Fractional powers of operators, interpolation theory and imbedding theorems

By Atsushi YOSHIKAWA*

(Communicated by H. Komatsu)

Introduction. In this paper we present an operator theoretical proof of the Hardy-Littlewood-Sobolev inequality (cf. [8], [21], [26]). More precisely, we formulate the problem in an abstract way and employ abstract interpolation methods, real and complex, to obtain a result including the desired proof of the above mentioned inequality.

We here sketch briefly our method. Let E and F be two Banach spaces contained in a Hausdorff topological linear space X. We are given a linear operator A acting in X, whose maximal restrictions A_E and A_F in E and in F, respectively, are non-negative in E and in F. That is, we have all positive reals in the resolvent sets of $-A_E$ and $-A_F$ and $\|r(r+A_E)^{-1}\|_{E\to E}\leqq M$, $\|r(r+A_F)^{-1}\|_{F\to F}\leqq M$, $r>0$, M being a positive constant. We introduce an assumption on A, $A\in(\sigma, m, E, F)$, $\sigma>0$, m a positive integer, that for each $r>0$, $(r+A)^{-m}$ is a well-defined bounded operator on E into F with $\|r^m(r+A)^{-m}\|_{E\to F}\leqq Lr^{\sigma}$. Requiring $A\in(\sigma, m, E, F)$, we already showed that $(E, D(A_E^k))_{\theta+\sigma/k,p}\subset(F, D(A_F^k))_{\theta,p}$, $1\leqq p\leqq\infty$, k a positive integer $>\sigma$, and $0<\theta<\theta+\sigma/k<1$ ([29]. cf. [2], [22], [26], [28]). Here $(Y, Z)_{\theta,p}$ denotes the mean space of Y and Z ([20], [5], [6], [19], [24]). We also presented an approach to treat the case $A^{-\sigma}$ is a bounded operator on E into F, assuming that A be of bounded inverse ([31]). As we remarked in [32], the just mentioned approach turned out to be quite close to contain the Hardy-Littlewood-Sobolev inequality. However, there was a gap.

In fact, in order to give a complete treatment, we must cover the case when A^{-1} is not bounded but merely densely defined. For that purpose we are led to study the inclusion relation of the mean spaces associated with the range of A. This is done in this paper (Theorem 5.2). We then combine this and our previous results with the theory of fractional powers of operators (Komatsu [13]-[17]. cf. [1], [10], [11], [33]). In order to achieve the proof of a generalized Hardy-Littlewood-Sobolev inequality, we need the inclusion relation between the domains of fractional powers of an operator and the real interpolation spaces associated with

* Partly supported by the Sakkokai Foundation.

the same operator. For operators in a certain class which contains the negative of the Laplacian in a finite dimensional Euclid space, we obtain the desired inclusion relation, employing Stein's result ([27]), and the complex interpolation method ([3], [18], [19]. cf. [4]). In this way, the Hardy-Littlewood-Sobolev inequality is completely covered by our method.

The contents of the paper are as follows: In §1 we review the abstract interpolation methods. §2 concerns the theory of fractional powers of operators. The statements in the first half can be found in Komatsu [13]-[17] with detailed discussions. In the second half of this section we discuss the domains of fractional powers and the complex interpolation method. In §3, we study the relation of the dual spaces of the mean spaces associated with a non-negative operator and the domains of (fractional powers of) the dual operator. In §4 we introduce the definition of (σ, m, E, F) and note some properties of the ranges of operators in (σ, m, E, F), thus refining our previous results ([29], [31]).

We give in §5 our imbedding theorems about the ranges of operators of (σ, m, E, F) with necessary remarks. In the final section, §6, we apply our results to prove a generalized Hardy-Littlewood-Sobolev inequality (cf. [27], [30]).

1. Interpolation theory. We review briefly two definitions of interpolation methods, the real and complex ones, and their fundamental properties that we shall need in this paper. For detailed discussions, see the cited articles.

Let E and F be two Banach spaces continuously imbedded in a common Hausdorff linear topological space X. Such a triplet E, F, X will be called an interpolation triplet. We can thus define two Banach spaces $E\cap F$ and $E+F$ in the following way. $E\cap F$ is the space of all $a\in E\in F$ furnished with the norm $\|a\|_{E\cap F}=\max(\|a\|_E, \|a\|_F)$. $E+F$ is the space of all $a=a'+a''$ (in X), $a'\in E$, $a''\in F$, furnished with the norm $\|a\|_{E+F}=\inf(\|a'\|_E+\|a''\|_F;\ a=a'+a'')$.

DEFINITION 1.1 (Mean spaces, Lions-Peetre [20]. cf. [30]). Let $1\leqq p\leqq\infty$, $0<\theta<1$. We denote by $(E, F)_{\theta,p}$ the space of means $a=\int_0^\infty u(t)dt/t$ with $t^\theta u(t)\in L_*^p(E)$, $t^{\theta-1}u(t)\in L_*^p(F)$. Here $L_*^p(E)$ denotes the space of all E-valued strongly measurable functions $u(t)$ defined on the positive real axis such that

$$\|u\|_{L_*^p(E)}\begin{cases} =\left(\int_0^\infty \|u(t)\|_E^p\,dt/t\right)^{1/p}<\infty & \text{if} \quad p<\infty\,, \\ =\text{ess. sup}_{t>0}\|u(t)\|_E<\infty & \text{if} \quad p=\infty\,. \end{cases}$$

$L_*^p(F)$ is defined similarly. The mean space $(E, F)_{\theta,p}$ is a Banach space with the norm

$$\|a\|_{(E,F)_{\theta,p}}=\inf \max (\|t^{\theta}u(t)\|_{L^p_*(E)}, \|t^{\theta-1}u(t)\|_{L^p_*(F)})$$

where the infimum is taken over all $u(t)$ such that $a=\int_0^\infty u(t)\,dt/t$.

DEFINITION 1.2 (Calderón [3], Lions [18], cf. [19], [20]). Let $0<\theta<1$. We denote by $[E, F]_\theta$ the space of all $a=f(\theta)$ with $f\in\mathscr{H}(E, F)$. Here $\mathscr{H}(E, F)$ denotes the space of all $E+F$-valued functions $f(z)$, continuous on $0\leqq\mathrm{Re}\,z\leqq1$, holomorphic in $0<\mathrm{Re}\,z<1$, such that f are E-valued bounded[1] continuous on $\mathrm{Re}\,z=0$ and that f are F-valued bounded[1] continuous on $\mathrm{Re}\,z=1$. $[E, F]_\theta$ is a Banach space with the norm

$$\|a\|_{[E,F]_\theta}=\inf \max (\sup \|f(iy)\|_E, \sup \|f(1+iy)\|_F)$$

where the infimum is taken over all f such that $f(\theta)=a$.

PROPOSITION 1.1. *Let two interpolation triplets E, F, X and E_1, F_1, X_1 be given. Let $0<\theta<1$ and $1\leqq p\leqq\infty$. If L is a linear operator from X to X_1 such that L continuously maps E into E_1 (with norm N_1) and F into F_1 (with norm N_2), then L maps continuously $(E, F)_{\theta,p}$ into $(E_1, F_1)_{\theta,p}$ (with norm N) and $[E, F]_\theta$ into $[E_1, F_1]_\theta$ (with norm N'). Furthermore, N, $N'\leqq\mathrm{const.}\,N_1^{1-\theta}N_2^{\theta}$.*

The real method is easier to handle. We give some of the properties of the mean spaces below (Lions-Peetre [20]).

PROPOSITION 1.2. *$E\cap F$ is dense in $(E, F)_{\theta,p}$, $1\leqq p<\infty$, $0<\theta<1$.*

PROPOSITION 1.3. *If $1\leqq p\leqq q\leqq\infty$, then $(E, F)_{\theta,p}\subset(E, F)_{\theta,q}$ with the continuous imbedding.*

PROPOSITION 1.4. *Let E_1, E_2, be two Banach spaces such that[2] $(E, F)_{\theta_1,1}\subset E_1\subset(E, F)_{\theta_1,\infty}$, $(E, F)_{\theta_2,1}\subset E_2\subset(E, F)_{\theta_2,\infty}$, $0<\theta_1<\theta_2<1$ and all the imbeddings being continuous. Then we have:*

(i) *if $F\subset E$ continuously, then $E_2\subset E_1$ continuously:*

(ii) *$(E_1, F_1)_{\theta,p}=(E, F)_{\lambda,p}$ if $\lambda=(1-\theta)\theta_1+\theta\theta_2$.*

PROPOSITION 1.5. *If $E\cap F$ is dense in both of E and F, then E^*, F^*, $(E\cap F)^*$ is an interpolation triplet. We have*

$$(E, F)^*_{\theta,p}=(E^*, F^*)_{\theta,p'},\ p'=p/(p-1),\ 1\leqq p<\infty,\ 0<\theta<1\,,$$

with equivalent norms. Here Y^* denotes the strong dual space of a Banach space Y.

If E and F are reflexive, we also have

$$[E, F]^*_\theta=[E^*, F^*]_\theta$$

---

1) Instead of "bounded", we may assume "of order exp (const. $|\mathrm{Im}\,z|$)" ([3], [19]).

2) E_i is said to be of class $\mathscr{K}_{\theta_i}(E, F)$, $i=1, 2$.

with equivalent norms ([3]).

For the sake of later convenience, we also note the following.

PROPOSITION 1.6. *If $E\cap F$ is dense in both of E and F, then we have*

$$(E\cap F)^*=E^*+F^*,\ (E+F)^*=E^*\cap F^*\ ,$$

each with equivalent norms.

PROOF. Let H be the closed subspace $\{x\oplus(-x);\ x\in E\cap F\}$ of the direct sum $E\oplus F$. Since $E\cap F$ and $E+F$ are canonically isomorphic to H and $E\oplus F/H$ respectively, we have $(E\cap F)^*=E^*\oplus F^*/H^0$ and $(E+F)^*=H^0$, where $H^0=\{e^*\oplus f^*\in E^*\oplus F^*;\ \langle x, e^*\rangle-\langle x, f^*\rangle=0,\ x\in E\cap F\}$. By the assumption E^* and F^* may be imbedded in $(E\cap F)^*$. Then H^0 coincides with $E^*\cap F^*$ (cf. Köthe [12], §§19, 22).

2. Fractional powers of operators. We briefly recall fundamental properties of fractional powers of non-negative operators. For detailed discussions, see Komatsu's series of works on fractional powers [13]-[17].

DEFINITION 2.1. Let E be a Banach space and B a closed linear operator defined in E. B is said to be non-negative (in E) if all positive reals are in the resolvent set $\rho(-B)$ of $-B$ with the estimate

$$\|r(r+B)^{-1}\|_{E\to E}\leqq M \quad \text{for all} \quad r>0\ ,$$

M being a positive constant independent of r.

The maximal restrictions of B to $\overline{D(B)}$, $\overline{R(B)}$, $\overline{D(B)}\cap\overline{R(B)}$ are denoted by B_+, B_-, B_0 respectively[3]. Namely, we have

$$B_+a=Ba \quad \text{for} \quad a\in D(B_+)=\{a\in D(B);\ Ba\in\overline{D(B)}\}\ ,$$

$$B_-a=Ba \quad \text{for} \quad a\in D(B_-)=D(B)\cap\overline{R(B)}\ ,$$

$$B_0a=Ba \quad \text{for} \quad a\in D(B_0)=\{a\in D(B)\cap\overline{R(B)};\ Ba\in\overline{D(B)}\}\ .$$

B_+, B_- and B_0 are non-negative in $\overline{D(B)}$, in $\overline{R(B)}$ and in $\overline{D(B)}\cap\overline{R(B)}$, respectively. We shall mostly consider the cases when $B=B_+=B_0$, $B=B_-=B_0$, or $B=B_0=B_+=B_-$. In fact, our main situation in this paper is that B is densely defined and densely ranged. Thus, $B=B_0=B_+=B_-$ so that B^{-1} is a well-defined non-negative operator in E.

A particular class of non-negative operators consists of the negatives of the infinitesimal generators of bounded continuons semi-groups of linear operators. $-B$ generates a bounded continuous semi-group $\exp(-tB)$, $t\geqq 0$, if B is densely

[3] For an operator B in E, $D(B)$ stands for its definition domain, $R(B)$ for its range and $N(B)$ for its null-space.

defined and if, for any $r>0$ and for any integer $m>0$,

$$\|r^m(r+B)^{-m}\|_{E\to E}\leqq M .$$

In particular, if $\exp(-tB)$ can be analytically and boundedly continued into the sector $S_\omega=\{t;\ |\arg t|<\omega, 0<\omega<\pi/2\}$, $\exp(-tB)$ is called a bounded holomorphic semi-group. If $\{z;\ |\arg z|<\pi-\omega\}$ is contained in $\rho(-B)$ with $\|z(z+B)^{-1}\|_{E\to E}\leqq M(\theta)$ for all z, $|\arg z|\leqq\theta$, $\theta<\pi-\omega$ being arbitrarily fixed, then $-B$ generates a bounded holomorphic semi-group.

PROPOSITION 2.1 ([14]). *Let $\alpha>0$, $1\leqq p\leqq\infty$ and m an integer $>\alpha$. The mean space $(E, D(B^m))_{\alpha/m,p}$ coincides with the space of all $x\in E$ such that $r^\alpha(B(r+B)^{-1})^m x\in L_*^p(E)$. $(E, D(B^m))_{\alpha/m,p}$ does not depend on $m>\alpha$. If $\alpha>\beta>0$, then $(E, D(B^m))_{\alpha/m,p}\subset(E, D(B^m))_{\beta/m,q}$, $1\leqq p$, $q\leqq\infty$. Furthermore,*

$$(E, D(B^m))_{\alpha/m,p}=(\overline{D(B)}, D(B_+^m))_{\alpha/m,p} .$$

In particular, if $-B$ generates a bounded continuous semi-group $\exp(-tB)$, *then $(E, D(B^m))_{\alpha/m,p}$ coincides with the space of all $x\in E$ such that*

$$t^{-\alpha}(1-\exp(-tB))^m x\in L_*^p(E) .$$

If $\exp(-tB)$ *is holomorphic, then $(E, D(B^m))_{\alpha/m,p}$ coincides with the space of all $x\in E$ such that $t^{m-\alpha}B^m$* $\exp(-tB)x\in L_*^p(E)$.

DEFINITION 2.2. Let $\operatorname{Re}\alpha>0$. B_+^α is defined to be the smallest closed extension of B_σ^α. Here B_σ^α is defined as follows:

$$B_\sigma^\alpha x=c_{m,\alpha}\int_0^\infty r^{\alpha-1}(B(r+B)^{-1})^m x\,dr,\quad c_{m,\alpha}=\Gamma(m)/\Gamma(\alpha)\Gamma(m-\alpha)) ,$$

for

$$x\in D(B_\sigma^\alpha)=(E, D(B^m))_{\alpha/m,1},\ 0<\operatorname{Re}\alpha<\sigma .$$

If $-B$ generates a bounded continuous semi-group $\exp(-tB)$, then

$$B_\sigma^\alpha x=K_{\alpha,m}^{-1}\int_0^\infty t^{-\alpha-1}(1-\exp(-tB))^m x\,dt,\quad x\in D(B_\sigma^\alpha) ,$$

where $K_{\alpha,m}=\int_0^\infty t^{-\alpha-1}(1-e^{-t})^m dt$. If $\exp(-tB)$ is holomorphic, then

$$B_\sigma^\alpha x=\Gamma(m-\alpha)^{-1}\int_0^\infty t^{m-\alpha-1}B^m\exp(-tB)x\,dt,\quad x\in D(B_\sigma^\alpha) .$$

This definition is well-defined and independent of σ, m or the expressions of B_σ^α (Komatsu [13], [14]).

PROPOSITION 2.2 ([14]). *If* $m>\operatorname{Re}\alpha>0$, *then we have*

$$(E, D(B^m))_{\operatorname{Re}\alpha/m,1}\subset D(B_+^\alpha)\subset(E, D(B^m))_{\operatorname{Re}\alpha/m,\infty},$$

all the imbeddings being continuous.

PROPOSITION 2.3 ([14]). *If* $D(B_+^\alpha)$ *is contained in (contains)* $(E, D(B^m))_{\operatorname{Re}\alpha/m,p}$ *for some* $\operatorname{Re}\alpha>0$, *then* $D(B_+^\alpha)$ *is contained in (contains)* $(E, D(B^m))_{\operatorname{Re}\alpha/m,p}$ *for any* $\operatorname{Re}\alpha>0$.

PROPOSITION 2.4 ([15]). *Let* $\alpha>0$, $1\leqq p\leqq\infty$, *and* m *an integer* $>\alpha$. *The mean space* $(E, R(B^m))_{\alpha/m,p}$ *coincides with the space of all* $x\in E$ *such that* $r^{-\alpha}(r(r+B)^{-1})^m x\in L_*^p(E)$. $(E, R(B^m))_{\alpha/m,p}$ *does not depend on* $m>\alpha$. *If* $\alpha>\beta>0$, *then* $(E, R(B^m))_{\alpha/m,p}\subset(E, R(B^m))_{\beta/m,q}$, $1\leqq p, q\leqq\infty$. *Furthermore*, $(E, R(B^m))_{\alpha/m,p}=(\overline{R(B)}, R(B^m))_{\alpha/m,p}$. *In particular, if* $-B$ *generates a bounded continuous semi-group* $\exp(-tB)$, *then* $(E, R(B^m))_{\alpha/m,p}$ *coincides with the space of all* $x\in E$ *such that* $t^\alpha\left(t^{-1}\int_0^t \exp(-sB)ds\right)^m x\in L_*^p(E)$. *If* $\exp(-tB)$ *is holomorphic, then* $(E, R(B^m))_{\alpha/m,p}$ *coincides with the space of all* $x\in E$ *such that* $t^\alpha\exp(-tB)x\in L_*^p(E)$.

DEFINITION 2.3. Let $\operatorname{Re}\alpha>0$. $B_-^{-\alpha}$ is defined to be the smallest closed extension of $B_{-\sigma}^{-\alpha}$. Here $B_{-\sigma}^{-\alpha}$ is defined as follows:

$$B_{-\sigma}^{-\alpha}x=c_{m,\alpha}\int_0^\infty r^{-\alpha-1}(r(r+B)^{-1})^m x\,dr$$

for $x\in D(B_{-\sigma}^{-\alpha})=(E, R(B^m))_{\alpha/m,1}$, $0<\operatorname{Re}\alpha<\sigma$. If $-B$ generates a bounded continuous semi-group $\exp(-tB)$, then

$$B_{-\sigma}^{-\alpha}x=K_{m-\alpha,m}^{-1}\int_0^\infty t^{\alpha-1}\left(t^{-1}\int_0^t \exp(-sB)ds\right)^m x\,dt,\quad x\in D(B_{-\sigma}^{-\alpha}),$$

where $K_{m-\alpha,m}=\int_0^\infty t^{\alpha-m-1}(1-e^{-t})^m dt$. If $\exp(-tB)$ is holomorphic, then

$$B_{-\sigma}^{-\alpha}x=\Gamma(\alpha)^{-1}\int_0^\infty t^{\alpha-1}\exp(-tB)x\,dt,\quad x\in D(B_{-\sigma}^{-\alpha}).$$

This definition is well-defined and independent of σ, m, or the expressions of $B_{-\sigma}^{-\alpha}$ (Komatsu [13], [15]).

PROPOSITION 2.5 ([15]). *Let* $m>\operatorname{Re}\alpha>0$. *Then*

$$(E, R(B^m))_{\operatorname{Re}\alpha/m,1}\subset D(B_-^{-\alpha})\subset(E, R(B^m))_{\operatorname{Re}\alpha/m,\infty},$$

all the imbeddings being continuous.

PROPOSITION 2.6 ([15]). *If* $D(B_-^{-\alpha})$ *is contained in (contains)* $(E, R(B^m))_{\operatorname{Re}\alpha/m,p}$ *for some* $\operatorname{Re}\alpha>0$, *then* $D(B_-^{-\alpha})$ *is contained in (contains)* $(E, R(B^m))_{\operatorname{Re}\alpha/m,p}$ *for any* $\operatorname{Re}\alpha>0$.

PROPOSITION 2.7 ([13]). *If* $\mathrm{Re}\,\alpha\cdot\mathrm{Re}\,\beta>0$, *then we have* $B_{\pm}^{\alpha}B_{\pm}^{\beta}=B_{\pm}^{\alpha+\beta}$ *in the sense of the product of operators.*

DEFINITION 2.4. Let α be any complex number. B_0^{α} is defined to be the smallest closed extension of $B_{\sigma,-\tau}^{\alpha}$, $-\tau<\mathrm{Re}\,\alpha<\sigma$. Here $B_{\sigma,-\tau}^{\alpha}$ is defined as follows: For $x\in D(B_{\sigma,-\tau}^{\alpha})=D(B_{+}^{\sigma})\cap D(B_{-}^{-\tau})$,

$$B_{\sigma,-\tau}^{\alpha}x=\begin{cases} x \quad \text{if} \quad \alpha=0\,, \\ -(\sin \pi\alpha/\pi)\Big\{(-1)^m\displaystyle\int_0^N r^{\alpha+m}(r+B_-)^{-1}B_-^{-m}x\,dr+ \\ \quad +\sum_{k=-m}^{n}(-1)^{k+1}\dfrac{N^{\alpha-k}}{\alpha-k}B_0^k x+(-1)^{n+1}\displaystyle\int_N^{\infty} r^{\alpha-n-1}B(r+B)^{-1}B^n x\,dr\Big\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad \alpha\neq 0\,. \end{cases}$$

Here N is an arbitrary fixed positive integer, and $\sigma>n$, $\tau>m$. B_0^{α} is well-defined and independent of σ, τ, N, n, m (Komatsu [13]).

PROPOSITION 2.8 ([13]). $B_0^{\alpha}=(B_0)_0^{\alpha}$. *If* $\mathrm{Re}\,\alpha>0$ ($\mathrm{Re}\,\alpha<0$), B_0^{α} *is the restriction of* B_{+}^{α} (B_{-}^{α}) *with the domain* $D(B_0^{\alpha})=D(B_{+}^{\alpha})\cap\overline{R(B)}$ ($D(B_{-}^{\alpha})\cap\overline{D(B)}$).

PROPOSITION 2.9 ([13]). *If* $x\in D(B_0^{\beta})\cap D(B_0^{\alpha+\beta})$, *then* $B_0^{\beta}x\in D(B_0^{\alpha})$ *and* $B_0^{\alpha}B_0^{\beta}x=B_0^{\alpha+\beta}x$. *In particular,* B_0^{α} *is one-to-one for any* α *and the inverse is* $B_0^{-\alpha}$.

PROPOSITION 2.10 ([13]). *Let* E, F *be two Banach spaces such that* $F\subset E$ *continuously. Let* B *be a non-negative operator in* E *and* B_F *be its maximal restriction to* F, *that is,* $B_F x=Bx$ *for* $x\in D(B_F)=\{x\in F\cap D(B)\,;\,Bx\in F\}$. *If* B_F *is non-negative and* $D(B_F)$ *is dense in* F, *then* $(B_{+}^{\alpha})_F=(B_{F+}^{\alpha})$ *for any* $\mathrm{Re}\,\alpha>0$.

In view of Proposition 2.8, we shall often omit suffices 0, +, or − and simply write B^{α} if we consider such B that $E=\overline{D(B)}=\overline{R(B)}$.

We have discussed in the above the relation between the domains $D(B_{\pm}^{\alpha})$ and real interpolation spaces. As for the complex method, we have the following propositions.

We begin by the following

PROPOSITION 2.11 ([13]). *Let* $\mathrm{Re}\,\alpha>0$. *If* B *is densely defined, then we have for any* $\nu>0$, $\mu>0$,

$$D((\nu+B)_{+}^{\alpha})=D(B_{+}^{\alpha})=R((\mu+B)_{-}^{-\alpha})\,.$$

Furthermore, for any $x\in E$, $(\mu+B)_{-}^{-\alpha}x$ *is analytic in* α, $\mathrm{Re}\,\alpha>0$.

PROPOSITION 2.12 (cf. [18] and [4]). *Let* E *be reflexive and* B *be densely defined and densely ranged. If for any* $r\in\mathbf{R}$, $B_0^{ir}\in\mathscr{L}(E,E)$ *and is strongly continuous in* r, *then*

$$[E, D(B^m)]_\theta = D(B_+^{m\theta}), \quad 0<\theta<1\,,$$

with equivalent norms.

PROOF. We first show that, for any fixed $\nu>0$, $(\nu+B)^{ir} \in \mathscr{L}(E, E)$, $r \in R$, and $\|(\nu+B)^{ir}\|_{E\to E} \leqq \text{const}\, e^{K|r|}$, K a constant. In fact, from the assumption, $D(B^{1+ir}) = D(B)$ for any $r \in R$. Hence, by Proposition 2.11,

$$D((\nu+B)^{1+ir}) = D(\nu+B) = D(B)\,. \tag{2.1}$$

On the other hand, we have two continuous bijections:

$$(\nu+B)^{1+ir} : D((\nu+B)^{1+ir}) \longrightarrow E\,,$$

$$(\nu+B) \quad : D(\nu+B) \longrightarrow E\,.$$

It follows from (2.1) that

$$(\nu+B)^{ir} = (\nu+B)^{1+ir}(\nu+B)^{-1} \in \mathscr{L}(E, E)\,.$$

In the same way, using Propositions 2.7, 2.9 and 2.11, we see that $((\nu+B)^{ir})^n = (\nu+B)^{inr}$. We have

$$\sup_{-1\leqq r\leqq 1} \|(\nu+B)^{ir}\|_{E\to E} \leqq M < \infty\,.$$

Hence,

$$\|(\nu+B)^{ir}\|_{E\to E} \leqq M^{[|r|]} \leqq Me^{|r|K}, \quad K = \max(\log M, 0)\,. \tag{2.2}$$

Here $[|r|]$ denotes the greatest integer $<|r|$.

Now let $x \in E$. Then $f(z) = (\nu+B)^{-mz}x \in \mathscr{H}_{\exp}(E, D(B^m))$ [4], and $f(\theta) \in D(B_+^{m\theta})$, $0<\theta<1$.

Hence, $D(B_+^{m\theta}) \subset [E, D(B^m)]_\theta$.

Conversely, let $g(z) \in \mathscr{H}(E, D(B^m))$. Take any $y \in D(B^{*m})$, and consider the expression:

$$G(z) = {}_E\langle g(z),\ (\nu+B^*)^{mz}y \rangle_{E^*}\,.$$

Then by (2.2) and the assumption on g, we have

$$|G(z)| \leqq \text{const}\, \|y\|_{E^*} e^K |\text{Im}\, z| \tag{2.3}$$

on $\text{Re}\, z=0$ and on $\text{Re}\, z=1$. Since E is reflexive, $D(B^{*m})$ is dense in E^*. Therefore, from (2.3) it follows that $h(z) = (\nu+B)^{mz}g(z)$ is weakly and hence strongly

[4] Let Y, Z, X be an interpolation triplet. $\mathscr{H}_{\exp}(Y, Z)$ denotes the space of $Y+Z$-valued functions $f(z)$, continuous on $0 \leqq \text{Re}\, z \leqq 1$, holomorphic in $0<\text{Re}\, z<1$, such that f are Y-valued continuous of order $\exp(K|\text{Im}\, z|)$ on $\text{Re}\, z=0$ and that f are Z-valued continuous of order $\exp(K|\text{Im}\, z|)$ on $\text{Re}\, z=1$ (Definition 1.2).

holomorphic in z. So $h\in\mathscr{H}_{\exp}(E, E)$. In particular, $h(\theta)\in E$ and $g(\theta)\in D(B_{+}^{m\theta})$. This proves $[E, D(B^{m})]_{\theta}\subset D(B_{+}^{m\theta})$. Q.E.D.

PROPOSITION 2.13. *Let E, F, X be an interpolation triplet such that E, F are reflexive and that $E\cap F$ is dense in both of E and F. Suppose furthermore $E^{*}\cap F^{*}$ is dense in both of E^{*} and F^{*}. Let B be a non-negative operator in $E+F$ such that its maximal restrictions B_{E} and B_{F} to E and F are densely ranged non-negative operators.*

If, for any $r\in\boldsymbol{R}$, $B_{E}^{ir}\in\mathscr{L}(E, E)$ and $B_{F}^{ir}\in\mathscr{L}(F, F)$ with $\|B_{E}^{ir}\|_{E\to E}$ and $\|B_{F}^{ir}\|_{F\to F}$ locally bounded in r, then we have for any $\alpha\geqq 0$, $\beta\geqq 0$,

$$[D(B_{E}^{\alpha}), D(B_{F}^{\alpha+\beta})]_{\theta}=D(B_{\theta}^{\alpha+\beta\theta}),\quad 0<\theta<1\,,$$

with equivalent norms. Here B_{θ} denotes the maximal restriction of B to $[E, F]_{\theta}$.

PROOF. It follows from the assumtions that B_{θ} is a densely defined and densely ranged non-negative operator (See Corollary 4.1). Hence, we can apply Proposition 2.10. On the other hand, as in the previous proposition, we see $(\nu+B_{E})^{ir}\in\mathscr{L}(E, E)$ and $(\nu+B_{F})^{ir}\in\mathscr{L}(F, F)$ for any $r\in\boldsymbol{R}$. In particular, $(\nu+B)^{ir}\in\mathscr{L}(E+F, E+F)$. Let $x\in D(B_{\theta}^{m\theta})$. Then $x=(\nu+B_{\theta})^{-m\theta}y$, $y\in[E, F]_{\theta}$. There is an $h(z)\in\mathscr{H}(E, F)$ such that $y=h(\theta)$ and that

$$F(z)=(\nu+B)^{-\alpha-\beta z}h(z)\in\mathscr{H}_{\exp}(D(B_{E}^{\alpha+\beta}))\,.$$

Furthermore, $F(\theta)=x$. Hence, $D(B_{\theta}^{m\theta})\subset[D(B_{E}^{\alpha}), D(B_{F}^{\alpha+\beta})]_{\theta}$.

Now we are going to prove the converse inclusion. Since $E+F$ is reflexive with $(E+F)^{*}=E^{*}\cap F^{*}$, $D(B^{*m})$ is dense in $E^{*}\cap F^{*}$. Hence, we can argue as in the previous proposition. In fact, let $g(z)\in\mathscr{H}(D(B_{E}^{\alpha}), D(B_{F}^{\alpha+\beta}))$ and consider the expression:

$$G(z)={}_{E+F}\langle g(z), (\nu+B^{*})^{\alpha+\beta z}y\rangle_{E^{*}\cap F^{*}}\quad\text{for any}\quad y\in D(B^{*\alpha+\beta})\,.$$

By the assumption on g, we have

$$|G(z)|\leqq\text{const}\,\|y\|_{E^{*}}e^{K|\operatorname{Im} z|}\quad\text{on}\quad\operatorname{Re}z=0\,,$$

$$|G(z)|\leqq\text{const}\,\|y\|_{F^{*}}e^{K|\operatorname{Im} z|}\quad\text{on}\quad\operatorname{Re}z=1\,.$$

Therefore, by the remark on B^{*} in the above, we see that $h(z)=(\nu+B)^{\alpha+\beta z}g(z)$ is weakly and hence strongly holomorphic in $E+F$. Hence, $h\in\mathscr{H}_{\exp}(E, F)$. Now, using Proposition 2.10, we see that $g(\theta)\in D(B_{\theta}^{\alpha+\beta\theta})$. This proves $[D(B_{E}^{\alpha}), D(B_{F}^{\alpha+\beta})]_{\theta}\subset D(B^{\alpha+\beta\theta})$. Q.E.D.

3. Duality of mean spaces associated with non-negative operators.

For a densely defined non-negative operator B in a Banach space E, we have noticed that $(E, D(B^m))_{\alpha/m,1}\subset D(B_+^\alpha)\subset(E, D(B^m))_{\alpha/m}$, for any $\alpha>0$, all the imbeddings being continuous. Later we shall discuss, in particular, such cases that $(E, D(B^m))_{\alpha/m,p}\subset D(B_+^\alpha)$ or $D(B_+^\alpha)\subset(E, D(B^m))_{\alpha/m,q}$, $1<p, q<\infty$ (cf. Propositions 2.2 and 2.3). Thus it will be interesting to discuss their duality relations. Namely, we here prove the following propositions.

THEOREM 3.1. *Let B be a densely defined non-negative operator in a Banach space E. If we have, for some p, $1\leqq p<\infty$,*

$$D(B_+^\alpha)\subset(E, D(B^m))_{\alpha/m,p},\ \alpha>0\ , \tag{3.1}$$

with the continuous imbedding, then for any positive integer k,

$$(E^*, D(B^{*m}))_{k/m,p'}\subset D(B^{*k}),\ p'=p/(p-1),\ m>k\ , \tag{3.2}$$

with the continuous imbedding. Here B^* denotes the adjoint operator of B and E^* the dual space of E.

COROLLARY 3.1. *Let E be reflexive. If* (3.1) *holds, then for any $\beta>0$,*

$$(E^*, D(B^{*m}))_{\beta/m,p'}\subset D(B_+^{*\beta}),\ p'=p/(p-1),\ m>\beta\ , \tag{3.2'}$$

with the continuous imbedding.

In fact, this is clear from (3.2) and from the fact that $B^*=B_+^*$ since E is reflexive.

THEOREM 3.2. *Let B, E, B^*, E^* be as in Theorem* 3.1. *If we have, for some q, $1\leqq q\leqq\infty$,*

$$(E, D(B^m))_{\alpha/m,q}\subset D(B_+^\alpha),\ \alpha>0\ , \tag{3.3}$$

with the continuous imbedding, then, for any positive integer k,

$$D(B^{*k})\subset(E^*, D(B^{*m}))_{k/m,q'},\ q'=q/(q-1),\ m>k\ , \tag{3.4}$$

with the continuous imbedding. In particular, for any $\beta>0$,

$$D(B_+^{*\beta})\subset(E^*, D(B^{*m}))_{\beta/m,q'},\ m>\beta\ , \tag{3.4'}$$

with the continuous imbedding.

PROOF OF THEOREM 3.1. Firstly we prove (3.2) for $k=1$. For any $r>0$, $(r+B)^{-m}$ continuously maps E into E and E onto $D(B^m)$, both in a one-to-one way. Hence, taking the adjoint, we see that $(r+B^*)^{-m}$ maps E^* into E^* continuously and $J_m^*(r)$, the adjoint of $(r+B)^{-m}$ from E onto $D(B^m)$, maps $D(B^m)^*$ onto E^* continuously. Furthermore, both adjoint operators are one-to-one. Since $D(B)$ is dense in E, $D(B^m)$ is dense in E. Thus we may take $E^*\subset D(B^m)^*$, the imbedding being continuous. In particular, $J_m^*(r)$ is the weak* closure of $(r+B^*)^{-m}$.

On the other hand, B may be considered as non-negative operator in $D(B)$. Taking the adjoint, we have a non-negative operator C in $D(B)^*$. We have $D(C)=E^*$. For, $(r+B)^{-1}$ maps $D(B)$ into $D(B)$, and E onto $D(B)$ for every $r>0$. Both are one-to-one. By the duality, E^* is imbedded in $D(B)^*$ and by this imbedding we see that the image of $(r+C)^{-1}$ coincides with E^*.

Now we show that $(r+C)^{-1}|_{E^*}=(r+B^*)^{-1}$. In fact, let $x\in E^*$. Then $x=I^*x\in D(B)^*$. Here I is the imbedding operator of $D(B)$ into E. Take any $y\in D(B)$. Noting that $y=Iy\in E$, we have

$$\begin{aligned}{}_{E^*}\langle(r+C)^{-1}x, y\rangle_E &= {}_{D(B)^*}\langle x, (r+B)^{-1}y\rangle_{D(B)}\\ &= {}_{D(B)^*}\langle I^*x, (r+B)^{-1}y\rangle_{D(B)}\\ &= {}_{E^*}\langle x, I(r+B)^{-1}y\rangle_E\\ &= {}_{E^*}\langle x, (r+B)^{-1}Iy\rangle_E\\ &= {}_{E^*}\langle(r+B^*)^{-1}x, y\rangle_E .\end{aligned}$$

On the other hand, $D(B^2)$ is dense in $D(B)$, and by the duality $D(B)^*$ is imbedded in $D(B^2)^*$. By this imbedding, we see that $J_2^*(r)$ is an extension of $(r+B^*)^{-1}(r+C)^{-1}$ in a similar way as in the above. Hence, we have three compatible continuous bijections:

$$J_2^*(r) : D(B^2)^* \longrightarrow E^* ,$$

$$(r+B^*)^{-2} : E^* \longrightarrow D(B^{*2}) ,$$

$$(r+B^*)^{-1}(r+C)^{-1} : D(B)^* \longrightarrow D(B^*) . \tag{3.5}$$

By the interpolation, we see that

(3.6) $J_2^*(r)$ maps $(E^*, D(B^2))_{1/2,p'}$ onto $(D(B^*), E^*)_{1/2,p'}$ in a one-to-one manner.

Now, since $p<\infty$, we see from (3.1) that $D(B)$ is dense in $(E, D(B^2))_{1/2,p}$. Thus taking the duals, we have

$$(E^*, D(B^2)^*)_{1/2,p'}\subset D(B)^* \tag{3.7}$$

with the continuous imbedding. Since $J_2^*(r)$ is an extension of $(r+B^*)^{-1}(r+C)^{-1}$, (3.7) implies $J_2^*(r)(E^*, D(B^2)^*)_{1/2,p'}\subset J_2^*(r)D(B)^*$. Thus from (3.5) and (3.6), we have

$$(E^*, D(B^{*2}))_{1/2,p'}=(D(B^{*2}), E^*)_{1/2,p'}\subset D(B^*) \tag{3.8}$$

with the continuous imbedding. This proves (3.2) for $k=1$. For the proof of general k, we use the bijections:

$$\begin{aligned}(r+B^*)^{-k+1}: D(B^*) &\longrightarrow D(B^{*k}) ,\\ E^* &\longrightarrow D(B^{*k-1}) ,\\ D(B^{*2}) &\longrightarrow D(B^{*k+1}) .\end{aligned}$$

By interpolation and from (3.8), we have

$$(E^*, D(B^{*m}))_{k/m, p'}=(D(B^{*k-1}), D(B^{*k+1}))_{1/2p'}\subset D(B^{*k})$$

with the continuous imbedding. Q.E.D.

PROOF OF THEOREM 3.2. (3.4) is proved exactly in the same way as (3.2). For the proof of (3.4′), we only note that $(E^*, D(B^{*m}))_{\theta,q'}=(\overline{D(B^*)}, D(B_+^{*m}))_{\theta,q'}$ (Proposition 2.1) and $D(B_+^*)$ is a closed subspace of $D(B^*)$. In fact, then we have

$$D(B_+^*)\subset D(B^*)\subset(E^*, D(B^*))_{1/2, q'}=(\overline{D(B^*)}, D(B_+^{*2}))_{1/2, q'} , \tag{3.9}$$

with the continuous imbeddings. (3.4′) follows immediately from (3.9) (Proposition 2.3). Q.E.D.

REMARK 3.1. A systematic discussion on dual fractional powers of non-negative operators is given in Komatsu [17]. He studies in [17] the relation between $(B^\alpha)^*$ and $B_+^{*\alpha}$. On the other hand, the space $(D(B^m)^*, E^*)_{\theta,p}$ was essentially considered in Grisvard [6].

Furthermore, using Proposition 2.7, we have the following corollaries by the same argument as in the proofs of Theorems 3.1 and 3.2.

COROLLARY 3.2. *If* (3.1) *holds, then, for any* $\beta>0$,

$$(E^*, D(B^{*m}))_{\beta/m, p'}\subset D(B_+^\beta)^*), \ p'=p/(p-1) ,$$

with the continuous imbedding.

COROLLARY 3.3. *If* (3.3) *holds, then, for any* $\beta>0$,

$$D((B_+^\beta)^*)\subset(E^*, D(B^*))_{\beta/m, q'}, \ q'=q/(q-1) ,$$

with the continuous imbedding.

4. **The class** (σ, m, E, F). Let E, F, X be an interpolation triplet. Thus we can define $E+F$ and $E\cap F$ in the usual manner. We consider a non-negative operator A in $E+F$. We denote by A_E the maximal restriction of A in E, that is

$$A_E x=Ax \quad \text{for} \quad x\in D(A_E)=\{x\in E\cap D(A);\ Ax\in E\} .$$

A_F is defined in a similar manner. A_E and A_F are clearly closed operators in E and in F, respectively. In this paper we consider only the case where A_E and A_F are non-negative operators in E and in F, respectively. As for this, we noted in

[31] the following

PROPOSITION 4.1. *Let A_1 and A_2 be non-negative operators densely defined in E and in F, respectively. If $(r+A_1)^{-1}x=(r+A_2)^{-1}x$ for $x\in E\cap F$, for all $r>0$, then the operator A in $E+F$ defined by*

$$(4.1)\qquad Ax=A_1x_1+A_2x_2 \quad \textit{for} \quad x=x_1+x_2\in D(A)=D(A_1)+D(A_2)$$

is well-defined. A is a densely defined non-negative operator in $E+F$, and $A_E=A_1$, $A_F=A_2$.

Since it is straightforward, we do not reproduce the proof.

We note however that from this proposition we have $D(A^m)=D(A_1^m)+D(A_2^m)$ for any integer $m>0$.

In fact, $x\in D(A^m)$ if and only if $(r+A)^{-m}y=x$ for some $y\in E+F$, $r>0$. Decomposing $y=y_1+y_2$, $y_1\in E$, $y_2\in F$, we have

$$x=(r+A)^{-m}y_1+(r+A)^{-m}y_2\ .$$

Since $A_E=A_1$ and $A_F=A_2$, we see that $x\in D(A_1^m)+D(A_2^m)$. That $D(A_1^m)+D_2(A_2^m)\subset D(A^m)$ can be shown similarly.

In this paper we pay much attention to the ranges of operators. First, we note

PROPOSITION 4.2[5]. *If $\overline{R(A_E)}=E$ and if $\overline{R(A_F)}=F$, then $\overline{R(A)}=E+F$.*

PROOF. It is enough to show that $\overline{R(A_E)}+\overline{R(A_F)}\subset\overline{R(A)}$. In fact, let $x=x'+x''\in\overline{R(A_E)}+\overline{R(A_F)}$, $x'\in\overline{R(A_E)}$, $x''\in\overline{R(A_F)}$. Then we can find $x'_n\in R(A_E)$ and $x''_n\in R(A_F)$ such that $\|x'-x'_n\|_E\to 0$ and $\|x''-x'_n\|_F\to 0$ as $n\to\infty$. Then $x'_n+x''_n\in R(A)$ and $\|x'_n+x''_n-x\|_{E+F}\leqq\|x'-x'_n\|_E+\|x''-x''_n\|_F\to 0$ as $n\to\infty$. Q.E.D.

REMARK 4.1. $\overline{R(A_E)}=E$, say, is realized if $N(A_E)=0$ and E is reflexive (Abelian ergodic theorem of Hille's type).

COROLLARY 4.1. *Assume that E, F are reflexive and $E\cap F$ is dense in E and in F. If A_E and A_F are densely defined and densely ranged, then A_θ* (or $A_{\theta,p}$), $0<\theta<1$, $1<p<\infty$, *is a densely defined and densely ranged non-negative operator. Here A_θ* (or $A_{\theta,p}$) *denotes the maximal restriction of A to $[E,F]_\theta$ (or to $(E,F)_{\theta,p}$).*

PROOF. A is densely ranged in $E+F$, in particular, $N(A)=0$. Hence, $N(A_\theta)=0$. The non-negativity of A_θ follows from the interpolation. Since $[E,F]_\theta$ is reflexive, A_θ is densely ranged by the Abelian ergodic theorem. A similar argument applied to A^{-1} shows that A_θ is densely defined. Similarly, we see that $A_{\theta,p}$ is densely

[5] $\overline{R(A_E)}$ is the closure of $R(A_E)$ in E. $\overline{R(A_F)}$ is the closure of $R(A_F)$ in F. $\overline{R(A)}$ is the closure of $R(A)$ in $E+F$. Similarly for $\overline{D(A_E)}$, $\overline{D(A_F)}$, $\overline{D(A)}$.

defined and densely ranged. Q.E.D.

Now we introduce the class of operators which plays the principal role in this paper. As we have shown previously ([29], [31]), many interesting operators in analysis fall in this class.

DEFINITION 4.1. A is said to be of class (σ, m, E, F) for some $\sigma>0$ and for some positive integer m if the following two conditions are satisfied:

(i) A_E and A_F are non-negative in E and in F, respectively.

(ii) For each $r>0$, $(r+A)^{-m}\in \mathscr{L}(E, F)$ with the norm

$$\|(r+A)^{-m}\|_{E\to F}\leqq L_m r^{\sigma-m},\ r>0\ ,$$

L_m being a positive constant independent of r.

We write $A\in(\sigma, m, E, F)$ if A is of class (σ, m, E, F).

PROPOSITION 4.3. *$A\in(\sigma, m, E, F)$ implies $A\in(\sigma, m+1, E, F)$. Conversely, $A\in(\sigma, m+1, E, F)$ implies $A\in(\sigma, m, E, F)$ if $m>\sigma$.*

The proof is easy [31].

PROPOSITION 4.4. *If $A\in(\sigma, m, E, F)$, $\sigma>0$, then $N(A_E)=0$.*

PROOF. Let $x\in N(A_E)$. Since $A\in(\sigma, m, E, F)$, we have

$$\|(r+A)^{-m}x\|_F\leqq L_m r^{\sigma-m}\|x\|_E \quad \text{for any} \quad r>0\ .$$

Hence, $r^m(r+A)^{-m}x\to 0$ in F as $r\to 0$. On the other hand, $r^m(r+A)^{-m}x=x$ in E. Thus, $x=0$ in $E+F$, and so $N(A_E)=0$.

PROPOSITION 4.5. *If $A\in(\sigma, m, E, F)$, then the image of $\overline{R(A_E)}$ by the mapping $(r+A)^{-m}$ for any $r>0$ is contained in $\overline{R(A_F)}$.*

PROOF. Let $x\in\overline{R(A_E)}$. Then $A(s+A)^{-1}x\to x$ in E as $s\to 0$, $s>0$. Since $A\in(\sigma, m, E, F)$, we have, for any $r>0$ and $s>0$,

$$\begin{aligned}&\|(r+A)^{-m}x-A_F(s+A_F)^{-1}(r+A)^{-m}x\|_F\\ &\quad=\|(r+A)^{-m}(x-A(s+A)^{-1}x)\|_F\leqq\|(r+A)^{-m}\|_{E\to F}\|x-A_E(s+A_E)^{-1}x\|_E\ .\end{aligned}$$

Hence, $A_F(s+A_F)^{-1}(r+A)^{-m}x\to(r+A)^{-m}x$ in F as $s\downarrow 0$. Thus, $x\in\overline{R(A_F)}$.

In a similar way, we can show (cf. Proposition 2.1 in [29]).

PROPOSITION 4.6. *If $A\in(\sigma, m, E, F)$, then the image of $\overline{D(A_E)}$ by the mapping $(r+A)^{-m}$ for any $r>0$ is contained in $\overline{D(A_F)}$.*

In view of these propositions, we may consistently assume, if necessary, that $E=\overline{R(A_E)}$, $F=\overline{R(A_F)}$, or $E=\overline{D(A_E)}$, $F=\overline{D(A_F)}$, or in particular, $E=\overline{R(A_E)}\cap\overline{D(A_E)}$, $F=\overline{R(A_F)}\cap\overline{D(A_F)}$. If E is reflexive, then $E=\overline{R(A_E)}$ is automatically realized in view of Proposition 4.4 and the Abelian ergodic theorem.

Let $G(t)=\exp(-tA)$, $t\geqq 0$, be a bounded continuous semi-group of operators in $E+F$. Denote by $G_E(t)$ and by $G_F(t)$ the maximal restrictions of $G(t)$ in E and in F, respectively. We only consider the case when $G_E(t)$ and $G_F(t)$ are bounded continuous semi-groups in E and in F, respectively.

PROPOSITION 4.7. *Let A_1 and A_2 be non-negative operators in E and in F, respectively, with the relation $(r+A_1)^{-1}x=(r+A_2)^{-1}x$ for $x\in E\cap F$, $r>0$. If $-A_1$ and $-A_2$ generate bounded continuous semi-groups $G_1(t)$ and $G_2(t)$ in E and in F, respectively, then $-A$ defined as in Proposition* 4.1 *generates a bounded continuous semi-group $G(t)$ in $E+F$, and $G_E(t)=G_1(t)$, $G_F(t)=G_2(t)$. If, in particular, $G_1(t)$ and $G_2(t)$ are bounded holomorphic, then so is $G(t)$.*

The proof is easy ([31]).

If we speak of $\exp(-tA)$ in the sequel, we consider only those cases that $-A_E$ and $-A_F$ generate bounded continuous (or holomorphic) semi-groups, and we simply say that $-A$ generates a bounded continuous (or holomorphic) semi-group instead of saying that $-A$, $-A_E$ and $-A_F$ generate bounded continuous (or holomorphic) semi-groups, respectively.

DEFINITION 4.2. $G(t)$ is said to be of class $S(\sigma, E, F)$ for some $\sigma>0$ if the following conditions are satisfied:

(i) $G_E(t)$ and $G_F(t)$ are bounded continuous semi-groups in E and in F, respectively;

(ii) For every $t>0$, $G(t)\in \mathscr{L}(E, F)$ with the norm

$$\|G(t)\|_{E\to F}\leqq Kt^{-\sigma},\ t>0\ ,$$

K being a positive constant independent of t.

We write $G(t)\in S(\sigma, E, F)$ if $G(t)$ is of class $S(\sigma, E, F)$.

The resolvent formula yields immediately the following ([29], [31]).

PROPOSITION 4.8. *If $\exp(-tA)\in S(\sigma, E, F)$, then $A\in(\sigma, m, E, F)$ for $m>\sigma$.*

We have shown previously ([31]):

PROPOSITION 4.9. *Suppose that $-A$, $-A_E$ and $-A_F$ generate bounded holomorphic semi-groups. Then $\exp(-tA)\in S(\sigma, E, F)$ if $A\in(\sigma, m, E, F)$.*

We have noted that this fails if $\exp(-tA)$ is not holomorphic, using the translation semi-groups in $L^p(R)$ or in $L^p(R_+)$ ([31]). In this respect, we make a supplementary remark. Namely,

PROPOSITION 4.10. *There is no bounded continuous group in $S(\sigma, E, F)$.*

PROOF. Assume that $\exp(-tA)$, $\exp(-tA_E)$, $\exp(-tA_F)$ are bounded continuous groups of operators. If $\exp(-tA)\in S(\sigma, E, F)$, then we have, for any $t>0$ and for any $x\in E$,

$$\|\exp(-tA)x\|_F\leqq Kt^{-\sigma}\|x\|_E,\ \sigma>0\ .$$

Now, $\exp(-sA)x\in F$ for any real s and for any $x\in E$. In fact, take any real

$t_0<s$. Then

$$\exp(-t_0A)x=\exp(-t_0A_E)x\in E$$

and

$$\exp(-sA)x=\exp(-(s-t_0)A)\exp(-t_0A)x\in F .$$

Hence,

$$\|\exp(-sA)x\|_F\leqq K(s-t_0)^{-\sigma}\|\exp(-t_0A)\|_E\leqq KM(s-t_0)^{-\sigma}\|x\|_E .$$

Letting $t_0\to-\infty$, we see $\exp(-sA)x=0$ in F. Since E, $F\subset E+F$ continuously, $\exp(-sA)x=0$ in E. This shows that $\exp(-sA_E)x=0$ for real s, and this contradicts the assumption that $\exp(-sA_E)$ is a bounded continuous group.

For later convenience, we state some of easy consequences of Definitions 4.1 and 4.2.

PROPOSITION 4.11. *Let* $A\in(\sigma, m, E, F)$. *Then*

(4.1) $$A\in(\sigma_1, m, X_1, F),\ \sigma_1=\sigma(1-\theta_1),\ 0<\theta_1<1;$$

(4.2) $$A\in(\sigma_2, m, E, X_2),\ \sigma_2=\sigma\theta_2,\ 0<\theta_2<1;$$

(4.3) $$A\in(\sigma_3, m, X_1, X_2),\ \sigma_3=\sigma(\theta_2-\theta_1),\ 0<\theta_1<\theta_2<1 .$$

Here

$$X_1=(E, F)_{\theta_1.p},\ 1\leqq p\leqq\infty,\ X_2=(E, F)_{\theta_2.q},\ 1\leqq q\leqq\infty .$$

PROOF. (4.1) and (4.2) follow from the non-negativeness of A_E and A_F and the interpolation theorem. (4.3) is a consequence of reiteration and (4.1), (4.2).

In an exactly same way, we see

PROPOSITION 4.12. *Let* $\exp(-tA)\in S(\sigma, E, F)$. *Then*

(4.1′) $$\exp(-tA)\in S(\sigma_1, x_1, E),\ \sigma_1=\sigma(1-\theta_1),\ 0<\theta_1<1;$$

(4.2′) $$\exp(-tA)\in S(\sigma_2, E, X_2),\ \sigma_2=\sigma\theta_2,\ 0<\theta_2<1;$$

(4.3′) $$\exp(-tA)\in S(\sigma_3, X_1, X_2),\ \sigma_3=\sigma(\theta_2-\theta_1),\ 0<\theta_1<\theta_2<1 .$$

Here

$$X_1=(E, F)_{\theta_1.p},\ 1\leqq p\leqq\infty,\ X_2=(E, F)_{\theta_2.q},\ 1\leqq q\leqq\infty .$$

As other trivial consequences of Definitions 4.1 and 4.2, we note three propositions.

PROPOSITION 4.13. *Let E, F, G be three Banach spaces continuously imbedded in a Hausdorff linear topological space X. Assume that a non-negative opera-*

tor in $E+F+G$ *be given. Then* $A\in(\sigma, m, E, F)$ *and* $A\in(\sigma', m', F, G)$ *imply* $A\in(\sigma+\sigma', m+m', E, G)$.

PROPOSITION 4.14. *Assume that* $E\cap F$ *is dense in* E *and in* F. *Then* $A\in(\sigma, m, E, F)$ *implies* $A^*\in(\sigma, m, E^*, F^*)$.

PROPOSITION 4.15. *If* $A\in(\sigma, m, E, F)$, $0<\sigma<m$, *then* $1+A\in(\sigma, m, E, F)$.

Finally we give a typical example.

PROPOSITION 4.16. *Consider an operator* A_p *in* $L^p(R)$, $1\leqq p<\infty$, *given by*

$$A_p u=-du/dx \quad \text{for} \quad u\in D(A_p)=W^{p,1}(R)\ .$$

A_∞ *in* $C_u(R)=B^0(R)$ *is given by*

$$A_\infty=-du/dx \quad \text{for} \quad u\in D(A_\infty)=B^1(R)\ .$$

We can construct A *in* $L^p(R)+L^q(R)$ *or* $L^p(R)+B^0(R)$, $1\leqq p<q<\infty$, *as in Proposition* 4.1. *Then*

$$A\in(1/p-1/q, 1, L^p(R), L^q(R)),\ 1\leqq p<q<\infty\ ,$$

$$A\in(1/p, 1, L^p(R),\ B^0(R)),\ 1\leqq p<\infty\ .$$

PROOF. This follows from the resolvent formula and the Hausdorff-Young inequality (see [29]).

Another example will be given in §6 (Corollary 6.1). We also note that many elliptic operators fall in this class as we have shown in [31].

5. Imbedding theorems. In this section we continue to investigate two Banach spaces E, F and a non-negative operator A, considered in the previous section. Our purpose here is to prove some imbedding results associated with operators of class (σ, m, E, F).

We have shown previously ([29], [31]):

THEOREM 5.1. *Let* $A\in(\sigma, \mathrm{m}, E, F)$. *Then*

$$(E, D(A_E^k))_{\theta+\sigma/k,p}\subset(F, D(A_F^k))_{\theta,p}$$

with the continuous imbedding. Here $0<\theta<\theta+\sigma/k<1$, $1\leqq p\leqq\infty$, *and* k *is any positive integer* $>\sigma/(1-\theta)$.

This theorem corresponds to an imbedding theorem of Besov-Nikol'skii type (see our discussions in §4 of [29]).

As to the imbedding relations associated with ranges of operators, we have the following theorem. As we shall see later, this theorem is important in our treatment of the Hardy-Littlewood-Sobolev inequality.

THEOREM 5.2. *Let* $A\in(\sigma, m, E, F)$. *Then*

$$(E, R(A_E^k))_{\theta,p} \subset (E+F, R(A^k))_{\theta+\sigma/k,p}$$

with the continuous imbedding. Here $0<\theta<\theta+\sigma/k<1$, $1 \leqq p<\infty$, *and* k *is any positive integer* $>\sigma/(1-\theta)$.

PROOF. $a \in (E, R(A_E^k))_{\theta,p}$ if and only if $a \in E$ and $r^{-k\theta}(r(r+A)^{-1})^k a \in L_*^p(E)$ (Proposition 2.4). In particular, $a \in E+F$. Now take $k>m$. We see

$$\begin{aligned} &\|r^{-k\theta-\sigma}(r(r+A)^{-1})^{k+m}a\|_F \\ &\quad \leqq r^{-k\theta-\sigma+k+m}\|(r+A)^{-m}\|_{E\to F}\|(r+A)^{-k}a\|_E \\ &\quad \leqq \text{const.}\, r^{-k\theta}\|(r(r+A)^{-1})^k a\|_E . \end{aligned}$$

Since $F \subset E+F$, we have

$$\|r^{-k\theta-\sigma}(r(r+A)^{-1})^{k+m}a\|_{E+F} \leqq \text{const.}\, r^{-k\theta}\|(r(r+A)^{-1})^k a\|_E .$$

It follows that $a \in E+F$ and $r^{-k\theta-\sigma}(r(r+A)^{-1})^{k+m}a \in L_*^p(E+F)$. Hence,

$$a \in (E+F,\ R(A^{k+m}))_{(k\theta+\sigma)/(k+m),p} = (E+F, R(A^k))_{\theta+\sigma/k,p} .$$

The continuity of the imbedding follows from the closed graph theorem.

Q.E.D.

As easy consequences of Theorems 5.1 and 5.2, we have

COROLLARY 5.1. *Let* $A \in (\sigma, m, E, F)$. *Then*

$$D(A_{E+}^{\alpha}) \subset D(A_{F+}^{\beta}),\ \mathrm{Re}\,\alpha > \mathrm{Re}\,\beta+\sigma>\sigma$$

with the continuous imbedding.

COROLLARY 5.2. *Let* $A \in (\sigma, m, E, F)$. *Then*

$$D(A_{E-}^{-\alpha}) \subset D(A_-^{-\beta}),\ 0<\mathrm{Re}\,\beta<\mathrm{Re}\,\alpha+\sigma ,$$

with the continuous imbedding.

In fact, these follow from the facts that

$$(E, D(A_E^m))_{\mathrm{Re}\,\alpha/m,1} \subset D(A_{E+}^{\alpha}) \subset (E, D(A_E^m))_{\mathrm{Re}\,\alpha/m,\infty} ,$$

$$(E, R(A_E^m))_{\mathrm{Re}\,\alpha/m,1} \subset D(A_{E-}^{-\alpha}) \subset (E, R(A_E^m))_{\mathrm{Re}\,\alpha/m,\infty} ,$$

$$(E, D(A_E^m))_{\theta,p} \subset (E, D(A_E^m))_{\theta',q},\ 0<\theta'<\theta<1,\ 1 \leqq p,\ q \leqq \infty ,$$

$$(E, R(A_E^m))_{\theta,p} \subset (E, R(A_E^m))_{\theta',q}$$

and similar relations for A_F and A.

Now we are to refine these results. Two directions are possible: Refinement with respect to the definition domains and that with respect to the ranges. As for the former, we introduced in our previous paper [31] a subclass $\Sigma(\sigma, E, F)$ of (σ, m, E, F). We reproduce its definition and its fundamental properties. As for the refinement with respect to the ranges, we can give a statement similar to the one about the domains.

DEFINITION 5.1. A is said to be of class $\Sigma(\sigma, E, F)$ for some $\sigma>0$ if the following conditions hold:

(i) A_E and A_F are densely defined non-negative operators with bounded inverses;

(ii) $A^{-\sigma}=A_0^{-\sigma}=A_-^{-\sigma}\in\mathscr{L}(E, F)$.

PROPOSITION 5.1. If $A\in\Sigma(\sigma, E, F)$, then $A\in(\sigma, m, E, F)$, $m\geqq\sigma$.

THEOREM 5.3. *Assume that* $\overline{D(A_E)}=E$, $\overline{D(A_F)}=F$, A_E^{-1}, A_F^{-1} *bounded. Then the following three conditions are mutually equivalent*:

$$A\in\Sigma(\sigma, E, F): \tag{5.2}$$

$$D(A_{E+}^{\sigma})\subset F; \tag{5.3}$$

$$D(A_{E+}^{\alpha+\sigma})\subset D(A_{F+}^{\alpha}) \quad \textit{for any} \quad \alpha, \quad \operatorname{Re}\alpha>0\ . \tag{5.4}$$

These two propositions are proved in [31].

We also note

PROPOSITION 5.2. $D(A_{E+}^{\sigma})\subset F$ *if and only if* $D(A_+^{\sigma})\subset F$.

PROOF. The if part is clear since $D(A_{E+}^{\sigma})\subset D(A_+^{\sigma})$. For the proof of the only if part, it is enough to prove $D(A^m)\subset D(A_{F+}^{m-\sigma})$ for some $m>\sigma$. Since $D(A^m)=D(A_E^m)+D(A_F^m)$ (Proposition 4.1), any $x\in D(A^m)$ is decomposed to $x=x_1+x_2$, $x_1\in D(A_E^m)$, $x_2\in D(A_F^m)$. Now $x_1\in D(A_{F+}^{m-\sigma})$ and $x_2\in D(A_F^m)\subset D(A_{F+}^{m-\sigma})$. Hence $x\in D(A_{F+}^{m-\sigma})$. Q.E.D.

The boundedness assumption on A_E^{-1}, A_F^{-1} is actually too strong, though it is enough for many applications. In order to remove this boundedness assumption, we are led to study the inclusion relations between the ranges.

THEOREM 5.4. *Assume that* $E=\overline{R(A_E)}=\overline{D(A_E)}$, $F=\overline{R(A_F)}=\overline{D(A_F)}$. *Then the following three conditions are mutually equivalent*:

$$E\subset R(A_+^{\sigma})=R(A_0^{\sigma}) \quad \textit{for some} \quad \sigma>0; \tag{5.5}$$

$$R(A_E)\subset R(A_0^{1+\sigma})=R(A_+^{1+\sigma})\ . \tag{5.6}$$

$$R(A_{E0}^{\alpha})=R(_{E+}^{\alpha})\subset R(A_0^{\alpha+\sigma})=R(A_+^{\alpha+\sigma}) \quad \textit{for any} \quad \alpha>0\ . \tag{5.7}$$

PROOF. From the assumption, we may omit the suffices $+$, $-$, 0. First we show (5.5) implies (5.7). Let $x\in R(A_E^{\alpha})=D(A_E^{-\alpha})$. Then $y=A^{-\alpha}x\in E\subset R(A^{\sigma})$. Thus there is a $z\in D(A^{\sigma})$ such that $y=A^{-\alpha}x=A^{\sigma}z$. Since A is densely defined and densely ranged, and since $A_-^{-\alpha}A_-^{-\sigma}=A_-^{-\sigma-\alpha}$ in the sense of the product of operators, it follows that $y\in R(A^{\sigma})$ and $z=A^{-\sigma}A^{-\alpha}x=A^{-\alpha-\sigma}x$. In particular, $z\in D(A^{\alpha+\sigma})$

and $A^{\alpha+\sigma}z=x$. That is, $x\in R(A^{\alpha+\sigma})$. Next, (5.7) implies (5.6) as a special case. Finally, we show that (5.6) implies (5.5). Set $B=A^{-1}$. Then B is a densely defined as well as B_E and B_F. We are to prove that $D(B_E)\subset D(B^{1+\sigma})$ implies $E\subset D(B_+^\sigma)$. Since $D(B_E)=D(1+A_E^{-1})$, $D(B^{1+\sigma})=D((1+A^{-1})^{1+\sigma})$, we may assume that B^{-1} and B_E^{-1} are bounded. Let $x\in E$. Then $z=B^{-1}x=B_E^{-1}x\in D(B_E)\subset D(B^{1+\sigma})$. Thus there exists a $y\in E+F$ such that $z=B^{-1}x=B^{-1-\sigma}y$. In particular, $z\in D(B)\cap D(B^{1+\sigma})$. Thus $Bz\in D(B^\sigma)$. This proves that $x\in D(B^\sigma)$ (see Proposition 2.9).

Q.E.D.

In view of studying whether $A^{-\sigma}\in\mathscr{L}(E,F)$, we prepare several propositions.

THEOREM 5.5. *Assume that* $E=\overline{R(A_E)}=\overline{D(A_E)}$, $F=\overline{R(A_F)}=\overline{D(A_F)}$. *If* $E\subset R(A^\sigma)$ *and* $D(A^\sigma)\subset F$, *then* $A^{-\sigma}\in L(E,F)$.

PROOF. From the assumption, $\overline{R(A)}=\overline{D(A)}=E+F$. Thus, $A_-^{-\sigma}=A_0^{-\sigma}=A^{-\sigma}$ is well-defined and $A^{-\sigma}\in\mathscr{L}(R(A^\sigma),\ D(A^\sigma))$. Q.E.D.

The converse statement of Theorem 5.5 also holds good.

THEOREM 5.6. *Assume that* $E=\overline{R(A_E)}=\overline{D(A_E)}$, $F=\overline{R(A_F)}=\overline{D(A_F)}$. *If* $A^{-\sigma}\in\mathscr{L}(E,F)$ *for* $\sigma>0$, *then* $E\subset R(A^\sigma)$ *and* $D(A^\sigma)\subset F$. *In particular, for any* $r>0$, $r+A\in\Sigma(\sigma,E,F)$, *and* $A\in(\sigma,m,E,F)$, $\sigma\leqq m$.

PROOF. $E\subset R(A^\sigma)$ is clear since $A^{-\sigma}x$ is well-defined and belongs to F for any $x\in E$. Next, we note that $(r+A)^{-\sigma}\subset\mathscr{L}(E+F,E+F)\cap\mathscr{L}(F,F)$ and that $(r+A)^{-\sigma}$ is an extension of the operator $A^\sigma(r+A)^{-\sigma}A^{-\sigma}$. On the other hand, it is clear that $(r+A)^{-\sigma}$ is closed as an operator from E into F, since E and F are both continuously imbedded in $E+F$, and since $(r+A)^{-\sigma}\in\mathscr{L}(E+F,E+F)$. Let $x\in E$. Then $A^{-\sigma}x$ is well-defined as an element in $F\subset E+F$, and we have

$$(r+A)^{-\sigma}x=A^\sigma(r+A)^{-\sigma}A^{-\sigma}x=A_F^\sigma(r+A)^{-\sigma}A^{-\sigma}x\in F\ .$$

Thus by the closed graph theorem, $(r+A)^{-\sigma}\in\mathscr{L}(E,F)$. Hence, by Theorem 5.3, we have $D(A_E^\sigma)=D((r+A_E)^\sigma)\subset F$. It follows from Proposition 5.2 that $D(A^\sigma)\subset F$. Similarly we see that $(r+A)^{-m}$, $m\geqq\sigma$, maps E continuously into F, and for any $x\in E$,

$$\|(r+A)^{-m}x\|_F=\|A^\sigma(r+A)^{-m}A^{-\sigma}x\|_F\leqq\text{const.}\ r^{\sigma-m}\|A^{-\sigma}\|_{E\to F}\|x\|_E\ .$$

This shows that $A\in(\sigma,m,E,F)$, $m\geqq\sigma$. The theorem is proved. Q.E.D.

REMARK 5.1. Assume that $E=\overline{R(A_E)}=\overline{D(A_E)}$, $F=\overline{R(A_F)}=\overline{D(A_F)}$. If $A\in(\sigma,m,E,F)$, then $D(A^{\sigma+\varepsilon})\subset F$ and $E\subset R(A^{\sigma-\varepsilon})$, $\sigma>\varepsilon>0$.

PROOF. Corollary 5.1 shows $D(A_E^{\alpha+\sigma+\varepsilon})\subset D(A_F^\alpha)$ for any $\alpha>0$, $\varepsilon>0$. Thus, by Theorem 5.3, we have $D(A_E^{\sigma+\varepsilon})=D((1+A_E)^{\sigma+\varepsilon})\subset F$. By Proposition 5.2, it follows that $D(A^{\sigma+\varepsilon})\subset F$.

On the other hand, Corollary 5.2 and Theorem 5.4 show $E\subset R(A^{\sigma-\varepsilon})$.

Q.E.D.

Thus we see that the case when we may take $\varepsilon=0$ is of a particular importance.

We shall discuss such a case in the next section.

REMARK 5.2. Assume that $E=\overline{R(A_E)}$ and $F=\overline{R(A_F)}$ so that $\overline{R(A)}=E+F$, and A_E^{-1}, A_F^{-1}, A^{-1} are non-negative.

If the completion F_m of $R(A_F^m)$ under the norm $|||x|||=\|A_F^{-m}x\|_F$ is continuously imbedded in X, then we can argue as in the proof of Theorem 5.1, using that $R(A_E^k)=D(A_E^{-k})$, etc., and we see that $(E, R(A_E^k))_{\theta,p}$ is imbedded in $(F, F_k)_{\theta',p}$. If A_F is of bounded inverse, then F_m coincides with the space of type $W^{-m,E}$ considered by Grisvard [6]. In particular, if F is reflexive, then $F_m=D(A_F^{*m})^*$.

THEOREM 5.2′. *Let* $A\in(\sigma, 1, E, F)$, $0<\sigma<1$. *Assume that* $E=\overline{R(A_E)}$, $F=\overline{R(A_F)}$ *and* $F_m\subset X$ *continuously. Then we have*

$$(E, R(A_E^k))_{\theta,p}\subset(F, F_k)_{\theta+\sigma/k,p}$$

with the continuous imbedding. Here $0<\theta<\theta+\sigma/k<1$, $1\leqq p\leqq\infty$, *and* k *is any positive integer* $>\sigma/(1-\theta)$.

PROOF. Noting that $(E, R(A_E^k))_{\theta,p}=(E, D(A_E^{-k}))_{\theta,p}$, we see that $a\in(E, R(A_E^k))_{\theta,p}$ implies that $a\in E$,

(5.8) $$t^{\tau}u_m(t)\in L_*^p(E),\ t^{\tau-m}A_E^{-m}u_m(t)\in L_*^p(E)\ ,$$

(5.8′) $$a=\int_0^\infty u_m(t)dt/t\ .$$

Here $\tau=k\theta$, m any integer $>k\theta+\sigma$, and

(5.8″) $$\begin{aligned}u_m(t)&=c_mt^m(A_E^{-1})^m(t+A_E^{-1})^{-2m}a\\&=c_mt^mA_E^m(tA_E+1)^{-2m}a,\quad c_m=\Gamma(2m)/\Gamma(m)^2\ .\end{aligned}$$

In fact, (5.8′) follows from the integration by parts. (5.8) holds since

$$(E, R(A_E^k))_{\tau/k,p}=(E, R(A_E^m))_{\tau/m,p}$$

(Proposition 2.4). Letting $s=1/t$, we have, by (5.8),

(5.9) $$v_m(s)=u_m(1/s)=c_ms^mA_E^m(s+A_E)^{-2m}a\ ,$$

(5.10) $$s^{-\tau}v_m(s)\in L_*^p(E),\ s^{-\tau+m}A_E^{-m}v_m(s)\in L_*^p(E)\ ,$$

(5.11) $$a=\int_0^\infty v_m(s)ds/s\ .$$

Now, since $A\in(\sigma, 1, E, F)$,

$$\|s^{m+1}A^{m+1}(s+A)^{-2m-2}a\|_F \leq \|sA(s+A)^{-2}\|_{E\to F}\|s^m A^m (s+A)^{-2m}a\|_E . \leq MLs^{\sigma+m}\|A^m(s+A)^{-2m}a\|_E .$$

Similarly,

$$\|s^{2m+1}A(s+A)^{-2m-2}a\|_F \leq MLs^{\sigma+2m}\|(s+A)^{-2m}a\|_E .$$

In other words, letting $m=k$, we see

$$s^{-\tau-\sigma}\|v_{k+1}(s)\|_F \leq \text{const. } s^{-\tau}\|v_k(s)\|_E ,$$

$$s^{-\tau-\sigma+k}\|\|v_{k+1}(s)\|\| \leq \text{ const. } s^{-\tau+k}\|A_E^{-k}v_k(s)\|_E .$$

By (5.10), it follows that

$$s^{-\tau-\sigma}v_{k+1}(s) \in L^p_*(F),\ s^{-\tau-\sigma+k}v_{k+1}(s) \in L^p_*(F_k) .$$

Hence,

$$b=\int_0^\infty v_{k+1}(s)ds/s \in (F, F_k)_{(\tau+\sigma)/k,p}=(F, F_k)_{\theta+\sigma/k,p} .$$

Since $a=b$ in X, $a\in(F, F_k)_{\theta+\sigma/k,p}$. The continuity of the imbedding follows from the closed graph theorem.

6. An application. An operator theoretical proof of the Hardy-Littlewood-Sobolev inequality. As an application of the results in the previous sections, we are going to prove the Hardy-Littlewood-Sobolev inequality. Namely, we prove

THEOREM 6.1. *Let (M, dm) be a sigma finite positive measure space. Let $\{T_t\}$, $t\geq 0$, be a family of linear operators mapping functions on M $(\subset L^1(M, dm)+L^\infty(M, dm))$ into functions on M. Assume that $\{T_t\}$ forms a semi-group in the sense that $T_tT_s=T_{t+s}$ for t, $s>0$, $T_0=$identity and that the restriction of $\{T_t\}$ in $L^2(M, dm)$ is a bounded continuous semi-group. Suppose furthermore the following five conditions hold:*[6]

(i) $\|T_tf\|_p\leq\|f\|_p$ *if* $f\in L^p(M, dm)$, $1\leq p\leq\infty$.

(ii) *Each* T_t, $t>0$, *is self-adjoint in* $L^2(M, dm)$.

(iii) $T_tf\geq 0$ *if* $f\geq 0$.

(iv) $t_t1=1$.

(v) *For each* $t>0$, $T_tf\in L^\infty(M, dm)$ *if* $f\in L^1(M, dm)$, *and* $\|T_tf\|_\infty\leq Kt^{-\sigma}\|f\|_1$ *for some* $\sigma>0$.

Then we have, for $1<p<q<\infty$,

6) $\|f\|_p=\|f\|_{L^p(M,dm)}$.

(6.1) $$A^{-\tau}\in\mathscr{L}(L^p(M, dm),\ L^q(M, dm)),\ \tau=\sigma(1/p-1/q)\ .$$

Here

$$A^{-\tau}f=\Gamma(\tau)^{-1}\int_0^\infty t^{\tau-1}T_t f dt\ .$$

COROLLARY 6.1 (The Hardy-Littlewood-Sobolev inequality). *Let* $f\in L^p(R^n)$, $g\in L^q(R^n)$, $1<p,\ q<\infty,\ 1/p+1/q>1$. *Then*

(6.2) $$\left|\iint_{R^n\times R^n}|x-y|^{-n\lambda}f(y)g(x)\,dx\,dy\right|<\infty,\ \lambda=2-1/p-1/q\ .$$

PROOF OF COROLLARY 6.1. Let (M, dm) be $(\boldsymbol{R}^n, dx)$. Define T_t by

$$(T_t f)(x)=\begin{cases} f(x) \text{ if } t=0\ ,\\ (4\pi t)^{-n/2}\int_{R^n} f(y)\exp(-|x-y|^2/4t)dy \text{ if } t>0\ .\end{cases}$$

Then it is clear that $\{T_t\}$ satisfies the conditions (i)-(iv) and (v) with $\sigma=n/2$. Thus by Theorem 6.1, we have

$$A^{-\tau}\in\mathscr{L}(L^p(\boldsymbol{R}^n),\ L^{q'}(\boldsymbol{R}^n)),\ q'=q/(q-1)\ .$$

Note that $q'>p$ since $1/p+1/q>1$. Here $\tau=n(1/p-1/q')/2$, and

$$\begin{aligned} A^{-\tau}f&=\text{const.}\int_{R^n}\left\{\int_0^\infty t^{\tau-1-n/2}\exp(-|x-y|^2/4t)dt\right\}f(y)dy\\ &=C\int_{R^n}f(y)|x-y|^{-n(1-1/p+1/q')}dy\\ &=C\int_{R^n}f(y)|x-y|^{-n(2-1/p-1/q)}dy\end{aligned}$$

with $C=2^{-2(\tau-n/2)}\Gamma(n/2-\tau)/((4\pi)^{n/2}\Gamma(\tau))$. This implies (6.2).

In the rest of the section, we prove Theorem 6.1. This proceeds in the line of our previous paper [32]. Since it appears that we only consider densely defined and densely ranged non-negative operators A below, we omit the suffices $+$, $-$, or 0 for fractional powers and we simply write A^α, etc.

First we summarize Stein's results [27].

PROPOSITION 6.1 (*Stein* [27]). *The conditions* (i) *and* (ii) *imply that* $\{T_t\}$ *in each* $L^p(M, dm)$, $1<p<\infty$, *is a bounded holomorphic semi-group, that is, the mapping* $t\longrightarrow T_t f$, $f\in L^p(M, dm)$, *is analytically continued to the sector*:

$$S_p=\{z\in\boldsymbol{C};\ |\arg z|<(1-|2/p-1|)\pi/2\}\ .$$

For the proof see Stein [27], also the footnote in [32].

PROPOSITION 6.2. *Let* $-A_p$ *be the infinitesimal generator of* $\{T_t\}$ *in* $L^p(M, dm)$, $1<p<\infty$. *Then* $N(A_p)=0$. *In particular,* $\overline{R(A_p)}=L^p(M, dm)$, $1<p<\infty$.

PROOF. This follows from the condition (v) combined with the interpolation and Proposition 4.4. The second assertion is a result of the Abelian ergodic theorem since $L^p(M, dm)$, $1<p<\infty$, is reflexive. Q.E.D.

Now define the Littlewood-Paley function (Stein [27]):

$$g_1(f)(\cdot)=\left(\int_0^\infty t|\partial T_t f(\cdot)/\partial t|^2 dt\right)^{1/2}. \tag{6.3}$$

PROPOSITION 6.3 (Stein [27]). *Let* $\{T_t\}$ *be the semi-group in Theorem* 6.1. *If* $f\in L^p(M, dm)$, *then* $g_1(f)\in L^p(M, dm)$, $1<p<\infty$, *and*

$$M_p^{-1}\|g_1(f)\|_p\leqq\|f\|_p\leqq M_p\|g_1(f)\|_p,\ 1<p<\infty .$$

Here M_p *is a constant depending only on* p.

PROOF. The result is due to Stein. In fact, he proved the first inequality and the second one in the form

$$\|f\|_p\leqq M_p(\|g_1(f)\|_p+\|E_0(f)\|_p) .$$

Here $E_0(f)=\operatorname{s-lim}_{t\to\infty} T_t f$ in $L^p(M, dm)$, $f\in L^p(M, dm)$. In view of Proposition 6.2, $E_0(f)=0$. Q.E.D.

PROPOSITION 6.4 (Stein [27]). *Write* $T_t=\int_0^\infty e^{-rt}dE(r)$ *in* $L^2(M, dm)$, *using the spectral decomposition. Define the operator*:

$$m(A)f=\int_0^\infty m(r)dE(r)f,\quad f\in L^2(M, dm) ,$$

where $m(r)$ *is a bounded measurable function on* $R_+=(0, \infty)$. *If, for a bounded measurable function* $M(t)$ *on* R_+, $m(r)=r\int_0^\infty e^{-rt}M(t)dt$, $r>0$, *then* $m(A)$ *is a bounded linear operator on all* $L^p(M, dm)$, $1<p<\infty$, *and*

$$\|m(A)f\|_p\leqq M_pL\|f\|_p,\ f\in L^p(M, dm) .$$

Here M_p *is constant depending only on* p, *and* $L=\sup|M(t)|$. *In particular,* A_p^{ir} *is a bounded linear operator on all* $L^p(M, dm)$, $1<p<\infty$, *for each real* r. *Furthermore, for* $f\in L^p(M, dm)$,

$$\|A_p^{ir}f\|_p\leqq M_p \exp(\pi|r|)\|f\|_p .$$

For the proof, see Stein [27]. Note that $A_p^{ir}=A_{p0}^{ir}$. Using Propositions 6.1, 6.3 and 6.4, we prove

PROPOSITION 6.5. *Let* $1<p\leqq2$. *Then, for any* $\alpha>0$,

$$R(A_p^\alpha)\subset(L^p(M, dm),\ R(A_p^n))_{\alpha/n,2}$$

with the continuous imbedding.

PROOF. It is enough to prove the proposition for $\alpha=1$. (Proposition 2.6). Let

$$|f|=\left(\int_0^\infty t\|T_tf\|_p^2dt\right)^{1/2} \quad\text{for}\quad f\in R(A_p)\ .$$

Then

$$|f|=\left(\int_0^\infty t\|AT_tA^{-1}f\|_p^2dt\right)^{1/2}=\left(\int_0^\infty \|t\partial T_tA^{-1}f/\partial t\|_p^2dt\right)^{1/2}\ .$$

By Minkowski's inequality,

$$\begin{aligned}|f|&\leqq\left\{\int_M\left(\int_0^\infty t^2|\partial T_th(m)/\partial t|^2dt/t\right)p/2\,dm\right\}^{1/p}\\&=\|g_1(A^{-1}f)\|_p\leqq\text{const.}\ \|A^{-1}f\|_p\ ,\qquad h=A_f^{-1}\ .\end{aligned}$$

Hence,

$$|f|+\|f\|_p\leqq\text{const.}\ (\|f\|_p+\|A^{-1}f\|_p)=\text{const.}\ \|f\|_{R(A_p)}\ . \qquad \text{Q.E.D.}$$

COROLLARY 6.2. *Let* $1<p\leqq 2$. *Then for any* $\alpha>0$,

$$R(A_{p\wedge 2}^\alpha)\subset(L^p(M, dm)\cap L^2(M, dm),\ R(A_{p\wedge 2}^n))_{\alpha/n,2}$$

with the continuous imbedding. Here $A_{p\wedge 2}$ *is the maximal restriction of* A *in* $L^p(M, dm)\cap L^2(M, dm)$.

PROOF. It is clear that $-A_{p\wedge 2}$ generates $\{T_t\}$ in $L^p(M, dm)\cap L^2(M, dm)$, and $N(A_{p\wedge 2})=0$. We give an equivalent norm

$$\|f\|_{p\wedge 2}=\|f\|_p+\|f\|_2 \quad\text{for}\quad f\in L^p(M, dm)\cap L_2(M, dm).$$

It is enough to prove the corollary for $\alpha=1$. Now,

$$\begin{aligned}|f|&=\left(\int_0^\infty \|tT_tf\|_{p\wedge 2}^2\,dt/t\right)^{1/2}\\&\leqq\left(\int_0^\infty \|tT_tf\|_p^2\,dt/t\right)^{1/2}+\left(\int_0^\infty \|tT_tf\|_2^2\,dt/t\right)^{1/2}\\&\leqq(\|g_1(A^{-1}f)\|_p+\|g_1(A^{-1}f)\|_2)\\&\leqq\text{const.}\ (\|A^{-1}f\|_p+\|A^{-1}f\|_2)\\&=\text{const.}\ \|A^{-1}f\|_{p\wedge 2}\ .\end{aligned}$$

Hence,

$$\begin{aligned}\|f\|_{p\wedge 2}+|f|&\leqq\text{const.}\ (\|f\|_{p\wedge 2}+\|A^{-1}f\|_{p\wedge 2})\\&=\text{const.}\ \|f\|_{R(A_{p\wedge 2})}\end{aligned} \qquad \text{Q.E.D.}$$

Applying the duality theorem (Theorem 3.1) for A and A^{-1}, and noting that L^p, $L^p\cap L^2$, $1<p\leqq 2$, are reflexive, we have

PROPOSITION 6.6. *Let $2\leqq p<\infty$. Then for any $\alpha>0$*

$$(L^p(M, dm),\ R(A_p^n))_{\alpha/n,2}\subset R(A_p^\alpha)$$

with the continuous imbedding.

COROLLARY 6.3. *Let $2\leqq p<\infty$. Then for any $\alpha>0$*

$$(L^2(M, dm)+L^p(M, dm),\ R(A_{p\vee 2}^n))_{\alpha/n,2}\subset R(A_{p\vee 2}^\alpha)$$

with the continuous imbedding. Here $A_{p\vee 2}$ is the operator in $L^2(M, dm)+L^p(M, dm)$ constructed as in Proposition 4.1.

PROOF OF THEOREM 6.1. We prove the theorem for $2\leqq p<q<\infty$. Then by the symmetry of the operator, the theorem holds also for $1<p<q\leqq 2$. Composing these two cases, the theorem is seen to hold for general $1<p<q<\infty$.

We first show, for $2<q<\infty$,

$$D(A_{2\vee q}^{\alpha+\tau'})\subset D(A_q^\alpha)\ \text{ for any }\ \alpha>0\,, \tag{6.4}$$

and

$$R(A_2^\alpha)\subset R(A_{2\vee q}^{\alpha+\tau'})\ \text{ for any }\ \alpha>0\,. \tag{6.5}$$

Here $\tau'=\sigma(1/2-1/q)$. In fact, from the condition (v), $T_t\in S(\tau', L^2, L^q)$ as a result of the interpolation. Hence, from Theorem 5.2, we have

$$(L^2(M, dm),\ R(A_2^n))_{\alpha/n,2}\subset(L^2(M, dm)+L^q(M, dm),\ R(A_{2\vee q}^n))_{(\alpha+\tau')/n,2}\,.$$

Using Propositions 6.5, 6.6 and Corollary 6.3, we see (6.5).

On the other hand, we have shown in [32]

$$D(A_2^{\alpha+\tau'})\subset D(A_q^\alpha)\,. \tag{6.4'}$$

Now (6.4) follows from (6.4′) by Proposition 5.2.

Let $X_\theta=[L^q(M, dm),\ L^2(M, dm)+L^q(M, dm)]_\theta$, and denote by A_θ the maximal restriction of A in X_θ. Then it is clear that A_θ is densely defined, densely ranged and non-negative with bounded pure imaginary powers. In fact, all these follow from the corresponding properties of A in $L^2(M, dm)+L^q(M, dm)$ and in $L^q(M, dm)$ (Corollary 4.1). Hence, interpolating (6.4) and $D(A_q^\alpha)=D(A_q^\alpha)$, we have, by Proposition 2.3,

$$D(A_\theta^{\alpha+\tau'\theta})=[D(A_q^\alpha),\ D(A_{2\vee q}^{\alpha+\tau'})]_\theta\subset D(A_q^\alpha)\,. \tag{6.6}$$

Similarly, interpolating (6.5) and $R(A_q^\alpha)=R(A_q^\alpha)$, we have

(6.7) $$R(A_p^\alpha)\subset R(A^{\alpha+\tau'\theta})=[R(A_q^\alpha),\ R(A_{2\vee q}^{\alpha+\tau'})]_\theta\ .$$

Here $1/p=\theta/2+(1-\theta)/q$, $0<\theta<1$.

An argument similar to the proof of Theorem 5.4 yields from (6.6) and (6.7) that

$$D(A^{\tau'\theta})\subset L^q(M, dm) \text{ and } L^p(M, dm)\subset R(A_\theta^{\tau'\theta})\ .$$

Since $X_\theta\supset L^p(M, dm)$, $X_\theta\supset L^q(M, dm)$ and $\tau'\theta=\sigma(1/p-1/q)=\tau$, it follows from Theorem 5.5 that $A^{-\tau}\in\mathscr{L}(L^p(M, dm),\ L^q(M, dm))$. The theorem is proved.

Q.E.D.

REMARK 6.1. There are many proofs of the Hardy-Littlewood-Sobolev inequality, using the rearrangement, or decomposition of given functions ([8], [9], [26]), or a group theoretical proof ([23]), or based on an elementary inequality [21]. The proof given here starts from a more general standpoint, but it relies on the Littlewood-Paley function. Stein [27] used the theory of martingales for the proof of Proposition 6.3. That is, the proof given here also depends on decompositions of functions.

References

[1] Balakrishnan, A. V., Fractional power of closed operators and the semi-groups generated by them, Pacific. J. Math. **10** (1960), 419-437.

[2] Besov, O. V., On embedding and extension theorems for some function spaces, Trudy Mat. Inst. Steklov, **60** (1960), 42-81 (in Russian).

[3] Calderón, A. P., Intermediate spaces and interpolation, the complex method, Studia Math. **24** (1964), 113-190.

[4] Fujiwara, D., On the asymptotic behaviour of the Green operator for elliptic boundary problems and the pure imaginary powers of some second order operators, J. Math. Soc. Japan, **21** (1969), 418-552.

[5] Gagliardo, E., Interpolazione di spazi di Banach e applicazioni, Ricerche Mat. **10** (1960), 58-61.

[6] Grisvard, P., Commutativité de deux foncteurs d'interpolation et applications, J. Math. Pures et Appl. **45** (1966), 143-290.

[7] Grisvard, P., Caractérisation de quelques espaces d'interpolation, Ark. Rat. Mech. Anal. **25** (1967), 40-63.

[8] Hardy-Littlewood-Polya, Inequalities, Cambridge University Press, 1964.

[9] Hörmander, L., Translation invariant operators in L^p, Acta. Math. **104** (1960), 93-140.

[10] Kato, T., Fractional powers of dissipative operators, J. Math. Soc. Japan **13** (1961), 246-274.

[11] Kato, T., Abstract evolution equations of parabolic type in Banach and Hilbert spaces, Nagoya Math. J. **19** (1961), 93-125.

[12] Köthe, G., Topological Vector Spaces, I, Springer, Berlin-Heidelberg-New York, 1969.

[13] Komatsu, H., Fractional powers of operators, Pacific J. Math. **19** (1966), 285-346.

[14] Komatsu, H., Fractional powers of operators, II, Interpolation spaces, Pacific J. Math. **21** (1967), 89-111.

[15] Komatsu, H., Fractional powers of operators, III, Negative powers, J. Math. Soc. Japan **21** (1969), 205-220.

[16] Komatsu, H., Fractional powers operators, IV, Potential operators, J. Math. Soc. Japan **21** (1969), 221-228.

[17] Komatsu, H., Fractional powers of operators, V, Dual operators, J. Fac. Sci. Univ. Tokyo Sect. IA **17** (1970), 373-396.

[18] Lions, J.-L., Remarques sur les espaces d'interpolation et les problèmes aux limites, Coll. C. N. R. S., Les Equations aux Derivées Partielles, 1962, pp. 75-86.

[19] Lions-Magenes, Problèmes aux Limites Non-Homogènes et Applications, I and II, Dunod, Paris, 1968.

[20] Lions-Peetre, Sur une classe d'espaces d'interpolation, Publ. Math. I. H. E. S. **19** (1964), 5-68.

[21] Muramatu, T., On imbedding theorems for Sobolev spaces and some of their generalization, Publ. RIMS. Kyoto Univ. Ser. A **3** (1968), 393-416.

[22] Nikolskii, S. M., Approximations of Functions in Several Variables and Embedding Theorems, Izd. Nauka, Moscow, 1969 (in Russian).

[23] O'Neil, R., Convolution operators and $L(p, q)$ spaces, Duke Math. J. **30** (1963), 120-142.

[24] Peetre, J., Espaces d'interpolation, généralisations, applications, Rend. Sem. Mat. Fis. Milano **34** (1964), 133-161.

[25] Peetre, J., Espaces d'interpolations et théorème de Soboleff, Ann. Inst. Fourier **16** (1966), 279-317.

[26] Sobolev, S. L., On a theorem of functional analysis, Mat. Sbornik **45** (1938), 471-496 (in Russian).

[27] Stein, E. M., Topics in Harmonic Analysis Related to the Littlewood-Paley Theory, Ann. of Math. Studies 63, Princeton University Press and The University of Tokyo Press, Princeton, 1970.

[28] Taibleson, M. H., On the theory of Lipschitz spaces of distributions on Euclidean n-space, I, Principal properties, J. Math. Mech. **13** (1964), 407-479.

[29] Yoshikawa, A., Remarks on the theory of interpolation spaces, J. Fac. Sci. Univ. Tokyo, Sect. I **15** (1968), 209-251.

[30] Yoshikawa, A., Remarques sur la théorie d'espaces d'interpolation, espaces de moyenne de plusieurs espaces de Banach, J. Fac. Sci. Univ. Tokyo, Sect. I **16** (1970), 407-468.

[31] Yoshikawa, A., An abstract formulation of Sobolev type imbedding theorems and its applications to elliptic boundary value problems, J. Fac. Sci. Univ. Tokyo Sect. IA **17** (1970), 543-558.

[32] Yoshikawa, A., An operator theoretical remark on the Hardy-Littlewood-Sobolev inequality, J. Fac. Sci. Univ. Tokyo Sect. IA **17** (1970), 559-580.

[33] Yosida, K., Functional Analysis, Springer, Berlin, 1965.

(Received September 28, 1970)

Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo
060 Japan