The intermediate logics on the second slice

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In [1], Hosoi defined the notion *slice* and proved that for any intermediate logic $\boldsymbol{M}$ there exists a unique slice $\mathscr{S}_n$ $(n \leqq \omega)$ such that $\boldsymbol{M} \in \mathscr{S}_n$. In this paper, we will give a complete description of the intermediate logics on $\mathscr{S}_2$ and those logics on $\mathscr{S}_3$ that contain the axiom $\neg p \vee \neg\neg p$. We assume a knowledge of the results and the notations in [1] and [2].

For a *model* $\boldsymbol{M}$, $\boldsymbol{M}$ means not only a logic as a set of formulas but a pseudo-Boolean algebra[1]. In [3], McKay proved that any intermediate logic can be represented as an intersection of some intermediate logics of the form $\boldsymbol{S}_1 \uparrow \boldsymbol{M}$ or $\boldsymbol{S}_1$. Using this result, we prove, in § 1, that if $\boldsymbol{M} \in \mathscr{S}_2$ then $\boldsymbol{M} \supset\subset \boldsymbol{S}_1 \uparrow \boldsymbol{S}_1^n$ for some n $(1 \leqq n \leqq \omega)$. Moreover we will give an axiomatization of each logic in $\mathscr{S}_2$. The results in § 2 resemble those in § 1.

§ 1.

We first cite a theorem due to McKay [3] without proof.

THEOREM 1.1 (McKay). *If $\boldsymbol{M}$ is an intermediate logic, then there are models $\boldsymbol{N}_\lambda$ $(\lambda \in \Lambda)$ such that $\boldsymbol{M} \supset\subset \bigcap_{\lambda \in \Lambda} (\boldsymbol{S}_1 \uparrow \boldsymbol{N}_\lambda)$.*

Remark that, if $\boldsymbol{M}$ is a finite model, then Λ and all $\boldsymbol{N}_\lambda$'s are finite. In general, the cardinal of $\boldsymbol{N}_\lambda$ is equal or less than that of $\boldsymbol{M}$.

Now, using the above theorem we investigate the intermediate logics on the second slice $\mathscr{S}_2$. By 1.1, if $\boldsymbol{M} \in \mathscr{S}_2$, then $\boldsymbol{M} \supset\subset \bigcap_{\lambda \in \Lambda} (\boldsymbol{S}_1 \uparrow \boldsymbol{N}_\lambda)$ for some $\boldsymbol{N}_\lambda$'s. Since $\boldsymbol{M} \subset \boldsymbol{S}_1 \uparrow \boldsymbol{N}_\lambda$ for any $\lambda \in \Lambda$, $\boldsymbol{S}_1 \uparrow \boldsymbol{N}_\lambda \in \mathscr{S}_m$ for some $m \leqq 2$ and therefore $\boldsymbol{S}_1 \uparrow \boldsymbol{N}_\lambda \in \mathscr{S}_2$ and $\boldsymbol{N}_\lambda \in \mathscr{S}_1$ by Theorem 6.2 in [1]. This means that each model $\boldsymbol{N}_\lambda$ is a Boolean algebra. Notice that each $\boldsymbol{S}_1^\sigma$ is isomorphic to the field of all subsets of $\{\mu | \mu < \sigma\}$, where σ is any cardinal number. $\boldsymbol{S}_1 \uparrow \boldsymbol{S}_1^\sigma$ is an example of a model in $\mathscr{S}_2$. We write $\boldsymbol{M}_\sigma$ for $\boldsymbol{S}_1 \uparrow \boldsymbol{S}_1^\sigma$ for a cardinal number σ. It is easy to see that if $m < \omega$ then the cardinal of $\boldsymbol{M}_m$ is $2^m + 1$.

Now, we have

LEMMA 1.2. *Let $\boldsymbol{M}$ be a Boolean algebra whose cardinal is σ. Then there exists a subalgebra $\boldsymbol{N}$ of $\boldsymbol{M}$ such that $\boldsymbol{N}$ is isomorphic to $\boldsymbol{S}_1^n$ $(n < \omega)$, if $2^n \leqq \sigma$.*

COROLLARY 1.3. *If $m < \omega$ and $m < n$, then $\boldsymbol{M}_n \subsetneqq \boldsymbol{M}_m$. ($n$ may be infinite.)*

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1) That is, $A \in \boldsymbol{M}$ means that a formula A is valid in $\boldsymbol{M}$. But $a \in \boldsymbol{M}$ means that a is an element of a pseudo-Boolean algebra $\boldsymbol{M}$. As for pseudo-Boolean algebras, see e.g. [5].

PROOF. By 1.2, there is a subalgebra N of S_1^n such that N is isomorphic to S_1^m. So M_m is isomorphic to a subalgebra of M_n. Thus $M_n \subset M_m$. Now let X_n be the formula $\bigvee_{1 \leq i < j \leq n+1} (q_i \equiv q_j)$[2]. Then by Corollary 6.10 in [1], $X_{2^m+1} \in M_m$ but $X_{2^m+1} \notin M_n$. For $r(M_n) \geqq r(M_{m+1}) = 2^{m+1}+1 > 2^m+1$. Thus $M_m \supsetneqq M_n$.

LEMMA 1.4. *If σ is an infinite cardinal number, then* $\bigcap_{m<\omega} M_m \subset M_\sigma$.

PROOF. Suppose that a formula A is not in M_σ. Then, there is an assignment f of M_σ such that $f(A) \neq 1_{M_\sigma}$[3]. Let $\{p_1, \cdots, p_n\}$ be the set of all propositional variables appearing in A. We write Q for the set $\{f(p_i) | f(p_i) \neq 1_{M_\sigma}\ 1 \leqq i \leqq n\}$. Let Q^* be the subalgebra of S_1^σ generated by Q. Since Q is finite, Q^* is also finite. So Q^* is isomorphic to S_1^m for some $m < \omega$. Thus f can be considered as an assignment of $S_1 \uparrow S_1^m$, or M_m. But $f(A) \neq 1_{M_\sigma} = 1_{M_m}$. So, A is not in M_m for some $m < \omega$.

COROLLARY 1.5. *Let M be a countably infinite Boolean algebra. Then* $S_1 \uparrow M \supset\subset M_\omega$.

PROOF. We can show that $M_\omega \supset\subset \bigcap_{m<\omega} M_m$ by 1.2 and 1.4. So $M_\omega \supset\subset \bigcap_{m<\omega} M_m \supset S_1 \uparrow M$ by 1.2. By the Stone representation theorem of Boolean algebra, M is isomorphic to a subalgebra of S_1^σ for some infinite cardinal number σ, since M is countably infinite. So $S_1 \uparrow M \supset S_1 \uparrow S_1^\sigma = M_\sigma \supset \bigcap_{m<\omega} M_m \supset\subset M_\omega$. Therefore $S_1 \uparrow M \supset\subset M_\omega$.

THEOREM 1.6. *If $M \in \mathscr{S}_2$, then $M \supset\subset M_n$ for some n $(1 \leqq n \leqq \omega)$. Hence $\mathscr{S}_2$ is linearly ordered by* $\subset$.

PROOF. Let M be in $\mathscr{S}_2$. We can take a countable model M_0 such that $M \supset\subset M_0$. By 1.1 and its remark, $M_0 \supset\subset \bigcap_{\lambda \in \Lambda} (S_1 \uparrow N_\lambda)$ for some N_λ's in $\mathscr{S}_1$ such that each N_λ is countable. If N_λ is finite, then $S_1 \uparrow N_\lambda = M_m$ for some $m < \omega$. Otherwise, $S_1 \uparrow N_\lambda \supset\subset M_\omega$ by 1.5. So, $M \supset\subset \bigcap_{\lambda \in \Lambda} M_{m_\lambda}$, where $1 \leqq m_\lambda \leqq \omega$. Let n be $\sup\{m_\lambda | \lambda \in \Lambda\}$. Then $1 \leqq n \leqq \omega$ and $\bigcap_{\lambda \in \Lambda} M_{m_\lambda} \supset\subset M_n$. Hence $M \supset\subset M_n$.

Next we will give an axiomatization of each logic in $\mathscr{S}_2$. Let T_k $(1 \leqq k \leqq \omega)$ be an intermediate logic obtained by adding two axiom schemata P_2 and X_k to the intuitionistic propositional logic L. (We write $T_k = L + P_2 + X_k$, following the notation in [1].) Let f be a function on natural numbers such that $f(k) = 2^k + 1$.

THEOREM 1.7. $M_k \supset\subset T_{f(k)}$ *if* $1 \leqq k < \omega$, *and* $M_\omega \supset\subset LP_2$.

PROOF. By Corollary 4.7 in [1], LP_2 is the minimal element in $\mathscr{S}_2$. So, $M_\omega \supset\subset LP_2$ by 1.6 and 1.3. By the proof of 1.3, $X_{f(k)} \in M_k$. Since $M_k \in \mathscr{S}_2$, $P_2 \in M_k$ and hence $T_{f(k)} \subset M_k$. By 1.6, $T_{f(k)} \supset\subset M_m$ for some m $(k \leqq m \leqq \omega)$. But

2) See [1].

3) 1_M denotes the designated element of a regular model M.

if $m>k$, then $r(M_m)=f(m)>f(k)$ hence $X_{f(k)}\in M_m$. Thus $m=k$. That is, $T_{f(k)}\supset\subset M_k$. We remark that if $f(k)\leqq m<f(k+1)$ then $T_m\supset\subset M_k$.

We next show that each logic $\mathscr{S}_2$ is normalizable (that is, X_k can be interdeducibly expressed by a formula only containing the logical operator $\rightarrow$)[4]. Define Q_{ij} by $Q_{ij}=(q_i\rightarrow q_j)\rightarrow((q_j\rightarrow q_i)\rightarrow q_0)$, and Y_n by $Y_n=Q_{12}\rightarrow(Q_{13}\rightarrow\cdots\rightarrow(Q_{1.n+1}\rightarrow (Q_{23}\rightarrow(Q_{24}\rightarrow\cdots\rightarrow(Q_{n.n+1}\rightarrow q_0)\cdots)))\cdots)$. Then it is easy to see that X_n and Y_n are interdeducible in L. Hence $M_k\supset\subset L+P_2+X_{f(k)}\supset\subset L+P_2+Y_{f(k)}$. Since both P_2 and Y_n contain only the connective $\rightarrow$, we have the following

COROLLARY 1.8. *Each intermediate logic in $\mathscr{S}_2$ is normalizable.*

§ 2.

Logics with the axiom $\neg p\vee\neg\neg p$ have been partially studied in [2]. Here we will show models for those logics on $\mathscr{S}_3$. (It has been known that only S_1 and S_2 are those in $\mathscr{S}_1\cup\mathscr{S}_2$ that contain $\neg p\vee\neg\neg p$).

We remark first that $S_1\uparrow S_1^m\uparrow S_1\in\mathscr{S}_3$ and $\neg p\vee\neg\neg p\in S_1\uparrow S_1^m\uparrow S_1$ for any cardinal number m (see 4.8 in [2]).

LEMMA 2.1. *If $\neg p\vee\neg\neg p\in S_1\uparrow N$ then for any a, $b\in S_1\uparrow N$ $a\wedge b<\omega$ if $a<\omega$ and $b<\omega$.*

PROOF. Suppose that $a<\omega$, $b<\omega$ and $a\wedge b=\omega$ for some a, $b\in S_1\uparrow N$. Let f be an assignment of $S_1\uparrow N$ such that $f(p)=a$. By the above assumption, $1_N< f(\neg p)\leqq b<\omega$. So $1_N<f(\neg\neg p)$ and $1_{S_1\uparrow N}\neq 1_N\leqq f(\neg p\vee\neg\neg p)$. This contradicts $\neg p\vee\neg\neg p\in S_1\uparrow N$.

We can prove the following lemma similarly as 1.3.

LEMMA 2.2. *If $m<n\leqq\omega$ then $S_1\uparrow S_1^m\uparrow S_1\supsetneqq S_1\uparrow S_1^n\uparrow S_1$.*

LEMMA 2.3. *Let N be a model in $\mathscr{S}_2$ such that $\neg p\vee\neg\neg p\in S_1\uparrow N$. Then $\bigcap_{m<\omega}(S_1\uparrow S_1^m\uparrow S_1)\subset S_1\uparrow N$. Furthermore, if N is infinite then $\bigcap_{m<\omega}(S_1\uparrow S_1^m\uparrow S_1)\supset\subset S_1\uparrow N$.*

PROOF. In this proof we write 2 for the minimal element 1_N of N. At first, we show that the relation

$$\forall a,\ b\ (\omega>b>a>2\rightarrow a\vee(a\supset b)=2)$$

holds in N. Assume that there exist a and b such that $\omega>b>a>2$ and $a\vee(a\supset b)>2$. We write c for $a\vee(a\supset b)$. Define an assignment g by $g(p_0)=\omega$, $g(p_1)=b$ and $g(p_2)=c$. Then $f(P_2)=c>2$ where $P_2=((p_2\rightarrow(((p_1\rightarrow p_0)\rightarrow p_1)\rightarrow p_1))\rightarrow p_2)\rightarrow p_2$. This contradicts that $N\in\mathscr{S}_2$. Now, for any fixed b such that $\omega>b>2$, we can show that the sublattice $\{a|b\geqq a\geqq 2\}$ of N is a Boolean algebra. Suppose that a

[4] See [4].

formula $A \notin S_1 \uparrow N$. Then there is an assignment f such that $f(A) \neq 1$. Let Q be the finite set $\{f(p_i) | 1 \leqq i \leqq m\} \cap (N - \{\omega\})$, where p_i's are all the propositional variables appearing in A. Let $b = \bigwedge_{a \in Q} a$. Since $\neg p \vee \neg\neg p \in S_1 \uparrow N$ and each $a \in Q$ is smaller than ω, we can show that $b < \omega$ by 2.1. Suppose that $b > 2$. We write $\boldsymbol{M}$ for the Boolean algebra $\{a | b \geqq a \geqq 2\}$ and $\boldsymbol{M}^*$ for the subalgebra of $\boldsymbol{M}$ generated by Q[5]. Since Q is finite, so is $\boldsymbol{M}^*$. Therefore $\boldsymbol{M}^*$ is of the form S_1^m $(m < \omega)$. Then it is easy to see that f is also an assignment of $S_1 \uparrow \boldsymbol{M}^* \uparrow S_1$. So, $A \notin S_1 \uparrow \boldsymbol{M}^* \uparrow S_1$ and $A \notin \bigcap_{m<\omega} (S_1 \uparrow S_1^m \uparrow S_1)$. If $b = 2$ then we can show that f can also be regarded as an assignment of S_3 and hence $A \notin S_3$ and $A \notin \bigcap_{m<\omega} (S_1 \uparrow S_1^m \uparrow S_1)$. Suppose N is infinite. Then we prove that $S_1 \uparrow S_1^m \uparrow S_1$ is a subalgebra of $S_1 \uparrow N$ for any $m < \omega$. If N is atomic, then there is only one atom by 2.1.[6] So N is of the form $N_0 \uparrow S_1$ where N_0 is an infinite Boolean algebra. By 1.2, $S_1 \uparrow S_1^m \uparrow S_1$ is a subalgebra of $S_1 \uparrow N_0 \uparrow S_1$. On the other hand, suppose N is not atomic. Now we can take a $(< \omega)$ in N such that there exists an ascending sequence of elements a_i in N. That is, $a < a_1 < a_2 \cdots < \omega$. So, for any $m < \omega$, there is a subalgebra of N isomorphic to $S_1^m \uparrow S_1$ since $\{b | a_i \geqq b \geqq 2\}$'s are Boolean algebras. Therefore $S_1 \uparrow N \subset S_1 \uparrow S_1^m \uparrow S_1$ for any $m < \omega$. This implies $S_1 \uparrow N \subset \bigcap_{m<\omega} (S_1 \uparrow S_1^m \uparrow S_1)$.

COROLLARY 2.4. $\bigcap_{m<\omega} (S_1 \uparrow S_1^m \uparrow S_1) \supset\subset S_1 \uparrow S_1^\omega \uparrow S_1$.

THEOREM 2.5. *Let* $\boldsymbol{M}$ *be a model in* $\mathscr{S}_3$ *such that* $\boldsymbol{M} \ni \neg p \vee \neg\neg p$. *Then there exists* m $(\leqq \omega)$ *such that* $\boldsymbol{M} \supset\subset S_1 \uparrow S_1^m \uparrow S_1$.

PROOF. By 1.1, $\boldsymbol{M} \supset\subset \bigcap_{\lambda \in \Lambda} (S_1 \uparrow N_\lambda)$. Suppose first that all N_λ's are finite. Then all N_λ's are atomic. As the proof of 2.3, each $S_1 \uparrow N_\lambda$ is of the form $S_1 \uparrow \boldsymbol{M}_\lambda \uparrow S_1$. Since $\boldsymbol{M}_\lambda \in \mathscr{S}_1$ and $\boldsymbol{M}_\lambda$ is finite, $\boldsymbol{M}_\lambda$ is of the form $S_1^{m_\lambda} (m_\lambda < \omega)$. Let $m = \sup\{m_\lambda | \lambda \in \Lambda\}$. Then by 2.2, $\boldsymbol{M} \supset\subset S_1 \uparrow S_1^m \uparrow S_1$. Suppose next that there is an infinite N_μ. Then by 2.3, $\bigcap_{m<\omega} (S_1 \uparrow S_1^m \uparrow S_1) \supset\subset S_1 \uparrow N_\mu \supset \boldsymbol{M}$. If $A \notin \boldsymbol{M}$ then there is an N_λ such that $A \notin S_1 \uparrow N_\lambda$. Then there is $m < \omega$ such that $A \notin S_1 \uparrow S_1^m \uparrow S_1$ whether N_λ is finite or not. So $\boldsymbol{M} \supset \bigcap_{m<\omega} (S_1 \uparrow S_1^m \uparrow S_1)$.

We know that the class $\{\boldsymbol{M} | \neg p \vee \neg\neg p \in \boldsymbol{M}, \boldsymbol{M} \in \mathscr{S}_3\}$ of logics on $\mathscr{S}_3$ is linearly ordered by $\supset$, by 2.2 and 2.5.

In [2], $\boldsymbol{LQ}_3$ is defined as $\boldsymbol{L} + Q + P_3$ where Q is a formula interdeducible with $\neg p \vee \neg\neg p$. Now it is easy to verify

THEOREM 2.6. $\boldsymbol{LQ}_3 \supset\subset S_1 \uparrow S_1^\omega \uparrow S_1$ *and* $\boldsymbol{LQ}_3 + X_{g(k)} \supset\subset S_1 \uparrow S_1^k \uparrow S_1$ *if* $k < \omega$, *where* $g(k) = 2^k + 2$.

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5) Remark that if $a < \omega$ then $\neg a = \omega$ by 2.1.

6) We say an element a in N is an atom, if $b \geqq a$ implies either $b = a$ or $b = \omega$. If for any $b < \omega$ there is an atom a such that $b < a$, we say N is atomic.

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