The existence of the trajectory field

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Introduction

In [1], we introduced the concept of the trajectory field, and showed that any topological manifold with dimension more than 6, has a Morse function with associated trajectory field. The trajectory field corresponds to the orthogonal vector field in the differentiable case. Therefore, this concept is useful to determine the structure of a topological manifold. From this point of view we have a problem whether an associated trajectory field exists for any given Morse function. In this paper, we give an answer to this problem, by making use of Edward-Kirby [5]. Our result is as follows.

THEOREM. *Let M^m be a compact topological manifold and let $f: M \to R$ be a topological Morse function. Then there exists a trajectory field associated to f.*

As a direct consequence of this theorem, we have the following corollaries.

COROLLARY 1. *If M^m has a Morse function with two critical points, then M is homeomorphic to S^m.*

COROLLARY 2. *If Morse function $f: M^m \to [a, b]$ has no critical points and $\partial M = f^{-1}(a) \cup f^{-1}(b)$, then $f^{-1}(a) \approx f^{-1}(b)$ and $M \approx f^{-1}(a) \times [a, b]$, where "$\approx$" stands for "is homeomorphic to".*

COROLLARY 3. *There exists a manifold which has no Morse functions.*

Corollary 1 has been proved in another way by Kuiper in [2]. Corollary 2 is an answer to the open problem presented by Cantwell [3]

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Proofs

Corollaries 1 and 2 are trivial by the theorem, therefore we omit the proof.

PROOF OF COROLLARY 3. Assume that M is a manifold which has no handlebody decomposition, given by Siebenmann [4]. If M has a Morse function f, then there is a trajectory field associated to f. By [1], a pair of Morse function and trajectory field gives a handlebody decomposition. Hence M has a handlebody decomposition. This contradicts the assumption. q.e.d.

To prove the theorem, we need the following lemma due to Edward-Kirby [5].

LEMMA. *Let M^n be a topological manifold and let C, C_1, C_2 be compact subsets of M, such that $C\subset \mathring{C}_1\subset C_1\subset \mathring{C}_2$. If $h_t: C_2\rightarrow M$, $-a\leqq t\leqq a$ is an isotopy such that h_0 is an inclusion, then there exist a positive number ε and an isotopy $\tilde{h}_t: C_2\rightarrow M$, $-\varepsilon\leqq t\leqq \varepsilon$ such that $\tilde{h}_t|_C=$*inclusion, *and $\tilde{h}_t|_{C_2-C_1}=h_t|_{C_2-C_1}$, for any t.*

For the proof, see [5] Theorem 5.1.

PROOF OF THE THEOREM. Let $c\in R$ be an arbitrary value. We need only to show that $f^{-1}([c-\varepsilon_c, c+\varepsilon_c])$ has a trajectory field associated to $f|_{f^{-1}([c-\varepsilon_c, c+\varepsilon_c])}$ for some $\varepsilon_c>0$. Then the theorem is proved as follows.

Since M is compact, $f(M)$ is compact subset of R. We can choose some finite covering of $f(M)$ from $\{[c-\varepsilon_c, c+\varepsilon_c]; c\in f(M)\}$. By chopping these intervals down if necessary, we have a family of intervals $\{[t_i, t'_i], i=1, \cdots, k\}$, such that $\cup_{i=1}^k[t_i, t'_i]=f(M)$ and $t_i=t'_{i-1}$ for $i=2, \cdots, k$. Then we have a trajectory field of M by pasting up the trajectory fields of $f^{-1}([t_i, t'_i])$.

Case 1. The case that c is a regular value.

We can choose a number $\varepsilon_1>0$ sufficiently small so that $[c-\varepsilon_1, c+\varepsilon_1]$ has no critical values. For any $x\in f^{-1}(c)$, there are an open set V_x in R^m and coordinate neighborhood (U_x, h_x), $h_x: V_x\rightarrow U_x\subset M$, around x, such that $f\circ h_x(x_1, x_2, \cdots, x_m)=x_m$. There is some into homeomorphism $g_x: 2B^{m-1}\rightarrow V_x\cap R_c^{m-1}$, such that $g_x(0)=h_x^{-1}(x)$, where $R_c^{m-1}=\{x; x=(x_1, \cdots, x_{m-1}, c)\}$. We denote $g_x(rB^{m-1})=B_x(r)$.

Since $f^{-1}(c)$ is compact, we can choose a positive number ε_2 with $\varepsilon_1\geqq\varepsilon_2>0$ and finite points $x_1, x_2, \cdots, x_n$, so that $V_{x_i}\supset B_{x_i}(2)\times[c-\varepsilon_2, c+\varepsilon_2]$ and $\cup_{i=1}^n h_{x_i}(B_{x_i}(1)\times[c-\varepsilon_2, c+\varepsilon_2])=f^{-1}([c-\varepsilon_2, c+\varepsilon_2])$. Each $h_{x_i}(B_{x_i}(2)\times[c-\varepsilon_2, c+\varepsilon_2])$ has a product structure but this structure may not be compatible each other. Now we adjust this structure by altering the homeomorphism h_{x_j} to h'_{x_j} inductively.

Fix a sequence of positive numbers $\{\gamma_i\}_{i=0}^{3n}$ which satisfies $2=\gamma_0>\gamma_1>\cdots>\gamma_{3n}>1$. Assume that $A_i=\cup_{j=1}^i h'_{x_j}(B_{x_j}(\gamma_{3i})\times[c-\varepsilon_3, c+\varepsilon_3])$ has a compatible structure. That is, there is a level preserving homeomorphism $h_i: C_i(\gamma_{3i})\times[c-\varepsilon_3, c+\varepsilon_3]\rightarrow A_i$ (where, $C_i(\gamma_k)=\cup_{j=1}^i h'_{x_j}(B_{x_j}(\gamma_k))$), such that the following diagram commutative.

$$\begin{array}{ccc} & A_i\subset M & \\ {\scriptstyle h'_{x_j}}\nearrow & & \nwarrow{\scriptstyle h_i} \\ B_{x_j}\times[c-\varepsilon_3, c+\varepsilon_3] & \xrightarrow[h_{x_j}\times \mathrm{id.}]{} & C_i(\gamma_{3_i})\times[c-\varepsilon_3, c+\varepsilon_3] \end{array} \qquad j=1, 2, \cdots, i$$

Consider the subset $D=h_{x_{i+1}}(B_{x_{i+1}}(\gamma_{3i+1}))\cap C_i(\gamma_{3i+1})$ of $C_i(\gamma_{3i})$. Then it is easy from the fact that D is compact and D is included in $\overset{\circ}{C}_i(\gamma_{3i})$, that there is some positive number ε_4 with $0<\varepsilon_4\leqq\varepsilon_3$, such that $h_{x_{i+1}}(h^{-1}_{x_{i+1}}(D)\times[c-\varepsilon_4, c+\varepsilon_4])\subset h_i(C_i(\gamma_{3i})\times[c-\varepsilon_4, c+\varepsilon_4])$.

Define an isotopy $k_t: D\to C_i(\gamma_{3i})$, $-\varepsilon_4\leqq t\leqq\varepsilon_4$, by $k_t(x)=\text{pr.}\circ h_i^{-1}\circ h_{x_{i+1}}(h^{-1}_{x_{i+1}}(x), c+t)$, where $\text{pr.}: C_i(\gamma_{3i})\times[c-\varepsilon_4, c+\varepsilon_4]\to C_i(\gamma_{3i})$ is a projection to the first factor. Then, by the lemma, there are a positive number ε_5 with $0<\varepsilon_5\leqq\varepsilon_4$, and an isotopy $\tilde{k}_t: D\to C_i(\gamma_{3i})$, $-\varepsilon_5\leqq t\leqq\varepsilon_5$, such that

i) $\tilde{k}_t|h_{x_{i+1}}(B_{x_{i+1}}(\gamma_{3i+3}))\cap C_i(\gamma_{3i+3})=\text{inclusion}$

and

ii) $\tilde{k}_t|D-h_{x_{i+1}}(B_{x_{i+1}}(\gamma_{3i+2}))\cap C_i(\gamma_{3i+2})=k_t$, for any t.

Now define new into homeomorphism $h'_{x_{i+1}}: B_{x_{i+1}}(\gamma_{3i+3})\times[c-\varepsilon_5, c+\varepsilon_5]\to M$ by

$$h'_{x_{i+1}}(x, c+t)=\begin{cases} h_i(\tilde{k}_t\circ h_{x_{i+1}}(x), c+t)\,; & x\in B_{x_{i+1}}(\gamma_{3i+3})\cap h^{-1}_{x_{i+1}}(D)\,. \\ h_{x_{i+1}}(x, c+t)\,; & x\in B_{x_{i+1}}(\gamma_{3i+3})-h^{-1}_{x_{i+1}}(C_i(\gamma_{3i+2}))\,. \end{cases}$$

Then by the compactness of $C_{i+1}(\gamma_{3i+3})=C_i(\gamma_{3i+3})\cup h'_{x_{i+1}}(B_{x_{i+1}}(\gamma_{3i+3}))$, there is a positive number ε_6 with $0<\varepsilon_6\leqq\varepsilon_5$, such that $A_{i+1}=\bigcup_{j=1}^{i+1}h'_{x_j}(B_{x_j}(\gamma_{3i+3})\times[c-\varepsilon_6, c+\varepsilon_6])$ has a compatible structure. This completes the proof of the case 1.

Case 2. The case that c is a critical value.

By the trivial observation, we may assume that $f^{-1}(c)$ has only single critical point. By the definition of the non-degenerate critical point, there is a neighborhood which has nice trajectory field. Starting from this neighborhood, we can adjust the product structure of the neighborhoods of the regular points inductively as the case 1. This completes the proof of the theorem.

References

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