Ample sheaves

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Let X be a noetherian scheme, and let $\mathcal{O}_X$ be its strucure sheaf. An invertible sheaf $\mathcal{L}$ on X is said to be ample, if for each coherent $\mathcal{O}_X$-module $\mathcal{F}$ there exists an integer n_0 such that for any $n \geqq n_0$ the $\mathcal{O}_X$-module $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by its global sections. Replacing $\mathcal{L}^{\otimes n}$ in the definition by the n-th symmetric product $\mathcal{S}^n(\mathcal{E})$ of a locally free sheaf of finite rank $\mathcal{E}$, R. Hartshorne [2] defined ample locally free sheaves of arbitrary rank. Then he tried to transport the theory of ample invertible sheaves to his case. In this paper we extend the theory to coherent sheaves of graded algebras. Let $\mathcal{A} = \mathcal{O}_X \oplus (\bigoplus_{n \geqq 1} \mathcal{A}_n)$ be a coherent graded $\mathcal{O}_X$-algebra which is generated by $\mathcal{A}_1$. $\mathcal{A}$ will be said to be ample, if for each coherent $\mathcal{O}_X$-module $\mathcal{F}$ there exists an integer n_0 such that for any $n \geqq n_0$ the $\mathcal{O}_X$-module $\mathcal{F} \otimes \mathcal{A}_n$ is generated by its global sections. This definition seems quite natural. In section 1, we show some conditions equivalent to the above condition (Theorem 1 and Theorem 2). They generalize the equivalent conditions for $\mathcal{A} = \bigoplus_{n \geqq 0} \mathcal{L}^{\otimes n}$ or $\mathcal{A} = \mathcal{S}(\mathcal{E}) = \bigoplus_{n \geqq 0} \mathcal{S}^n(\mathcal{E})$. In section 2, we next show some elementary properties of ample graded $\mathcal{O}_X$-algebras.

We make extensive use of notations, conventions, and results in [1]. If we read, say II. 4. 6. 8, it refers to the paragraph 4. 6. 8 in chapter II of [1]. We recall in section 0 some fundamental results in [1].

0. Preliminaries.

1) Let X and S be preschemes, and let $f: X \to S$ be a morphism. The direct image of an $\mathcal{O}_X$-module $\mathcal{F}$ is denoted by $f_*(\mathcal{F})$, and $f^*(\mathcal{G})$ denotes the inverse image of an $\mathcal{O}_S$-module $\mathcal{G}$. f_* is a left exact covariant functor, and f^* is a right exact covariant functor. We have a canonical bijection between $\mathrm{Hom}_{\mathcal{O}_X}(f^*(\mathcal{G}), \mathcal{F})$ and $\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{G}, f_*(\mathcal{F}))$. Hence, there are canonical homomorphisms $f^* f_*(\mathcal{F}) \to \mathcal{F}$ and $\mathcal{G} \to f_* f^*(\mathcal{G})$ which are corresponding to identical homomorphisms of $f_*(\mathcal{F})$ and $f^*(\mathcal{G})$ respectively. (See 0.3 and 0.4.)

2) Let X, S, and f be as above. Let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_X$-algebra, and let $\mathcal{R}$ be a quasi-coherent $\mathcal{O}_S$-algebra. Then $\mathrm{Spec}(f^*(\mathcal{R})) = \mathrm{Spec}(\mathcal{R}) X_S X$,

and we have canonical bijections among $\mathrm{Hom}_{\mathcal{O}_X}(f^*(\mathscr{R}), \mathscr{A})$, $\mathrm{Hom}_X(\mathrm{Spec}(\mathscr{A}), \mathrm{Spec}(f^*(\mathscr{R})))$, and $\mathrm{Hom}_S(\mathrm{Spec}(\mathscr{A}), \mathrm{Spec}(\mathscr{R}))$. A morphism $\mathrm{Spec}(\mathscr{A})\to \mathrm{Spec}(\mathscr{R})$ which corresponds to a homomorphism $f^*(\mathscr{R})\to\mathscr{A}$, is the composition of $\mathrm{Spec}(\mathscr{A})\to\mathrm{Spec}(f^*(\mathscr{R}))$ and the projection $\mathrm{Spec}(f^*(\mathscr{R}))\to\mathrm{Spec}(\mathscr{R})$. (See II. 1.)

If $\mathscr{A}=\mathcal{O}_X\oplus(\bigoplus_{n\geq 1}\mathscr{A}_n)$ is a quasi-coherent graded $\mathcal{O}_X$-algebra on a prescheme X, we have a closed immersion $X\to\mathrm{Spec}(\mathscr{A})$ which corresponds to the augmentation i.e. the natural surjection $\mathscr{A}\to\mathcal{O}_X$. This closed immersion is called the zero-section of $\mathrm{Spec}(\mathscr{A})$.

3) Let X, S, and f be as above. Let $\mathscr{A}=\bigoplus_{n\geq 0}\mathscr{A}_n$ and $\mathscr{R}=\bigoplus_{n\geq 0}\mathscr{R}_n$ be a quasi-coherent graded $\mathcal{O}_X$-algebra and a quasi-coherent graded $\mathcal{O}_S$-algebra respectively. With a homomorphism (of degree zero) of graded $\mathcal{O}_X$-algebras $\varphi: f^*(\mathscr{R})\to\mathscr{A}$, we can associate, in a canonical way, an open subset G of $\mathrm{Proj}(\mathscr{A})$ and a morphism $G\to\mathrm{Proj}(f^*(\mathscr{R}))$. Since $\mathrm{Proj}(f^*(\mathscr{R}))=\mathrm{Proj}(\mathscr{R})X_SX$, we have another morphism $G\to\mathrm{Proj}(\mathscr{R})$ which is the composition of $G\to\mathrm{Proj}(f^*(\mathscr{R}))$ and the projection $\mathrm{Proj}(f^*(\mathscr{R}))\to\mathrm{Proj}(\mathscr{R})$. These two morphisms are called morphisms associated with φ.

4) Let X be a prescheme, and let $\mathscr{A}=\mathcal{O}_X\oplus(\bigoplus_{n\geq 1}\mathscr{A}_n)$ be a quasi-coherent graded $\mathcal{O}_X$-algebra. Put $\hat{\mathscr{A}}=\mathscr{A}[u]$, where u is an indeterminate. We consider $\hat{\mathscr{A}}$ as a graded $\mathcal{O}_X$-algebra whose homogeneous part of degree n is $\bigoplus_{k=0}^{n}\mathscr{A}_k\cdot u^{n-k}$. Put $Y=\mathrm{Proj}(\mathscr{A})$, $V=\mathrm{Spec}(\mathscr{A})$, and $\hat{V}=\mathrm{Proj}(\hat{\mathscr{A}})$. Then, we have a closed immersion $X\to V$ which corresponds to the augmentation $\mathscr{A}\to\mathcal{O}_X$ (the zero-section of V), and an open immersion $V\to\hat{V}$ by which V is identified with an open subset $D_+(u)$ of $\hat{V}$. The composition of these two morphisms is a closed immersion $X\to\hat{V}$ which is associated with a natural surjection of graded $\mathcal{O}_X$-algebras $\hat{\mathscr{A}}\to\mathcal{O}_X[u]$. The canonical morphism associated with the injection $\mathscr{A}\to\hat{\mathscr{A}}$, is $\hat{V}-X\to Y$ which is surjective and affine. Let $V_0=V-X$ and $\hat{V}_0=\hat{V}-X$ be open subsets of V and $\hat{V}$ respectively, then $V_0=V\cap\hat{V}_0$.

$$\begin{array}{ccccccc} X & \longrightarrow & V & \longrightarrow & \hat{V} & \qquad & \hat{V} & & \\ & & \cup & & \cup & & \cup & & \\ & & V_0 & & \hat{V}_0 & & \hat{V}_0 & \longrightarrow & Y \end{array}$$

Now, assume that X is an affine scheme, and put $A=\Gamma(X,\mathscr{A})=\bigoplus_{n\geq 0}\Gamma(X,\mathscr{A}_n)$. If E is a subset of homogeneous elements of A such that the radical of the ideal generated by E is $A_+=\bigoplus_{n\geq 1}A_n$, then $V_0=\bigcup_{f\in E}D(f)$ and $\hat{V}_0=\bigcup_{f\in E}D_+(f)$. In the latter case, we consider a section f as an element of $\Gamma(X,\hat{\mathscr{A}})$. (See II.8.3.)

5) Besides the above notation, let X' be a prescheme, and let $\mathscr{A}'=\bigoplus_{n\geq 0}\mathscr{A}'_n$

be a quasi-coherent graded $\mathcal{O}_{X'}$-algebra such that $\mathcal{A}'_0=\mathcal{O}_{X'}$. Put $\hat{\mathcal{A}}'=\mathcal{A}'[u]$, $Y'=\mathrm{Proj}(\mathcal{A}')$, $V'=\mathrm{Spec}(\mathcal{A}')$, and $\hat{V}'=\mathrm{Proj}(\hat{\mathcal{A}}')$. Let $f: X\to X'$ be a morphism, and let $\varphi: f^*(\mathcal{A}')\to\mathcal{A}$ be a homomorphism (of degree zero) of graded $\mathcal{O}_X$-algebras. We can extend φ, naturally, to a homomorphism $\hat{\varphi}: f^*(\hat{\mathcal{A}}')\to\hat{\mathcal{A}}$. If the morphism associated with φ is everywhere defined; $Y\to Y'$, then the morphism associated with $\hat{\varphi}$ is also everywhere defined; $\hat{V}\to\hat{V}'$. Denote this morphism by $\hat{\Phi}$, and denote the morphism $V\to V'$ which corresponds to φ by Φ. Then, $\hat{V}_0=\hat{V}-X=\hat{\Phi}^{-1}(\hat{V}'_0)=\hat{V}'_0 X_{Y'} Y$ (where $\hat{V}'_0=\hat{V}'-X'$), and $V_0=V-X=\Phi^{-1}(V'_0)$ (where $V'_0=V'-X'$). Hence, if $Y\to Y'$ is an isomorphism (an open immersion resp.), $\hat{\Phi}|_{\hat{V}_0}:\hat{V}_0\to\hat{V}'_0$ and $\Phi|_{V_0}:V_0\to V'_0$ are isomorphisms (open immersions resp.). (See II. 8. 5.)

6) Let X be a noetherian scheme, and let $\mathcal{A}=\bigoplus_{n\geq 0}\mathcal{A}_n$ be a quasi-coherent graded $\mathcal{O}_X$-algebra of finite type. Let $p: Y=\mathrm{Proj}(\mathcal{A})\to X$ be the canonical projective morphism. Then, for any coherent $\mathcal{O}_Y$-module $\mathcal{F}$, a canonical homomorphism $f^*f_*(\mathcal{F}\otimes\mathcal{O}_Y(n))\to\mathcal{F}\otimes\mathcal{O}_Y(n)$ is surjective if n is large enough (III. 2. 2. 1). If a coherent $\mathcal{O}_Y$-module $\mathcal{F}$ is associated with a quasi-coherent graded $\mathcal{A}$-module $\mathcal{M}=\bigoplus_{n\in Z}\mathcal{M}_n$ such that $\bigoplus_{n\geq k}\mathcal{M}_n$ is an $\mathcal{A}$-module of finite type for k large enough, a canonical homomorphism $\mathcal{M}_n\to p_*(\mathcal{F}\otimes\mathcal{O}_Y(n))$ is isomorphic for n large enough (III. 2. 3. 1).

1. Definition of ample sheaves.

First we recall the following theorem without proof.

THEOREM-DEFINITION. Let X and S be noetherian schemes, and let $f: X\to S$ be a separated morphism of finite type. For an invertible sheaf $\mathcal{L}$ on X, the following three conditions are equivalent. We say $\mathcal{L}$ is f-ample or S-ample, if it satisfies them.

1) For each coherent $\mathcal{O}_X$-module $\mathcal{F}$, there exists an integer n_0 such that $f^*f_*(\mathcal{F}\otimes\mathcal{L}^{\otimes n})\to\mathcal{F}\otimes\mathcal{L}^{\otimes n}$ is surjective for $n\geqq n_0$ (II. 4. 6. 8).

2) Put $\mathcal{R}=\mathcal{O}_S\oplus(\bigoplus_{n\geq 1}f_*(\mathcal{L}^{\otimes n}))$, then the morphism associated with a canonical homomorphism $f^*(\mathcal{R})\to\bigoplus_{n\geq 0}\mathcal{L}^{\otimes n}$ is everywhere defined and a dominating open immersion $X\cong\mathrm{Proj}(\bigoplus_{n\geq 0}\mathcal{L}^{\otimes n})\to T=\mathrm{Proj}(\mathcal{R})$ (II. 4. 6. 3).

3) There exist an S-scheme C, an S-section $S\to C$ (which is necessarily a closed immersion), and an S-morphism $v: V=\mathrm{Spec}(\bigoplus_{n\geq 0}\mathcal{L}^{\otimes n})\to C$ such that a diagram

$$\begin{array}{ccc} X & \longrightarrow & V \\ \downarrow & & \downarrow v \\ S & \longrightarrow & C \end{array}$$

is commutative, and that the restriction of v to $V-X$ is an open immersion into $C-S$. Where $X\to V$ is the zero-section of V, and we identify S and X with their images in C and V respectively. (II. 8. 9. 1).

Now let $\mathscr{A}=\bigoplus_{n\geqq 0}\mathscr{A}_n$ be a coherent graded $\mathscr{O}_X$-algebra. We say $\mathscr{M}=\bigoplus_{n\in Z}\mathscr{M}_n$ is a *TF* graded $\mathscr{A}$-module, if it satisfies the condition (*TF*) in (II. 3. 4. 2), i.e. a graded $\mathscr{A}$-module $\bigoplus_{n\geqq k}\mathscr{M}_n$ is finite type for k large enough.

We generalize the above theorem as follows.

THEOREM 1. *Let $f: X\to S$ be a separated morphism of finite type between noetherian schemes X and S, and let $\mathscr{A}=\mathscr{O}_X\oplus(\bigoplus_{n\geqq 1}\mathscr{A}_n)$ be a coherent graded $\mathscr{O}_X$-algebra generated by $\mathscr{A}_1$. Let $p: Y=\mathrm{Proj}(\mathscr{A})\to X$ be the canonical projective morphism, and put $g=f\cdot p: Y\to S$. Then the following five conditions for $\mathscr{A}$ are equivalent.*

0) *The tautological invertible sheaf $\mathscr{O}_Y(1)$ on Y is g-ample.*

1) *For each TF graded $\mathscr{A}$-module $\mathscr{M}=\bigoplus_{n\in Z}\mathscr{M}_n$, a canonical homomorphism $f^*f_*(\mathscr{M})\to\mathscr{M}$ is TN-surjective, i.e. $f^*f_*(\mathscr{M}_n)\to\mathscr{M}_n$ is surjective for all n large enough.*

1′) *For each coherent $\mathscr{O}_X$-module $\mathscr{F}$, a canonical homomorphism $f^*f_*(\mathscr{F}\otimes\mathscr{A})\to\mathscr{F}\otimes\mathscr{A}$ is TN-surjective.*

2) *Put $\mathscr{R}=\mathscr{O}_S\oplus(\bigoplus_{n\geqq 1}f_*(\mathscr{A}_n))$, then the morphism associated with a canonical homomorphism $f^*(\mathscr{R})\to\mathscr{A}$ is everywhere defined and a dominating open immersion $Y\to T=\mathrm{Proj}(\mathscr{R})$.*

3) *There exist an S-scheme C, an S-section $S\to C$ (which is necessarily a closed immersion), and an S-morphism $v: V=\mathrm{Spec}(\mathscr{A})\to C$ such that a diagram*

$$\begin{array}{ccc} X & \longrightarrow & V \\ \downarrow & & \downarrow v \\ S & \longrightarrow & C \end{array}$$

is commutative, and that the restriction of v to $V-X$ is an open immersion into $C-S$. Where $X\to V$ is the zero-section of V, and we identify S and X with their images in C and V respectively.

PROOF. 1)$\Longleftrightarrow$1′). Let $\mathscr{M}=\bigoplus_{n\in Z}\mathscr{M}_n$ be a *TF* graded $\mathscr{A}$-module. We know that, for k large enough, $\mathscr{M}_k$ is a coherent $\mathscr{O}_X$-module, and that a canonical homomorphism $\mathcal{M}_k\otimes\mathcal{A}\to\mathcal{M}(k)=\bigoplus_{n\in Z}\mathcal{M}_{n+k}$ is TN-surjective (II. 2. 1. 6). Hence, if $f^*f_*(\mathcal{M}_k\otimes\mathcal{A})\to\mathcal{M}_k\otimes\mathcal{A}$ is TN-surjective, a commutative diagram of homomorphisms

$$\begin{array}{ccc} f^*f_*(\mathcal{M}_k\otimes\mathcal{A}) & \longrightarrow & \mathcal{M}_k\otimes\mathcal{A} \\ \downarrow & & \downarrow \\ f^*f_*(\mathcal{M}(k)) & \longrightarrow & \mathcal{M}(k) \end{array}$$

shows that $f^*f_*(\mathcal{M}(k))\to\mathcal{M}(k)$ is also TN-surjective, which tells us that 1′) implies 1). The converse is obvious.

1)$\Longrightarrow$0). Assume 1), and let $\mathcal{F}$ be a coherent $\mathcal{O}_Y$-module. A canonical homomorphism $p^*p_*(\mathcal{F}\otimes\mathcal{O}_Y(n))\to\mathcal{F}\otimes\mathcal{O}_Y(n)$ is surjective for n large enough. On the other hand, since $p_*(\bigoplus_{n\geq 0}(\mathcal{F}\otimes\mathcal{O}_Y(n))$ is a TF graded $\mathcal{A}$-module, $f^*f_*p_*(\mathcal{F}\otimes\mathcal{O}_Y(n))\to p_*(\mathcal{F}\otimes\mathcal{O}_Y(n))$ is surjective for n large enough by 1). Hence the surjection for large n of $g^*g_*(\mathcal{F}\otimes\mathcal{O}_Y(n))\to\mathcal{F}\otimes\mathcal{O}_Y(n)$ which is decomposed to $p^*f^*f_*p_*(\mathcal{F}\otimes\mathcal{O}_Y(n))\to p^*p_*(\mathcal{F}\otimes\mathcal{O}_Y(n))\to\mathcal{F}\otimes\mathcal{O}_Y(n)$ is proved.

0)$\Longrightarrow$1). Let $\mathcal{O}_Y(1)$ be g-ample, and let k be a positive integer such that $\mathcal{O}_Y(k)$ is g-very-ample. We have a projective morphism $P\to S$ and a dominating open immersion $j:Y\to P$ such that $j^*(\mathcal{O}_P(1))=\mathcal{O}_Y(k)$. Let $\mathcal{M}=\bigoplus_{n\in Z}\mathcal{M}_n$ be a TF graded $\mathcal{A}$-module, and let $\mathcal{F}$ be the associated coherent $\mathcal{O}_Y$-module (See II. 3. 2). Then there exists a coherent $\mathcal{O}_P$-module $\mathcal{H}$ such that $j^*(\mathcal{H})=\mathcal{F}$ (I. 9. 4. 2). Let $\mathcal{H}$ be associated with a TF graded module $\mathcal{N}=\bigoplus_{n\in Z}\mathcal{N}_n$ on S (See II. 3. 3). Since $i=(j, p):Y\to PX_SX$ is an immersion (I. 5. 3. 13) and a proper morphism (II. 5. 4. 4), it is a closed immersion. Hence, a canonical homomorphism $h^*(\mathcal{H})\to i_*i^*h^*(\mathcal{H})=i_*(\mathcal{F})$ is surjective. Where h is the projection $PX_SX\to P$. On the other hand, $h^*(\mathcal{H})$ and $i_*(\mathcal{F})$ are associated with TF graded modules $f^*(\mathcal{N})$ and $\mathcal{M}^{(k)}=\bigoplus_{n\in Z}\mathcal{M}_{nk}$ respectively. If we decompose $f^*(\mathcal{N}_n)\to\mathcal{M}_{nk}$ to $f^*(\mathcal{N}_n)\to f^*f_*(\mathcal{M}_{nk})\to\mathcal{M}_{nk}$, we can see that $f^*f_*(\mathcal{M}_{nk})\to\mathcal{M}_{nk}$ is surjective for n large enough. Taking $\mathcal{M}(i)=\bigoplus_{n\in Z}\mathcal{M}_{n+i}$ $i=1, 2, \cdots, k-1$ in place of the above $\mathcal{M}$, we know that a canonical homomorphism $f^*f_*(\mathcal{M})\to\mathcal{M}$ is TN-surjective.

2)$\Longleftrightarrow$0). The morphism $Y\to T$ in 2) is also considered to be associated with a canonical homomorphism $g^*(\mathcal{R})\to\bigoplus_{n\geq 0}\mathcal{O}_Y(n)$. $\mathcal{R}=\mathcal{O}_S\oplus(\bigoplus_{n\geq 1}f_*(\mathcal{A}_n))$ is TN-isomorphic to $\mathcal{O}_S\oplus(\bigoplus_{n\geq 1}g_*(\mathcal{O}_Y(n)))$, since $\mathcal{A}$ is TN-isomorphic to $p_*(\bigoplus_{n\geq 0}\mathcal{O}_Y(n))$. Thus, we have the equivalence of 2) and 0).

3)$\Longrightarrow$0). We have a morphism $W=\mathrm{Spec}(\bigoplus_{n\geq 0}\mathcal{O}_Y(n))\to V$ which is corresponding to a canonical homomorphism $p^*(\mathscr{A})\to\bigoplus_{n\geq 0}\mathscr{O}_Y(n)$. Its restriction to $W-Y$ is an isomorphism to $V-X$, since $\mathrm{Proj}(\bigoplus_{n\geq 0}\mathscr{O}_Y(n))\cong Y=\mathrm{Proj}(\mathscr{A})$. (See 5) in section 0.) Hence 3) implies 0).

2)$\Longrightarrow$3). We have a morphism $V\to C=\mathrm{Spec}(\mathscr{R})$ corresponding to a canonical homomorphism $f^*(\mathscr{R})\to\mathscr{A}$. If we assume 2), its restriction to $V-X$ is an open immersion into $C-S$, which proves 3). (See again 5) in section 0.)

Q.E.D.

DEFINITION. Let $f: X\to S$ be a separated morphism of finite type between noetherian schemes, and let $\mathscr{A}=\mathscr{O}_X\oplus(\bigoplus_{n\geq 1}\mathscr{A}_n)$ be a coherent graded $\mathscr{O}_X$-algebra generated by $\mathscr{A}_1$. We call $\mathscr{A}$ is *f-ample* or *S-ample*, if it satisfies the equivalent conditions in Theorem 1. A coherent $\mathscr{O}_X$-module $\mathscr{F}$ is called f-ample, if its symmetric algebra $\mathscr{S}(\mathscr{F})=\bigoplus_{n\geq 0}\mathscr{S}^n(\mathscr{F})$ is f-ample.

Let X be a closed subscheme of a noetherian scheme Z which is of finite type over a noetherian scheme S, and let $\mathscr{I}$ be the $\mathscr{O}_Z$-ideal which defines X. Put $\mathscr{A}=\bigoplus_{n\geq 0}\mathscr{I}^n/\mathscr{I}^{n+1}$, then $V=\mathrm{Spec}(\mathscr{A})$ is the *normal bundle* of X in Z. Theorem 1 says that X is contractible to S in its normal bundle if and only if $\mathscr{A}$ is S-ample.

We have another theorem as follows.

THEOREM 2. *Let $f: X\to S$ and $\mathscr{A}$ be as in Theorem 1, and moreover let f be proper. Then, $\mathscr{A}$ is f-ample if and only if:*

4) *For each TF graded $\mathscr{A}$-module $\mathscr{M}=\bigoplus_{n\in Z}\mathscr{M}_n$, there exists an integer n_0 such that $R^qf_*(\mathscr{M}_n)=0$ for all $n\geq n_0$ and all $q>0$.*

If $\mathscr{A}_n$ is, further, $\mathscr{O}_X$-flat for all n large enough, the above condition is equivalent to:

4') *For each coherent $\mathscr{O}_X$-module $\mathscr{F}$, there exists an integer n_0 such that $\mathscr{R}^qf_*(\mathscr{F}\otimes\mathscr{A}_n)=0$ for all $n\geq n_0$ and all $q>0$.*

PROOF. Let a coherent $\mathscr{O}_Y$-module $\mathscr{F}$ be associated with a TF graded $\mathscr{A}$-module $\mathscr{M}=\bigoplus_{n\in Z}\mathscr{M}_n$. Then, a canonical homomorphism $\mathscr{M}\to\bigoplus_{n\in Z}p_*(\mathscr{F}\otimes\mathscr{O}_Y(n))$ is TN-isomorphism. On the other hand, there exists a spectral sequence

$$E_2^{p,q}=R^qf_*(R^pp_*(\mathscr{F}\otimes\mathscr{O}_Y(n)))\Rightarrow E^r=R^rg_*(\mathscr{F}\otimes\mathscr{O}_Y(n))\ .$$

Since $R^pp_*(\mathscr{F}\otimes\mathscr{O}_Y(n))=0$ for n large enough and $p>0$ (III. 2. 2. 1: Serre's Theorem), we have an isomorphism $R^qf_*(\mathscr{M}_n)\simeq R^qg_*(\mathscr{F}\otimes\mathscr{O}_Y(n))$ for n large enough and $q\geq 0$. $\mathscr{O}_Y(1)$ is g-ample if and only if there exists an integer n_0 such that $R^qg_*(\mathscr{F}\otimes\mathscr{O}_Y(n))=0$ for all $n\geq n_0$ and all $q>0$ (III. 2. 6. 1). Hence, we have the equivalence of 4) and 0) in Theorem 1.

Now, assume 4') where $\mathscr{A}_n$ is $\mathscr{O}_X$-flat for n large enough. Let x be a closed point of X, and let $\mathscr{I}$ be the $\mathscr{O}_X$-ideal which induces the reduced structure on the closed subset $\{x\}$ of X. Let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module, and let n be an integer such that $\mathscr{A}_n$ is $\mathscr{O}_X$-flat and that $R^1f_*(\mathscr{I}\mathscr{F}\otimes\mathscr{A}_n)=0$. Then, an exact sequence

$$0\to\mathscr{I}\mathscr{F}\otimes\mathscr{A}_n\to\mathscr{F}\otimes\mathscr{A}_n\to\mathscr{F}\otimes\mathscr{A}_n\otimes\kappa(x)\to 0$$

gives a surjection $f_*(\mathscr{F}\otimes\mathscr{A}_n)\to f_*(\mathscr{F}\otimes\mathscr{A}_n\otimes\kappa(x))$. Hence, we have a surjection

$$(f^*f_*(\mathscr{F}\otimes\mathscr{A}_n))_x\to(f^*f_*(\mathscr{F}\otimes\mathscr{A}_n\otimes\kappa(x)))_x=\mathscr{F}\otimes\mathscr{A}_n\otimes\kappa(x)\,.$$

$(f^*f_*(\mathscr{F}\otimes\mathscr{A}_n))_x\to(\mathscr{F}\otimes\mathscr{A}_n)_x$ is, therefore, surjective by Nakayama's Lemma. Since $(f^*f_*(\mathscr{F}\otimes\mathscr{A}_n))_x=(f_*(\mathscr{F}\otimes\mathscr{A}_n))_{f(x)}\otimes_{\mathscr{O}_{f(x)}}\mathscr{O}_x$, and since $(\mathscr{F}\otimes\mathscr{A}_n)_x$ is finitely generated $\mathscr{O}_x$-module, there exists an open neighbourhood U of $f(x)$ such that $(\mathscr{F}\otimes\mathscr{A}_n)_x$ is generated by a finite number of sections of $f_*(\mathscr{F}\otimes\mathscr{A}_n)$ over U, say $t_i\in\Gamma(U,f_*(\mathscr{F}\otimes\mathscr{A}_n))=\Gamma(f^{-1}(U),\mathscr{F}\otimes\mathscr{A}_n)$ $i=1,2,\cdots,\nu$. Hence, there exists a neighbourhood U_0 of x $(U_0\subset f^{-1}(U))$ such that $t_i|_{U_0}$ $i=1,2,\cdots,\nu$ generate $(\mathscr{F}\otimes\mathscr{A}_n)_{x'}$ at any point x' of U_0 (0. 5. 2. 2). This means that $f^*f_*(\mathscr{F}\otimes\mathscr{A}_n)\to\mathscr{F}\otimes\mathscr{A}_n$ is surjective in a neighbourhood U_0 of x. Hence we know that $f^*f_*(\mathscr{F}\otimes\mathscr{A}_n)\to\mathscr{F}\otimes\mathscr{A}_n$ is surjective, since X is quasi-compact. This proves 1′) in Theorem 1. Obviously 4) implies 4′).

Q.E.D.

2. Properties of ample sheaves.

In this section, we always assume that $f\colon X\to S$ is a separated morphism of finite type between noetherian schemes.

PROPOSITION 1. *Let* $\mathscr{A}=\mathscr{O}_X\oplus(\bigoplus_{n\geq 1}\mathscr{A}_n)$ *and* $\mathscr{B}=\mathscr{O}_X\oplus(\bigoplus_{n\geq 1}\mathscr{B}_n)$ *be coherent graded* $\mathscr{O}_X$*-algebras generated by* $\mathscr{A}_1$ *and* $\mathscr{B}_1$ *respectively. If* $\mathscr{A}$ *is f-ample, and if* $\mathscr{B}$ *is a graded* $\mathscr{A}$*-algebra which satisfies the condition* (TF) *as a graded* $\mathscr{A}$*-module, then* $\mathscr{B}$ *is also f-ample. In particular, if* $\mathscr{A}$ *is f-ample, and if there exists a TN-surjection* $\mathscr{A}\to\mathscr{B}$, $\mathscr{B}$ *is f-ample* (*cf.* [2], *Prop.* 2. 2).

PROOF. If $\mathscr{M}=\bigoplus_{n\in Z}\mathscr{M}_n$ is a TF graded $\mathscr{B}$-module, it satisfies the condition (TF), too, as a graded $\mathscr{A}$-module. Hence, Proposition 1 is easily verified.

PROPOSITION 2 (*cf.* II. 4. 5. 6 *and* [2], *Prop.* 2. 4).

Let $\mathscr{A}=\mathscr{O}_X\oplus(\bigoplus_{n\geq 1}\mathscr{A}_n)$ *be a coherent graded* $\mathscr{O}_X$*-algebra generated by* $\mathscr{A}_1$. *Then we have the followings.*

i) *For each positive integer d,* $\mathscr{A}^{(d)}=\bigoplus_{n\geq 0}\mathscr{A}_{nd}$ *is f-ample, if and only if* $\mathscr{A}$ *is f-ample.*

ii) *If* $\mathscr{A}_d$ *or the symmetric algebra* $\mathscr{S}(\mathscr{A}_d)$ *is f-ample for some integer* d, $\mathscr{A}$ *is f-ample.*

PROOF. There exists a canonical isomorphism $Y=\mathrm{Proj}(\mathscr{A})\simeq Y'=\mathrm{Proj}(\mathscr{A}^{(d)})$ by which $\mathscr{O}_Y(d)$ and $\mathscr{O}_{Y'}(1)$ are identified. Hence, we have i) by (II. 4. 5. 6) which says that $\mathscr{O}_Y(d)$ is f-ample if and only if $\mathscr{O}_Y(1)$ is f-ample. Since there exists a surjection $\mathscr{S}(\mathscr{A}_d)\to\mathscr{A}^{(d)}$, i) and Proposition 1 prove ii).

PROPOSITION 3. *Let* $\mathscr{A}=\mathscr{O}_X\oplus(\bigoplus_{n\geq 1}\mathscr{A}_n)$ *and* $\mathscr{B}=\mathscr{O}_X\oplus(\bigoplus_{n\geq 1}\mathscr{B}_n)$ *be coherent graded* $\mathscr{O}_X$*-algebras generated by* $\mathscr{A}_1$ *and* $\mathscr{B}_1$ *respectively.*

i) $\mathscr{A}\oplus\mathscr{B}$ *is f-ample, if and only if* $\mathscr{A}$ *and* $\mathscr{B}$ *are both f-ample.*

ii) $\mathscr{A}\otimes\mathscr{B}$ *is f-ample, if and only if* $\mathscr{A}$ *and* $\mathscr{B}$ *are both f-ample* (*cf.* [2], *Prop.* 2. 2).

iii) *Let a graded* $\mathscr{O}_X$*-algebra* $\bigoplus_{n\geq 0}(\mathscr{A}_n\otimes\mathscr{B}_n)$ *be denoted by* $\Delta(\mathscr{A}\otimes\mathscr{B})$. *Then* $\Delta(\mathscr{A}\otimes\mathscr{B})$ *is f-ample, if* $\mathscr{A}$ *and* $\mathscr{B}$ *are both f-ample* (*cf.* II. 4. 5. 7 *and* [2], *Cor.* 2. 3).

PROOF. To have i), it is sufficient to see the *if* part, by Proposition 1. Let $\mathscr{M}=\bigoplus_{n\in Z}\mathscr{M}_n$ be a TF graded $(\mathscr{A}\oplus\mathscr{B})$-module. For k large enough, $\mathscr{M}_k$ is a coherent $\mathscr{O}_X$-module, and a canonical homomorphism $\mathscr{M}_k\otimes(\mathscr{A}\oplus\mathscr{B})\to\mathscr{M}(k)$ is TN-surjective. Hence, a commutative diagram

$$\begin{array}{ccc} f^*f_*(\mathscr{M}_k\otimes\mathscr{A})\oplus f^*f_*(\mathscr{M}_k\otimes\mathscr{B}) & \longrightarrow & \mathscr{M}_k\otimes(\mathscr{A}\oplus\mathscr{B}) \\ \downarrow & & \downarrow \\ f^*f_*(\mathscr{M}(k)) & \longrightarrow & \mathscr{M}(k) \end{array}$$

shows that $\mathscr{A}\oplus\mathscr{B}$ is f-ample, if $\mathscr{A}$ and $\mathscr{B}$ are both f-ample.

Let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module, and put

$$\sigma_{p,q}: f^*f_*(\mathscr{F}\otimes\mathscr{A}_p\otimes\mathscr{B}_q)\to\mathscr{F}\otimes\mathscr{A}_p\otimes\mathscr{B}_q\,.$$

If $\mathscr{A}$ and $\mathscr{B}$ are both f-ample, we have; a) there exists an integer n_q for each $q\geq 0$, such that $\sigma_{p,q}$ is surjective if $p\geq n_q$ (since $\mathscr{A}$ is f-ample). b) there exists an integer m_p for each $p\geq 0$, such that $\sigma_{p,q}$ is surjective if $q\geq m_p$ (since $\mathscr{B}$ is f-ample). c) there exists an integer k such that $f^*f_*(\mathscr{F}\otimes\mathscr{A}_p)\to\mathscr{F}\otimes\mathscr{A}_p$, $f^*f_*(\mathscr{B}_q)\to\mathscr{B}_q$, and hence $\sigma_{p,q}$ are surjective if $p\geq k$ and $q\geq k$ (since $\mathscr{A}$ and $\mathscr{B}$ are both f-ample). By a), b), and c) $\sigma_{p,q}$ is surjective if $p+q\geq\max_{0\leq s,t<k}(s+m_s, n_t+t)$. This proves the *if* part of ii). The *only if* part is an immediate consequence of Proposition 1. The assertion c) proves iii).

PROPOSITION 4. *Let* $\mathscr{A}$ *and* $\mathscr{B}$ *be as above. If* $\mathscr{A}$ *is f-ample, and if a canonical homomorphism* $f^*f_*(\mathscr{B})\to\mathscr{B}$ *is TN-surjective,* $\Delta(\mathscr{A}\otimes\mathscr{B})$ *is f-ample* (*cf.* II. 4. 5. 6 *and* [2], *Cor.* 2. 3).

PROOF. The assertion c) in the above proof, also holds in this case. Hence we have the proposition.

PROPOSITION 5 (*cf.* II. 4. 5. 8). *Let* $\mathscr{A}$ *and* $\mathscr{B}$ *be as above. If* $\mathscr{A}$ *is f-ample, we have the followings.*

i) *There exists a positive integer* d_0 *such that a canonical homomorphism* $f^*f_*(\Delta(\mathscr{A}^{(d)}\otimes\mathscr{B}))\to\Delta(\mathscr{A}^{(d)}\otimes B)$ *is surjective if* $d\geqq d_0$.

ii) *There exists a positive integer* d_0 *such that* $\Delta(\mathscr{A}^{(d)}\otimes\mathscr{B})$ *is f-ample if* $d\geqq d_0$.

PROOF. Since $\mathscr{A}$ is f-ample, $f^*f_*(\mathscr{A}_d\otimes\mathscr{B}_1)\to\mathscr{A}_d\otimes\mathscr{B}_1$ is surjective for d large enough. Hence $f^*f_*((\mathscr{A}_d\otimes\mathscr{B}_1)^{\otimes n})\to(\mathscr{A}_d\otimes\mathscr{B}_1)^{\otimes n}$ is surjective for d large enough. On the other hand, we have the following commutative diagram:

$$\begin{array}{ccc} f^*f_*((\mathscr{A}_d\otimes\mathscr{B}_1)^{\otimes n}) & \longrightarrow & (\mathscr{A}_d\otimes\mathscr{B}_1)^{\otimes n} \\ \downarrow & & \downarrow \\ f^*f_*(\mathscr{A}_{nd}\otimes\mathscr{B}_n) & \longrightarrow & \mathscr{A}_{nd}\otimes\mathscr{B}_n\,. \end{array}$$

Since $(\mathscr{A}_d\otimes\mathscr{B}_1)^{\otimes n}\cong\mathscr{A}_d^{\otimes n}\otimes\mathscr{B}_1^{\otimes n}$ and $\mathscr{A}_d^{\otimes n}\otimes\mathscr{B}_1^{\otimes n}\to\mathscr{A}_{nd}\otimes\mathscr{B}_n$ is surjective, we have i). ii) is given by i) and Proposition 4.

PROPOSITION 6. *Let* $g: Y\to S$ *be a separated morphism of finite type, and let* $j: Y\to X$ *be an immersive S-morphism. If* $\mathscr{A}$ *is an f-ample coherent graded* $\mathscr{O}_X$*-algebra on X,* $\mathscr{B}=j^*(\mathscr{A})$ *is g-ample (cf.* [2], *Prop.* 4. 1).

PROOF. Let $\mathscr{N}=\bigoplus_{n\in Z}\mathscr{N}_n$ be a TF graded $\mathscr{B}$-module on Y. If j is an open immersion, there exists a TF graded $\mathscr{A}$-module $\mathscr{M}=\bigoplus_{n\in Z}\mathscr{M}_n$ such that $j^*(\mathscr{M})=\mathscr{N}$. If j is a closed immersion, $\mathscr{M}=j_*(\mathscr{N})$ is a TF graded $\mathscr{A}$-module and $j^*(\mathscr{M})=\mathscr{N}$. Hence, for any immersion j, we can find a TF graded $\mathscr{A}$-module $\mathscr{M}=\bigoplus_{n\in Z}\mathscr{M}_n$ such that $j^*(\mathscr{M})=\mathscr{N}$. If $\mathscr{A}$ is f-ample, $f^*f_*(\mathscr{M})\to\mathscr{M}$ is TN-surjective, and therefore $g^*g_*(\mathscr{N})\to\mathscr{N}$ is also TN-surjective. This proves the proposition.

$$\begin{array}{ccc} j^*f^*f_*(\mathscr{M}) & \longrightarrow & j^*(\mathscr{M}) \\ \downarrow & & \| \\ g^*g_*(\mathscr{N}) & \longrightarrow & \mathscr{N} \end{array}$$

PROPOSITION 7. *Let* $\mathscr{A}=\mathscr{O}_X\oplus(\bigoplus_{n\geqq 1}\mathscr{A}_n)$ *be a coherent graded* $\mathscr{O}_X$*-algebra generated by* $\mathscr{A}_1$. *Let Y be a closed subscheme of X defined by a nilpotent* $\mathscr{O}_X$*-ideal. Then* $\mathscr{A}$ *is f-ample, if and only if* $\mathscr{A}\otimes\mathscr{O}_Y$ *is* $f|_Y$*-ample.*

PROOF. Passing to Proj$(\mathscr{A})$ and Proj$(\mathscr{A}\otimes\mathscr{O}_Y)$, we reduce to the case of an invertible sheaf (II. 4. 5. 13).

PROPOSITION 8. *Let* $f: X\to S$ *be a proper morphism, and let* $\mathscr{A}=\mathscr{O}_X\oplus$

$(\bigoplus_{n\geq 1} \mathscr{A}_n)$ *be a coherent graded* $\mathcal{O}_X$*-algebra generated by* $\mathscr{A}_1$. *If, for a point* s *of* S, *the restriction* $\mathscr{A}\otimes\mathcal{O}_{X_s}$ *of* $\mathscr{A}$ *to the fibre* X_s *is* $\kappa(s)$*-ample, there exists an open neighbourhood* U *of* s *such that the restriction* $\mathscr{A}\otimes\mathcal{O}_{f^{-1}(U)}$ *of* $\mathscr{A}$ *is* U*-ample.*

PROOF. By passing to Proj$(\mathscr{A})$ again, we reduce to the case of an invertible sheaf, which is (III. 4. 7. 1).

PROPOSITION 9. *Let* $g: Y\to S$ *be a separated morphism of finite type, and let* $h: X\to Y$ *be a finite (i.e. proper and affine)* S*-morphism. Let* $\mathscr{A}=\mathcal{O}_X\oplus(\bigoplus_{n\geq 1}\mathscr{A}_n)$ *be a coherent graded* $\mathcal{O}_X$*-algebra generated by* $\mathscr{A}_1$. *Then,* $\mathscr{B}=\mathcal{O}_Y\oplus(\bigoplus_{n\geq 1}h_*(\mathscr{A}_n))$ *is a coherent graded* $\mathcal{O}_Y$*-algebra generated by* $\mathscr{B}_1$. *If* $\mathscr{B}$ *is* g*-ample,* $\mathscr{A}$ *is* f*-ample.*

PROOF. Since h is proper, $\mathscr{B}$ is coherent. And since h is affine, $\mathscr{B}$ is generated by $\mathscr{B}_1$. Let $\mathscr{M}=\bigoplus_{n\in Z}\mathscr{M}_n$ be a TF graded $\mathscr{A}$-module, and put $\mathscr{N}=h_*(\mathscr{M})$. If a canonical homomorphism $g^*g_*(\mathscr{N})\to\mathscr{N}$ is TN-surjective, $f^*f_*(\mathscr{M})\to h^*h_*(\mathscr{M})$ is TN-surjective. On the other hand, $h^*h_*(\mathscr{M})\to\mathscr{M}$ is surjective, since h is affine. Hence, if $\mathscr{B}$ is g-ample $f^*f_*(\mathscr{M})\to\mathscr{M}$ is TN-surjective, which proves the assertion.

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