, y) d_2 u_1^n(y, s) u_3^{n-1}(y, s) dy ,
\end{aligned}\right. \tag{4.5}$$

$$\left|\begin{array}{l} u_3^n(x, t)=\varphi_3(x)+\int_0^t d_3 u_2^{n-1}(x, s) u_4^n(x, s) ds-\int_0^t d_2 u_1^{n-1}(x, s) u_3^n(x, s) ds , \\ u_4^n(x, s)=\varphi_4(x)-\int_0^t d_1 u_1^{n-1}(x, s) u_4^n(x, s) ds-\int_0^t d_3 u_2^{n-1}(x, s) u_3^n(x, s) ds . \end{array}\right.$$

We now consider the difference $u_i^{n+1}(x, t)-u_i^n(x, t)$. Putting

$|u_i^{n+1}-u_i^n|_{\mathscr{B}_t}=\sup_{x\in R^n}|u_i^{n+1}(x, t)-u_i^n(x, t)|$, $k=\max(d_1, d_2, d_3)$ and $M=\|\Phi\|_{\mathscr{B}}$,

we can derive

$$(4.6)\quad \left\{\begin{array}{l} |u_1^{n+1}-u_1^n|_{\mathscr{B}_t}\leqq 2kM\int_0^t\{2|u_1^{n+1}-u_1^n|_{\mathscr{B}_s}+|u_3^n-u_3^{n-1}|_{\mathscr{B}_s} \\ \qquad\qquad +|u_4^n-u_4^{n-1}|_{\mathscr{B}_s}\}ds , \\ |u_2^{n+1}-u_2^n|_{\mathscr{B}_t}\leqq 2kM\int_0^t\{|u_1^{n+1}-u_1^n|_{\mathscr{B}_s}+|u_2^{n+1}-u_2^n|_{\mathscr{B}_s} \\ \qquad\qquad +u_3^n-u_3^{n-1}|_{\mathscr{B}_s}+|u_4^n-u_4^{n-1}|_{\mathscr{B}_s}\}ds , \\ |u_3^{n+1}-u_3^n|_{\mathscr{B}_t}\leqq 2kM\int_0^t\{|u_1^n-u_1^{n-1}|_{\mathscr{B}_s}+|u_2^n-u_2^{n-1}|_{\mathscr{B}_s}+|u_3^{n+1}-u_3^n|_{\mathscr{B}_s} \\ \qquad\qquad +|u_4^{n+1}-u_4^n|_{\mathscr{B}_s}\}ds , \\ \text{and} \\ |u_4^{n+1}-u_4^n|_{\mathscr{B}_t}\leqq 2kM\int_0^t\{|u_1^n-u_1^{n-1}|_{\mathscr{B}_s}+|u_2^n-u_2^{n-1}|_{\mathscr{B}_s} \\ \qquad\qquad +2|u_4^{n+1}-u_4^n|_{\mathscr{B}_s}\}ds . \end{array}\right.$$

Adding these inequalities, we can get

$$\sum_{i=1}^4|u_i^{n+1}-u_i^n|_{\mathscr{B}_t}\leqq 6kM\int_0^t\sum_{i=1}^4(|u_i^{n+1}-u_i^n|_{\mathscr{B}_s}+|u_i^n-u_i^n|_{\mathscr{B}_s})ds .$$

For $t\in[0, T_0)$, where $T_0=\dfrac{1}{18kM}$, we have

$$\sum_{i=1}^4|u_i^{n+1}-u_i^n|_{\mathscr{B}_t}\leqq 6kMT_0(\|U^{n+1}-U^n\|_{\mathscr{B}_{T_0}}+\|U^n-U^{n-1}\|_{\mathscr{B}_{T_0}}) .$$

Therefore

$$(1-6kMT_0)\|U^{n+1}-U^n\|_{\mathscr{B}_{T_0}}\leqq 6kMT_0\|U^n-U^{n-1}\|_{\mathscr{B}_{T_0}} ,$$

$$\|U^{n+1}-U^n\|_{\mathscr{B}_{T_0}}\leqq\frac{1}{2}\|U^n-U^{n-1}\|_{\mathscr{B}_{T_0}} .$$

This implies that the sequence $U^n(x, t)$ converges to a limit $U(x, t)$ in $\mathscr{B}_{T_0}$.

We now verify that $U(x, t)$ is a solution of (I.V.P). Letting $n\to\infty$ in the system of integral equations (4.5), $U(x, t)$ satisfies (3.1). On account of Lemma 2.3, $u_1(x, t)$ and $u_2(x, t)$ are continuously differentiable in x. Hence, noting that $\varphi_3(x)$ and $\varphi_4(x)$ are locally Hölder continuous and making use of Lemma 2.3 again, we find that $u_1(x, t)$ and $u_2(x, t)$ satisfy the parabolic equations,

$$\frac{\partial u_1}{\partial t}=\Delta u_1-d_1u_1u_4-d_2u_1u_3 ,$$

$$\frac{\partial u_2}{\partial t}=\Delta u_2-d_3u_2u_4+d_2u_1u_3 .$$

Obviously $u_3(x, t)$ and $u_4(x, t)$ satisfy the equations,

$$\frac{\partial u_3}{\partial t}=d_3u_2u_4-d_2u_1u_3 ,$$

$$\frac{\partial u_4}{\partial t}=-d_1u_1u_4-d_3u_2u_4 ,$$

and $U(x, t)$ satisfies the initial condition $U(x, 0)=\Phi(x)$. Q.E.D.

§ 5. Global existence of nonnegative solutions

To extend the local (in time) solution obtained in the previous section to a global solution, we shall need an *a priori* bound.

LEMMA 5.1 (*A priori* bound). *Suppose that a nonnegative solution* $U(x, t)$ *exists in* $0\leqq t<\tilde{T}$. *Then we have an estimate*

(5.1) $$\|U(x, t)\|_{\infty \tilde{T}}\leqq 2\|U(x, 0)\|_{\infty} .$$

PROOF. Since all $u_i(x, t)$ are nonnegative,

$$\frac{\partial(u_1+u_2)}{\partial t}-\Delta(u_1+u_2)\leqq 0 ,$$

$$\frac{\partial(u_3+u_4)}{\partial t}\leqq 0 .$$

Hence we can easily get the estimate (5.1) from these differential inequalities. Q.E.D.

Now we are going a study global existence of nonnegative solutions.

PROOF OF MAIN THEOREM. We shall prove that the nonnegative solution $U(x, t)$ assured to exist by Proposition 4.2 can be extended to a larger interval. It is clear that we can construct a solution $V(x, t)$ of (I.V.P) starting at the initial moment $T_0-\varepsilon$, $\varepsilon>0$ arbitrary small, and with the initial value $U(x, T_0-\varepsilon)$. From the foregoing argument, it follows that $V(x,t)$ is defined in the interval $\left[T_0-\varepsilon, \frac{3}{2}T_0-\varepsilon\right)$, because of $\|U(x, T_0-\varepsilon)\|_{\mathscr{B}}\leqq 2\|U(x,0)\|_{\mathscr{B}}=2M$.

We easily verify that $\tilde{U}(x,t)$ defined by

$$\tilde{U}(x,t)=\begin{cases} U(x,t)\,, & 0\leqq t<T_0\,,\\ V(x,t)\,, & T_0-\varepsilon\leqq t<\frac{3}{2}T_0-\varepsilon\,,\end{cases}$$

is a solution of original (I.V.P) in $\mathscr{B}_{(3/2)T_0-\varepsilon}$, because for $t\in[T_0-\varepsilon, T_0)$ $U(x,t)$ and $V(x,t)$ are coincident in view of the uniqueness. $\varepsilon>0$ is arbitrary, so we have a solution of (I.V.P) in $\mathscr{B}_{(3/2)T_0}$. By writing $U(x,t)$ instead of $\tilde{U}(x,t)$, we have established that the solution $U(x,t)\in\mathscr{B}_{T_0}$ can be extended from $[0, T_0)$ to $\left[0, \frac{3}{2}T_0\right)$ and in $\mathscr{B}_{(3/2)T_0}$.

Moreover, the *a priori* estimate (Lemma 5.1)

$$\|U(x,t)\|_{\mathscr{B}_{(3/2)T_0}}\leqq 2\|U(x,0)\|_{\mathscr{B}}\,,$$

holds.

Since this procedure can be continued indefinitely, $U(x,t)$ is extended to $\left[0, T_0+\frac{n}{2}T_0\right)$ $(n=1,2,3,\cdots)$ within the class $\mathscr{B}_{T_0+(n/2)T_0}$.

Again recalling the uniqueness, we have proved Main Theorem.

§ 6. Concluding remarks

In the previous sections we have only discussed the Cauchy problem. But as mentioned in the introduction, even for mixed problems global existence of non-negative solutions can be shown by the same method. In fact, in order to show existence of a global nonnegative solution of (I.V.P), we have made use of elementary properties of parabolic equations. Such properties are well-known for mixed problems (the Dirichlet problem, or the Neumann problem) of parabolic equations. Iteration procedure to construct a nonnegative solution is the same, and the corresponding *a priori* bound can be obtained similarly.

Also we can treat the Cauchy problem for the following system of an immobilizing reaction of high order:

$$\text{(6.1)}\quad \begin{cases} \dfrac{\partial u_1}{\partial t}=\Delta u_1-d_2u_1^{p_1+1}u_3^{p_3+1}-d_1u_1^{p_5+1}u_4^{p_6+1}\,,\\[2ex] \dfrac{\partial u_2}{\partial t}=\Delta u_2+d_2u_1^{p_1+1}u_3^{p_3+1}-d_3u_2^{p_2+1}u_4^{p_4+1}\,,\end{cases}$$

$$
\left\{
\begin{aligned}
\frac{\partial u_3}{\partial t} &= \quad -d_2 u_1^{p_1+1} u_3^{p_3+1} + d_3 u_2^{p_2+1} u_4^{p_4+1}, \\
\frac{\partial u_4}{\partial t} &= \quad -d_3 u_2^{p_2+1} u_4^{p_4+1} - d_1 u_1^{p_5+1} u_4^{p_6+1},
\end{aligned}
\right.
$$

with the initial conditions $u_i(x, 0)=\varphi_i(x)$, where p_i $(i=1, 2, \cdots, 6)$ are nonnegative integers.

In fact, we can construct a nonnegative solution of the Cauchy problem of the system (6.1) by the following iteration:

(i) $n=0$

$$
\begin{aligned}
\frac{\partial u_1^0}{\partial t} &= \Delta u_1^0, \\
\frac{\partial u_2^0}{\partial t} &= \Delta u_2^0, \\
\frac{\partial u_3^0}{\partial t} &= 0, \\
\frac{\partial u_4^0}{\partial t} &= 0,
\end{aligned}
$$

with the initial data $u_i^0(x, 0)=\varphi_i(x)$ $(i=1, 2, 3, 4)$.

(ii) $n=1, 2, 3, \cdots,$

$$
\begin{aligned}
\frac{\partial u_1^n}{\partial t} &= \Delta u_1^n - d_2 (u_1^{n-1})^{p_1} (u_3^{n-1})^{p_3+1} u_1^n - d_1 (u_1^{n-1})^{p_5} (u_4^{n-1})^{p_6+1} u_1^n, \\
\frac{\partial u_2^n}{\partial t} &= \Delta u_2^n + d_2 (u_1^{n-1})^{p_1} (u_3^{n-1})^{p_3+1} u_1^n - d_3 (u_2^{n-1})^{p_2} (u_4^{n-1})^{p_4+1} u_2^n, \\
\frac{\partial u_3^n}{\partial t} &= \quad -d_2 (u_1^{n-1})^{p_1+1} (u_3^{n-1})^{p_3} u_3^n + d_3 (u_2^{n-1})^{p_2+1} (u_4^{n-1})^{p_4} u_4^n, \\
\frac{\partial u_4^n}{\partial t} &= \quad -d_3 (u_2^{n-1})^{p_2+1} (u_4^{n-1})^{p_4} u_4^n - d_1 (u_1^{n-1})^{p_5+1} (u_4^{n-1})^{p_6} u_4^n,
\end{aligned}
$$

with the initial data $u_i^n(x, 0)=\varphi_i(x)$ $(i=1, 2, 3, 4)$.

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