On existence of periodic weak solutions of the Navier-Stokes equations in regions with periodically moving boundaries

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§ 0. Introduction

Recently, H. Fujita and N. Sauer ([5], [6]) have studied the Navier-Stokes initial-boundary value problem in region $\Omega(t)$ which varies as time t goes on, and have proved the existence of weak solutions of Hopf's class by the so-called penalty method. In this paper, we shall show that, if $\Omega(t)$, the external force and the boundary value all depend periodically on t with period T, then there exist similar weak periodic solutions with the same period. In this connection, we note that for the case $\Omega(t)\equiv\Omega$, the existence of weak periodic solutions has been proved by several authors (see, for instance, Prodi [11], Lions [9]), and the existence of strong periodic solution for 2-dimensional case, for instance, by Takeshita [12].

In § 1, we fix the notations and present the results. Some smoothness assumptions on the lateral boundary of $\bigcup_{-\infty<t<\infty} t\times\Omega(t)$ are found in **1.1**, and the definition of several function spaces in **1.2**. In **1.3**, we formulate the problem and introduce our approximation based on the penalty method. The main result is stated in **1.4** (Theorem 1.7).

In **2.1**, we recall some properties of the Stokes operator. A few investigations on solenoidal function spaces are found in **2.2**, and basic inequalities in **2.3**. In **2.4**, we try to extend the boundary value to the whole domain with a certain nice property, by means of a refined form of Hopf's technique [7] (Lemma 2.7). **2.5** is devoted to a modification of Aubin's theorem concerning the compactness (Lemma 2.11).

In § 3, we prove the existence of weak periodic solutions of the penalized equation $(AP)_n$ by Galerkin's method.

Our main theorem is proved in § 4.

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§ 1. Notations and results

1.1. Assumptions on the domain

The domain $\Omega(t)$ occupied by the fluid at time t is assumed to be a bounded domain in R^m, m being equal to two or three, although all our arguments go through with a slight modification for larger m. When t goes on, $\Omega(t)$ generates a (t, x)-domain $\hat{\Omega}_\infty = \bigcup_{-\infty<t<\infty} t\times\Omega(t)$, and the boundary $\Gamma(t)$ of $\Omega(t)$ generates a (t, x)-hypersurface $\hat{\Gamma}_\infty = \bigcup_{-\infty<t<\infty} t\times\Gamma(t)$. Let T be a positive number. We assume that the lateral boundary is periodic and smooth in the following sense.

ASSUMPTION 1.1.

i) $\Omega(t+T)=\Omega(t)$ and $\Gamma(t+T)=\Gamma(t)$ for all $t\in R^1$.

ii) At every t, $\Gamma(t)$ consists of a finite fixed number of simple closed surfaces $\Gamma_\alpha(t)$ of class C^3.

iii) There exists a positive number δ_0 such that dist. $(\Gamma_\alpha(t), \Gamma_{\alpha'}(t))$, m-dimensional distance between $\Gamma_\alpha(t)$ and $\Gamma_{\alpha'}(t)$ $(\alpha\neq\alpha')$, is never smaller than δ_0.

iv) As t varies, each $\Gamma_\alpha(t)$ changes smoothly in the sense that the hypersurface $\hat{\Gamma}_\alpha = \bigcup_{0\leq t\leq T} t\times\Gamma_\alpha(t)$ is covered by a finite number of open patches and the portion of $\hat{\Gamma}_\infty$ lying in each patch can be represented by $x_1'=\varphi(x_2', \cdots, x_m', t)$ in terms of a C^3-function φ of m variables under a suitable choice of coordinates in R^m.

When we restrict ourselves to the closed time interval $[0, T]$ for one period, we use the symbol $\hat{\Omega}$ and $\hat{\Gamma}$ for $\bigcup_{0\leq t\leq T} t\times\Omega(t)$ and $\bigcup_{0\leq t\leq T} t\times\Gamma(t)$, respectively.

We fix an auxiliary bounded domain B in R^m such that $\Omega(t)\subset B$ at every t, the boundary ∂B of B is smooth and dist. $(\partial B, \Gamma(t))\geqq\delta_0$ for every t. We put $\hat{B}=[0, T]\times B$, $\hat{B}_\infty=R^1\times B$.

Later we need the symbols for boundary strips. Let δ be any small positive number. For each t, $\omega_i(t; \delta)$ means the interior boundary strip of $\Omega(t)$ with width δ, i.e.,

$$\omega_i(t; \delta)=\{x\in\Omega(t);\ \text{dist.}\,(x, \Gamma(t))<\delta\}\ ,$$

or simply $\omega_i(\delta)$ when t is understood. Similarly $\omega_e(t; \delta)$ is the exterior boundary strip of $\Omega(t)$ with width δ. Moreover we put

$$\hat{\omega}_i(\delta)=\{(t, x);\ t\in[0, T],\ x\in\omega_i(t; \delta)\}$$

$$\hat{\omega}_e(\delta)=\{(t, x),\ t\in[0, T],\ x\in\omega_e(t; \delta)\}\ .$$

1.2. Some function spaces

In this subsection, Ω stands for an arbitrary bounded domain in R^m with boundary of class C^3. The functions considered in this paper are all real. $L_p(\Omega)$

and the Sobolev space $W_p^l(\Omega)$ of order l are defined as usual. (See, for instance, Yosida [13].) Whether the elements of these spaces are scalar or vector functions is understood from the contexts, unless stated explicitly.

When $p=2$, we write sometimes $\|u\|_\Omega$ or simply $\|u\|$ instead of $\|u\|_{L_2(\Omega)}$. Similar abbreviation is used for the inner product in $L_2(\Omega)$.

Now we define the solenoidal function spaces. $D_\sigma(\Omega)$ consists of all vector functions φ in $C_0^\infty(\Omega)$ with $\operatorname{div}\varphi=0$ in Ω. $H_\sigma(\Omega)$ is the completion of $D_\sigma(\Omega)$ under $L_2(\Omega)$-norm. $H_\sigma^1(\Omega)$ is the completion of $D_\sigma(\Omega)$ under $W_2^1(\Omega)$-norm.

Next we consider periodic functions. A (vector) function defined on R^1 is called a T-periodic function if it is a periodic function of period $T>0$. $\hat{G}_\infty$ (resp. $\hat{G}$) stands for $\hat{\Omega}_\infty$ or $\hat{B}_\infty$ (resp. $\hat{\Omega}$ or $\hat{B}$). Then $\hat{D}_\sigma(\hat{G}_\infty;\pi)$ is the set of all T-periodic functions φ in $C^\infty(\hat{G}_\infty)$ such that the support of $\varphi\subset\hat{G}_\infty$ and $\operatorname{div}\varphi=0$ in $\hat{G}_\infty$. Thus, any φ in $\hat{D}_\sigma(\hat{G}_\infty;\pi)$ vanishes identically near the lateral boundary of $\hat{G}_\infty$. The completion of $\hat{D}_\sigma(\hat{G}_\infty;\pi)$ under the $L_2(\hat{G})$-norm, or equivalently under the L_2-norm over any portion of $\hat{G}_\infty$ which corresponds to one period is denoted by $\hat{H}_\sigma(\hat{G}_\infty;\pi)$. As is easily seen, any function in $\hat{H}_\sigma(\hat{G}_\infty;\pi)$ belongs to $L_2^{\mathrm{loc}}(\hat{G}_\infty)$, and $f(t+T,x)=f(t,x)$ for almost every $(t,x)\in\hat{G}_\infty$. By definition, we have

$$\|f\|^2_{\hat{H}_\sigma(\hat{B};\pi)}=\int_0^T\|f(t)\|_B^2dt=\int_\alpha^{\alpha+T}\|f(t)\|_B^2dt\ ,$$

$$\|f\|^2_{\hat{H}_\sigma(\hat{\Omega};\pi)}=\int_0^T\|f(t)\|^2_{\Omega(t)}\,dt=\int_\alpha^{\alpha+T}\|f(t)\|^2_{\Omega(t)}\,dt\ .$$

Let us define $\nu(u)$ for the function u defined in $\hat{G}$:

$$\nu(u)\geqq 0\ ,$$

$$\nu(u)^2=\iint_{\hat{G}}|\nabla u|^2dx\,dt=\|\nabla u\|^2_{\hat{G}}\ .$$

Then $\nu(u)$ is a norm in $\hat{D}_\sigma(\hat{G}_\infty;\pi)$. The completion of $\hat{D}_\sigma(\hat{G}_\infty;\pi)$ under the norm $\nu(u)$ is denoted by $\hat{H}_\sigma^1(\hat{G}_\infty;\pi)$.

We proceed to vector valued functions. Let X be a Banach space, and let $1\leqq p\leqq\infty$. $L_p(X;\pi)$ denotes the L_p-space of X-valued functions which are periodic in t. Namely, $L_p(X;\pi)$ consists of $f\in L_p^{\mathrm{loc}}(R^1;X)$ such that $f(t+T)=f(t)$ for almost every t in R^1. It is a Banach space with norm

$$\left\{\int_0^T\|f(x)\|_X^p dt\right\}^{1/p}\ .$$

1.3. Formulation of the problem and the approximation by the penalty method

We are concerned with the Navier-Stokes equations in $\Omega(t)$ with periodically moving boundaries. The classical formulation of this problem is as follows:

$$
(1.1) \qquad \begin{cases} u_t = \Delta u - \nabla p - (u\cdot\nabla)u + f_0 & \text{in} \quad \hat{\Omega}_\infty\,, \\ \operatorname{div} u = 0 & \text{in} \quad \hat{\Omega}_\infty\,, \\ u = \beta & \text{on} \quad \hat{\Gamma}_\infty\,. \end{cases}
$$

Here $u=u(t,x)$ denotes the velocity field, $p=p(t,x)$ is the pressure. When the external force $f_0=f_0(t,x)$ and the boundary value $\beta=\beta(t,x)$ are of period T, the problem to find a T-periodic solution u and p satisfying (1.1) is called (Pr. π).

For f_0 and β, we make

ASSUMPTION 1.2. f_0 belongs to $L_2(\hat{\Omega}_\infty;\pi)\equiv\{f\in L_2(L_2(B);\pi); f|_{\hat{E}_\infty}=0\}$. β can be extended to a vector function $b=b(t,x)$ of the form $b=\operatorname{rot} c$, where $c(t,x)$, the function defined in $\hat{B}_\infty$, is of class C^3 and T-periodic.

We approximate the equation (1.1) by the following (1.2), n being an arbitrary positive integer.

$$
(1.2) \qquad \begin{cases} u^n_t = \Delta u^n - \nabla p^n - (u^n\cdot\nabla)u^n - n\chi(u^n - b) + \tilde{f}_0 & \text{in} \quad \hat{B}_\infty\,, \\ \operatorname{div} u^n = 0 & \text{in} \quad \hat{B}_\infty\,, \\ u^n = 0 & \text{on} \quad (-\infty,\infty)\times\partial B\,. \end{cases}
$$

Here $\chi=\chi(t,x)$ is the characteristic function of $\hat{E}_\infty\equiv\hat{B}_\infty-\hat{\Omega}_\infty$, $\tilde{f}_0$ is the extension of f_0 to $\hat{B}_\infty$ which is zero outside of $\hat{\Omega}_\infty$. The problem to find a T-periodic solution of (1.2) is denoted by $(AP)_n$. The term $n\chi(u^n-b)$ is called the penalizing term. Remark that the first equation in (1.2) coincides with that of (1.1) when restricted to $\hat{\Omega}_\infty$.

Now let us define some function classes in which we seek the solution.

DEFINITION 1.3.

$$
\mathscr{U}(\hat{\Omega}_\infty;\pi)=\{u\in \hat{H}^1_o(\hat{\Omega}_\infty;\pi);\ \operatorname*{ess.sup}_{0\le t\le T}\|u(t)\|_{\Omega(t)}<+\infty\}
$$

$$
\mathscr{U}(\hat{B}_\infty;\pi)=\{u\in \hat{H}^1_o(\hat{B}_\infty;\pi);\ \operatorname*{ess.sup}_{0\le t\le T}\|u(t)\|_{B}<+\infty\}\,.
$$

DEFINITION 1.4. $u=u(t,x)$ defined in $\hat{\Omega}_\infty$ is called a weak solution of (Pr. π) if i) and ii) hold:

i) $u-b\in\mathscr{U}(\hat{\Omega}_\infty;\pi)$.

ii) For all φ in $\hat{D}_o(\hat{\Omega}_\infty;\pi)$, u satisfies the following equality

(1.3) $$F(u, \varphi)\equiv\int_0^T\{(u, \varphi_t)+(u, \Delta\varphi)+(u, (u\cdot\nabla)\varphi)\}dt=-\int_0^T(f_0, \varphi)dt\ ,$$

where the domain of integration in the inner product is understood.

DEFINITION 1.5. $u^n=u^n(t, x)$ defined in $\hat{B}_\infty$ is called a weak solution of $(AP)_n$ if i) and ii) hold:

i) $u^n\in \mathscr{U}(\hat{B}_\infty; \pi)$.

ii) For all $\varphi\in\hat{D}_o(\hat{B}_\infty; \pi)$, u^n satisfies

(1.4) $$F(u^n, \varphi)=n\int_0^T(\chi(u^n-b), \varphi)dt-\int_0^T(\tilde{f}_0, \varphi)dt\ .$$

REMARK 1.6. If u (resp. u^n) is a weak solution of (Pr. π) (resp. $(AP)_n$), then $v=u-b$ (resp. $v^n=u^n-b$) satisfies the following equality (1.5) (resp. (1.6)).

(1.5) $$F(v, \varphi)=\int_0^T\{((v\cdot\nabla)b, \varphi)-(v, (b\cdot\nabla)\varphi)-(f, \varphi)\}dt$$

(1.6) $$F(v^n, \varphi)=\int_0^T\{((v^n\cdot\nabla)b, \varphi)-(v^n, (b\cdot\nabla)\varphi)-(\tilde{f}, \varphi)+n(\chi v^n, \varphi)\}dt\ ,$$

where $f=f_0+\Delta b-(b\cdot\nabla)b-b_t$ and $\tilde{f}=\tilde{f}_0+\Delta b-(b\cdot\nabla)b-b_t$.

1.4. Results

THEOREM 1.7. *Under Assumptions* 1.1 *and* 1.2, *there exists a weak solution of* (Pr. π) *such that* $u(t)=u(t, \cdot)$ *is defined for every* t *in* R^1, *and* $(u(t), \varphi(t))_{\Omega(t)}$ *is a continuous T-periodic function for every* φ *in* $\hat{D}_o(\hat{\Omega}_\infty; \pi)$.

REMARK 1.8. We have the energy inequality for the weak solution u of (Pr. π). Namely, there exist constants R and C_0 such that

(1.7) $$\|u(t)-b(t)\|^2+\int_0^t\|\nabla u(s)-\nabla b(s)\|^2ds\leqq R^2+C_0\int_0^t\|\tilde{f}(s)\|_{\hat{h}}^2ds$$

holds for every t in $[0, t]$, where $\tilde{f}=\tilde{f}_0+\Delta b-(b\cdot\nabla)b-b_t$.

Furthermore, we can show that

(1.8) $$(u(t), \varphi(t))_{\Omega(t)}=(u(0), \varphi(0))$$
$$+\int_0^t\{(u, \varphi_s)+(u, \Delta\varphi)+(u, (u\cdot\nabla)\varphi)\}ds+\int_0^t(f_0, \varphi)ds$$

holds for all φ in $\hat{D}_o(\hat{\Omega}_\infty; \pi)$ and t in $[0, T]$.

§ 2. Preliminaries

2.1. The Stokes operator

In this section, Ω is a fixed bounded domain in R^m and the boundary $\partial\Omega$ of Ω

is smooth. Let $P=P(\Omega)$ be the orthogonal projection from $L_2(\Omega)$ of vector functions onto $H_\sigma(\Omega)$. Then the operator $-P\Delta$ with the domain of definition $D_\sigma(\Omega)$ is a positive symmetric operator in $H_\sigma(\Omega)$. The Friedrichs extension of $-P\Delta$ is called the Stokes operator and is denoted $A(\Omega)$, which we may write A when Ω is understood. A is a self-adjoint positive operator in $H_\sigma(\Omega)$, A^{-1} is a bounded operator on $H_\sigma(\Omega)$ (in fact, completely continuous), and we have

$$(2.1)\qquad (Au, v)_\Omega=(\nabla u, \nabla v)_\Omega$$

for any $u\in D(A)$ and $v\in H_\sigma^1(\Omega)$ (see, e.g., Ladyzhenskaya [8]). It is known (Cattabriga [3], Ladyzhenskaya [8], Agmon-Douglis-Nirenberg [1]) that $D(A)$, the domain of definition of A, is $W_2^2(\Omega)\cap H_\sigma^1(\Omega)$, $D(A^{1/2})=H_\sigma^1(\Omega)$ and $\|A^{1/2}u\|=\|\nabla u\|$ $(u\in D(A^{1/2}))$. Let $\{\varphi_j\}$ be the complete orthonormal system of eigenfunctions of A, $\{\lambda_j\}$ corresponding eigenvalues (>0), λ_0 the smallest eigenvalue. It is easy to verify the following

$$(2.2)\qquad \|u\|_{L_2(\Omega)}^2\leqq\frac{1}{\lambda_0}\|\nabla u\|_{L_2(\Omega)}^2 \quad \text{for} \quad \forall u\in H_\sigma^1(\Omega)\,.$$

Furthermore, for u in $D(A)$, we have

LEMMA 2.1. *There exists a constant C depending only on Ω such that*

$$\|\nabla u\|_{L_p(\Omega)}\leqq C\|Au\|_{L_2(\Omega)}$$

holds for $u\in D(A)$ and $2\leqq p\leqq 6$.

PROOF. Since $D(A)=W_2^2(\Omega)\cap H_\sigma^1(\Omega)$, Sobolev's imbedding theorem yields the conclusion. Q.E.D.

Next lemma is necessary for the successful application of Galerkin's method for $(AP)_n$. In the lemma, $C^\infty(R^1;\pi)$ stands for the set of smooth periodic scalar functions:

$$C^\infty(R^1;\pi)=\{\eta\in C^\infty(R^1);\ \eta(t+T)=\eta(t) \text{ for } \forall t\in R^1\}\,.$$

LEMMA 2.2. *Let $\hat{W}$ be the set of all finite linear combinations of φ_j with coefficients in $C^\infty(R^1;\pi)$. Then $\hat{W}$ is dense in $L_2(D(A);\pi)$.*

PROOF. $D(A)$ is considered as a Hilbert space with the inner product $((u, v))=(Au, Av)_\Omega$. The conclusion follows from the completeness of the eigenfunctions. Q.E.D.

REMARK 2.3. Let us apply Lemma 2.2 for the operator $A=A(B)$. Then we see that any function in $\hat{D}_\sigma(\hat{B}_\infty;\pi)$ can be approximated by the element of $\hat{W}$ under the norm of $L_2(D(A);\pi)$.

2.2. Remarks on $\hat{D}_\sigma$, $\hat{H}_\sigma^1$

$\hat{D}_\sigma(\hat{\Omega}_\infty; \pi)$ is separable in the following sense:

LEMMA 2.4. *There exists a countable subset $\hat{M}_\sigma$ of $\hat{D}_\sigma(\hat{\Omega}_\infty; \pi)$ such that every η in $\hat{D}_\sigma(\hat{\Omega}_\infty; \pi)$ can be uniformly approximated in $\hat{\Omega}$ together with η_t, η_{x_j}, $\eta_{x_i x_j}$ $(i, j=1, \cdots, m)$, by elements of $\hat{M}_\sigma$.*

The proof is similar to that of Lemma 4.2 of Fujita-Sauer [6] and is omitted.

REMARK 2.5. We fix $t_0 \in (-\infty, \infty)$. Then $\{\eta(t_0, \cdot) | \eta \in \hat{M}_\sigma\}$ is dense in $H_\sigma^1(\Omega(t_0))$ as well as in $H_\sigma(\Omega(t_0))$. In fact, if $h \in D_\sigma(\Omega(t_0))$, we can find a scalar function $\eta(t) \in C^\infty(R^1; \pi)$, such that $\eta(t_0)=1$ and $\eta(t)h \in \hat{D}_\sigma(\hat{\Omega}_\infty; \pi)$.

LEMMA 2.6. *Let v be in $\hat{H}_\sigma^1(\hat{B}_\infty; \pi)$ with $\hat{\gamma}(v) \equiv v|_{\hat{\Gamma}}=0$. Then u, the restriction of v to $\hat{\Omega}_\infty$, belongs to $\hat{H}_\sigma^1(\hat{\Omega}_\infty; \pi)$.*

The proof is omitted, because it is similar to that of Lemma 4.5 of Fujita-Sauer [6].

2.3. Some inequalities

For any functions u in $W_2^1(\Omega)$, we can define its boundary value $\gamma u = u|_{\partial\Omega}$. The trace operator γ is continuous (in fact completely continuous) from $W_2^1(\Omega)$ to $L_2(\partial\Omega)$. (See, for instance, Lions-Magenes [10] Chap. 1). Moreover, the following estimates hold for any small positive number δ. Let φ be an arbitrary element of $H_\sigma^1(B)$. Since $H_\sigma^1(B) \subset W_2^1(B)$, we have,

$$\|\varphi\|_{\omega(\delta)}^2 \leqq C\delta\{\|\varphi\|_{\Gamma(t)}^2 + \|\varphi\|_{\omega(\delta)} \|\nabla\varphi\|_{\omega(\delta)}\}, \tag{2.3}$$

$$\delta\|\varphi\|_{\Gamma(t)}^2 \leqq C\{\|\varphi\|_{\omega(\delta)}^2 + \delta\|\varphi\|_{\omega(\delta)} \cdot \|\nabla\varphi\|_{\omega(\delta)}\}. \tag{2.4}$$

Here and in what follows, the symbol C stands for various constants independent of φ, t and δ. $\omega(\delta)$ is either $\omega_i(t; \delta)$ or $\omega_e(t; \delta)$. From these inequalities, after simple calculation, we have

$$\|\varphi\|_{\omega(\delta)}^2 \leqq C\{\delta\|\varphi\|_{\Gamma(t)}^2 + \delta^2\|\nabla\varphi\|_{\omega(\delta)}^2\}, \tag{2.5}$$

$$\|\varphi\|_{\Gamma(t)}^2 \leqq C\left\{\frac{1}{\delta}\|\varphi\|_{\omega(\delta)}^2 + \delta\|\nabla\varphi\|_{\omega(\delta)}^2\right\}. \tag{2.6}$$

Under our assumptions on $\hat{\Gamma}_\infty$, the restriction map $\hat{\gamma}_\infty$: $\hat{\gamma}_\infty u = u|_{\hat{\Gamma}_\infty}$ can be defined for any u in $\hat{H}_\sigma^1(\hat{B}_\infty, \pi)$, and by the above estimates, we have

$$\delta\|u\|_{\hat{\Gamma}}^2 \leqq C\{\|u\|_{\hat{\omega}(\delta)}^2 + \delta\|u\|_{\hat{\omega}(\delta)} \|\nabla u\|_{\hat{\omega}(\delta)}\}, \tag{2.7}$$

$$\|u\|_{\hat{\omega}(\delta)}^2 \leqq C\{\delta\|u\|_{\hat{\Gamma}}^2 + \delta^2\|\nabla u\|_{\hat{\omega}(\Gamma)}^2\}. \tag{2.8}$$

Let ρ be the distance from the point $x \in \Omega(t)$ to the boundary $\Gamma(t)$. Put

$$\omega(t;\varepsilon,\delta)=\{x\in\Omega(t);\ \varepsilon<\mathrm{dist.}(x,\Gamma(t))<\delta\}\ .$$

Then the estimate

$$\left\|\frac{\varphi}{\rho}\right\|^2_{\omega(t;\varepsilon,\delta)}\leqq C\left\{\frac{\|\varphi\|^2_{\Gamma(t)}}{\varepsilon}+\|\nabla\varphi\|^2_B\right\} \tag{2.9}$$

holds for any φ in $H^1_\sigma(B)$. The proof of these inequalities will be found in Appendix 1.

2.4. Extension of the boundary value

LEMMA 2.7. *For any given* $\varepsilon>0$, *there exists a function* $b=b_\varepsilon$ *subject to Assumption* 1.2 *and a constant* $C=C_\varepsilon$ *independent of* t *such that*

$$|((\varphi\cdot\nabla)\varphi, b(t,\cdot))_B|\leqq\varepsilon\|\nabla\varphi\|^2_B+C(\chi(t,\cdot)\varphi,\varphi)_B$$

holds for any φ *in* $H^1_\sigma(B)$.

PROOF. We construct the required b, following the argument of Hopf [7] (see also Fujita [4]). Let b^* be any function subject to Assumption 1.2. Then $b^*(t,x)=\mathrm{rot}\,c(t,x)$ for some function c. Put $b(t,x)=\mathrm{rot}\,\{h(\rho(t,x))c(t,x)\}$, where $\rho=\rho(t,x)$ is the distance in R^m between $x\in B$ and $\Gamma(t)$, and $h(s)$ is a function involving positive parameters κ $(0<\kappa<1/4)$ and γ as follows. Let $j(s)$ be a function in $C^\infty[0,\infty)$ such that $0\leqq j(s)\leqq s^{-1}$ for $s>0$, $j(s)\equiv0$ for $s\in[0,\kappa\gamma]$ and $s\in[(1-\kappa)\gamma,\infty)$, $j(s)=s^{-1}$ for $s\in[2\kappa\gamma,(1-2\kappa)\gamma]$. Putting

$$h(s)\equiv1-\int_0^s j(\sigma)d\sigma\Big/\int_0^\infty j(\sigma)d\sigma\ ,$$

we try to determine the parameters κ and γ appropriately.

Thanks to the formula $b=h$ rot $c+\mathrm{grad}\,h\times c$, it is enough to check $b_1=h$ rot c and $b_2=\nabla h\times c$ separately. We notice that the support of b_1 is contained in $\hat\omega_i(\gamma)\cup\hat\omega_e(\gamma)$, that of b_2 is contained in $\hat\omega_i(\kappa\gamma,\gamma)\cup\hat\omega_e(\kappa\gamma,\gamma)$, and ρb_2 tends to zero as κ tends to zero. Let φ be any element in $H^1_\sigma(B)$. Then we have

$$|((\varphi\cdot\nabla)\varphi, b_1)|\leqq\sup_{x,t}|b_1(t,x)|\cdot\|\varphi\|_{\omega_i(\gamma)\cup\omega_e(\gamma)}\|\nabla\varphi\|_{\omega_i(\gamma)\cup\omega_e(\gamma)}\ .$$

Using (2.5), the right hand side is estimated by $C\sup|b_1|(\|\varphi\|^2_{\Gamma(t)}+\gamma\|\nabla\varphi\|^2_B)$. The application of (2.6) with $\omega(\gamma)=\omega_e(\gamma)$ yields

$$|((\varphi\cdot\nabla)\varphi, b_1)|\leqq C\sup|b_1|\left\{\frac{1}{\gamma}(\chi\varphi,\varphi)+\gamma\|\nabla\varphi\|^2_B\right\}\ . \tag{2.10}$$

On the other hand,

$$|((\varphi\cdot\nabla)\varphi, b_2)| \leqq \|\nabla\varphi\|_B \left\| \rho b_2 \cdot \frac{\varphi}{\rho} \right\|_{\omega_i(\kappa\gamma,\gamma)\cup\omega_e(\kappa\gamma,\gamma)}$$

$$\leqq \sup|\rho b_2| \cdot \|\nabla\varphi\|_B \cdot \left\| \frac{\varphi}{\rho} \right\|_{\omega_i(\kappa\gamma,\gamma)\cup\omega_e(\kappa\gamma,\gamma)} .$$

Appling (2.9) and making use of (2.6) with $\omega(\kappa\gamma)=\omega_e(\kappa\gamma)$, we have

$$|((\varphi\cdot\nabla)\varphi, b_2)| \leqq C \sup|\rho b_2| \left\{ \frac{(\chi\varphi, \varphi)}{(\kappa\gamma)^2} + \|\nabla\varphi\|_B^2 \right\} . \tag{2.11}$$

The assertion of the lemma is obtained from (2.10) and (2.11) since $C(\gamma \sup|b_1| + \sup|\rho b_2|)$ can be made smaller than any ε if κ and γ are sufficiently small.

Q.E.D.

REMARK 2.8. The inequality

$$|((\varphi\cdot\nabla)\varphi, b)_\Omega| \leqq \varepsilon \|\nabla\varphi\|_\Omega^2 \text{ for any } \varphi \in H_\sigma^1(\Omega)$$

is used in the case of the inhomogeneous boundary value problem for stationary solution in a fixed domain. Our preceding lemma is an extension of this inequality.

From now on, we fix the function b for $\varepsilon=1/4$. Let n_0 be the smallest integer which is greater than $C=C_{1/4}$. Then we have

$$|((\varphi\cdot\nabla)\varphi, b)_B| \leqq 1/4\|\nabla\varphi\|_B^2 + n_0(\chi\varphi, \varphi)_B . \tag{2.12}$$

The next two lemmas are elementary but are used frequently in § 3. The proof is omitted since it is obvious.

LEMMA 2.9. *Let $g(t)$, $h(t)$ be nonnegative continuons functions on the interval $[0, T]$. If $g(t)$ is differentiable and if $\frac{d}{dt}g(t)+cg(t) \leqq h(t)$ on $[0, T]$ for nonzero constant c, then we have*

$$g(t) \leqq \left\{ g(0) + \int_0^t e^{cs} h(s) ds \right\} e^{-ct} \quad \textit{for} \quad t \in [0, T] .$$

LEMMA 2.10. *Let u, v and w be in $W_2^1(\Omega)$, at least one of which vanishes on $\partial\Omega$, and* $\operatorname{div} u=0$ *in Ω. Then*

$$((u\cdot\nabla)v, w)_\Omega = -((u\cdot\nabla)w, v)_\Omega .$$

In particular,

$$((u\cdot\nabla)v, v)_\Omega = 0 .$$

2.5. A modification of Aubin's compactness theorem

As in Fujita-Sauer [6], we later need a modified version of Aubin's compactness theorem.

Let M_i $(i=0, 1, 2)$ be three Hilbert spaces. We consider the operators $P: M_0 \to M_1$ and $S: M_0 \to M_2$ with the following properties:

i) P and S are compact linear.

ii) $Sv=0$ implies $Pv=0$ for $v \in M_0$.

By the standard argument we can show that for any $\varepsilon>0$ there exists a constant C_ε such that

$$\|Pv\|_{M_1} \leqq \varepsilon \|v\|_{M_0} + C_\varepsilon \|Sv\|_{M_2} \tag{2.13}$$

holds for all v in M_0.

Now we introduce some classes of M_i-valued functions. $X_i = L_2((\alpha, \beta); M_i)$ $(i=0, 1, 2)$ are the set of M_i-valued functions defined on the finite interval (α, β), and form the Hilbert space under the inner product

$$(u, v)_{X_i} = \int_\alpha^\beta (u(t), v(t))_{M_i} dt \ .$$

Define operators $\hat{P}: X_0 \to X_1$ and $\hat{S}: X_0 \to X_2$ as follows:

$$(\hat{P}v)(t) \equiv Pv(t) \ , \quad (\hat{S}v)(t) \equiv Sv(t) \ .$$

Then we have the following lemma concerning the compactness, which has been obtained by Fujita-Sauer [6] as a modification of Aubin's theorem [2] (See also Lions [9].).

LEMMA 2.11. *Under the notations and assumptions above, suppose that* $v_n \in X_0$, $n=1, 2, \cdots$ *be such that*

i) $\{v_n\}$ *is a bounded set in* X_0,

ii) $\left\{\frac{d}{dt} \hat{S} v_n\right\}$ *(derivative in the sense of distribution) is a bounded set in* X_2.

Then we can choose a subsequence of $\{\hat{P} v_n\}$, *which converges strongly in* X_1.

The proof is in Appendix 2.

§ 3. Approximating equations

3.1. A priori estimates

In this subsection, we seek a periodic solution of the following equation (3.1) in the finite dimensional vector space. Note that n is fixed throughout this section.

Let $\{\varphi_j\}$ be the family of eigenfunctions of the Stokes operator $A=A(B)$. We consider the following initial value problem which is denoted by $(IVP)_m^n$.

$$\frac{d}{dt}(w_m(t), \varphi_j)+(\nabla w_m(t), \nabla\varphi_j)-((w_m(t)\cdot\nabla)\varphi_j, w_m(t))$$
$$+n(\chi w_m(t), \varphi_j)-((b(t)\cdot\nabla)\varphi_j, w_m(t))+((w_m(t)\cdot\nabla)b(t), \varphi_j)$$
$$=(\bar{f}(t), \varphi_j)\,, \qquad j=1, \cdots, m. \tag{3.1}$$

$w_m(t)\in\Phi_m\equiv$vector space generated by $\varphi_1, \cdots, \varphi_m$.

$w_m(0)=w_{m0}$ given in Φ_m.

Here (,) denotes the inner product in $L_2(B)$, b is chosen as in (2.12) and $n\geqq n_0$. The solution of $(IVP)_m^n$ should be represented as

$$w_m(t)=\sum_{j=1}^{m} g_{jm}(t)\varphi_j$$

with $g_{jm}(t)$, the scalar functions in t. Multiplying (3.1) by $g_{jm}(t)$ and summing up with respect to j, we have

$$\frac{1}{2}\frac{d}{dt}\|w_m(t)\|^2+\|\nabla w_m(t)\|^2+n(\chi w_m(t), w_m(t))$$
$$=((w_m(t)\cdot)\nabla w_m(t), b(t))+(\bar{f}(t), w_m(t))\,, \tag{3.2}$$

where we have used Lemma 2.10. By (2.2) and (2.12) the right hand side of (3.2) is dominated by

$$\frac{1}{2}\|\nabla w_m(t)\|^2+n_0(\chi w_m(t), w_m(t))+\frac{1}{\lambda_0}\|\bar{f}(t)\|^2\,,$$

where $\lambda_0=\lambda_0(B)$. Thus we have

$$\frac{d}{dt}\|w_m(t)\|^2+\|\nabla w_m(t)\|^2+2(n-n_0)(\chi w_m(t), w_m(t))\leqq\frac{2}{\lambda_0}\|\bar{f}(t)\|^2\,. \tag{3.3}$$

Using again (2.2),

$$\frac{d}{dt}\|w_m(t)\|^2+\lambda_0\|w_m(t)\|^2+2(n-n_0)(\chi w_m(t), w_m(t))\leqq\frac{2}{\lambda_0}\|\bar{f}(t)\|^2\,. \tag{3.4}$$

Since $n\geqq n_0$, we get, applying Lemma 2.9,

$$\|w_m(t)\|^2\leqq\left\{\|w_m(0)\|^2+\frac{2}{\lambda_0}\int_0^t e^{\lambda_0\tau}\|\bar{f}(\tau)\|^2 d\tau\right\}e^{-\lambda_0 t}\,, \tag{3.5}$$

for any $t\in[0, T]$.

3.2. Periodic solutions of (3.1)

If we substitute $\sum_{j=1}^{m} g_{jm}(t)\varphi_j$ for $w_m(t)$ in (3.1), the problem $(IVP)_m^n$ is transformed into the initial value problem for a system of nonlinear ordinary differential equations and we have the local existence of the solution, that is, the existence of a continuous solution in an interval $[0, t_0]$ for some positive t_0. However, thanks to the estimate (3.5), we can take $t_0=T$. Moreover, the solution is unique and depends continuously on the initial data. Let $w_m(t)$ be the solution of $(IVP)_m^n$ obtained as above. By τ_m, we denote the mapping which carries w_{m0} to $w_m(T)$, that is,

$$\tau_m : H_\sigma(B) \to H_\sigma(B) ,$$

$$\tau_m w_{m0} = w_m(T) .$$

If we choose a constant R so that

$$R^2 \geqq \frac{2}{\lambda_0(e^{\lambda_0 T}-1)} \int_0^T e^{\lambda_0 t} \|\bar{f}(t)\|^2 dt \tag{3.6}$$

holds, then τ_m maps the closed ball $\mathscr{B}_R$ of radius R in the m-dimensional vector space Φ_m into itself:

$$\tau_m : \mathscr{B}_R = \{\varphi \in \Phi_m, \|\varphi\| \leqq R\} \to \mathscr{B}_R .$$

Since τ_m is continuous, we can apply Brouwer's fixed point theorem, and we see that there exist fixed points w_{m0} of τ_m in $\mathscr{B}_R$. Hence the existence of a periodic solution $w_m^n(t)$ of period T follows. We sometimes write $w_m(t)$ instead of $w_m^n(t)$, if n is understood. Furthermore, $\|w_m^n(0)\| \leqq R$. Integrating (3.3), we have

$$\begin{aligned} \|w_m(t)\|^2 + \int_0^t \|\nabla w_m(s)\|^2 ds + 2(n-n_0) \int_0^t (\chi w_m(s), w_m(s)) ds \\ \leqq \|w_m(0)\|^2 + \frac{2}{\lambda_0} \int_0^t \|\bar{f}(s)\|^2 ds , \quad \text{for} \quad 0 \leqq \forall t \leqq T . \end{aligned} \tag{3.7}$$

Thus we have proved

PROPOSITION 3.1. *There exists a continuous periodic solution $w_m^n(t)$ of the equation* (3.1), *the period being T. Moreover, there exists a constant R independent of m and n, such that*

$$\|w_m^n(t)\|^2 \leqq R^2 + \frac{2}{\lambda_0} \int_0^t \|\bar{f}(s)\|^2 ds ,$$

$$\int_0^t \|\nabla w_m^n(s)\|^2 ds \leqq R^2 + \frac{2}{\lambda_0} \int_0^t \|\bar{f}(s)\|^2 ds ,$$

$$\int_0^t (\chi w_m^n(s), w_m^n(s)) ds \leqq \frac{1}{2(n-n_0)} \left\{ R^2 + \frac{2}{\lambda_0} \int_0^t \|\bar{f}(s)\|^2 ds \right\}$$

for any $t \in [0, T]$ and $n > n_0$.

In the following we shall show that for every fixed n the set $\left\{ \frac{d}{dt} w_m^n \right\}_{m=1}^\infty$

forms a bounded set in a certain space. Let $D(A)$ be the domain of the Stokes operator $A=A(B)$, $D(A)'$ its dual space. The norms of these spaces are defined as follows:

$$\|u\|_{D(A)}=\|A(B)u\|_{L_2(B)} ,$$

$$\|f\|_{D(A)'}=\sup_{u\in D(A)} \frac{|\langle f, u\rangle|}{\|u\|_{D(A)}} .$$

PROPOSITION 3.2. *Let $w_m^n(t)$ be the periodic solutions of* (3.1) *obtained in Proposition* 3.1. *Then for every fixed n, $\left\{\frac{d}{dt}w_m^n(t)\right\}_{m=1}^{\infty}$ forms a bounded set of $L_2(D(A)'; \pi)$.*

PROOF. Let P_m be the orthogonal projection from $H_\sigma(B)$ onto the vector space Φ_m. Consider the restriction of P_m to $D(A)$. P_m is a bounded operator from $D(A)$ into $D(A)$, and

$$\|P_m\|_{\mathcal{L}(D(A), D(A))}\leqq 1 ,$$

because $\{\varphi_j\}$ are eigenfunctions of A and, in particular, A commutes with P_m. By the transposition, we have for the dual operator P_m' of $P_m : D(A)\to D(A)$,

$$\|P_m'\|_{\mathcal{L}(D(A)', D(A)')}\leqq 1 .$$

Let J be a canonical injection from $L_2(B)$ into $D(A)'$. Then, from (3.1), we obtain

$$J\frac{d}{dt}w_m=-P_m'JAw_m-P_m'J(w_m\cdot\nabla)w_m -P_m'J\{n\chi w_m+(b\cdot\nabla)w_m+(w_m\cdot\nabla)b-\bar{f}\} . \tag{3.8}$$

Here we have used the relation (3.1) and Lemma 2.10. Take an arbitrary element h in $D(A)$. We have

$$\begin{aligned}|\langle JAw_m(t), h\rangle| &=|(Aw_m(t), h)_{L_2(B)}|\leqq\|\nabla w_m(t)\|_{L_2(B)}\|\nabla h\|_{L_2(B)}\\ &\leqq c\|w_m(t)\|_{H_\sigma^1(B)}\|Ah\|_{L_2(B)} ,\end{aligned}$$

$$\begin{aligned}|\langle J(w_m(t)\cdot\nabla)w_m(t), h\rangle| &=|((w_m(t)\cdot\nabla)w_m(t), h)_B|\\ &\leqq\|w_m(t)\|_{L_2(B)}\|w_m(t)\|_{L_6(B)}\|\nabla h\|_{L_3(B)}\\ &\leqq c\|w_m(t)\|_{L_2(B)}\|w_m(t)\|_{H_\sigma^1(B)}\|Ah\|_{L_2(B)} ,\end{aligned}$$

where we have used Lemma 2.1. Therefore the following estimates hold:

$$\|P_m'JAw_m\|_{L_2(D(A)';\pi)}\leqq c\|w_m\|_{\hat{H}_\sigma^1(\hat{B}_\infty;\pi)} ,$$

$$\|P_m'J(w_m\cdot\nabla)w_m\|_{L_2(D(A)';\pi)}\leqq c(\sup_t\|w_m(t)\|_{L_2(B)})\cdot\|w_m\|_{\hat{H}_\sigma^1(\hat{B}_\infty;\pi)} .$$

The other terms in (3.8) can be estimated similarly and the conclusion of the proposition follows from Proposition 3.1. Q.E.D.

3.3. Solutions of $(AP)_n$

We shall prove, with the aid of Propositions 3.1, 3.2 and Lemma 2.11, that a subsequence of the periodic solutions $w_m=w_m^n$ of the equation (3.1) converges to a function $v^n\in\mathscr{U}(\hat{B}_\omega;\pi)$ satisfying (1.6) as $m\to\infty$.

PROPOSITION 3.3. *There exists a weak periodic solution u^n of $(AP)_n$ for every $n\geqq n_0$, with the following properties*:

i) *for every h in $H_\sigma(B)$, the inner product $(u^n(t), h)$ is a continuous and T-periodic function in t.*

ii) *For every φ in $\hat{D}_\sigma(\hat{B}_\omega;\pi)$ and for any t in $[0, T]$,*

$$(u^n(t),\varphi(t))=(u^n(0),\varphi(0))+\int_0^t\{(u^n(s),\varphi_s(s))+(u^n(s),\Delta\varphi(s))+(u^n(s),(u^n\cdot\nabla)\varphi(s))\}ds+\int_0^t(\tilde{f}_0(s),\varphi(s))ds-n\int_0^t(\chi(u^n-b),\varphi)ds\,. \tag{3.9}$$

iii) *The following energy inequality for $v^n(t)=u^n(t)-b(t)$ holds for $\forall t\in[0, T]$, where $\tilde{f}=\tilde{f}_0+\Delta b-(b\cdot\nabla)b-b_t$:*

$$\|v^n(t)\|^2+\int_0^t\|\nabla v^n(s)\|^2ds+2(n-n_0)\int_0^t(\chi v^n, v^n)ds\leqq R^2+\frac{2}{\lambda_0}\int_0^t\|\tilde{f}(s)\|^2ds\,. \tag{3.10}$$

PROOF. First of all, we shall prove the existence of the subsequence $\{w_\mu\}$ of $\{w_m\}$ such that, for every h in $H_\sigma(B)$, $(w_\mu(t), h)_B$ converges uniformly on $[0, T]$ to $(w(t), h)_B$ where $w(t)\in H_\sigma(B)$. For a moment we fix the suffix j, and integrate (3.1) with respect to t. Then, considering the estimates obtained in Proposition 3.1, we see that $\{(w_m(t),\varphi_j)\}_{m=1}^\infty$ are uniformly bounded and equicontinuous on $[0, T]$. By the Ascoli-Arzelà theorem, we can select a subsequence of $\{(w_m(t),\varphi_j)\}_m$ converging in $C[0, T]$. By means of the diagonal argument, we get a subsequence $\{w_\nu\}$ for which $(w_\nu(t),\varphi_j)$ converges in $C[0, T]$ for every $j=1, 2,\cdots$. Since $\{\varphi_j\}$ are dense in $H_\sigma(B)$ and $\{\|w_\nu(t)\|\}$ are bounded on $[0, T]$, it holds that $(w_\nu(t), h)$ converges in $C[0, T]$, for every h in $H_\sigma(B)$. We write its limit as $J_h(t)$. Since $|J_h(t)|\leqq\overline{\lim_\nu}\|w_\nu(t)\|\cdot\|h\|<C\|h\|$, Riesz' theorem asserts that there exists $w(t)\in H_\sigma(B)$ such that $J_h(t)=(w(t), h)$, and hence for every $h\in H_\sigma(B)$,

$$(w_\nu(t), h)\to(w(t), h)\quad\text{uniformly in } t\in[0, T]\,. \tag{3.11}$$

Now, let us apply Lemma 2.11 with $M_0=H_\sigma^1(B)$, $M_1=H_\sigma(B)$, $M_2=(D(A))'$, P=injection from $H_\sigma^1(B)$ into $H_\sigma(B)$, S=injection from $H_\sigma^1(B)$ into $(D(A))'$. It is easy to verify that the hypothesis in Lemma 2.11 is satisfied, and there exists a subsequence $\{w_{\nu'}\}$ of $\{w_\nu\}$ such that $\{w_{\nu'}\}$ converges strongly in $L_2(H_\sigma(B);\pi)$.

Choosing a subsequence of $\{\nu'\}$ if necessary, we establish the following convergence as μ tends to infinity along a suitable subsequence of $N=\{1, 2, \cdots\}$:

$$w_\mu \to w \text{ strongly in } L_2(H_\sigma(B); \pi)\,, \tag{3.12}$$

$$\frac{d}{dt} w_\mu \to \frac{d}{dt} w \text{ weakly in } L_2(D(A)'; \pi)\,, \tag{3.13}$$

$$w_\mu \to w \text{ weakly in } L_2(H^1_\sigma(B); \pi)\,, \tag{3.14}$$

and

$$w_\mu \to w \text{ weakly* in } L_\infty(H_\sigma(B); \pi)\,. \tag{3.15}$$

It follows from (3.14) and (3.15) that w belongs to $\mathscr{U}(\hat{B}_\infty; \pi)$.

Consequently, if we can show that w satisfies the equation (1.6), then $w+b$ is a weak periodic solution of $(AP)_n$. To this end, take an arbitrary scalar function $\eta(t)$ in $C^\infty(R^1; \pi)$. Multiplying (3.1) by η and integrating with respect to t, we have

$$\begin{aligned} G_{w_\mu}(\phi) \equiv \int_0^T \{ &\langle w'_\mu, \phi\rangle + (\nabla w_\mu, \nabla\phi) - ((w_\mu\cdot\nabla)\phi, w_\mu) \\ &+ n(\chi w_\mu, \phi) - ((b\cdot\nabla)\phi, w_\mu) + ((w_\mu\cdot\nabla)b, \phi) - (\tilde{f}, \phi)\}dt = 0\,, \end{aligned}$$

where we have put $\phi=\phi(t, x)=\eta(t)\varphi_j(x)$. The convergence of $\{w_\mu\}$ established above yields that for any ϕ in $\hat{W}$, we have

$$\begin{aligned} G_w(\phi) = -\int_0^T (w, \phi_t)dt + \int_0^T \{&(\nabla w, \nabla\phi) - ((w\cdot\nabla)\phi, w) \\ &- ((b\cdot\nabla)\phi, w) + ((w\cdot\nabla)b, \phi) + n(\chi w, \phi) - (\tilde{f}, \phi)\}dt = 0\,. \end{aligned} \tag{3.16}$$

On the other hand, it follows from Lemma 2.1, Propositions 3.1 and 3.2 that there exists a constant c independent of μ such that

$$|G_{w_\mu}(\phi)| \leqq c\|\phi\|_{L_2(D(A);\pi)} \tag{3.17}$$

holds for any ϕ in $\hat{W}$. Hence, we see that $G_w(\phi)=0$ for any ϕ in $L_2(D(A); \pi)$, and in particular, for any ϕ in $\hat{D}_\sigma(\hat{B}_\infty; \pi)$ (Remark 2.3). Therefore w satisfies (1.6), and hence, $w+b$ satisfies (1.4). Thus $w+b$ is a weak solution of $(AP)_n$. At this stage, we put $v^n=w$ and $u^n=w+b$, making the dependence on n explicit. The conclusion i) follows from (3.11), ii) has been already proved, and iii) is obtained from Proposition 3.1 in consideration of the manner of convergence (3.11), (3.12), (3.14) of w_μ. Q.E.D.

§ 4. Proof of Theorem 1.7

In this section, we shall prove our main theorem. With the aid of the results so far obtained, the argument goes parallel to that of Fujita-Sauer [6].

Since $(\chi(u^n-b), \varphi)=0$ for any φ in $\hat{D}_\sigma(\hat{\Omega}_\infty; \pi)$, it follows from (3.9) that for any φ in $\hat{D}_\sigma(\hat{\Omega}_\infty; \pi)$, u^n satisfies

$$(4.1)\qquad (u^n(t), \varphi(t))=(u^n(0), \varphi(0)) + \int_0^t \{(u^n(s), \varphi_s(s))+(u^n(s), \Delta\varphi(s))+((u^n\cdot\nabla)\varphi, u^n)\}ds+\int_0^t (f_0, \varphi)ds\,,$$
$$n=n_0,\ n_0+1,\ n_0+2,\ \cdots.$$

We also need the following identity for $v^n=u^n-b$:

$$(4.2)\qquad (v^n(t), \varphi(t))=(v^n(0), \varphi(0)) + \int_0^t (v^n, \varphi_s)+(v^n, \Delta\varphi)+((v^n\cdot\nabla)\varphi, v^n)ds + \int_0^t ((b\cdot\nabla)\varphi, v^n)ds-\int_0^t ((v^n\cdot\nabla)b, \varphi)ds+\int_0^t (f, \varphi)ds\,,$$

where $f=f_0+\Delta b-(b\cdot\nabla)b-b_t$.

LEMMA 4.1. *We can select a subsequence* $\{v^\nu; \nu=\nu(m), m=1, 2, \cdots\}$ of $\{v^m\}_{m=1}^\infty$ *such that* $(v^\nu(t), \varphi(t))_{\Omega(t)}$ *converges to* $V_\varphi(t)$ *uniformly on* $[0, T]$ *for each* φ *in* $\hat{D}_\sigma(\hat{\Omega}_\infty; \pi)$ *as* $\nu\to\infty$. $V_\varphi(t)$ *is a continuous periodic function in* t, *which is represented as*

$$(4.3)\qquad V_\varphi(t)=(v(t), \varphi(t))_{\Omega(t)}$$

with a $v(t)\in H_\sigma(\Omega(t))$ *subject to*

$$(4.4)\qquad \|v(t)\|_{\Omega(t)}^2\leqq R^2+\frac{2}{\lambda_0}\int_0^t \|\bar{f}(\tau)\|_B^2 d\tau$$

for all t *in* $[0, T]$.

PROOF. For a moment we fix $\varphi\in\hat{D}_\sigma(\hat{\Omega}_\infty; \pi)$. Then the integral representation (4.2) shows with the aid of the energy inequality (3.10) that $\{(v^n(t), \varphi(t))_{\Omega(t)}\}$ are uniformly bounded and equicontinuous on $[0, T]$. Therefore $\{(v^n(t), \varphi(t))_{\Omega(t)}\}$ form a compact subset of $C[0, T]$. Now we range φ over the countable subset $\hat{M}_\sigma$ of $\hat{D}_\sigma(\hat{\Omega}_\infty; \pi)$. By the diagonal argument we can select a subsequence $\{v^\nu\}$ for which $(v^\nu(t), \varphi(t))_{\Omega(t)}$ converges uniformly on $[0, T]$ for any φ in $\hat{M}_\sigma$. According to Lemma 2.4, any element of $\hat{D}_\sigma(\hat{\Omega}_\infty; \pi)$ can be uniformly approximated in $\hat{\Omega}$ by an element of $\hat{M}_\sigma$. Hence, in consideration of the estimate (3.10), we see that for each φ in $\hat{D}_\sigma(\hat{\Omega}_\infty; \pi)$, $(v^\nu(t), \varphi(t))$ converges uniformly to a continuous periodic function $V_\varphi(t)$. Moreover

$$|V_\varphi(t)|\leqq \overline{\lim_\nu}\, \|v^\nu(t)\|\cdot\|\varphi(t)\|\leqq\|\varphi(t)\|_{\Omega(t)}\left(R^2+\frac{2}{\lambda_0}\int_0^t \|\bar{f}(\tau)\|^2 d\tau\right)^{1/2}.$$

As was noted in Remark 2.5, $\{\varphi(t, \cdot)|\varphi\in\hat{D}_\sigma(\hat{\Omega}_\infty; \pi)\}$ is dense in $H_\sigma(\Omega(t))$. Therefore, by Riesz' theorem, we see that there exists a unique $v(t)$ in $H_\sigma(\Omega(t))$ satisfying (4.3) and (4.4). Q.E.D.

LEMMA 4.2. *From the subsequence $\{v^\nu\}$ in the preceding lemma, we can select a new subsequence, which we denote again by v^ν, such that*

$$v^\nu \to v^* \text{ weakly in } \hat{H}_\sigma^1(\hat{B}_\infty; \pi) ,$$

and

$$v^\nu \to v^* \text{ weakly* in } L_\infty(H_\sigma(B); \pi)$$

for some v^* in $\mathscr{U}(\hat{B}_\infty; \pi)$. Moreover, $\|v^*\|_{L_2(\hat{E})}=0$. If v^{**} stands for the restriction of v^* to $\hat{\Omega}_\infty$, then v^{**} belongs to $\hat{H}_\sigma^1(\hat{\Omega}_\infty; \pi)$ and $v^{**}(t, \cdot)=v(t, \cdot)$ for almost every t.

PROOF. By the energy inequality (3.10), we see that the set $\{v^\nu\}$ is bounded not only in $\hat{H}_\sigma^1(\hat{B}_\infty; \pi)$ but also in $L_\infty(H_\sigma(B); \pi)$. Hence, after choosing a subsequence of $\{v^\nu\}$ if necessary, we see that

$$v^\nu \to v^* \text{ weakly in } L_2(L_2(B); \pi) ,$$

$$v^\nu \to v^* \text{ weakly in } \hat{H}_\sigma^1(\hat{B}_\infty; \pi) ,$$

and

$$v^\nu \to v^* \text{ weakly* in } L_\infty(H_\sigma(B); \pi) .$$

Consequently v^* belongs to $\mathscr{U}(\hat{B}_\infty; \pi)$. Moreover,

$$\|v^*\|_{L_2(\hat{E})} \leqq \varliminf_{\nu} \|v^\nu\|_{L_2(\hat{E})}^2 \leqq \lim_{\nu} \frac{1}{2(\nu-n_0)} \left\{R^2+\frac{2}{\lambda_0}\int_0^T \|\tilde{f}(t)\|^2 dt\right\}=0 .$$

If we use the estimate (2.7) with $\hat{\omega}(\delta)=\hat{\omega}_e(\delta)$, then we see that $\hat{\gamma}_\infty(v^*)=0$ since $\|v^*\|_{\hat{E}}=0$. Hence, according to Lemma 2.6, $v^{**}=v^*|_{\hat{\Omega}_\infty}$ belongs to $\hat{H}_\sigma^1(\hat{\Omega}_\infty; \pi)$. Now we take an arbitrary function φ in $\hat{D}_\sigma(\hat{\Omega}_\infty; \pi)$, and consider the integral

$$\iint_{\hat{B}} v^\nu(t, x)\varphi(t, x)dt\,dx \equiv \int_0^T (v^\nu(t), \varphi(t))_B dt .$$

The weak convergence of v^ν to v^* in $L_2(L_2(B); \pi)$ and the uniform convergence of $(v^\nu(t), \varphi(t))$ to $(v(t), \varphi(t))$ yield the equality

$$\int_0^T (v^{**}(t), \varphi(t))dt = \int_0^T (v(t), \varphi(t))dt .$$

By the arbitrariness of φ, we have

$$\int_0^T \eta(t)(v^{**}(t)-v(t), \varphi(t))dt=0$$

for any $\eta \in C^{\infty}(R^1; \pi)$. Consequently $(v^{**}(t)-v(t), \varphi(t))_{\Omega(t)}$ is zero for almost every t. Since $\{\varphi(t, \cdot) | \varphi \in \hat{M}_\sigma\}$ is dense in $H_\sigma(\Omega(t))$ (Remark 2.5), and since $v^{**}(t, \cdot)$ belongs to $H_\sigma(\Omega(t))$ for almost every t, we see that $v^{**}(t)=v(t)$ for almost every t.

Q.E.D.

REMARK 4.3. For such t that $v^{**}(t)=v(t)$ holds, $v(t)$ belongs to $H_\sigma^1(\Omega(t))$ and hence v belongs to $\hat{H}_\sigma^1(\hat{\Omega}_\infty; \pi)$. Therefore, if we redefine $v^*=0$ on $\hat{E}_\infty$, and $v^*=v$ on $\hat{\Omega}_\infty$, then v^* belongs to $\hat{H}_\sigma^1(\hat{B}_\infty; \pi)$, and the convergence in Lemma 4.2 occurs.

We shall finally prove that $v=v^*|_{\hat{\Omega}_\infty}$ satisfies the weak equation (1.5). Once this is established, $u=v+b$ will satisfy (1.3) and the proof of Theorem 1.7 will be completed. Let us examine the proof of Proposition 3.3. Then we find that v satisfies (1.5) if $\{v^\nu\}$ contains a subsequence which converges strongly in $L_2(\hat{\Omega}_\infty; \pi)$. The remaining part of this section is devoted to establish this fact.

Let us apply Lemma 2.11 in the following manner. Ω is a bounded fixed domain in R^m with smooth boundary $\partial\Omega$. Let M_i $(i=0, 1, 2)$ be three Hilbert spaces as follows:

$$M_0=\{u \in W_2^1(\Omega) | \operatorname{div} u=0\} ,$$

$$M_1=H_\sigma(\Omega) ,$$

and

$$M_2=(D(A))', \ A \text{ being the Stokes operator } A(\Omega) .$$

The norms are defined as

$$\|u\|_{M_0}=(\|u\|_{L_2(\Omega)}^2+\|\nabla u\|_{L_2(\Omega)}^2)^{1/2} ,$$

$$\|u\|_{M_1}=\|u\|_{L_2(\Omega)} ,$$

$$\|u\|_{M_2}=\sup_{\varphi \in D(A)} \frac{|\langle u, \varphi \rangle|}{\|A\varphi\|_{L_2(\Omega)}} .$$

The operator $P: M_0 \to M_1$ is defined as

$$Pu=P(\Omega)u \qquad \text{for} \quad u \in M_0 ,$$

where $P(\Omega)$ is the orthogonal projection from $L_2(\Omega)$ onto $H_\sigma(\Omega)$. The operator $S: M_0 \to M_2$ is defined as

$$Su(\varphi)=\langle Su, \varphi \rangle=(u, \varphi)_{L_2(\Omega)} \qquad (\varphi \in D(A))$$

for any u in M_0. It is easy to see that these operators satisfy the hypothesis of Lemma 2.11.

Let (α, β) be an arbitrary finite real interval. We put $X_i=L_2((\alpha, \beta); M_i)$

$(i=0, 1, 2)$. $\hat{P}: X_0 \to X_1$ and $\hat{S}: X_0 \to X_2$ are defined in terms of P and S as those in Lemma 2.11. Now we choose α, β and Ω so that $(\alpha, \beta)\times\Omega$ is contained in $\hat{\Omega}$.

We return to the consideration of the weak solutions u^n of $(AP)_n$.

Let us deal with the restriction of $v^n = u^n - b$ to the domain $(\alpha, \beta)\times\Omega$, which we denote also by v^n.

LEMMA 4.4. *$\{\hat{P}v^n\}$ is compact in X_1, and hence, $\{P(\Omega)v^n\}$ is a compact subset of $L_2((\alpha, \beta)\times\Omega)$.*

PROOF. According to Proposition 3.3, $\{v^n\}$ forms a bounded set in X_0. Put $w^n = \frac{d}{dt}\hat{S}v^n$. Then it is easy to verify, using the expression (3.9), that

$$\begin{aligned}(4.5)\qquad \langle w^n(t), \varphi\rangle = &-(\nabla v^n(t), \nabla\varphi)_\Omega + (v^n(t), (v^n(t)\cdot\nabla)\varphi)_\Omega \\ &+(v^n(t), (b\cdot\nabla)\varphi)_\Omega - ((v^n(t)\cdot\nabla)b, \varphi)_\Omega + (f(t), \varphi)_\Omega\end{aligned}$$

for almost every t in (α, β) and for all φ in $D(A)$. Let us estimate $\|w^n\|_{X_2}$. Below c denotes various constants independent of t and φ.

$$|(\nabla v^n(t), \nabla\varphi)_\Omega| \leqq \|\nabla v^n(t)\|_\Omega \|\nabla\varphi\|_\Omega \leqq c\|\nabla v^n(t)\|_B \|A\varphi\|_\Omega ,$$

$$\begin{aligned}|(v^n(t), (v^n(t)\cdot\nabla)\varphi)_\Omega| &\leqq \|v^n(t)\|_{L_2(\Omega)} \|v^n(t)\|_{L_6(\Omega)} \|\nabla\varphi\|_{L_3(\Omega)} \\ &\leqq c\|v^n(t)\|_B \|\nabla v^n(t)\|_B \|A\varphi\|_\Omega ,\end{aligned}$$

$$|(v^n(t), (b\cdot\nabla)\varphi)_\Omega| \leqq \|b\|_{L_\infty(\Omega)} \|v^n(t)\|_{L_2(\Omega)} \|\nabla\varphi\|_{L_2(\Omega)} \leqq c\|v^n(t)\|_B \|A\varphi\|_\Omega ,$$

where we have used Lemma 2.1 and Sobolev's inequality. The other terms in (4.5) can be estimated similarly and we have, in view of (3.10),

$$\|w^n(t)\|_{M_2} = \sup_{\varphi\in D(A)} \frac{|\langle w^n(t), \varphi\rangle|}{\|A\varphi\|_\Omega} \leqq c(\|\nabla v^n(t)\|_B + \|\tilde{f}(t)\|_B) .$$

Using (3.10) again, we see that $\{w^n\}$ forms a bounded set of X_2. Thus we can apply Lemma 2.11 and see that $\{\hat{P}v^n\}$ is a compact set of X_1. Q.E.D.

Let $\{t_j\}$ be a countable dense subset of $[0, T]$. For any integer j, k, l, we put $G_{j,k,l} = (t_j, t_k)\times\Omega^l(t_j)$ where

$$\Omega^l(t_j) = \Omega(t_j) - \hat{\omega}_i\left(t_j; \frac{1}{l}\right) = \left\{x\in\Omega(t_j) \mid \text{dist.}\,(x, \Gamma(t_j)) > \frac{1}{l}\right\} .$$

DEFINITION 4.5.

$$\mathfrak{G} = \{G_{j,k,l} \mid G_{j,k,l} \text{ is an open non-void subset of } \hat{\Omega}\} .$$

An element G of $\mathfrak{G}$ is called a slab of type $\mathfrak{G}$.

LEMMA 4.6. *We can choose a subsequence v^ν of v^n such that $P(\Omega)v^\nu$ converges strongly in $L_2(G)$ for any slab $G = (\alpha, \beta)\times\Omega$ of type $\mathfrak{G}$.*

PROOF. Since $\mathfrak{G}$ is a countable set, this Lemma follows directly from the preceding lemma. Q.E.D.

The following two lemmas can be obtained in the same way as Lemmas 5.8 and 5.10 of Fujita-Sauer [6].

LEMMA 4.7. *Let* $G=(\alpha, \beta)\times\Omega$ *be a slab of type* $\mathfrak{G}$ *and let* w *belong to* $\hat{H}_\sigma^1(\hat{B}_\infty; \pi)$. *Then there exists a positive constant* c *independent of* G *and* w *such that*

$$\int_\alpha^\beta \|w(t)-P(\Omega)w(t)\|_L^2 dt \leqq c\int_\alpha^\beta \|w\|_{\partial G}^2 dt .$$

LEMMA 4.8. *Let* $G=(\alpha, \beta)\times\Omega$ *be a slab of type* $\mathfrak{G}$, *and let* δ *be a small positive number. Suppose that the lateral boundary* $(\alpha, \beta)\times\partial\Omega$ *lies in the interior boundary strip* $\hat{\omega}_i(\delta)$. *Then for any* $w\in\hat{H}_\sigma^1(\hat{B}_\infty; \pi)$, *we have*

$$\int_\alpha^\beta \|w\|_{\partial\Omega}^2 dt \leqq c\left\{\int_\alpha^\beta \|w\|_{\Gamma(t)}^2 dt+\delta\int_\alpha^\beta \|\nabla w\|_B^2 dt\right\} ,$$

where c *is a constant independent of* G, δ *and* w.

We now return to the consideration of the weak solution u^n of $(AP)_n$. We put $v^n=u^n-b$.

LEMMA 4.9. *There exists a constant* c *independent of* n *such that*

$$\|v^n\|_{\hat{\Gamma}}\leqq cn^{-1/4} ,$$

for all $n\geqq 2n_0$.

PROOF. Using (2.7) with $\hat{\omega}(\delta)=\hat{\omega}_e(\delta)$, we have

$$\|v^n\|_{\hat{\Gamma}}^2 \leqq c\{\delta^{-1}\|v^n\|_{\hat{E}}^2+\|v^n\|_{\hat{E}}\|\nabla v^n\|_{\hat{B}}\} .$$

By (3.10), we have

$$\|v^n\|_{\hat{\Gamma}}^2 \leqq c\{\delta^{-1}(n-n_0)^{-1}+(n-n_0)^{-1/2}\}\left\{R^2+\frac{2}{\lambda_0}\int_0^T \|\bar{f}(s)\|^2 ds\right\}$$

for any $n>n_0$. This yields the assertion of the lemma. Q.E.D.

LEMMA 4.10. *Let* v^ν *be the subsequence chosen in Lemma* 4.6. *Then* v^ν *converges strongly in* $L_2(\hat{\Omega})$.

Using the results so far obtained, the lemma can be proved in the same way as Lemma 5.13 of Fujita-Sauer [6] and the proof is omitted.

Lemmas 4.4~4.10 show that there exists a subsequence of $\{v^n\}$, converging strongly in $L_2(\hat{\Omega}_\infty; \pi)$. Its limit coincides with $v^*|_{\hat{\Omega}_\infty}=v$, and hence, as was mentioned in Remark 4.3, v satisfies (1.5) and $u\equiv v+b$ is a weak solution of (Pr. π). Thus we have proved Theorem 1.7. Q.E.D.

Appendix 1. Proof of the inequalities in Subsection 2.3

We prove the inequalities (2.3) and (2.9) only, since the proof of (2.4) is similar to that of (2.3), (2.5) (resp. (2.6)) follows from (2.3) (resp. (2.4)) by the inequality

$$2ab \leqq \varepsilon a^2 + \frac{1}{\varepsilon} b^2 \qquad (\varepsilon > 0)\,, \tag{A.1}$$

and, since (2.7) (resp. (2.8)) is obtained from (2.4) (resp. (2.5)) by integration with respect to t.

We begin with some geometric preparation. As is assumed in Assumption 1.1, $\hat{\Gamma}_\alpha$ is covered by a finite number of patches $\mathcal{O}_\alpha^j$ $(j=1, \cdots, j(\alpha))$ and $\hat{\Gamma}_\alpha \cap \mathcal{O}_\alpha^j$ is represented as $x_1'=\varphi(x_2', \cdots, x_m', t)$ by some function φ of C^3-class under a suitable choice of the coordinate $(x_1' \cdots x_m')$ in R^m. Let (x, y, z, t) be any point in $\mathcal{O}_\alpha^j$, (x_0, y_0, z_0, t) the point on $\Gamma_\alpha(t)$ such that the normal at (x_0, y_0, z_0) to the m-dimensional hypersurface $\Gamma_\alpha(t)$ passes through (x, y, z). We define ρ as follows:

$$\rho^2=(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\,,$$

$$\rho>0 \quad \text{if} \quad (x, y, z) \in \Omega(t)\,,$$

$$\rho \leqq 0 \quad \text{if} \quad (x, y, z) \in B-\Omega(t)\,.$$

If we restrict ourselves to the domain $\hat{\omega}(\delta) \cap \mathcal{O}_\alpha^j$ sufficiently close to the boundary $\hat{\Gamma}_\alpha \cap \mathcal{O}_\alpha^j$, then we can apply the implicit function theorem and we see that there exist functions F, G, H of C^2 class such that

$$x_0=F(x, y, z, t)\,,$$

$$y_0=G(x, y, z, t)\,,$$

$$\rho^2=\left[1+\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2\right]_{(x_0, y_0, t)}(z-\varphi(x_0, y_0, t))^2=H(x, y, z, t)^2\,.$$

Moreover, there exists a constant c independent of t such that $c>\left|\frac{\partial H}{\partial z}\right|>\frac{1}{c}$ holds in the neighborhood $\hat{\omega}(\delta) \cap \mathcal{O}_\alpha^j$ of $\hat{\Gamma}_\alpha \cap \mathcal{O}_\alpha^j$. Let Φ be the coordinate transformation from (x, y, z, t) to (x, y, ρ, t). Then Φ is of class C^2 in $\hat{\omega}(\delta) \cap \mathcal{O}_\alpha^j$, Φ^{-1} is also of class C^2, and $\hat{\Gamma}_\alpha$ is mapped by Φ into the plane $\rho=0$.

Let $\{a_\alpha^j(x, y, z, t)\}_j$ be a partition of unity subordinate to the covering $\{\mathcal{O}_\alpha^j\}_j$. From now on we omit the suffix α.

PROOF OF (2.3). Let $\varphi(x, y, z)$ be a smooth function defined in B. We consider $a_j(x, y, z, t)\varphi(x, y, z)^2=\tilde{a}_j(x, y, \rho, t)\tilde{\varphi}(x, y, \rho)^2$. Differentiating with respect to ρ, we have

$$\frac{\partial}{\partial\rho}(\tilde{a}_j\tilde{\varphi}^2)=\left(\frac{\partial}{\partial\rho}\tilde{a}_j\right)\tilde{\varphi}^2+2\tilde{a}_j\tilde{\varphi}\frac{\partial}{\partial\rho}\tilde{\varphi}\ .$$

Integrating the both sides with respect to ρ,

$$\text{(A.2)}\qquad \tilde{a}_j(x, y, \rho, t)\tilde{\varphi}(x, y, \rho)=\tilde{a}_j(x, y, 0, t)\tilde{\varphi}(x, y, 0)^2 +2\int_0^\rho \tilde{a}_j\tilde{\varphi}\frac{\partial}{\partial\rho'}\tilde{\varphi}d\rho'+\int_0^\rho\left(\frac{\partial}{\partial\rho'}\tilde{a}_j\right)\tilde{\varphi}^2 d\rho'\ .$$

Consequently we have

$$\text{(A.3)}\qquad \iint dx\, dy\int_0^\delta \tilde{a}_j(x, y, \rho, t)\tilde{\varphi}(x, y, \rho)^2 d\rho$$
$$=\delta\iint \tilde{a}_j(x, y, 0, t)\tilde{\varphi}(x, y, 0)^2 dx\, dy$$
$$+2\iint dx\, dy\int_0^\delta d\rho\int_0^\rho \tilde{a}_j\tilde{\varphi}\frac{\partial}{\partial\rho'}\tilde{\varphi}d\rho'$$
$$+\iint dx\, dy\int_0^\delta d\rho\int_0^\rho\left(\frac{\partial}{\partial\rho'}\tilde{a}_j\right)\tilde{\varphi}^2 d\rho'\ .$$

Thanks to Schwarz' inequality, the second term of the right hand side is dominated by

$$2\delta\left[\iint dx\, dy\int_0^\delta \tilde{a}_j\tilde{\varphi}d\rho'\cdot\iint dx\, dy\int_0^\delta \tilde{a}_j\left(\frac{\partial}{\partial\rho'}\tilde{\varphi}\right)^2 d\rho'\right]^{1/2},$$

and the third term by

$$\delta\iint dx\, dy\int_0^\delta\left|\frac{\partial}{\partial\rho'}\tilde{a}_j\right|\cdot\tilde{\varphi}^2 d\rho'\ .$$

In consideration of the boundedness of $\left|\frac{\partial H}{\partial z}\right|$, we have

$$c^{-1}\iiint_{\omega(\delta)} a_j\varphi^2\, dx\, dy\, dz\leqq\iint dx\, dy\int_0^\delta \tilde{a}_j\tilde{\varphi}^2 d\rho\leqq c\iiint_{\omega(\delta)} a_j\varphi^2\, dx\, dy\, dz\ .$$

Summing up (A.3) with respect to j, we have

$$\iiint_{\omega(t;\delta)}\varphi^2\, dx\, dy\, dz\leqq c\delta\|\varphi\|^2_{\Gamma(t)}+2c^2\delta\|\varphi\|_{\omega(t;\delta)}\|\nabla\varphi\|_{\omega(t;\delta)} +c^2\delta\sup_{t,x}\sum_j|\nabla a_j(t, x)|\cdot\|\varphi\|^2_{\omega(t;\delta)}\ .$$

If we choose δ sufficiently small, say, so small that the coefficient of $\|\varphi\|^2_{\omega(t;\delta)}$ is less than 1/2, then we see that (2.3) holds.

PROOF OF (2.9). If we differentiate $\tilde{a}_j\tilde{\varphi}^2/\rho$ with respect to ρ, then we have

$$\frac{\partial}{\partial\rho}(\tilde{a}_j\tilde{\varphi}^2/\rho)+\tilde{a}_j\tilde{\varphi}^2/\rho^2=2\tilde{a}_j\frac{\partial\tilde{\varphi}}{\partial\rho}\frac{\tilde{\varphi}}{\rho}+\frac{\partial\tilde{a}_j}{\partial\rho}\frac{\tilde{\varphi}^2}{\rho}\ .$$

Integrating the both sides with respect to ρ from ε to δ, we have

$$\tilde{a}_j(x, y, \delta, t)\tilde{\varphi}(x, y, \delta)^2/\delta+\int_\varepsilon^\delta \tilde{a}_j\tilde{\varphi}^2/\rho^2 d\rho$$
$$=\tilde{a}_j(x, y, \varepsilon, t)\tilde{\varphi}(x, y, \varepsilon)^2/\varepsilon+2\int_\varepsilon^\delta \tilde{a}_j \frac{\partial\tilde{\varphi}}{\partial\rho}\frac{\tilde{\varphi}}{\rho}d\rho+\int_\varepsilon^\delta \frac{\delta\tilde{a}_j}{\partial\rho}\frac{\tilde{\varphi}^2}{\rho}d\rho .$$

Furthermore, we integrate the both sides with respect to x and y. Then

$$\iint dx\,dy\int_\varepsilon^\delta \tilde{a}_j\frac{\tilde{\varphi}^2}{\rho^2}d\rho \leqq \frac{1}{\varepsilon}\iint \tilde{a}_j(x, y, \varepsilon, t)\tilde{\varphi}(x, y, \varepsilon)^2 dx\,dy$$
$$+2\left[\iint dx\,dy\int_\varepsilon^\delta \tilde{a}_j\left(\frac{\partial\tilde{\varphi}}{\partial\rho}\right)^2 d\rho\cdot\iint dx\,dy\int_\varepsilon^\delta \tilde{a}_j\left(\frac{\tilde{\varphi}}{\rho}\right)^2 d\rho\right]^{1/2}$$
$$+\iint dx\,dy\int_\varepsilon^\delta \rho\left|\frac{\partial\tilde{a}_j}{\partial\rho}\right|\frac{\tilde{\varphi}^2}{\rho^2}d\rho .$$

By means of the inequality (A.1), we have

$$\text{(A.4)}\qquad \iint dx\,dy\int_\varepsilon^\delta \tilde{a}_j\left(\frac{\tilde{\varphi}}{\rho}\right)^2 d\rho \leqq \frac{2}{\varepsilon}\iint \tilde{a}_j(x, y, \varepsilon, t)\tilde{\varphi}(x, y, \varepsilon)^2 dx\,dy$$
$$+4\iint dx\,dy\int_\varepsilon^\delta \tilde{a}_j\left(\frac{\partial\tilde{\varphi}}{\partial\rho}\right)^2 d\rho+2\iint dx\,dy\int_\varepsilon^\delta \rho\left|\frac{\partial\tilde{a}_j}{\partial\rho}\right|\left(\frac{\tilde{\varphi}}{\rho}\right)^2 d\rho .$$

On the other hand, if we put $\rho=\varepsilon$ in (A.2) and integrate the both sides with respect to x and y, then we have

$$\iint \tilde{a}_j(x, y, \varepsilon, t)\tilde{\varphi}(x, y, \varepsilon)^2 dx\,dy \leqq \iint \tilde{a}_j(x, y, 0, t)\tilde{\varphi}(x, y, 0)^2 dx\,dy$$
$$+\frac{1}{\varepsilon}\iint dx\,dy\int_0^\varepsilon \tilde{a}_j\tilde{\varphi}^2 d\rho+\varepsilon\iint dx\,dy\int_0^\varepsilon \tilde{a}_j\left(\frac{\partial\tilde{\varphi}}{\partial\rho}\right)^2 d\rho+\iint dx\,dy\int_0^\varepsilon\left|\frac{\partial\tilde{a}_j}{\partial\rho}\right|\tilde{\varphi}^2 d\rho .$$

Substituting this estimate into (A.4) and taking the summation with respect to j, we have

$$\text{(A.5)}\qquad \frac{1}{c}\left\|\frac{\varphi}{\rho}\right\|^2_{\omega(\varepsilon;\delta)} \leqq \frac{2}{\varepsilon}\|\varphi\|^2_{\Gamma(t)}+\frac{2}{\varepsilon^2}c\|\varphi\|^2_{\omega(t:\varepsilon)}$$
$$+4cc_1\|\nabla\varphi\|^2_{\omega(t:\delta)}+\frac{cc^2}{\varepsilon}\|\varphi\|^2_{\omega(t:\delta)}+2\delta cc_2\left\|\frac{\varphi}{\rho}\right\|^2_{\omega(\varepsilon,\delta)} ,$$

where the boundedness of $\left|\dfrac{\partial H}{\partial z}\right|$ has been used. The constants c_1 and c_2 do not depend on φ, t, ε nor on δ. By (2.5), the right hand side of (A.5) is dominated by

$$\frac{c_3}{\varepsilon}\|\nabla\varphi\|^2_{\Gamma(t)}+c_4\|\varphi\|^2_{\omega(t;\delta)}+c_5\delta\left\|\frac{\varphi}{\rho}\right\|^2_{\omega(\varepsilon,\delta)}$$

with the constants c_i independent of φ, t, ε or δ. In this way we see that, if we choose δ sufficiently small, then we obtain the estimate (2.9).

Appendix 2. Proof of Lemma 2.11

Although an indication of the proof is found in Fujita-Sauer [6], we give here a self-contained proof of Lemma 2.11 for the sake of completeness (Cf. Lions [9]).

We begin with the proof of (2.13). Suppose that this does not hold. Then for some $\eta>0$ there exists a sequence $\{v_n \in M_0\}$ such that $\|Pv_n\|_{M_1} \geqq \eta\|v_n\|_{M_0}+n\|Sv_n\|_{M_2}$ holds. We put $w_n=v_n/\|v_n\|_{M_0}$. Then

$$\|Pw_n\|_{M_1} \geqq \eta+n\|Sw_n\|_{M_2} . \tag{A.6}$$

Since $\{Pw_n\}$ is a bounded set in M_1, (A.6) yields that $\|Sw_n\|_{M_2} \to 0$ as $n\to\infty$. By the compactness of P and S, we can find a subsequence $\{w_{n'}\}$ of $\{w_n\}$, which converges in the following manner:

$$w_{n'} \to w \quad \text{weakly in} \quad M_0 ,$$

$$Pw_{n'} \to Pw \quad \text{strongly in} \quad M_1 ,$$

and

$$Sw_{n'} \to Sw \quad \text{strongly in} \quad M_2 .$$

Therefore $Sw=0$, and hence, according to the assumption, $Pw=0$ holds, that is, $Pw_{n'}$ converges to zero strongly in M_1. This contradicts (A.6) and hence, (2.13) must hold.

Now we are going to prove the lemma. Under the assumptions, we can find a subsequence $\{v_{n'}\}$ of $\{v_n\}$ such that

$$v_{n'} \to v \quad \text{weakly in} \quad X_0 ,$$

$$\hat{P}v_{n'} \to \hat{P}v \quad \text{weakly in} \quad X_1 ,$$

and

$$\frac{d}{dt}\hat{S}v_{n'} \to \frac{d}{dt}\hat{S}v \quad \text{weakly in} \quad X_2 .$$

Taking v_n-v instead of v_n, the lemma is reduced to the following form:

"If $\{v_n\}$ $\left(\text{resp.} \left\{\frac{d}{dt}\hat{S}v_n\right\}\right)$ converges weakly to zero in X_0 (resp. X_2), then $\{\hat{P}v_n\}$ has a subsequence which converges strongly to zero in X_1".

It follows from (2.13) that for any $\eta>0$ there exists a constant $d_\eta>0$ for which $\|\hat{P}v_n\|_{X_1} \leqq \eta\|v_n\|_{X_0}+d_\eta\|\hat{S}v_n\|_{X_2}$ holds for all n. Let ε be an arbitrary positive number. Then we can find η subject to

$$\eta\|v_n\|_{X_0}<\frac{\varepsilon}{2} \qquad \text{for all} \quad n ,$$

since $\{v_n\}$ is bounded in X_0. Consequently, if we establish that $\|\hat{S}v_n\|_{X_2}\to 0$ as $n\to\infty$, then we can conclude that $\|\hat{P}v_n\|_{X_1}\to 0$ as $n\to\infty$.

It is obvious that $\{\hat{S}v_n\}$ and $\left\{\frac{d}{dt}\hat{S}v_n\right\}$ are bounded in X_2. Therefore $\{\hat{S}v_n\}$ forms a bounded set in $C^0([\alpha,\beta]:M_2)=$ the set of M_2-valued continuous functions on $[\alpha,\beta]$ (See e.g. Lions [9]). Since Lebesgue's bounded convergence theorem is applicable in order to show $\|\hat{S}v_n\|_{X_2}\to 0$, it suffices to prove that $\|Sv_n(t)\|_{M_2}$ converges to zero as n tends to infinity for each t in $[\alpha,\beta]$. It is enough to prove this for $t=\alpha$. Put $w_n(t)=v_n(\lambda t+\alpha)$ for $t\geqq 0$, where λ is a positive number to be determined later. Then we have

$$w_n(0)=v_n(\alpha)\ ,$$

$$\|\hat{S}w_n\|_{L_2(0,\beta-\alpha:M_2)}=\frac{1}{\sqrt{\lambda}}\|\hat{S}v_n\|_{X_2}\ ,$$

and

$$\left\|\frac{d}{dt}\hat{S}w_n\right\|_{L_2(0,\beta-\alpha:M_2)}=\sqrt{\lambda}\left\|\frac{d}{dt}\hat{S}v_n\right\|_{X_2}.$$

We take an arbitrary φ in $C^1(0,\beta-\alpha)$ with $\varphi(0)=-1$ and $\varphi(\beta-\alpha)=0$. Note

$$Sw_n(0)=\int_0^{\beta-\alpha}(\varphi(t)Sw_n(t))'dt=\beta_n+\gamma_n\ ,$$

where

$$\beta_n=\int_0^{\beta-\alpha}\varphi(t)(Sw_n(t))'dt\ ,$$

and

$$\gamma_n=\int_0^{\beta-\alpha}\varphi'(t)Sw_n(t)dt\ .$$

Then

$$\|Sw_n(0)\|_{M_2}\leqq c\sqrt{\lambda}\left\|\frac{d}{dt}\hat{S}v_n\right\|_{X_2}+\|\gamma_n\|_{M_2}\ ,$$

where c stands for a constant depending only on φ. If we take λ sufficiently small, then $c\sqrt{\lambda}\left\|\frac{d}{dt}\hat{S}v_n\right\|_{X_2}$ is less than $\frac{\varepsilon}{2}$ for all n. On the other hand, $\int_0^{\beta-\alpha}\varphi' w_n dt$ converges weakly to zero in M_0, since w_n converges weakly to zero in $L_2(0,\beta-\alpha;M_0)$. Thanks to the compactness of S, $\gamma_n=S\int_0^{\alpha-\beta}\varphi' w_n dt$ converges, for each fixed λ, strongly to zero in M_2 as $n\to\infty$. Thus we can make $\|Sw_n(0)\|=\|Sv_n(\alpha)\|$ less than ε. This completes the proof. Q.E.D.

Bibliography

[1] Agmon, S., A. Douglis and L. Nirenberg: Estimates near the boundary for solutions of elliptic differential equations satisfying general boundary conditions. I: Comm. Pure Appl. Math. **12** (1959), 623-727; II: Comm. Pure Appl. Math. **17** (1964), 35-92.
[2] Aubin, J. P.: Un théorème de compacité, C. R. A. S. Paris **256** (1963), 5042-5044.
[3] Cattabriga, L.: Su un problema al contorno relativo al sistema di equazioni di Stokes, Rend. Sem. Mat. Univ. Padova **31** (1961), 308-340.
[4] Fujita, H.: On the existence and regularity of the steady-state solutions of the Navier-Stokes equation, J. Fac. Sci. Univ. Tokyo Sect. I **9** (1961), 59-102.
[5] Fujita, H. and N. Sauer: Construction of weak solutions of the Navier-Stokes equations in a non-cylindrical domain, Bull. Amer. Math. Soc. **75** (1969), 465-468.
[6] Fujita, H. and N. Sauer: On existence of weak solutions of the Navier-Stokes equations in regions with moving boundary, J. Fac. Sci. Univ. Tokyo Sect. IA **17** (1970), 403-420.
[7] Hopf, E.: On nonlinear partial differential equation, Lecture series of symposium on partial differential equations, Univ. of Kansas, 1957, 1-32.
[8] Ladyzhenskaya, O. A.: The Mathematical Theory of Viscous Incompressible Flow, Gordon and Breach, New York, 1963.
[9] Lions, J. L.: Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod, Paris, 1969.
[10] Lions, J. L. et E. Magenes: Problèmes aux limites non homogènes, Vol. 1, Dunod, Paris, 1968.
[11] Prodi, G.: Qualche risultato riguardo alle equazioni di Navier-Stokes nel caso bidimensionale, Rend. Sem. Mat. Univ. Padova **30** (1960), 1-15.
[12] Takeshita, A.: On the reproductive property of 2-dimensional Navier-Stokes equations, J. Fac. Sci. Univ. Tokyo Sect. I **16** (1970), 297-311.
[13] Yosida, K.: Functional Analysis, Springer, Berlin, 1965.

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