

# On the Pontrjagin square and the signature

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## §1. Statement of results.

Let  $V$  be a  $Z_2$ -vector space and let

$$\mu: V \otimes V \rightarrow Z_2$$

be a non-singular symmetric pairing. A function  $\eta: V \rightarrow Z_4$  is said to be quadratic with respect to  $\mu$  if

$$\eta(x+y) = \eta(x) + \eta(y) + j\mu(x \otimes y) \quad \text{for all } x, y \in V,$$

where  $j: Z_2 \rightarrow Z_4$  is the non-trivial homomorphism. Then according to E.H. Brown [1], we define the Arf invariant  $\sigma(V, \eta) \in Z_8$  as follows.

If we put  $\alpha(\eta) = \sum_{x \in V} i^{\eta(x)} \in C$ , (where  $i^2 = -1$ , and  $Z_4$  acts naturally on  $\{1, i, -1, -i\}$ ), then it can be shown that (see the proof of Prop. 2-3)

$$\alpha(\eta)^8 = (\sqrt{2}^{\dim V})^8;$$

therefore

$$\alpha(\eta) = \sqrt{2}^{\dim V} \cdot \left( \frac{1+i}{\sqrt{2}} \right)^m$$

for some  $m \in Z_8$ . We put  $\sigma(V, \eta) = m$ .

Now let  $M^{4n}$  be an oriented Poincaré complex of formal dimension  $4n$ , then the Pontrjagin square

$$P_2: H^{2n}(M; Z_2) \rightarrow H^{4n}(M; Z_4) = Z_4$$

is quadratic with respect to the cup-product

$$\mu: H^{2n}(M; Z_2) \otimes H^{2n}(M; Z_2) \rightarrow H^{4n}(M; Z_2) = Z_2.$$

Hence we can define

$$\sigma(M, P_2) = \sigma(H^{2n}(M; Z_2), P_2) \in Z_8.$$

Our results are

**THEOREM 1-1.** *If  $M^{4n}$  is an oriented Poincaré complex of formal dimension  $4n$ , then*

$$\sigma(M, P_2) = \text{signature } M \pmod{8}.$$

This theorem was conjectured by E.H. Brown in [2].

COROLLARY 1-2. *Let  $M^{4n}$  be an oriented Poincaré complex, then signature  $M \equiv 0 \pmod{4}$  if and only if*

$$P_2(v_{2n}) = 0.$$

Here  $v_{2n}$  is the  $2n$ -th Wu class of  $M$ .

Theorem 1-1 and Corollary 1-2 will be proved in §4.

I would like to express my gratitude to Professor A. Hattori for many advices and encouragement.

*Added in proof.* I have heard from G. Brumfiel that he and E. H. Brown, E. Thomas also have proved Theorem 1-1.

## §2. Some remarks on the Arf invariant.

The following proposition is due to E. H. Brown.

PROPOSITION 2-1. (i). *Let  $\gamma_i : V_i \rightarrow Z_4$  ( $i=1, 2$ ) be two quadratic functions with respect to  $\mu_i$ . If we define*

$$\gamma_1 + \gamma_2 : V_1 \oplus V_2 \rightarrow Z_4$$

*by  $(\gamma_1 + \gamma_2)(x_1, x_2) = \gamma_1(x_1) + \gamma_2(x_2)$ , then  $\gamma_1 + \gamma_2$  is quadratic with respect to  $\mu_1 + \mu_2$  and*

$$\sigma(V_1 + V_2, \gamma_1 + \gamma_2) = \sigma(V_1, \gamma_1) + \sigma(V_2, \gamma_2).$$

(ii). *If  $L : V \rightarrow Z_4$  is linear, then*

$$\begin{aligned} \alpha(L) &= 2^{\dim V} && \text{if } L=0, \\ \alpha(L) &= 0 && \text{if } L \neq 0. \end{aligned}$$

(iii). *If  $\eta : U \rightarrow Z$  is a unimodular quadratic form over  $Z$ ,  $\eta : U/2U \rightarrow Z_4$  is well defined and quadratic and*

$$\sigma(U/2U, \eta) = \text{signature } \eta \pmod{8}.$$

COROLLARY 2-2. *If  $V = A \oplus B$ ,  $\dim A = \dim B$  and  $\mu$  is zero on  $A$ , then for any quadratic function  $\eta : V \rightarrow Z_4$  with respect to  $\mu$  such that  $\eta(A) = 0$ ,*

$$\sigma(V, \eta) = 0.$$

PROOF. Let  $A_b = \{a+b; a \in A\}$  for any  $b \in B$ , and we give a  $Z_2$ -vector space structure on  $A_b$  by

$$(a_1+b) + (a_2+b) = a_1 + a_2 + b.$$

Consider the function  $\eta_b : A_b \rightarrow Z_4$  defined by

$$\gamma_b(a+b) = \gamma(a+b) - \gamma(b) = j\mu(a \otimes b),$$

then  $\gamma_b$  is linear with respect to  $+$ , hence by Proposition 2-1 if  $b \neq 0$ , then  $\alpha(\gamma_b) = 0$ , and if  $b=0$ , then  $\alpha(\gamma_0) = 2^{\dim A}$ . Therefore

$$\alpha(\gamma) = \sum_{b \in B} \alpha(\gamma|A_b) = \sum_{b \in B} \alpha(\gamma_b) \cdot i^{\gamma(b)} = \alpha(\gamma_0) = 2^{\dim A}.$$

Hence

$$\sigma(V, \gamma) = 0. \quad \text{Q.E.D.}$$

Now let  $M^{2n}$  be an oriented Poincaré complex and let  $\gamma : H^n(M; Z_2) \rightarrow Z_4$  be a quadratic function with respect to the cup-product, then we prove a formula relating  $\sigma(M, \gamma)$  with  $\gamma(v_n)$ , where  $v_n$  is the  $n$ -th Wu class of  $M$ .

**PROPOSITION 2-3.** *In the above situation,  $\sigma(M, \gamma) = \gamma(v_n) \pmod{4}$ .*

**PROOF.** We write  $V$  for  $H^n(M; Z_2)$ . Consider

$$\gamma + \gamma : V \oplus V \rightarrow Z_4.$$

Let  $V_v = \{(u+v, u); u \in V\}$  and we give  $V_v$  a  $Z_2$ -vector space structure by

$$(u_1+v, u_1) + (u_2+v, u_2) = (u_1+u_2+v, u_1+u_2).$$

Consider

$$2\gamma|V_v : V_v \rightarrow Z_4,$$

we have

$$2\gamma(u+v, u) = \gamma(u+v) + \gamma(u) = \gamma(v) + 2 \cdot \gamma(u) + j\mu(u \otimes v) = \gamma(v) + j\mu(u \otimes u) + j\mu(u \otimes v).$$

Hence if we define  $\gamma_v : V_v \rightarrow Z_4$  by

$$\gamma_v(u+v, u) = 2\gamma(u+v, u) - \gamma(v),$$

then  $\gamma_v$  is linear with respect to  $+$ . Now if

$$u^2 + uv = 0 \quad \text{for all } u \in V,$$

then  $v = v_n$ , the  $n$ -th Wu class. Hence if  $v \neq v_n$ , then  $\alpha(\gamma_v) = 0$ , and  $\alpha(\gamma_{v_n}) = 2^{\dim V}$  (by Prop. 2-1). Therefore

$$\begin{aligned} \alpha(2\gamma) &= \sum_{u \in V} \alpha(2\gamma|V_v) = \sum_{u \in V} \alpha(\gamma_v) \cdot i^{\gamma(v)} = \alpha(\gamma_{v_n}) \cdot i^{\gamma(v_n)} = 2^{\dim V} \cdot i^{\gamma(v_n)} \\ &= 2^{\dim V} \cdot \left( \frac{1+i}{\sqrt{2}} \right)^{2\gamma(v_n)}. \end{aligned}$$

Hence

$$2\sigma(M, \gamma) = 2\sigma(V, \gamma) = \sigma(V \oplus V, \gamma + \gamma) = 2\gamma(v_n),$$

hence

$$\sigma(M, \gamma) \equiv \gamma(v_n) \pmod{4}. \quad \text{Q.E.D.}$$

COROLLARY 2-4. *Let  $M^{2n}$  be an oriented Poincaré complex,  $n$ ; odd, and let  $\gamma: H^n(M; Z_2) \rightarrow Z_4$  be a quadratic function, then*

$$\sigma(M, \gamma) \equiv 0 \pmod{4}$$

i.e.,

$$\sigma(M, \gamma) \equiv 0 \text{ or } 4.$$

PROOF. By Proposition 2-3,

$$\sigma(M, \gamma) \equiv \gamma(v_n) \pmod{4},$$

but since  $n$  is odd and  $M$  is orientable,  $v_n = 0$ , hence

$$\sigma(M, \gamma) \equiv 0 \pmod{4}. \quad \text{Q.E.D.}$$

### § 3. The Bockstein Spectral Sequence.

Let  $M^{4n}$  be an oriented Poincaré complex and let  $\{E_r^*, d_r\}$  and  $\{E_r^*, d_r\}$  be the mod 2 Bockstein spectral sequence in cohomology and homology respectively. Then by Browder [1], we have

PROPOSITION 3-1. (i).  $\{E_r^*, d_r\}$  and  $\{E_r^*, d_r\}$  are dual each other by the Kronecker index.

$$(ii) \quad E_r^* = H^*(M)/\text{Tor} \otimes Z_2, \quad E_r^* = H_*(M)/\text{Tor} \otimes Z_2.$$

Now since  $M^{4n}$  is orientable, we have

$$E_1^{4n} = E_2^{4n} = \cdots = E_\infty^{4n} = Z_2$$

and

$$E_{4n}^1 = E_{4n}^2 = \cdots = E_{4n}^\infty = Z_2.$$

Let  $\mu_2 \in E_{4n}^1 = \cdots = E_{4n}^\infty$  be the mod 2 fundamental class. Then we can prove

PROPOSITION 3-2. *The Poincaré duality holds for  $\{E_r^*, d_r\}$  and  $\{E_r^*, d_r\}$ ;*

(i) *The cup-product*

$$\mu: E_r^a \otimes E_r^b \rightarrow E_r^{a+b}$$

*is well defined and similarly for the cap-product.*

(ii)  $\cap \mu_2: E_r^k \rightarrow E_{4n-k}^r$  *is an isomorphism for all  $k$  and  $r$ , and*

$$d^r(x \cap \mu_2) = d^r x \cap \mu_2 \text{ for all } x \in E_r^k.$$

(iii) *The cup-product*

$$\mu : E_r^k \otimes E_r^{4n-k} \rightarrow E_r^{4n} = Z_2$$

is non singular.

PROOF. (i) If  $r=1$ , then it is clear. Assume the statement holds for  $r \leq m$ , then we define

$$\mu : E_{m+1}^a \otimes E_{m+1}^b \rightarrow E_{m+1}^{a+b}$$

by

$$\mu([x] \otimes [y]) = [\mu(x \otimes y)],$$

where  $x \in E_m^a$ ,  $d_m x = 0$  and  $y \in E_m^b$ ,  $d_m y = 0$ . Since

$$d_m(x \cdot y) = d_m x \cdot y + x \cdot d_m y = 0,$$

and

$$d_m x' \cdot y = d_m(x' \cdot y) \quad \text{for } x' \in E_m^{a-1},$$

$\mu$  is well defined.

The proof for the cap-product is similar.

(ii) Clearly  $\cap \mu_2 : E_1^k \rightarrow E_{4n-k}^1$  is an isomorphism. And

$$d^1(x \cap \mu_2) = d_1 x \cap \mu_2 + x \cap d^1 \mu_2,$$

but since  $M$  is orientable,  $d^1 \mu_2 = 0$  and

$$d^1(x \cap \mu_2) = d_1 x \cap \mu_2.$$

Now suppose

$$\cap \mu_2 : E_r^k \xrightarrow{\simeq} E_{4n-k}^r, \quad \text{and } d^r(x \cap \mu_2) = d_r x \cap \mu_2$$

for all  $r \leq m$  and  $x \in E_r^k$ .

Then we define

$$\cap \mu_2 : E_{m+1}^k \rightarrow E_{4n-k}^{m+1}$$

as follows.

Let  $x \in E_m^k$  and suppose  $d_m x = 0$ , then  $[x] \in E_{m+1}^k$  and  $d^m(x \cap \mu_2) = d_m x \cap \mu_2 = 0$ .

Hence

$$[x \cap \mu_2] \in E_{4n-k}^{m+1}$$

and we define

$$[x] \cap \mu_2 = [x \cap \mu_2].$$

Now since

$$\begin{aligned} [x+d_mx'] \cap \mu_2 &= [(x+d_mx') \cap \mu_2] = [x \cap \mu_2] + [d_mx' \cap \mu_2] \\ &= [x \cap \mu_2] + [d^m(x' \cap \mu_2)] = [x \cap \mu_2] \end{aligned}$$

$\cap \mu_2$  is well defined.

(a)  $\cap \mu_2$  is epimorphic.

Take any  $[y] \in E_{4n-k}^{m+1}$ ,  $y \in E_{4n-k}^m$ ,  $d^m y = 0$ , then there is an element  $x \in E_m^k$  such that

$$x \cap \mu_2 = y.$$

Then

$$0 = d^m y = d^m(x \cap \mu_2) = d_m x \cap \mu_2,$$

hence  $d_m x = 0$  and  $[x] \in E_{m+1}^k$ .

Therefore

$$[x] \cap \mu_2 = [y]$$

and  $\cap \mu_2$  is epimorphic.

(b)  $\cap \mu_2$  is monomorphic.

Suppose

$$[x] \cap \mu_2 = 0,$$

then  $x \cap \mu_2 = d^m y$  for some  $y \in E_{4n-k-1}^m$ . Let  $x' \in E_m^{k+1}$  be such that

$$x' \cap \mu_2 = y,$$

then

$$d_m x' \cap \mu_2 = d^m(x' \cap \mu_2) = d^m y.$$

Hence

$$d_m x' \cap \mu_2 = x \cap \mu_2, \quad x = d_m x', \quad [x] = 0.$$

(c)  $d^{m+1}([x] \cap \mu_2) = d_{m+1}[x] \cap \mu_2$ , for

$$d^{m+1}([x] \cap \mu_2) = d_{m+1}[x] \cap \mu_2 + [x] \cap d^{m+1}\mu_2 = d_{m+1}[x] \cap \mu_2.$$

(iii) Suppose

$$x \cdot y = 0 \quad \text{for all } x,$$

then

$$\langle x \cdot y, \mu_2 \rangle = 0, \quad \langle x, y \cap \mu_2 \rangle = 0$$

hence  $y = 0$ .

Q.E.D.

§4. Proof of Theorem 1-1 and Corollary 1-2.

Let  $\{E_r^*, d_r\}$  and  $\{E_r^*, d_r\}$  be the mod 2 Bockstein spectral sequence for  $H^*(M)$  and  $H_*(M)$ , where  $M^{4n}$  is an oriented Poincaré complex. We prove the following statement by induction on  $r$ .

$(Q_r); P_2^{(r)} : E_r^{2^n} \rightarrow Z_4$  can be defined so as to be quadratic with respect to the cup-product

$$\mu : E_r^{2^n} \otimes E_r^{2^n} \rightarrow E_r^{4^n} = Z_2$$

and  $\sigma(E_r^{2^n}, P_2^{(r)}) = \sigma(E_{r-1}^{2^n}, P_2^{(r-1)})$ .

If  $r=1$ , then  $(Q_1)$  is trivial and

$$\sigma(E_1^{2^n}, P_2^{(1)}) = \sigma(M, P_2).$$

Now assume  $(Q_r)$  holds for  $r \leq m$ , then we define

$$P_2^{(m+1)} : E_{m+1}^{2^n} \rightarrow Z_4$$

as follows. If  $[x] \in E_{m+1}^{2^n}$ ,  $x \in E_m^{2^n}$ ,  $d_m x = 0$ , then we set

$$P_2^{(m+1)}([x]) = P_2^{(m)}(x).$$

To show that  $P_2^{(m+1)}$  is well defined, it suffices to show that

$$P_2^{(m)}(\text{im } d_m) = 0,$$

for

$$\begin{aligned} P_2^{(m)}(x + d_m y) &= P_2^{(m)}(x) + P_2^{(m)}(d_m y) + j(x \cdot d_m y) \\ &= P_2^{(m)}(x) + P_2^{(m)}(d_m y) + j d_m(x \cdot y) \\ &= P_2^{(m)}(x) + P_2^{(m)}(d_m y), \end{aligned}$$

where  $x \in E_m^{2^n}$ ,  $d_m x = 0$ ,  $y \in E_m^{2^{n-1}}$ . Here we have used the fact that  $d_m | E_m^{2^{n-1}} = 0$  which follows from the orientability of  $M$ .

Now let  $d_m x \in E_m^{2^n}$ ,  $x \in E_m^{2^{n-1}}$ , then if  $x$  is represented by an integral cochain  $u$ ,

$$\delta u = 2^m \cdot a$$

and  $d_m x$  is represented by  $1/2^m \cdot \delta u = a$ ,  $[a] \in H^{2^n}(M; Z)$  and

$$P_2^{(m)}(d_m x) = P_2[a].$$

But clearly  $2^m [a] = 0$ , hence  $[a]^2$  is a torsion element in  $H^{4n}(M; Z) = Z$ , therefore

$$[a]^2 = 0, \quad P_2[a] = [a]^2 \text{ mod } 4 = 0.$$

Clearly  $P_2^{(m+1)}$  is quadratic with respect to the cup-product.

Next we prove  $\sigma(E_{m+1}^{2n}, P_2^{(m+1)}) = \sigma(E_m^{2n}, P_2^{(m)})$ . Let  $[x_1], [x_2], \dots, [x_p]$  be a basis for  $E_{m+1}^{2n}$  and let

$V =$  the subspace of  $E_m^{2n}$ , spanned by  $x_1, x_2, \dots, x_p$ ,  $x_i \in E_m^{2n}$ ,  $d_m x_i = 0$ .

Let

$$\bar{V} = \{y \in E_m^{2n}; x \cdot y = 0 \text{ for all } x \in V\},$$

then

LEMMA 4-1.  $E_m^{2n} = V \oplus \bar{V}$ .

PROOF. If  $x \in V \cap \bar{V}$ , then since  $x \in V$ , we have  $d_m x = 0$  and  $[x] \in E_{m+1}^{2n}$ . Let  $[y] \in E_{m+1}^{2n}$  be any element, then we may take  $y$  in  $V$  and

$$[x] \cdot [y] = [x \cdot y] = 0.$$

Hence  $[x] = 0$  and since  $V \cap \text{im } d_m = \{0\}$ , we have

$$x = 0.$$

By counting dimension, we obtain the lemma. Q.E.D.

Now we have

$$\sigma(E_m^{2n}, P_2^{(m)}) = \sigma(V, P_2^{(m)} | V) + \sigma(\bar{V}, P_2^{(m)} | \bar{V}),$$

but clearly

$$\sigma(V, P_2^{(m)} | V) = \sigma(E_{m+1}^{2n}, P_2^{(m+1)}),$$

hence we have only to prove

$$\sigma(\bar{V}, P_2^{(m)} | \bar{V}) = 0.$$

To show this, it suffices to show that there is a subspace  $A \subset \bar{V}$  such that

- (i)  $P_2^{(m)}(A) = 0$ ,
- (ii) The cup-product is trivial on  $A$ ,
- (iii)  $\dim A = 1/2 \cdot \dim \bar{V}$ . (See Cor. 2-2).

Now we claim that the subspace

$$\text{im}(d_m : E_m^{2n-1} \rightarrow E_m^{2n}) \subset \bar{V}$$

satisfies the conditions above.

- (i) has already been proved (well definedness of  $P_2^{(m+1)}$ ).
- (ii) If  $d_m x, d_m y \in \text{im}(d_m : E_m^{2n-1} \rightarrow E_m^{2n})$ , then

$$d_m x \cdot d_m y = d_m(x \cdot d_m y) = 0,$$

since  $M$  is orientable.

- (iii) First we prove

LEMMA 4-2.  $\dim E_m^{2n} = \dim \ker (d_m : E_m^{2n} \rightarrow E_m^{2n+1}) + \dim \text{im} (d_m : E_m^{2n-1} \rightarrow E_m^{2n})$ .

PROOF. We have

$$(1) \dim E_m^{2n} - \dim \ker (d_m : E_m^{2n} \rightarrow E_m^{2n+1}) = \dim \text{im} (d_m : E_m^{2n} \rightarrow E_m^{2n+1}).$$

Now we prove that

$$(\text{im} (d_m : E_m^{2n} \rightarrow E_m^{2n+1}))^\perp = \ker (d^m : E_{2n+1}^m \rightarrow E_{2n}^m).$$

In fact if  $\alpha \in (\text{im} (d_m : E_m^{2n} \rightarrow E_m^{2n+1}))^\perp$ , then

$$\langle d_m x, \alpha \rangle = 0 \text{ for all } x \in E_m^{2n}.$$

Hence

$$\langle x, d^m \alpha \rangle = 0 \text{ for all } x.$$

Therefore

$$d^m \alpha = 0 \text{ i.e., } \alpha \in \ker (d^m : E_{2n+1}^m \rightarrow E_{2n}^m).$$

Conversely, assume  $d^m \alpha = 0$ , then

$$\langle d_m x, \alpha \rangle = \langle x, d^m \alpha \rangle = 0 \text{ for all } x.$$

Hence  $\alpha \in (\text{im} (d_m : E_m^{2n} \rightarrow E_m^{2n+1}))^\perp$ .

Therefore

$$(2) \dim \text{im} (d_m : E_m^{2n} \rightarrow E_m^{2n+1}) = \dim E_{2n+1}^m - \dim \ker (d^m : E_{2n+1}^m \rightarrow E_{2n}^m).$$

Consider the following diagram, which is commutative by Proposition 3-2.

$$\begin{array}{ccc} E_{2n+1}^m & \xrightarrow{d^m} & E_{2n}^m \\ \uparrow \lambda \cap \mu_2 & & \uparrow \lambda \cap \mu_2 \\ E_m^{2n-1} & \xrightarrow{d_m} & E_m^{2n} \end{array}$$

We have

$$(3) \dim E_{2n+1}^m - \dim \ker (d^m : E_{2n+1}^m \rightarrow E_{2n}^m) = \dim \text{im} (d_m : E_m^{2n-1} \rightarrow E_m^{2n}).$$

Combining (1), (2) and (3), we obtain the lemma. Q.E.D.

Proof of (iii);

$$\begin{aligned} \dim \bar{V} &= \dim E_m^{2n} - \dim E_{m+1}^{2n} = \dim \ker d_m + \dim \text{im} d_m \\ &\quad - (\dim \ker d_m - \dim \text{im} d_m) = 2 \cdot \dim \text{im} (d_m : E_m^{2n-1} \rightarrow E_m^{2n}). \end{aligned}$$

This proves (iii).

Thus we have proved

$$\sigma(E_{m+1}^{2n}, P_2^{(m+1)}) = \sigma(E_m^{2n}, P_2^{(m)})$$

for all  $m$ . Therefore

$$\sigma(M, P_2) = \sigma(E_1^{2n}, P_2^{(1)}) = \sigma(E_\infty^{2n}, P_2^{(\infty)}) = \text{signature } M \text{ mod } 8 \text{ (by Prop. 2-1).}$$

This proves Theorem 1-1.

PROOF OF COROLLARY 1-2. By Theorem 1-1

$$\text{signature } M \bmod 8 = \sigma(M, P_2)$$

and by Proposition 2-3,

$$\sigma(M, P_2) \equiv P_2(v_{2n}) \pmod{4},$$

hence

$$\text{signature } M \equiv P_2(v_{2n}) \pmod{4}.$$

Q.E.D.

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(Received March 4, 1971)

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