The existence of topological Morse function and its application to topological h-cobordism theorem

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Introduction

The h-cobordism theorem (dim$\geqq 6$) in the differentiable and the P. L. cases has been proved by Smale (Smale [1], Milnor [1]) and Mazur (Mazur [1]). In this paper, we show that the topological h-cobordism theorem also holds for dimensions $\geqq 7$.

Our main result is the following

THEOREM. *Suppose the triad of the topological manifold* $(W^n; V, V')$ *has the properties*

1) W, V *and* V' *are simply connected,*
2) $H_*(W, V)=0$,
3) dim $W=n\geqq 7$.

Then W *is homeomorphic to* $V\times[0.1]$.

It is well known in the differentiable case, that the handlebody decompositions correspond to the Morse functions. Our proof starts from analogous fact in topological case. To perform the proof, we newly introduce a concept of "topological trajectory field", in addition to the topological Morse function defined by M. Morse (Morse [1]). This corresponds to the orthogonal vector field in the differentiable case.

Some powerful tools are obtained in the recent few years. Kirby [1] showed that any topological manifold, dim$\geqq 7$, has a handlebody decomposition, and gave conditions to extend a P. L. structure of a submanifold to the manifold. Rushing [1] showed that a flatly embedded triangulable manifold can be considered as P. L. embedded, locally, if codimension is more than 3.

Making use of them, we carry out the proof of the topological h-cobordism theorem by the slightly modified Milnor's method (Milnor [1]). We adopt the terminologies of Milnor [1], if the situation is clear from the context. In §1, the concepts of critical point, non-degeneracy, topological Morse function and topological trajectory field are defined. In §2, we obtain that any manifold of dim$\geqq 6$ have a Morse function and a trajectory field by using Kirby's result. In §3, we show so called rearrangement theorem (cf. Theorem 3.1). Using Rushing's result, this theorem is reduced to the P. L. general position theorem. By the grace of the rearrangement theorem, we can define the k-skeleton $(M, N)_k$ of the handle decomposition of M relative to N. $(M, N)_3$ plays very important role in this paper, because $(M, N)_3$ has the following properties: (1) $\pi_1((M, N)_3) \cong \pi_1(M)$, (2) if N has a P. L. structure, then $(M, N)_3$ has a P. L. structure which is the extension of the P. L. structure of N. In § 4, topological "transverse intersection" of two manifolds is defined. It is shown that intersection of the left hand sphere and the right hand sphere may have this property under some conditions, by the annulus theorem, (Kirby [1]), and by introducing P. L.-structure on the neighborhood of union of the right-hand sphere and the left-hand sphere. So called cancellation theorem (cf. Theorem 4.4, 4.9) is reduced to the P. L. case, using the property of the transverse intersection. (See Hudson [1].) In § 5, h-cobordism theorem is completely proved.

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§ 1. Definitions

We let $\boldsymbol{R}^n$, S^n, B^n denote Euclidean n-space, n-sphere, n-ball respectively. We denote the interior of a manifold M by either Int (M) or $\mathring{M}$ and the boundary of M by ∂M.

Let M^m be a compact topological manifold with or without boundary and let $f: M \to \boldsymbol{R}$ be a continuous map. If M has a boundary ∂M, we assume that the restriction of f on each connected component of ∂M is a constant map. A point $x \in M$ is called a *regular point* of f, if there exists a coordinate neighborhood (U, h) around x

$$h: U \to V$$

such that $U \cap f^{-1}(f(x))$ is an $(m-1)$-dimensional submanifold of U and $f \circ h^{-1}(x_1, \cdots, x_m) = x_m$, where $V \subset \boldsymbol{R}^m$ for $x \in \text{Int}\, M$, and $V \subset \boldsymbol{R}^m_{f(x)+} = \{x \in \boldsymbol{R}^m;\ x_m \geqq f(x)\}$ or $\boldsymbol{R}^m_{f(x)-} = \{x \in \boldsymbol{R}^m;\ x_m \leqq f(x)\}$ for $x \in \partial M$ respectively. A point $x \in M$ is a topological *critical point* if it is not a topological regular point. A point $x \in M$ is called a *non-degenerate critical point of index* λ, if there exists a coordinate neighborhood (U, h) such that, $h(x) = 0$ and $f \circ h^{-1} = f_{(x)} - (x_1^2 + \cdots + x_\lambda^2) + (x_{\lambda+1}^2 + \cdots x_n^2)$. Note that λ is uniquely determined for each non-degenerate critical point, by M. Morse [1].

1.1. DEFINITION. A map $f: M \to \boldsymbol{R}$ is called a *Morse function* if

(1) f has no critical point on ∂M and

(2) f has only the non-degenerate critical points.

Let $(W;\ V_0, V_1)$ be a triad of the topological manifold and let $f: W \to I = [0, 1]$ be a Morse function satisfying $f^{-1}(1) = V_1$ and $f^{-1}(0) = V_0$, then f is called a *Morse function of triad* $(W;\ V_1, V_0)$.

Let f be a Morse function of M and let $\mathscr{H}_f$ be the set of continuous maps $g: [t, t'] \to M$ such that $f \circ g = \mathrm{id.}$, where $[t, t']$ is a closed interval.

A *trajectory field* $\mathscr{T}_{M.f}$ *of* M *associated to* f is a subset of $\mathscr{H}_f$ which satisfies the following conditions.

(1) For any point $x \in M$, there exists $g \in \mathscr{T}_{M.f}$ such that $\mathrm{Im}\, g \ni x$, and if x is regular, then this g is unique. Hereafter we denote such g by g_x.

(2) For any $g: [t, t'] \to M$ in $\mathscr{T}_{M.f}$, $g(t)$, $g(t') \in \{\text{critical points}\} \cup \partial M$, and $g((t, t')) \cap \{\text{critical points}\} = \phi$.

(3) For any regular point $x \in M$, there exists an open subset U in W and an into homeomorphism $h: U \to (U \cap f^{-1}f(x)) \times \boldsymbol{R}$ (if $x \in \partial M$, then replace $\boldsymbol{R}$ with $\boldsymbol{R}_{f(x)+}$ or $\boldsymbol{R}_{f(x)-}$), such that $h(y) = (b, f(y))$, where $b \in f^{-1}f(x) \cap \mathrm{Im}\, g_y$.

(4) For any critical point $x \in M$, there exists some coordinate neighborhood (U, h) around x such that $h(x) = 0$ and $f \circ h^{-1}(x) = f(x) - (x_1^2 + \cdots + x_\lambda^2) + x_{\lambda+1}^2 + \cdots + x_n^2$, and hg is an orthogonal trajectory of fh^{-1}, in differentiable sense with respect to the Riemann metric in $h(U)$, for any $g \in \mathscr{T}_{M.f}$. Note that fh^{-1} can be regarded as differentiable function on $h(U) \subset R^n$. An element of the trajectory field is called *trajectory*.

Hereafter we denote a Morse function on M by f_W, and a trajectory of W associated to f by $\mathscr{T}_{M.f}$. If f is understood from the context, we denote simply $\mathscr{T}_M$ instead of $\mathscr{T}_{M.f}$.

§2. Existence theorem of the topological Morse function

In this section, we show that for any manifold which has handlebody structure, there exists a pair of Morse function and associated trajectory field on it.

Let N, N' be two topological m-dimensional manifolds such that N' is obtained from N by adding a handle of index k with an attaching homeomorphism $h: \partial B^k \times B^{n-k} \to \partial N$.

2.1. LEMMA. Suppose N has a Morse function f_N and an associated trajectory field $\mathscr{T}_N$ such that $(1)_N$ $f_N(\partial N) = 1$, $f_N(N - \partial N) < 1$ and that $(2)_N$ there exists δ_N for which $f_N^{-1}([1 - \delta_N, 1])$ is a collar neighborhood of ∂N with the product structure given by the trajectories passing through the boundary of N. Then N' has a

Morse function $f_{N'}$, an associated trajectory field $\mathscr{T}_{N'}$ and a positive number $\delta_{N'}$ which satisfy the conditions $(1)_{N'}$ and $(2)_{N'}$.

REMARK. Above condition $(2)_N$ is required in order to add the handle to the collar of ∂N and to adjust the attaching homeomorphism in the collar.

PROOF. We define a real valued function of handle of index k, $g: B^k\times B^{m-k}\to \boldsymbol{R}$ by $g(x_1, \cdots, x_k, x_{k+1}, \cdots, x_m)=-(x_1^2+\cdots+x_k^2)+(x_{k+1}^2+\cdots+x_m^2)(1+x_1^2+\cdots+x_k^2)$. Consider $g'=\varepsilon g+(1-\varepsilon)$, where ε is a positive number such that $\delta_N>3\varepsilon$ is hold, then there exist differentiable orthogonal trajectories on $B^k\times B^{m-k}$ associated to g'. Let a manifold $(N-Q)\cup_{g''}(B^k\times B^{m-k})$ be obtained by adding $B^k\times B^{m-k}$ to $N-Q$ with the into homeomorphism $g'': \partial B^k\times B^{m-k}\to h(\partial B^k\times B^{m-k})\times[1-\delta_N, 1]$, where $Q=\{(y, t);\ y\in h(\partial B^k\times B^{m-k}),\ t\in[1-\delta_N, 1],\ t\gneqq g'(y)\}$ and $g''(x)=(h(x), g'(x))$. Then by an easy observation, we have that $N-Q\cup_{g''}B^k\times B^{m-k}$ is homeomorphic to $M\cup_h(B^k\times B^{m-k})$.

Define a map $f_{N'}: N'\to \boldsymbol{R}$ by $f_{N'}|N-Q=f_N|N-Q$ and $f_{N'}|B^k\times B^{m-k}=g'$, then it is easy to see that $f_{N'}$ is a Morse function which satisfies the condition. Next, we have a set $\mathscr{T}_{N'}$ by

$$\mathscr{T}_{N'}=\left\{\begin{array}{l}\alpha\cup\alpha'_{g''^{-1}(x)}\ ;\ \alpha\in\mathscr{T}_N,\ \alpha'_{g''^{-1}(x)}\in\mathscr{T}_{B^k\times B^{m-k}},\ \text{if Im}\,(\alpha)\cap\\ \qquad\qquad g''(\partial B^k\times B^{m-k})=\{x\}\\ \alpha\ ;\ \alpha\in\mathscr{T}_N,\ \text{if Im}\,(\alpha)\cap g''(B^k\times B^{m-k})=\phi\\ \alpha'\ ;\ \alpha'\in\mathscr{T}_{B^k\times B^{m-k}},\ \text{if Im}\,(\alpha')\cap\partial B^k\times B^{m-k}=\phi\end{array}\right\},$$

where $\alpha\cup\alpha'|[f(x), 1]=\alpha'$, $\alpha\cup\alpha'|(-\infty, f(x)]=\alpha|(-\infty, f(x)]$. This set $\mathscr{T}_{N'}$ obviously satisfies the conditions. Q.E.D.

2.2. THEOREM. *Let M^m be a compact topological manifold. If M^m admits a topological handlebody decomposition, then M has a Morse function f_M and associated trajectory field $\mathscr{T}_M$.*

PROOF. If M^m is m-ball, then define f_M by $f_M(x)=x_1^2+\cdots+x_m^2$ and $\mathscr{T}_M$ is given canonically. These f_M and $\mathscr{T}_M$ satisfy the condition of Lemma 2.1, hence we now construct f_M and $\mathscr{T}_M$ inductively in the general case. Q.E.D.

2.3. COROLLARY. *Let M^m be a compact topological manifold, $m\geqq 6$, then there exist a Morse function and an associated trajectory field.*

PROOF. By Kirby [1] M^m is obtained from $\partial M\times[0, 1]$ by attaching handles since $m\geqq 6$. Take a canonical Morse function and a canonical associated trajectory cn $\partial M\times[0, 1]$, and apply Lemma 2.1. Q.E.D.

Let $(W;\ V_0, V_1)$ be a triad of topological manifold with $\dim\geqq 6$, then W has a Morse function f_W by 2.3. However, since $f_W(V_0\cup V_1)=0$, this f_W is not a Morse function of triad (See 1.1). Now, we show the existence theorem of the

Morse function of triad by modifying of f_W.

2.4. THEOREM. *Let $(W^n;\ V_0,\ V_1)$ be a triad of the topological manifold. Suppose* that *W be obtained from $(V\cup V')\times I$ by adding handles, and that $V\times I$ be obtained from $V\times\partial I$ by adding handles. Then, there exist Morse function of triad and an associated trajectory.*

PROOF. Let $V_1\times I$ and $V_2\times I$ be two copies of $V\times I$. Let $f_{V_1\times I}$ be a canonical Morse function of $V_1\times I$ and $\mathscr{T}_{V_1\times I}$ be a canonical associated trajectory field. On the other hand $V_2\times I$ and W have Morse functions $f_{V_2\times I}$ and f_W and associated trajectory fields $\mathscr{T}_{V_2\times I}$ and $\mathscr{T}_W$ respectively, given by 2.3. Replacing f_W with $f'_W=a_1f_W+b_1$ and $f_{V_2\times I}$ by $f'_{V_2\times I}=a_2f_{V_2\times I}+b_2$, where a_i, b_i are some real numbers, then we assume that (1) $f'^{-1}_W(1)=(V\cup V')$, $f'_W(W)=(0,1]$, (2) there exists some δ_W, such that δ_W satisfy the condition $(2)_W$ of 2.1, (3) $f'_{V_2\times I}(V_2\times I)\subset[1-\delta_W,1)$, $f'^{-1}_{V_2\times I}(1-\delta_W)=V_2\times\partial I$.

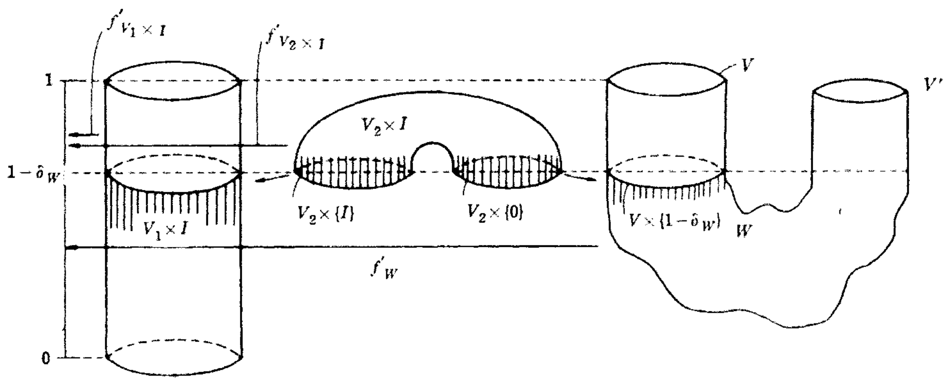

Fig. 2.1.

Paste $V_2\times\{0\}$ to $V\times\{1-\delta_W\}\subset W-V\times(1-\delta_W,1]$, and $V_2\times\{1\}$ to $V_1\times\{1-\delta_W\}\subset V_1\times I-V_1\times(1-\delta_W,1]$. It is easy to see that an obtained manifold is homeomorphic to W, and has a Morse function and an associated trajectory field. Q.E.D.

2.5. COROLLARY. *Let $(W^n;\ V_0,\ V_1)$ be a triad of the topological manifold with $n\geqq 6$. Then there exists a pair of Morse function of triad and an associated trajectory field.*

PROOF. By Kirby [1], W^n and $V_0\times I$ satisfy the condition of 2.4. Q.E.D.

The method of the proofs of 2.2 and 2.4 implies the following.

2.6. LEMMA. *Let W be a manifold with or without boundary and let $f\colon W\to[0,1]$ be a Morse function which is given by Theorem 2.2 (or 2.4), with critical points $p_1,\cdots,p_k$, then f can be approximated by a Morse function g with the same critical points such that $g(p_i)\neq g(p_j)$ for $i\neq j$.*

PROOF. In the proof of 2.1, we can shift ε so that the condition is satisfied.

2.7. DEFINITION. The *Morse number* μ of $(W;\ V_0, V_1)$ is the minimum of the number of critical points of f over all Morse functions f which has an associated trajectory field.

Hereafter by "cobordism" we mean the compact cobordism with a Morse function and an associated trajectory field.

2.8. COROLLARY. *Any cobordism can be expressed as a composition of cobordisms with Morse number* 1.

PROOF. Let a be a regular value. We need to prove that $f_W^{-1}[0, a]$ is a cobordism. But it is obvious by the definition of the regular value.

The following theorem is obtained by the standard argument using the topological trajectory field.

2.9. THEOREM. *If a Morse number of* $(W;\ V_0, V_1)$ *is* 0, *then* $(W;\ V_0, V_1)$ *is a product cobordism.*

As the differentiable case the critical points are in one to one correspondence with the handles (Milnor [1] Theorem 3.14).

2.10. THEOREM. *Let* $(W;\ V_0, V_1)$ *be an elementary* λ*-index cobordism with a critical point* p, *then* $W-(V_0\times[0, \varepsilon])$ *is a* λ*-handlebody, where* $0<\varepsilon<f(p)$.

§3. Rearrangement theorem

Let $(W;\ V_0, V_1)$ be a triad of the topological manifold with a Morse function f, an associated trajectory field $\mathscr{T}_W$, having two critical points p, p', with index λ, λ' respectively such that $f(p)<f(p')$.

3.1. THEOREM. *Let* $\lambda\geqq\lambda'$, dim $W\geqq 6$, *and let* $V=f^{-1}(a)$, *where* $f(p)<a<f(p')$. *Then it is possible to alter the trajectory field associated to f on a prescribed small neighborhood of V so that the corresponding new spheres* $\overline{S_R}$ *(right hand sphere of* p*) and* $\overline{S'_L}$ *(left hand sphere of* p'*) in* V, *do not intersect.*

REMARK. We use the terminologies of Milnor [1] in this paper for the *left hand sphere* and the *right hand sphere* and so on.

To prove 3.1 we need the following

3.2. LEMMA. *Let* M^m, N^n *be two compact triangulable submanifolds of the closed manifold* V^v. *If* M, N *have the flat neighborhoods, and* $m+n<v$, $v\geqq 5$, *then there exists a homeomorphism* $h: V\to V$, *which is isotopic to the identity, such that* $h(M)\cap N=\phi$.

The proof of 3.2 will be based on the following theorem which is due to Rushing [1].

3.3. THEOREM (Rushing). *Let P^k be a polyhedron, Q^n be a* P. L. *manifold, and $n-k\geqq 3$, $n\geqq 5$. If $f: P\to Q$ is a locally flat embedding on the open simplex of some triangulation, then f is ε-tame. (An embedding f is called ε-tame if for any $\varepsilon>0$, there is an isotopy e_t of Q such that $e_0=1$ and $e_1 f$ is* P. L., *distance $(x, e_t(x))<\varepsilon$ for all $x\in Q$ and $e_{t|Q-N_\varepsilon(f(p))}=1$.)*

PROOF OF LEMMA 3.2. Since $v-n\geqq 3$ or $v-m\geqq 3$, we consider the case where $v-n\geqq 3$. Let $f: R^{v-m}\times M\to V$ be an embedding given by flatness. We identify $R^{v-m}\times M$ with $f(R^{v-m}\times M)$ and introduce a P. L. structure on $f(R^{v-m}\times M)$ in the trivial way. Triangulate N and take sufficiently fine subdivision such that $2\mathring{B}^{v-m}\times M\supset N'\supset (B^{v-m}\times M)\cap N$, where N' is a subpolyhedron of N which consists of all simplices included in $2\mathring{B}^{v-m}\times M$. By Theorem 3.3, there exists an ambient ε-isotopy h_t such that $h_1(N')$ is a subcomplex of $R^{v-m}\times M$ and h_0 is the identity. Apply the general position theorem, then we get the result. Q.E.D.

PROOF OF THEOREM 3.1. Since S_R and S'_L can be regarded as flatly embedded submanifold, and $\dim S_R+\dim S'_L=(v-1)+\lambda'-\lambda$, the proof follows from Lemma 3.2. Q.E.D.

3.4. THEOREM (*rearrangement theorem*). *Any cobordism c of* $\dim c\geqq 6$ *may be expressed as a composition*

$$c=c_0c_1\cdots c_n\ ,$$

where each cobordism c_k admits a Morse function with just one critical point, and $\operatorname{index}(c_i)\geqq\operatorname{index}(c_j)$ *if $i\geqq j$.*

PROOF. Compare Milnor [1], Theorem 4.1 and Theorem 4.8. Q.E.D.

Let W^n be a compact topological manifold $n\geqq 6$ and let $V_0\subset\partial W$ be a union of components of ∂W. Assume that W be obtained from $V_0\times I$ by adding handles, then W has a Morse function $f_W: W\to[0,1]$, $f_W^{-1}(0)=V_0$ by 2.2. f_W can be rearranged to g_W by 3.4, therefore for any positive integer k, there exists some regular value a such that all critical points of g in $g^{-1}([0,a])$ have indices less than k and all critical points of g in $g^{-1}([a,1])$ have indices more than $k+1$. A submanifold $g^{-1}([0,a])$ is called *k-skeleton of this handlebody decomposition relative* to V.

It is easy to see that for any handle decomposition, k-skeleton of the handlebody decomposition is well defined.

§4. Cancellation theorems

Let f be a Morse function on the triad $(W^n; V_0, V_1)$ for cc', where cc' is a composition of the cobordisms of index $\lambda: c=(W; V_0, V)$, and index $\lambda+1: c'=$

$(W;\ V, V_1)$. p is a critical point of c, p' is a critical of c', S_R is the right hand sphere of p, S'_L is the left hand sphere of p', $\mathscr{T}_W$ is a trajectory field of W associated with f. Note that $\dim S_R + \dim S'_L = (n-\lambda-1)+\lambda = n-1 = \dim V$.

We define the concept of "topological transversal intersection" of two embeddings, analogous to the differentiable and the P. L. cases.

4.1. DEFINITION. Two embeddings $f: M^m \times B^n \to V^{m+n}$ and $g: B^m \times N^n \to V^{m+n}$ are called *transversely intersect* at $q \in f(M\times\{0\}) \cap g(\{0\}\times N)$, if there is a homeomorphism $\gamma: B^m \times B^n \to f(M\times B^n) \cap g(B^m \times N)$, s.t. $\gamma(0)=q$ and there are maps $\phi: B^n \to N$, $\psi: B^m \to M$, s.t. $f\circ(\psi\times 1_{B^n}) = \gamma = g\circ(1_{B^m}\times\phi)$.

Since p is a non-degenerate critical point, we have some neighborhood $U\subset W$ and homeomorphism $\varphi: U \to \tilde{U} \subset R^n$ such that $\tilde{U} \ni 0$, and $f\varphi^{-1} = f(x) - (x_1^2 + \cdots + x_\lambda^2) + x_{\lambda+1} + \cdots + x_n^2$. We may assume that $\tilde{U} \supset 3B^n$. An *attaching neighborhood* NS_R (resp. $2NS_R$) *of* S_R is the intersection of V with all trajectories passing through $\varphi^{-1}(B^\lambda \times B^{n-\lambda}) \subset U$ (resp. $\varphi^{-1}(2B^\lambda \times 2B^{n-\lambda}) \subset U$). The definition of the attaching neighborhood NS_L $(2NS_L)$ of S_L is similar. Note that there are canonical homeomorphisms $h: 2B^\lambda \times S^{n-\lambda-1} \to 2NS_R \subset V$ and $h': S^\lambda \times 2B^{n-\lambda-1} \to 2NS'_L$, where NS_R is an attaching neighborhood of S_R, and NS'_L an attaching neighborhood of S'_L.

4.2. THEOREM. *Let W^n be a simply connected manifold, $n \geqq 7$, $n-2 \geqq \lambda \geqq 1$. When $\lambda=2$ and $\lambda=n-3$, we add the assumption that V_0, V_1 are simply connected. Then we can alter the associated trajectory field near V, so that h and h' transversely intersect at each intersections.*

We need the following lemma to prove the above theorem in the case $\lambda=2$.

4.3. LEMMA. *If W, V_0 and V_1 are all simply connected, and $\lambda=2$, then there exists a compact triangulable submanifold K of V such that NS_R is included in K as* P.L. *submanifold of K, and $K-S_R$ is simply connected.*

PROOF OF 4.3. Since $\lambda=2$, $\dim V_0 - \dim S_L \geqq 3$, where S_L is the left hand sphere of the critical point p. By introducing a natural P. L. structure on NS_L and using the general position theorem, it is easy to see that $V_0 - S_L$ is simply connected, hence $V-S_R$ is simply connected. The submanifold $V - \operatorname{Int} NS_R$ is obtained from $\partial(V-\operatorname{Int} NS_R)\times I = \partial(NS_R)\times I$ by adding handles. Let L be a 3-skeleton of this handlebody decomposition relative to $\partial(NS_R)$. Then $V - \operatorname{Int} NS_R$ is obtained from L by adding handles with indices more than 4, therefore $\pi_1(L) \cong \pi_1(V - \operatorname{Int} NS_R) \cong 1$, by Van Kampen's theorem. On the other hand it is easy to see that $H^4(L, \partial(NS_R); Z_2) = 0$, by the universal coefficient theorem and the assumption that L is 3-skeleton. By Kirby [1], L has a P.L. structure which is the extension of the P.L. structure of $\partial(NS_R)$, accordingly $K = L \cup NS_R$ has P.L. structure. The submanifold K satisfies the condition. Q.E.D.

PROOF OF THEOREM 4.2. We will show that there is an ambient isotopy $\Gamma_t : V \to V$, $0 \leqq t \leqq 5$, such that $\Gamma_5 h'$ and h transversely intersect. We consider the case $\lambda \leqq n-\lambda-1$, since another case is similar.

At first, making use of Lemma 3.3 and P.L. general position theorem, S_R and S'_L can be altered by ambient isotopy, so that they intersect at finite points $a_1, \cdots, a_k$, and for each a_i there exists some coordinate system (g_i, U_i) around a_i such that $U_i \cap S_R$ is mapped to $R^{n-\lambda-1} \times 0$, and $U_i \cap S'_L$ is mapped to $0 \times R^\lambda$ by g_i.

Choose some into homeomorphism $\phi : 2B^{n-\lambda-1} \to S_R$ and $\psi : 2B^\lambda \to S'_L$ and identify $2B^{n-\lambda-1}$ with $\phi(2B^{n-\lambda-1})$, and $2B^\lambda$ with $\psi(2B^\lambda)$. By adjusting some attaching neighborhood ϕ and ψ, we may assume that $h(2\mathring{B}^\lambda \times 2\mathring{B}^{n-\lambda-1}) \supset h'(B^\lambda \times B^{n-\lambda-1}) \supset h'(\mathring{B}^\lambda \times \mathring{B}^{n-\lambda-1}) \supset h(B^\lambda \times B^{n-\lambda-1})$ and that $h'(2\mathring{B}^\lambda \times 0) \supset h(2B^\lambda \times 0)$. Hence we only need to prove that $h^{-1}h'|_{2B^\lambda \times B^{n-\lambda-1}}$ can be altered to intersect transversely with $\mathrm{id.}|_{B^\lambda \times 2B^{n-\lambda-1}}$.

By using annulus theorem (Kirby [1]) to $\partial(h^{-1}h'(B^\lambda \times B^{n-\lambda-1}))$ and $\partial(B^\lambda \times B^{n-\lambda-1})$, we have an isotopy $\gamma_t : 2B^\lambda \times 2B^{n-\lambda-1} \to 2B^\lambda \times 2B^{n-\lambda-1}$; $0 \leqq t \leqq 1$, such that (1) $\gamma_0 = \mathrm{id.}$, (2) $\gamma_t|_{\partial(2B^\lambda \times 2B^{n-\lambda-1})} = \mathrm{id.}$, (3) $\gamma_1 \circ (h^{-1}h')|_{\partial(B^\lambda \times B^{n-\lambda-1})} = \mathrm{id}$.

By Alexander trick there exists an isotopy $\Gamma_t : V \to V$; $1 \leqq t \leqq 2$, such that $\Gamma_2 h^{-1}h'|_{B^\lambda \times B^{n-\lambda-1}} = \mathrm{id.}$, and Γ_t; $1 \leqq t \leqq 2$, leaves $(\gamma_1 h^{-1}h'(B^\lambda \times B^{n-\lambda-1}))^c$ fixed. Therefore we have an ambient isotopy Γ_t; $0 \leqq t \leqq 2$, such that $\tilde{h} = \Gamma_2 h^{-1}h'|_{2B^\lambda \times B^{n-\lambda-1}}$ transversely intersect with $\mathrm{id.}|_{B^\lambda \times B^{n-\lambda-1}}$ at $h^{-1}(a_i)$. However, by this ambient isotopy, it may happen that new intersections of $\tilde{h}(2B^\lambda \times 0)$ and $0 \times 2B^{n-\lambda-1}$ in $\mathrm{Int}\,(2B^\lambda \times 2B^{n-\lambda-1} - \mathring{B}^\lambda \times \mathring{B}^{n-\lambda-1})$ are generated.

We will eliminate these intersections. Let $V' = 2B^\lambda \times 2B^{n-\lambda-1} - \mathring{B}^\lambda \times \mathring{B}^{n-\lambda-1}$ and let $D = \xi B^\lambda \times 0 - \zeta \mathring{B}^\lambda \times 0$, $2 > \xi > \zeta > 1$, be a subset of $2B^\lambda \times 0 - \mathring{B}^\lambda \times 0$, which satisfy $\tilde{h}((2B^\lambda \times 0 - \mathring{B}^\lambda \times 0) - D) \cap (0 \times 2B^{n-\lambda-1}) = \phi$. When $\lambda \neq 2$, every components of $V' - S_R \cong S^{\lambda-1} \times B^{n-\lambda-1}$ are simply connected. By Theorem 3.3 there exists an ε-isotopy $e_t : V' \to V'$, $2 \leqq t \leqq 3$, such that $e_3 \tilde{h}|_D$ is P.L. We choose ε so small that $e_3(\tilde{h}(2B^\lambda \times 0 - \mathring{B}^\lambda \times 0) - D) \cap 0 \times 2B^{n-\lambda-1} = \phi$, and $e_t|_{\partial V'} = \mathrm{id}$.

By the P.L. general position theorem there exists an ambient isotopy $e_t : V' \to V'$; $3 \leqq t \leqq 4$, such that $e_t|_{\partial V'} = \mathrm{id.}$, $e_4(\tilde{h}(D)) \cap 0 \times 2B^{n-\lambda-1}$ consists of finite points, and that $e_4(\tilde{h}(D))$ and $0 \times 2B^{n-\lambda-1}$ intersect transversely at each point in the P.L. sense (see Hudson [1]).

It is easy to see that the intersection number of $e_4(h(D))$ and $0 \times 2B^{n-\lambda-1}$ is 0, hence there is an ambient isotopy $e_t : V' \to V'$; $4 \leqq t \leqq 5$, such that $e_t|_{\partial V'} = \mathrm{id.}$, and $e_5 \tilde{h}(2B^\lambda \times 0 - \mathring{B}^\lambda \times 0) \cap 0 \times 2B^{n-\lambda-1} = \phi$. Extend this isotopy $e_t : V' \to V'$; $2 \leqq t \leqq 5$, to the isotopy $\Gamma_t : V \to V$; $2 \leqq t \leqq 5$, in a canonical way, then we get the desired sotopy Γ_t; $0 \leq t \leq 5$.

When $\lambda = 2$, we take K instead of V', where K is given by Lemma 4.3.

Q.E.D.

We obtain the following corollary as a direct consequence of the above theorem.

4.5. COROLLARY. *With W, V_0, V_1, NS'_L, and NS_R as above, we can alter the trajectory field near V, and can adjust NS'_L, NS_R so that $NS'_L \cup NS_R$ has the canonical differentiable structure (therefore, has a P.L. structure, too).*

PROOF. Move S'_L by ambient isotopy in V so that h and h' transversely intersect by 4.4. And adjust NS'_L and NS_R so that $(NS'_L - h'(B_1^\lambda \times B_1^{n-\lambda-1} \cup \cdots \cup B_m^\lambda \times B_m^{n-\lambda-1})) \cap NS_R = \phi$, where $B_i^\lambda \times B_i^{n-\lambda-1} \ni h'^{-1}(a_i)$ is given in the proof of 4.2. We can induce a differentiable structure to $NS'_L \cup NS_R$ from $S^\lambda \times B^{n-\lambda-1}$ and $S^{n-\lambda-1} \times B^\lambda$ in a canonical way.

Q.E.D.

The proof of the "first cancellation theorem" is similar to the P.L. case.

4.6. THEOREM (*first cancellation theorem*). *If S_R and S'_L transversely intersect at a single point q, then the cobordism is a product cobordism.*

For the proof, we need some notations and lemma.

We say that a topological manifold G is a standard model of $(\lambda, \lambda+1)$-pair if G is a disjoint union of $2B'^{\lambda+1} \times (1/2\, B'^{n-\lambda-1})$ and $B^\lambda \times B^{n-\lambda}$ by identifying $(x_1, \cdots, x_\lambda, x_{\lambda+1}, x_{\lambda+2}, \cdots, x_n) \in \partial(2B'^{\lambda+1}) \times (1/2\, B'^{n-\lambda-1})$, $x_{\lambda+1} > 0$, with $(x_1, \cdots, x_\lambda, x'_{\lambda+1}, x_{\lambda+2}, \cdots, x_n) \in B^\lambda \times \partial B^{n-\lambda}$, $x'_{\lambda+1} > 0$.

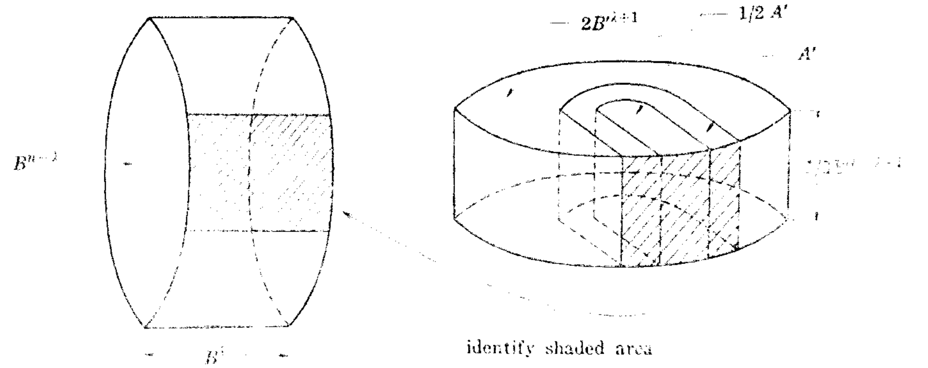

Fig. 4.1.

Define Morse function $f_{B^\lambda \times B^{n-\lambda}}$ on $B^\lambda \times B^{n-\lambda}$ by $f_{B^\lambda \times B^{n-\lambda}}(x_1, \cdots, x_n) = -(x_1^2 + \cdots + x_\lambda^2) + x_{\lambda+1}^2 + \cdots + x_n^2$, and f' on $2B'^{\lambda+1} \times (1/2\, B'^{n-\lambda-1})$ by $f'(x_1, \cdots, x_n) = -x_1^2 - \cdots - x_{\lambda+1}^2 + x_{\lambda+2}^2 + \cdots + x_n^2$, then we have the differentiable orthogonal trajectories on $B^\lambda \times B^{n-\lambda}$ and on $2B'^{\lambda+1} \times (1/2\, B'^{n-\lambda-1})$. We obtain family of curves on G by pasting trajectories at identifying points. Let A, $1/2A$ be subsets of $2B'^{\lambda+1}\times 1/2\,B'^{n-\lambda-1}$ defined as follows; $2B'^{\lambda+1}\supset A'=B'^{\lambda+1}\cup\{x;\ x_1^2+\cdots+x_\lambda^2\leq 1, x_{\lambda+1}>0\}$ $1/2\,A'=1/2\,B'^{\lambda+1}\cup\{x;\ x_1^2+\cdots+x^2\leq 1/4\}$, then $A=A'\times 1/2\,B'^{n-\lambda-1}$ $1/2\,A=1/2\,A'\times 1/2\,B'^{n-\lambda-1}$.

4.7. LEMMA. *There exists a homeomorphism* $h: G-(1/2\,\mathring{B}^\lambda\times B^{n-\lambda}\cup 1/2\,\mathring{A})\rightarrow G$ *such that* $h|\partial B^\lambda\times B^{n-\lambda}\cup(\partial(2B'^{\lambda+1})\times 1/2\,B'^{n-\lambda-1}-\mathring{A})$ *is the identity.*

PROOF. It is easy to see that G collapses to $G-1/2\,\mathring{B}^\lambda\times B^{n-\lambda}\cup 1/2\,\mathring{A}$. Therefore the lemma is trivial (see Hudson [1]). Q.E.D.

PROOF OF THE THEOREM. Let G be a standard model of the $(\lambda, \lambda+1)$-pair, and let $NS_R=B_1^\lambda\times S_R$ and $NS'_L=S'_L\times B_2^{n-\lambda-1}$, then there exists a homeomorphism $\varphi: E=D_R\times B_1^\lambda\cup D'_L\times B_2^{n-\lambda-1}\rightarrow G$ which corresponds the trajectories of $D_R\times B_1^\lambda\cup D'_L\times B_2^{n-\lambda-1}$ with the curves of G, and $\varphi(p)=(0,0)\in B^\lambda\times B^{n-\lambda}$, $\varphi(p')=(0,0)\in B'^{\lambda+1}\times B'^{n-\lambda-1}$.

By 2.9 and thickening handle, cc' is obtained from c by adding $\lambda+1$ handle $D'_L\times B_2^{n-\lambda-1}$. By 4.7, $cc'=(cc'-\mathring{E})\cup_{\text{id.}}E$ is homeomorphic to $(cc'-\mathring{E})\cup_{\text{id.}}(E-\varphi^{-1}(1/2\,B^\lambda\times B^{n-\lambda}\cup 1/2\,A))=cc'-\varphi^{-1}(1/2\,B^\lambda\times B^{n-\lambda}\cup 1/2\,A)$. It is easy to see that f has no critical points in it and that each trajectory intersect at single point with its boundary. Q.E.D.

Now we will show that the "stronger cancellation theorem" is reduced to the P.L. case.

Assume that S'_L and S_R are transversely intersect in V with respect to the attaching neighborhood. As it was proved (Corollary 4.5), $NS'_L\cup NS_R$ has differentiable structure, in which S'_L and S_R are differentiable submanifold. An intersection number (in topological category) $S'_L\cdot S_R$ of S'_L and S_R is the intersection number $S'_L\cdot S_R$ in differentiable sense (see Milnor [1]). Thus it coincides with algebraic intersection number.

4.9. THEOREM. *Let* $(W^n;\ V_0, V_1)$ *be a triad, and let* W, V_0 *and* V_1 *are simply connected and let* $\lambda\geqq 2$, $\lambda+1\leqq n-3$, $n\geqq 7$. *If* $S_R\cdot S'_L=\pm 1$, *then we can alter an associated trajectory field near* V, *so that right-hand and left-hand spheres in* V *intersect in a single point.*

PROOF. Choose a handlebody decomposition of $V-\text{Int}\,(NS'_L\cup NS_R)$, and take the 3-skeleton K' of this handlebody decomposition relative to $\partial(V-\text{Int}\,(NS'_L\cup NS_R))$. Then the theorem is reduced to the P.L. case by introducing a P.L. structure on $K'\cup(NS'_L\cup NS_R)$. Q.E.D.

§5. Cancellation of the critical points

We will cancel the critical points of the middle dimensions by the method essentially same as differentiable case (see Milnor [1]).

5.1. the THEOREM. *Let* $(W;\ V,\ V')$ *be a triad of dimension* $n\geqq 7$, *possessing a Morse function with no critical points of indices* 0,1, *or* $n-1$, n, *and let* W, V *and* V' *be all simply connected. If* $H_*(W, V)=0$, *then* $(W;\ V,\ V')$ *is product cobordism.*

PROOF. We adopt the method essentially the same as Milnor [1] except Lemma 7.7 of Milnor [1]. We need only the following lemma to complete the proof.

Q.E.D.

Let V_0^{n-1} be a topological manifold, and $h: S^{\lambda-1}\times R^{n-\lambda}\to V_0$ and $h_i: R^{\lambda}\times S^{n-\lambda-1}\to V_0$; $i=1, \cdots, k$, embeddings, such that, $S_i\cap S_j=\phi$, for any $i\neq j$, where $S_0=h(S^{\lambda-1}\times 0)$, $S_i=h_i(0\times S^{n-\lambda-1})$, and let $a\in S_0$ and $b\in S_1$ be given points.

5.2. LEMMA. *Assume that* $n\geqq 7$, $2\leqq\lambda\leqq n-2$, *then there exists an embedding* $\varphi: (0, 3)\times R^{\lambda-1}\times R^{n-\lambda-1}\to V_0$ *such that*

1) $\varphi|(0, 3)\times 0\times 0$ *is a curve which intersects each of* S_0 *and* S_1 *once orthogonally in* $\varphi(1, 0, 0)=a$, $\varphi(2, 0, 0)=b$.

2) $\varphi^{-1}(S_0)=1\times R^{\lambda-1}\times 0$, $\varphi^{-1}(S_1)=2\times 0\times R^{n-\lambda-1}$.

3) *The image of* φ *misses the other spheres. Moreover,* φ *can be chosen so that* $\varphi((0, 3)\times R^{\lambda-1}\times 0)$ *intersect* S_1 *at* b *with intersection number* $+1$.

PROOF. At first, we choose some locally flat curve $\varphi_1: (0, 3)\to V_0$, such that $\varphi_1(0, 3)$ intersects each of S_0 and S_1 once orthogonally in $\varphi_1(1)=a$, $\varphi_1(2)=b$. By M. Brown and H. Gluck [1], an arbitrary locally flat simple closed curve in V_0 has a trivial tubular neighborhood. Hence we can choose an embedding $\varphi': (0, 3)\times R^{\lambda-1}\times R^{n-\lambda-1}\to V_0$, such that $\varphi'(s, 0, 0)=\varphi_1(s)$. Then applying the method of Theorem 4.2, we can construct an ambient isotopy $\Gamma_t: V_0\to V_0$; $0\leqq t\leqq 1$, such that $\Gamma_0=\mathrm{id.}$, and that $\Gamma_1\varphi'|(0, 3)\times R^{\lambda-1}\times 0$ transversely intersects with $h_1|_{B^\lambda S^{n-\lambda-1}}$ at b with intersection number 1, and $\Gamma_1\varphi'|_{(0,3)\times 0\times R^{n-\lambda-1}}$ transversely intersects with $h|_{S^{\lambda-1}\times B^{n-\lambda}}$ at a. Eliminate new intersections of $\Gamma_1\varphi'|_{(0,3)\times 0\times 0}$ with S_0 and S_i, by the general position theorem and Theorem 3.3, then we get a map $\varphi: (0, 3)\times R^{\lambda-1}\times R^{n-\lambda-1}=(0, 3)\times \mathring{B}^{\lambda-1}\times \mathring{B}^{n-\lambda-1}\to V_0$ which satisfies the conditions. Q.E.D.

Now, we will state the theorem we have been striving to prove,

5.3. THEOREM (*h-cobordism theorem in the topological case*).
Suppose the triad of topological manifold $(W^n;\ V,\ V')$ *has the properties*

1) W, V *and* V' *are simply connected,*

2) $H_*(W, V)=0$

3) $\dim W=n\geqq 7$.

Then W *is homeomorphic to* $V\times[0, 1]$.

For the proof we show the following

5.4. LEMMA. *Let V^v be a closed manifold, $v \geqq 6$. If V is simply connected, then arbitrary flat embeddings $f,\ g: S^1 \to V$ are ambient isotopic.*

PROOF. We denote $S_1 = f(S^1)$. $S_2 = g(S^1) \subset V$. Then we may regard as $S_1 \times R^{v-1}$, $S_2 \times R^{v-1} \subset V$. By the theorem 3.3 and the general position theorem, we can assume that $S_1 \cap S_2 = \phi$, and therefore, that $S_1 \times R^{v-1} \cap S_2 \times R^{v-1} = \phi$.

Fix a handlebody decomposition of $V - (S_1 \times \mathring{B}^{v-1} \cup S_2 \times \mathring{B}^{v-1})$ and take the 3-skeleton V' of this handlebody decomposition relative to $\partial(V - (S_1 \times \mathring{B}^{v-1} \cup S_2 \times \mathring{B}^{v-1}))$, then the lemma is reduced to the P.L. case by introducing a P.L. structure on $V' \cup S_1 \times B^{v-1} \cup S_2 \times B^{v-1}$. Q.E.D.

PROOF OF 5.3. By the above lemma, Theorem 4.2 and the same argument with Milnor [1], we can eliminate critical points of indices 0, 1, $n-1$ and n. Now Theorem 5.1 gives the desired conclusion. Q.E.D.

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