On non-homogeneous Siegel domains

By Masaki SUDO
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§1. **Introduction.** The purpose of this note is to make supplementary comments to our former paper [2]. Since we consider here exclusively Siegel domains $\mathscr{D}(V, F)$ of second kind as in [2], we refer to them omitting the specification "of second kind". We note that, due to a recent result [3], a Siegel domain $\mathscr{D}(V, F)$ is homogeneous under the group $\mathscr{G}_h$ of all holomorphic automorphisms of $\mathscr{D}(V, F)$ if and only if $\mathscr{D}(V, F)$ is homogeneous under the group $\mathscr{G}_a$ of all affine automorphisms of $\mathscr{D}(V, F)$. A Siegel domain $\mathscr{D}(V, F)$ is called homogeneous in the following if $\mathscr{G}_a$ is transitive on $\mathscr{D}(V, F)$.

Now we have proved in [2] that, for a homogeneous Siegel domain $\mathscr{D}(V, F)$, the group $\mathscr{G}_a$ coincides with the affine automorphism $\mathscr{G}(S)$ of the Šilov boundary S if and only if $\mathscr{D}(V, F)$ is non-degenerate. Futhermore we have proved in [2] that, without assuming the homogeneity of $\mathscr{D}(V, F)$, $\mathscr{G}(S) = \mathscr{G}_a$ implies that $\mathscr{D}(V, F)$ is non-degenerate.

In this note, after the preliminaries on terminologies in §2, we will show in §3 that one can always associate a non-homogeneous Siegel domain to every non-degenerate homogeneous Siegel domain (Proposition 1). However, the construction used in Proposition 1 may associate a homogeneous Siegel domain if one starts from a degenerate homogeneous Siegel domain. Such an example is given in §3 (Example 1).

Finally, we will show that the converse of the implication stated above, namely the implication

$$\mathscr{G}(S) = \mathscr{G}_a \Longrightarrow \mathscr{D}(V, F) \quad \text{is non-degenerate}$$

is false, by constructing a non-homogeneous and non-degenerate Siegel domain $\mathscr{D}(V, F)$ such that $\mathscr{G}(S) \supsetneqq \mathscr{G}_a$.

§2. **Preliminaries.** Let R be a real vector space of dimension n, R_C its complexification and V an open convex cone with vertex 0 in R which contains no entire straight lines. (For brevity such a cone will be called a *convex cone* in R.) Let W be a complex vector space of dimension $m(\geqq 1)$. A hermitian mapping F of $W \times W$ into R_C is called a *V-hermitian form*, if the following two conditions are satisfied;

1) $F(u, u) \in \bar{V}$ for all $u \in W$. ($\bar{V}$ means the closure of V.)

2) $F(u, u) \neq 0$ for all $u \neq 0$.

Let V be a convex cone and F a V-hermitian form. Then we define a Siegel domain $\mathscr{D}(V, F)$ in the complex vector space $R_C \times W$ by putting

$$\mathscr{D}(V, F)=\{(x+\sqrt{-1}\,y, u) \in R_C \times W ; y-F(u, u) \in V\} .$$

For a Siegel domain $\mathscr{D}(V, F)$ in $R_C \times W$, we know that its Šilov boundary S is the subset of $R_C \times W$ defined by

$$S=\{(x+\sqrt{-1}\,y, u) ; y-F(u, u)=0\} .$$

(See [1], [2]).

We know also that an affine transformation g of $R_C \times W$ belongs to the affine automorphism group $\mathscr{G}(S)$ of S, if and only if g is of the form

$$\begin{cases} z \to Az+a+2\sqrt{-1}\,F(Bu, c)+\sqrt{-1}\,F(c, c) \\ u \to Bu+c \end{cases}$$

where $a \in R$, $c \in W$, A is an $\boldsymbol{R}$-linear transformation of R and B is a $\boldsymbol{C}$-linear transformation of W such that

$$AF(u, u)=F(Bu, Bu)\ , \quad \text{for all} \quad u \in W .$$

We denote the set of all such transformations A, B by $\tilde{\mathscr{A}}$, $\mathscr{B}$ respectively. In particular, an element $g \in \mathscr{G}(S)$ belongs to the group $\mathscr{G}_a$ of affine automorphisms of $\mathscr{D}(V, F)$, if and only if A is an automorphism of the cone V. We denote the set of all such transformations A by $\mathscr{A}$ ([1]).

DEFINITION. A Siegel domain $\mathscr{D}(V, F)$ is called *non-degenerate* if the $\boldsymbol{R}$-linear closure of the set $\{F(u, u) ; u \in W\}$ coincides with R. Otherwise $\mathscr{D}(V, F)$ is called *degenerate*.

§ 3. Non-homogeneous Siegel domains.

PROPOSITION 1. *Let $\mathscr{D}(V, F)$ be a non-degenerate homogeneous Siegel domain in $R_C \times W$. Let V' be a convex cone in R with a common vertex 0 which properly contains V. Then $\mathscr{D}(V', F)$ is a non-homogeneous Siegel domain.*

PROOF. By the choice of V', $\mathscr{D}(V', F)$ contains properly $\mathscr{D}(V, F)$ and the Šilov boundary S' of $\mathscr{D}(V', F)$ coincides with the Šilov boundary S of $\mathscr{D}(V, F)$. If $\mathscr{D}(V', F)$ were homogeneous, we must have $\mathscr{G}(S)=\mathscr{G}_a=\mathscr{G}'_a$ by the above cited result. Thus $\mathscr{D}(V', F)$, which is the $\mathscr{G}'_a$-orbit through an interior point of $\mathscr{D}(V, F)$, coincides with $\mathscr{D}(V, F)$. This is a contradiction. Therefore $\mathscr{D}(V', F)$ is non-homogeneous. q.e.d.

The following example shows that in degenerate case above Proposition 1 is not always valid.

EXAMPLE 1. Put $W=C_1$, $R=R^3$, $V=\{(y_1, y_2, y_3); y_1>0, y_1\cdot y_2-y_3^2>0\}$, $V'=\{(y_1, y_2, y_3); y_1>0, 2y_1\cdot y_2-y_3^2>0\}$, $F(u, v)=(u\bar{v}, 0, 0)$ for any $u, v\in W$. Then $\mathscr{D}(V, F)$ and $\mathscr{D}(V', F)$ are degenerate Siegel domains and we know that $\mathscr{D}(V, F)$ is homogeneous [1]. But there is an affine isomorphism between $\mathscr{D}(V, F)$ and $\mathscr{D}(V', F)$ by sending (z_1, z_2, z_3, u) to $\left(\frac{1}{2}z_1, z_2, z_3, \frac{1}{\sqrt{2}}u\right)$. Therefore $\mathscr{D}(V', F)$ is also homogeneous.

In general we have

PROPOSITION 2. *Let $\mathscr{D}(V, F)$, $\mathscr{D}(V', F)$ be Siegel domains in the same space $R_C\times W$ such that $\mathscr{D}(V, F)$ is homogeneous. Suppose that there exists a linear isomorphism $\phi: V\to V'$ belonging to $\widetilde{\mathscr{A}'}$. (Thus ϕ also belongs to $\widetilde{\mathscr{A}}=\widetilde{\mathscr{A}'}$). Then $\mathscr{D}(V', F)$ is also homogeneous.*

PROOF. As we have $\phi(V)=V'$ and $\phi\in\widetilde{\mathscr{A}'}$ by assumption, there exists φ belonging to $\mathscr{B}$ such that the relation $\phi F(u, u)=F(\varphi(u), \varphi(u))$ is valid for all $u\in W$. By sending $(x+\sqrt{-1}\,y, u)$ to $(\phi(x)+\sqrt{-1}\,\phi(y), \varphi(u))$, we get an affine injection from $\mathscr{D}(V, F)$ into $\mathscr{D}(V', F)$. Also we have an affine injection from $\mathscr{D}(V', F)$ into $\mathscr{D}(V, F)$ by sending $(x'+\sqrt{-1}\,y', u')$ to $(\phi^{-1}(x')+\sqrt{-1}\,\phi^{-1}(y'), \varphi^{-1}(u'))$. Now it is easy to see that the pair (ϕ, φ) gives an affine equivalence between $\mathscr{D}(V, F)$ and $\mathscr{D}(V', F)$. Hence $\mathscr{D}(V', F)$ is homogeneous. q.e.d.

In the following example, we shall construct a non-homogeneous and non-degenerate Siegel domain $\mathscr{D}(V, F)$ such that $\mathscr{G}(S)$ properly contains $\mathscr{G}_a$.

EXAMPLE 2. Put $W=C^2$, $R=R^3$, $V=\{(y_1, y_2, y_3); y_1>0, y_1\cdot y_2-y_3^2>0\}$, $V'=\{(y_1, y_2, y_3); y_1>0, 2y_1\cdot y_2-y_3^2>0\}$, $F(u, v)=\left(u_1\bar{v}_1, u_2\bar{v}_2, \frac{1}{2}(u_1\bar{v}_2+u_2\bar{v}_1)\right)$ for any $u=(u_1, u_2)$, $v=(v_1, v_2)\in W$. Then we know that $\mathscr{D}(V, F)$ is non-degenerate and homogeneous ([1], [2]) and so by Proposition 1, $\mathscr{D}(V', F)$ is a non-degenerate and non-homogeneous Siegel domain. For $A=\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, $B=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, the relation $A^t(|u_1|^2, |u_2|^2, \mathrm{Re}\, u_1\bar{u}_2)=F(B^t u, B^t u)$ is valid for all $u=(u_1, u_2)\in W$. Therefore $A\in\widetilde{\mathscr{A}'}(=\widetilde{\mathscr{A}}$ in this case). If we take $\left(1, \frac{3}{5}, -1\right)\in V'$, we have $A^t\left(1, \frac{3}{5}, -1\right)={}^t\left(-\frac{2}{5}, \frac{3}{5}, -\frac{2}{5}\right)\notin V'$. Therefore this Siegel domain $\mathscr{D}(V', F)$ is non-degenerate and non-homogeneous and we have $\mathscr{G}(S)\supsetneqq\mathscr{G}_a$.

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Department of Mathematics
Tokyo Metropolitan University
Fukazawa, Setagaya-ku, Tokyo
158 Japan