Spectral theory of Laplacian in an asymptotically cylindrical domain

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§ 1. Introduction

In the present paper we shall consider the scattering theory for a pair of self-adjoint operators determined by the negative Laplacian with the Dirichlet boundary condition, the one being defined in a semi-cylinder S and the other in a perturbed semi-cylinder. The main problems we shall consider are the unitary equivalence of these operators and the construction of perturbed eigenfunction expansions. Throughout the present paper, our cylinder S is supposed to have the form

$$S=l\times[0, \infty)=\{x=(\tilde{x}, x_n)\in R^n; \tilde{x}\in l, x_n\geqq 0\} ,$$

where l is an $(n-1)$-dimensional bounded domain with sufficiently smooth boundary.

Recently this problem was investigated by C. Goldstein [1], [2] under the condition that Ω coincides with S for sufficiently large x_n. In treating this problem he constructed in [1] a set of generalized eigenfunctions by solving the boundary value problems with the aid of the principle of limiting absorption. In this paper we shall prove some of his results under different assumptions which are considerably weaker than his assumptions in some respects.

We treat this problem by transforming the Laplacian in $L^2(\Omega)$ unitarily to a second order elliptic differential operator in $L^2(S)$ by means of a suitable coordinate transformation. Then we apply a method of smooth perturbation in the scattering theory. Goldstein [2] also uses a coordinate transformation from Ω to S. By means of that transformation, however, he reduced the problem to nonselfadjoint perturbation. On the contrary, we will always remain with selfadjoint operators. In § 2 we formulate and prove our results for differential operators in $L^2(S)$, where we shall apply a method of smooth perturbation due to Kato and Kuroda ([3], [4], [6]).

In § 3 we consider the negative Laplacian in $L^2(\Omega)$.

§ 2. Perturbation of negative Laplacian in a semifinite cylinder

We begin with some notations. $\mathscr{E}^k_{L^2}(S)$ is the Sobolev space of order k over S and $\mathscr{D}^k_{L^2}(S)$ is the completion of $\mathscr{D}(S)$, the Schwartz space of test functions, in $\mathscr{E}^k_{L^2}(S)$. We define the closed Hermitian form h_1 as follows:

$$h_1[u, v]=\sum_{i=1}^{n}(\partial_i u, \partial_i v)_S \quad \text{for} \quad u, v\in \mathscr{D}^1_{L^2}(S)$$

where $(f, g)_S=\int f(x)\overline{g(x)}dx$ and $\partial_i=\partial/\partial x_i$. Thus, the form h_1 is defined on the domain $\mathscr{D}(h_1)=\mathscr{D}^1_{L^2}(S)$. We denote by H_1 the selfadjoint operator associated with h_1. Then, one has $H_1=K_1\otimes K_2$, where K_1 is the negative Laplacian with the zero Dirichlet condition in $L^2(l)$, and $K_2u=-\dfrac{d^2u}{dx_n^2}$ in $L^2(0, \infty)$ with the zero Dirichlet condition at $x_n=0$.

Following Goldstein [1] we shall first give a spectral representation for H_1. We denote by ν_n the eigenvalues of K_1, ordered increasingly with repetitions according to multiplicity, and by $\eta_n(\tilde{x})$ the corresponding orthonormal systems of eigenfunctions of K_1, and introduce the Hilbert space

$$\mathscr{H}'=\sum\oplus L^2(0, \infty)\ , \quad \|\{f_n\}\|^2_{\mathscr{H}'}=\sum_{n=1}^{\infty}\int_0^{\infty}|f_n(\xi)|^2d\xi\ .$$

For each $f(x)\in L^2(S)$, we set

$$(Tf)_m(\xi)=\underset{L\to\infty}{\text{l.i.m.}}\int_l\int_0^L\overline{w_m(x, \xi)}f(x)dx\ , \qquad \xi>0\ ,$$

$$w_m(x, \xi)=\left(\frac{2}{\pi}\right)^{1/2}\sin \xi x_n\cdot\eta_m(\tilde{x})\ ,$$

where l.i.m. signifies that the limit is to be taken in the sense of the Hilbert space $\mathscr{H}'$. Then it can be shown that T is a unitary operator from $L^2(S)$ onto $\mathscr{H}'$, and $TH_1f=\{(\xi^2+\nu_m)(Tf)_m(\xi)\}$. Next we set

$$Ff=\left\{(Tf)_m((\mu-\nu_m)^{1/2})\cdot\frac{1}{2^{1/2}(\mu-\nu_m)^{1/4}}\right\}$$

for each $f(x)\in L^2(S)$. Then F is a unitary transformation from $L^2(S)$ onto

$$\mathscr{H}=\sum\oplus L^2(\nu_m, \infty)\left(\|\{f_m\}\|^2_{\mathscr{H}}=\sum_{m=1}^{\infty}\int_{\nu_m}^{\infty}|f_m(\mu)|^2d\mu\right)$$

and

$$FH_1f=\{\mu(Ff)_m(\mu)\}\ .$$

In other words FH_1F^{-1} is the operator of multiplication by μ. Thus F gives a spectral representation of the selfadjoint operator H_1.

Next we define the Hermitian form h_2 as follows:

$$h_2[u, v]=\sum_{i,j}(a_{ij}(x)\partial_j u, \partial_i v)_S+\sum_j\{(b_j(x)\partial_j u, v)_S+(u, b_j(x)\partial_j v)_S\}+(q(x)u, v)_S$$
$$\text{for}\quad u, v\in\mathscr{D}^1_{L^2}(S)\ .$$

We impose following conditions on $a_{ij}(x)$, $b_j(x)$, $q(x)$:

(C.1) (i) $a_{ij}(x)$, $b_j(x)$, $q(x)$ are real; $b_j(x)\in C^1(S)$, $a_{ij}(x)\in C^2(S)$;

(ii) There exist $\alpha>1$ and $C>0$ such that

$$|\alpha_{ij}(x)|\leqq C(1+x_n)^{-\alpha}\ ,\quad \text{where}\quad \alpha_{ij}(x)=\delta_{ij}-a_{ij}(x)\ ,$$
$$|b_j(x)|\leqq C(1+x_n)^{-\alpha}\ ,\quad |q(x)|\leqq C(1+x_n)^{-\alpha}\ ;$$

(iii) There exists $\delta>0$ such that

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\bar{\xi}_j\geqq\delta\cdot\xi^2\quad\text{for every}\quad x\in S\quad\text{and}\quad \xi\in R^n\ .$$

Under condition (C.1) h_2 is a closed Hermitian form bounded from below. We denote by H_2 the selfadjoint operator associated with h_2, by $H_{2,ac}$ the absolutely continuous part of H_2, and by $\mathfrak{H}_{2,ac}$ the subspace of absolute continuity with respect to H_2. We denote by Λ the set of all real numbers $\lambda>\nu_1$ which are not equal to ν_m, $m=2, 3, \cdots$.

Now we state the theorems of this section.

THEOREM 1. *Under condition* (C.1)[1], $H_{2,ac}$ *is unitarily equivalent to* H_1, *namely there exists an isometric operator* W *with final set* $\mathfrak{H}_{2,ac}$ *such that* $WH_1\subset H_2W$.

REMARK. In the proof of Theorem 1 the operators $W_\pm$ having the properties of W described in Theorem 1 will be constructed by applying a general way of constructing such operators given by Kato and Kuroda (Cf. references given in § 1). In particular, it is known that these operators $W_\pm$ coincide with the time dependent wave operators. Namely in the situation of Theorem 1 we have

$$W_\pm=\text{s-}\lim_{t\to\pm\infty} e^{itH_2}e^{-itH_1}\ ,$$

the limit being shown to exist. More generally, the invariance principle for wave operators hold.

THEOREM 2. *Under condition* (C.1), *there exists a closed null set* $\Gamma_0\subset\Lambda$ *such that the following assertions hold.*

1) *For every* ξ, m *with* $\xi^2+\nu_m\notin\Gamma_0$, *there exist* $w_m^\pm(x, \xi)$ *having the following properties* $(m=1, 2, \cdots)$

a) $w_m^\pm(x, \xi)\cdot(1+x_n)^{-1-\varepsilon}\in\mathscr{D}^1_{L^2}(S)\cap\mathscr{E}^2_{L^2(\mathrm{loc})}(S)$ *for any* $\varepsilon>0$

[1] For this theorem the regularity assumption on $a_{jk}(x)$ and $b_j(x)$ can be weakened.

b) $-\sum_{i,k}^{n} \partial_i a_{ik}(x)\partial_k w_m^{\pm}(x, \xi)+(-\sum_{j}^{r} \partial_j b_j(x)+q(x))w_m^{\pm}(x, \xi)=(\xi^2+\nu_m)w_m^{\pm}(x, \xi)$

$m=1, 2, \cdots$.

2) *For each* $u(x)\in \mathfrak{H}_{2.ac}$, *we set*

$$T_{\pm}u=\{(T_{\pm}u)_m(\xi)\}=\underset{L\to\infty}{\text{l.i.m.}} \left\{\int_0^L \int_l u(x)\overline{w_m^{\pm}(x, \xi)}dx\right\}$$

where l.i.m. *signifies that the limit is to be taken in the sense of the Hilbert space* $\mathscr{H}'$. *Then* $T_{\pm}$ *is a unitary transformation from* $\mathfrak{H}_{2.ac}$ *onto* $\mathscr{H}'$, *and*

$$(T_{\pm}W_{\pm}u)_m(\xi)=(Tu)_m(\xi) .$$

3) *For each* $u(x)\in \mathfrak{H}_{2.ac}\cap \mathscr{D}(H_2)$

$$(T_{\pm}H_2u)_m(\xi)=(\xi^2+\nu_m)(T_{\pm}u)_m(\xi) .$$

To prove Theorems 1 and 2 we use the factorization method, namely, we use the fact that formally H_2 can be written as $H_2\sim H_1+\sum_{\nu} A_{\nu}^{*}C_{\nu}B_{\nu}$, where A_{ν}, B_{ν} are closed operators in $L^2(S)$ and C_{ν} is bounded operators in $L^2(S)$. More precisely we introduce operators in $L^2(S)$ determined by

$$A_0u(x)=B_0u(x)=(1+x_n)^{-\alpha/2}u(x) ,$$

$$A_ju(x)=A_{jk}u(x)=B_j'u(x)=B_{kj}u(x)=(1+x_n)^{-\alpha/2}\partial_ju(x) , \qquad 1\leqq j,\ k\leqq n ,$$

$$A_j'=B_j=A_0 ,$$

$$C_0u(x)=q(x)\cdot(1+x_n)^{\alpha}u(x) ,$$

$$C_ju(x)=C_j'u(x)=b_j(x)\cdot(1+x_n)^{\alpha}u(x) ,$$

$$C_{jk}u(x)=a_{jk}(x)\cdot(1+x_n)^{\alpha}u(x) .$$

A_0, A_j', B_0, B_j, C_j, C_j', C_{jk} are bounded in $L^2(S)$, and the domains of A_j, B_j', A_{jk}, B_{jk} are $\mathscr{D}^1_{L^2}(S)$. Then it can be shown that for $u,\ v\in \mathscr{D}^1_{L^2}(S)$

$$h_2[u, v]=h_1[u, v]+\sum_{j,k=1}^{n}(C_{jk}B_{jk}u, A_{jk}u)_S+\sum_{j=1}^{n}(C_jB_ju, A_jv)_S$$
$$+\sum_{j=1}^{n}(C_j'B_j'u, A_jv)_S+(C_0B_0u, A_0v)_S .$$

Here and in what follows we denote the closure of an operator T by T^a, the adjoint of T by T^*, the resolvent of H_1 by $R_1(z)=(H_1-z)^{-1}$, and spectral resolution of the selfadjoint operator H_1 by $E_1(\lambda)$. To prove Theorem 1, it suffices to prove the following assertions (see [4] Lemma 7.2).

Assertion. There exist bounded operators $M_{jk}(\lambda)$ in $L^2(S)$ for $\lambda\in\Lambda$ such that:

i) $M_{jk}(\lambda)$ is locally Hölder continuous in Λ with respect to the operator norm $0\leqq j,\ k\leqq n$:

ii) For any $j,\ k$ and any $N>\nu_1$, $\int_{\nu_1}^{N}\|M_{j,k}(\lambda)\|d\lambda<\infty$;

iii) For any compact interval $I\subset\Lambda$

$$[A_j E_1(I)A_k^*]^a=\int_I M_{jk}(\lambda)d\lambda\ ,$$

where $A_j=(1+x_n)^{-\alpha/2}\partial_j$, $\partial_0=I$; and

iv) For any $\lambda\in\Lambda$, $M_{jk}(\lambda)$ is a compact operator.

In order to prove the assertions we shall construct $M_{jk}(\lambda)$ in the form $M_{jk}(\lambda)=T_j(\lambda)^*T_k(\lambda)$ with suitably defined operators $T_j(\lambda)$ from $L^2(S)$ to l^2. Let us begin with introducing some more notations.

For each $y\in Y\equiv\{\{y_1, y_2, \cdots\}\in\mathscr{H}; y_n\in C(\nu_n, \infty)\}$ and $\lambda\in\Lambda$ we set $V(\lambda)y=\{y_1(\lambda), \cdots, y_N(\lambda), 0, 0, \cdots\}$, where $N=N(\lambda)$ is determined by the relation $\nu_N<\lambda<\nu_{N+1}$. Then $V(\lambda)$ is an operator from Y into l^2. For each $u(x)\in C_0^\infty(S)$, we set $T_i(\lambda)u=V(\lambda)(FA_i^*u)$. Furthermore, we write

$$\|u\|_{\mathfrak{H}}=\left[\int_S|u(x)|^2dx\right]^{1/2}\ ,$$

for each $u(x)\in L^2(S)$, and $\|\alpha\|_{l^2}=(\sum_{i=1}^{\infty}|\alpha_i|^2)^{1/2}$ for each $\alpha\in l^2$. In the following we suppose that condition (C.1) is satisfied.

LEMMA 1. *There exists $C_N>0$, which may depend on N but not on $\lambda\in(\nu_N, \nu_{N+1})$ such that*

$$\|T_i(\lambda)u\|_{l^2}\leqq\frac{C_N}{(\lambda-\nu_N)^{1/4}}\|u\|_{\mathfrak{H}}\quad for\quad \lambda\in(\nu_N, \nu_{N+1})\ ,\qquad 0\leqq i\leqq n\ .$$

PROOF. For any $u\in C_0^\infty(S)$ we have

$$\|T_i(\lambda)u\|_{l^2}^2=\sum_{m=1}^{N}|(FA_i^*u)_m(\lambda)|^2\ ,$$

$$|(FA_i^*u)_m(\lambda)|^2=\frac{1}{2(\lambda-\nu_m)^{1/2}}\left|\int w_m(x;(\lambda-\nu_m)^{1/2})\cdot A_i^*u(x)dx\right|^2\ .$$

First, let $1\leqq i\leqq n$. Then by means of integration by parts we obtain

$$|(FA_i^*u)_m(\lambda)|^2=\begin{cases}\dfrac{1}{2(\lambda-\nu_m)^{1/2}}\left|\displaystyle\int\left(\frac{2}{\pi}\right)^{1/2}\sin(\lambda-\nu_m)^{1/2}x_n\cdot\partial_i\eta_m(\tilde{x})(1+x_n)^{-\alpha/2}u(x)dx\right|^2 & \text{for}\quad 1\leqq i\leqq n-1\\ \dfrac{1}{2(\lambda-\nu_m)^{1/2}}\left|\displaystyle\int\left(\frac{2}{\pi}\right)^{1/2}(\lambda-\nu_m)^{1/2}\cos(\lambda-\nu_m)^{1/2}x_n\cdot\eta_m(\tilde{x})(1+x_n)^{-\alpha/2}u(x)dx\right|^2 & \text{for}\quad i=n\ .\end{cases}$$

Noting that $\alpha>1$ and using Schwarz's inequality, we obtain

$$\|T_i(\lambda)u\|_{l^2}\leqq\frac{C_N}{(\lambda-\nu_m)^{1/4}}\|u\|_{\mathfrak{H}}\ .$$

For $T_0(\lambda)$ we can show this inequality similarly without partial integration. Thus we get the lemma.

From the above lemma it follows that for each $\lambda\in\Lambda$, $T_i(\lambda)$ can be extended to a bounded linear transformation from $L^2(S)$ into l^2, which we will denote by the same notation $T_i(\lambda)$.

LEMMA 2. *For each $\lambda\in\Lambda$, $T_i(\lambda)$ is compact.*

PROOF. $T_i(\lambda)$ transforms $L^2(S)$ into a finite dimensional subspace of l^2.

We shall use the following lemma whose proof we omit.

LEMMA 3. *For each $f(x)$ such that $(1+x)^\beta f(x)\in L^2(R^1)$, $\beta>1/2$, we set*

$$\hat{f}(\xi)=\int_0^\infty \sin\xi x\cdot f(x)dx\ ,$$

then there exists $C>0$ and $\delta>0$ such that

$$|\hat{f}(\xi+h)-\hat{f}(\xi)|\leqq Ch^\delta\left[\int_0^\infty|f(x)|^2(1+x)^{2\beta}dx\right]^{1/2}\ .$$

An immediate application of Lemma 2 yields the following lemma.

LEMMA 4. *For each compact interval I_N contained in (ν_N,ν_{N+1}) there exist $C>0$ and $\delta>0$ such that*

$$\|T_i(\lambda)-T_i(\mu)\|\leqq C|\lambda-\mu|^\delta\quad \textit{for each}\quad \lambda,\ \mu\in I_N\ .$$

Since $T_i(\lambda)$ is continuous in Λ (Lemma 4) and $\|T_i(\lambda)\|$ is locally square integrable in (ν_1,∞) (Lemma 1) we see that $\|T_j(\lambda)^*T_k(\lambda)\|$ is integrable on any compact interval. Now we have the following lemma.

LEMMA 5. *For any compact interval I*

$$[A_jE_1(I)A_k^*]^a=\int_I T_j(\lambda)^*T_k(\lambda)d\lambda\ .$$

The proof of this lemma can be done in essentially the same way as in the proof of Theorem 3.1 of Kuroda [6] and will be omitted.

We now put $M_{jk}(\lambda)=T_j(\lambda)^*T_k(\lambda)$. Then $M_{jk}(\lambda)$ satisfies the properties of the assertion. Thus we get Theorem 1.

To prove Theorem 2 we begin with some definitions. If the following condition is satisfied, we say that we have generalized eigenfunction expansions with respect to a selfadjoint operator H (see [4]).

(C.2) (i) There are a σ-finite measure space (Ω, Σ, ρ), a partial isometry Φ of $\mathfrak{H}$ onto $L^2(\rho)$ with initial set $\mathfrak{H}_{ac}$ ($\mathfrak{H}_{ac}$ denotes the subspace of absolute continuity with respect to H) and a measurable function $\omega; \Omega \to R^1$ such that

$$(\Phi E(\Delta)u)(\xi)=\chi_\Delta(\omega(\xi))(\Phi u)(\xi)\ \rho\text{-a.e.}\quad \xi\in\Omega \text{ for each } u\in\mathfrak{H} \text{ and } \Delta\subset R^1 .$$

(ii) There are a subspace $\mathscr{X}$ of $\mathfrak{H}$, which is dense in $\mathfrak{H}$ and has its own norm (but is not necessarily complete), and a mapping $\phi: \Omega \to \mathscr{X}^*$ such that

$$(\Phi x)(\xi)=\langle x, \phi(\xi)\rangle\ \rho\text{-a.e.}\quad \xi\in\Omega \text{ for each } x\in X .$$

In order to show that the eigenfunction expansions in terms of $w_m(x, \xi)$ mentioned in the beginning of this section can be considered as a generalized eigenfunction expansions in the above sense, we define $\mathscr{X}\subset L^2(S)$ as follows. Let $\mathfrak{H}'$ be the Hilbert space $\mathfrak{H}'=\underbrace{L^2(S)\oplus\cdots\oplus L^2(S)}_{(n+1)^2}$ and define the operator $\mathscr{A}$ from $L^2(S)$ to $\mathfrak{H}'$ as follows;

$$\mathscr{D}(\mathscr{A}^0)=\mathscr{D}^1_{L^2}(S)$$

$$\mathscr{A}^0u=\{A_0u, A_1u, \cdots, A_nu, A_1'u, \cdots, A_n'u, A_{11}u, A_{12}u, \cdots, A_{nn}u\} .$$

We denote by $\mathscr{A}$ the closure of $\mathscr{A}^0$. We take $\mathscr{X}$ to be the range of $\mathscr{A}^*$, i.e. $\mathscr{X}=\mathscr{R}(\mathscr{A}^*)$, equipped with the norm $\|x\|_{\mathscr{X}}=\inf_{\mathscr{A}^*u=x}\|u\|_{\mathfrak{H}'}$. In order to characterize $\mathscr{X}^*$, we use the following lemma. We denote by $\mathscr{N}(T)$ the null space of an operator T, by $\mathscr{D}(T)$ the domain of T, and by $\tilde{\mathscr{X}}$ the completion of a normed space $\mathscr{X}$. $\mathscr{X}_1\simeq\mathscr{X}_2$ means that $\mathscr{X}_1$ can be identified with $\mathscr{X}_2$ isometrically.

LEMMA 6. *Suppose that $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are Hilbert spaces and that A is a densely defined closed operators from $\mathfrak{H}_1$ to $\mathfrak{H}_2$. We set $\mathscr{X}=\mathscr{R}(A^*)$ equipped with the norm $\|x\|_{\mathscr{X}}=\inf_{\mathscr{A}^*u=x}\|u\|_{\mathfrak{H}_2}$ and $\mathscr{B}=\mathscr{D}(A)/\mathscr{N}(A)$ equipped with the norm $\|b\|_{\mathscr{B}}=\|Ab\|_{\mathfrak{H}_2}$.*

Then

$$\mathscr{X}^*\simeq\tilde{\mathscr{B}} ,$$

where the identification is given by the relation $\langle b, x\rangle=(Ab'\, u)_{\mathfrak{H}_2}$ with $b=[b']\in\mathscr{D}(A)/\mathscr{N}(A)$, $b'\in\mathscr{D}(A)$, and $x=A^*u$.

PROOF. We denote by $\widetilde{\mathscr{R}(A)}$ the closure of the range of A in $\mathfrak{H}_2$. Then it is easily shown that $\widetilde{\mathscr{R}(A)}$ is identified with $\tilde{\mathscr{B}}$ isometrically by means of A, i.e. $\widetilde{\mathscr{R}(A)}\simeq\tilde{\mathscr{B}}$. On the other hand we have $\tilde{\mathscr{X}}\simeq[\mathscr{D}(A^*)/\mathscr{N}(A^*)]^{\sim}$. Since $\mathscr{D}(A^*)$ is dense in $\mathfrak{H}_2$, we have $[\mathscr{D}(A^*)/\mathscr{N}(A^*)]^{\sim}\simeq\mathfrak{H}_2/\mathscr{N}(A^*)\simeq\widetilde{\mathscr{R}(A)}$. Thus we get $\tilde{\mathscr{X}}\simeq\tilde{\mathscr{B}}$, so that we get $\mathscr{X}^*\simeq\tilde{\mathscr{B}}$ by identifying the Hilbert space $\tilde{\mathscr{X}}$ with its dual $\mathscr{X}^*$.

In our case we take $L^2(S)$ as $\mathfrak{H}_1$, $\mathfrak{H}'$ as $\mathfrak{H}_2$, and $\mathscr{A}$ as A. Then obviously the condition of Lemma 6 is satisfied. In this case $\mathscr{N}(\mathscr{A})=\{0\}$ and we get

$$\mathscr{X}^*=\widetilde{\mathscr{D}(A)}=\{f(x);\ (1+x_n)^{-1-\varepsilon}\cdot f(x)\in \mathscr{D}^1_{L^2}(S) \text{ for any } \varepsilon>0\}\ .$$

Thus we see $w_m(x,\xi)\in\mathscr{X}^*$, so that we obtain the generalized eigenfunction expansions with respect to H_1.

Then, following a general method due to Kuroda, eigenfunction expansions with respect to H_2 can be obtained as follows. Using the operator $W_\pm$ appearing in Remark after Theorem 1 we define $T_\pm=TW_\pm^*$. Then $T_\pm$ are partially isometric operators from $L^2(S)$ onto $\mathscr{H}'$ with initial set $\mathfrak{H}_{2,ac}$ and satisfy the relation

$$(T_\pm E_2(\Delta)u)_m(\xi)=\chi_\Delta(\xi^2+\nu_m)(T_\pm u)_m(\xi)$$

for each $u\in L^2(S)$ and $\Delta\subset\Lambda$, where χ_Δ is a characteristic function of Δ.

On the other hand it was shown that there is a closed null set Γ_0 of Λ such that the limit[2)]

$$G_2(\lambda\pm i0)=1-\lim_{\varepsilon\downarrow 0} C[BR_2(\lambda\pm i\varepsilon)A^*]^a$$

exist for $\lambda\in\Gamma_0$, $\lambda\in\Lambda$, in the operator norm in $\mathfrak{H}'$. Then it is not difficult to infer that $G_2(\lambda\pm i0)$ induces naturally a bounded operator $\widetilde{G}_2(\lambda\pm i0)$ acting in $\mathscr{X}$. General results now tell us that if we define $w_m^\pm(x,\xi)=\widetilde{G}_2(\xi^2+\nu_m\pm i0)^*w_m(x,\xi)$, then we have

$$(T_\pm u)_m(\xi)=\int_S u(x)\overline{w_m^\pm(x,\xi)}dx\ .$$

Thus we have proved Theorem 2 except b) of 1).

The proof of b) of 1) of Theorem 2 will proceed as follows. For any $y\in C_0^\infty(\Omega)$ we have

$$(H_2y,\ w_m^\pm(x,\xi))_S=(\xi^2+\nu_m)\cdot(y,\ w_m^\pm(x,\xi))_S\ .$$

For such y we know that H_2 is a differential operator of the form

$$H_2y=-\sum_{i,k}^n \partial_i a_{ik}(x)\partial_k y(x)+\Big(-\sum_{j=1}^n \partial_j b_j(x)+q(x)\Big)\cdot y(x)\ .$$

Then $w_m^\pm$ is a weak solution of H_2, so that we have b) of 1) of Theorem 2 in virtue of ellipticity of H_2.

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2) B and C are defined as follows:
$Bu=(B_0u,\ B_1u,\ \cdots,\ B_nu,\ B_1'u,\ \cdots,\ B_n'u,\ B_{11}u,\ B_{12}u,\ \cdots,\ B_{nn}u)$ for $u\in\mathscr{D}^1_{L^2}(S)$.
$Cu=(C_0u_0,\ C_1u_1,\ \cdots,\ C_nu_n,\ C_1'u_{n+1},\ \cdots,\ C_n'u_{2n},\ C_{11}u_{2n+1},\ C_{12}u_{2n+2},\ \cdots,\ C_{nn}u_{n^2-1})$
for $\boldsymbol{u}=(u_0,\ u_1,\ \cdots,\ u_n,\ u_{n+1},\ \cdots,\ u_{2n},\ u_{2n+1},\ \cdots,\ u_{n^2-1})\in\mathfrak{H}'$.

§ 3. Negative Laplacian in $L^2(\Omega)$.

We consider a perturbed cylindrical domain Ω with smooth boundaries. In order to characterize Ω as a perturbed cylinder which is asymptotically equal to S, we introduce several assumptions on Ω successively. We denote a generic point of Ω by x and that of S by X. First of all we suppose that there exists a C^3-diffeomorphism ψ from Ω onto S. For each $f(x)\in L^2(\Omega)$, we set

$$(Uf)(X_1, \cdots, X_n)=\left|\frac{\partial(x_1, \cdots, x_n)}{\partial(X_1, \cdots, X_n)}\right|^{1/2}\cdot f(\psi^{-1}(X)) ,$$

where we denote by $\dfrac{\partial(x_1, \cdots, x_n)}{\partial(X_1, \cdots, X_n)}$ the Jacobian. Then U is a unitary transformation from $L^2(\Omega)$ onto $L^2(S)$. Furthermore, we suppose that ψ can be chosen in such a way that U transforms $\mathscr{D}^1_{L^2}(\Omega)$ onto $\mathscr{D}^1_{L^2}(S)$ homeomorphically.

To consider negative Laplacian in $L^2(\Omega)$, we define a Hermitian closed form $\hat{h}$ as follows:

$$\hat{h}[u, v]=\sum_{i=1}^{n}(\partial_i u, \partial_i v)_\Omega \quad \text{for} \quad u,\ v\in\mathscr{D}^1_{L^2}(\Omega) ,$$

where

$$(f, g)_\Omega=\int_\Omega f(x)\overline{g(x)}dx .$$

We denote by $\hat{H}$ the selfadjoint operator associated with $\hat{h}$. Now we state the theorem in this section.

THEOREM 3. *Let Ω be such a domain that there exists a coordinate transformation ψ which satisfies the conditions stated above and the following conditions*[3]:

Condition. There exist $C>0$, $\alpha>1$ such that

$$\left|\frac{\partial X_j}{\partial x_i}(\psi^{-1}(X))-\delta_{ij}\right|\leqq\frac{C}{(1+X_n)^\alpha} \qquad 1\leqq i,\ j\leqq n$$

$$\left|\frac{\partial d}{\partial x_i}(\psi^{-1}(X))\right|\leqq\frac{C}{(1+X_n)^\alpha} \qquad 1\leqq i\leqq n$$

where

$$d(X)=\frac{\partial(X_1, \cdots, X_n)}{\partial(x_1, \cdots, x_n)}(\psi^{-1}(X)) .$$

Then the following assertions hold.

[3] Examples of Ω having such a ψ will be given later.

1) *$\hat{H}_{ac}$ is unitary equivalent to H_1.*

2) *There exists a closed null set $\Gamma_0 \subset \Lambda$ such that the following statements hold.*

i) *For every ξ, m with $\xi^2+\nu_m \in \Gamma_0$, there exist $w_m^{\pm}(x, \xi)$ having the following properties ($m=1, 2, \cdots$).*

a) $w_m^{\pm}(x, \xi)\cdot(1+x_n)^{-1-\varepsilon} \in \mathscr{D}^1_{L^2}(\Omega) \cap \mathscr{E}^2_{L^2(loc)}(\Omega)$ *for any* $\varepsilon>0$.

b) $-\Delta w_m^{\pm}(x, \xi)=(\xi^2+\nu_m)w_m^{\pm}(x, \xi)$.

ii) *For each $u(x) \in \mathfrak{H}_{ac}$ ($\mathfrak{H}_{ac}$ denotes the absolutely continuous subspace of $L^2(\Omega)$ with respect to $\hat{H}$), we set*

$$T^{\pm}u=\{(T^{\pm}u)_m(\xi)\}=\underset{L\to\infty}{\mathrm{l.i.m.}}\left\{\int_{\Omega_L} u(x)\overline{w_m^{\pm}(x, \xi)}\,dx\right\},$$

where l.i.m. *signifies that the limit is to be taken in the sense of the Hilbert space $\mathscr{H}'$, and Ω_L represent that part of Ω in which $x_n \leqq L$. Then $T^{\pm}$ are unitary transformations from $\mathfrak{H}_{ac}$ onto $\mathscr{H}'$.*

iii) *For each $u(x) \in \mathfrak{H}_{ac} \cap \mathscr{D}(\hat{H})$*

$$(T^{\pm}\hat{H}u)_m(\xi)=(\xi^2+\nu_m)(T^{\pm}u)_m(\xi).$$

PROOF. In order to transform $\hat{H}$ unitarily to some selfadjoint operator in $L^2(S)$, we define a closed Hermitian form h_2 by using the coordinate transformation ϕ as follows.

$$\begin{aligned}
h_2[u, v]&=\hat{h}[U^{-1}u, U^{-1}v]\\
&=\sum_{i,j}\left(d(X)^2\sum_{l=1}^{n}\frac{\partial X_i}{\partial x_l}(\phi^{-1}(X))\frac{\partial X_j}{\partial x_l}(\phi^{-1}(X))\frac{\partial u}{\partial X_i},\ \frac{\partial v}{\partial X_j}\right)_S\\
&+\frac{1}{2}\sum_{j=1}^{n}\left\{\left(d(X)\sum_{i=1}^{n}\frac{\partial d}{\partial x_i}(\phi^{-1}(X))\frac{\partial X_j}{\partial x_i}(\phi^{-1}(X))\frac{\partial u}{\partial x_j}, v\right)_S\right.\\
&\qquad\left.+\left(u, d(X)\sum_{i=1}^{n}\frac{\partial d}{\partial x_i}(\phi^{-1}(X))\frac{\partial X_j}{\partial x_i}(\phi^{-1}(X))\frac{\partial v}{\partial X_j}\right)_S\right\}\\
&+\frac{1}{4}\left(\sum_{i=1}^{n}\left(\frac{\partial d}{\partial x_i}\right)^2(\phi^{-1}(X))\cdot u, v\right)_S \quad \text{for} \quad u,\ v\in\mathscr{D}^1_{L^2}(S).
\end{aligned}$$

We denote by H_2 the selfadjoint operator associated with h_2. Then it can be shown that $H_2=U\hat{H}U^{-1}$.

We now apply Theorems 1 and 2 to the pair H_1 and H_2. Then, since H_2 and $\hat{H}$ are unitarily equivalent, we get 1) of Theorem 3. Next, we construct eigenfunction expansions for H_2 by means of Theorem 2 and then transform it by ϕ^{-1} to the domain Ω. Then, it is easy to see that the transformed functions satisfy the requirement of 2) of Theorem 3.

Example. We will consider the problem in R^3. In terms of the cylindrical coordinates we suppose that S and Ω have the following form:

$$S=\{(R, \Theta, X_3);\ R<f_1(\Theta),\ X_3>0\}\ ,$$

$$\Omega=\{(r, \theta, x_3);\ r<f_2(\theta, x_3),\ x_3>0\}\ ,$$

where $f_1(\Theta)$ and $f_2(\theta; x_3)$ are sufficiently smooth positive functions.

PROPOSITION. *If the above $f_1(\theta)$ and $f_2(\theta; x_3)$ satisfy the following condition, the conclusions of Theorem 3 holds*:

Condition. 1. $f_2(\theta; x_3)=f_1(\theta)$ *for* $0<x_3\leqq 1$;

2. *There exist* $C>0$ *and* $\alpha>1$ *such that*

$$|f_1(\theta)-f_2(\theta; x_3)|\leqq \frac{C}{(1+x_3)^{\alpha}}\ ,$$

$$\left|\frac{df_1(\theta)}{d\theta}-\frac{\partial f_2}{\partial\theta}(\theta; x_3)\right|\leqq \frac{C}{(1+x_3)^{\alpha}}\ ,$$

$$\left|\frac{\partial f_2}{\partial x_3}(\theta; x_3)\right|\leqq \frac{C}{(1+x_3)^{\alpha}}\ .$$

(We assumed Condition 1 to avoid problems of transforming the smooth domain onto the domain having corners. Further analysis may make Condition 1 unnecessary.)

PROOF. We set the coordinate transformation ψ from Ω onto S as follows.

$$\begin{cases} X_3=x_3\ , \\ \Theta=\theta\ , \\ R=\left\{\dfrac{f_1(\theta)}{f_2(\theta; x_3)}-1\right\} r\cdot\chi(r)+r\ , \end{cases}$$

where $\chi(r)\in \mathscr{E}^{\infty}$ such that $\chi(r)=0$ if $r\leqq\varepsilon_0/2$ and $\chi(r)=1$ if $r\leqq\varepsilon_0$ with ε_0 a sufficiently small positive number. By easy calculations we can show that the above ψ satisfies the condition of Theorem 3, so that the conclutions of Theorem 3 hold.

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